Fujita, H.
Osaka J. Math.
55 (2018), 619–645

A DANILOV-TYPE FORMULA FOR TORIC ORIGAMI MANIFOLDS VIA LOCALIZATION OF INDEX

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(Received May 16, 2016, revised March 30, 2017)

Abstract
We give a direct geometric proof of a Danilov-type formula for toric origami manifolds by using the localization of Riemann-Roch number.

Contents

2010 Mathematics Subject Classification. Primary 53D50; Secondary 53C27, 58J20, 57S25.
Partly supported by Grant-in-Aid for Young Scientists (B) 26800045.

1. Introduction

A *symplectic toric manifold* is a symplectic manifold on which a half dimensional torus T acts in an effective Hamiltonian way. A famous theorem of Delzant [6] says that there is one-to-one correspondence between the set of (compact connected) symplectic toric manifolds and the set of simple polytopes called *Delzant polytopes* (see [13]) via moment maps. Therefore, several properties of symplectic toric manifolds, such as the symplectic volume and the ring structure of the (equivariant) cohomology and so on, can be detected from the Delzant polytopes. In view of the geometric quantization of symplectic manifolds we are interested in the *Riemann-Roch numbers*. The Riemann-Roch number $RR(M, L)$ is an invariant of a compact symplectic manifold (M, ω) with a *pre-quantizing line bundle* (L, ∇), a pair consisting of a Hermitian line bundle L and a Hermitian connection ∇ whose curvature form is equal to $-\sqrt{-1}\omega$, which is defined as follows. We fix an ω-compatible almost complex structure and then it determines a spin^c-structure of M and we have a spin^c-Dirac operator D with coefficients in L. We define an integer $RR(M, L)$ as the analytic index of the spin^c-Dirac operator:

$$RR(M, L) := \mathrm{ind}(D).$$

If a compact Lie group G acts on M preserving all the data, ω, (L, ∇) and D, then the index becomes a virtual representation of G, an element of the character ring $R(G)$. In this case the Riemann-Roch number is called the *Riemann-Roch character* or the *G-equivariant Riemann-Roch number* and is denoted by $RR_G(M, L)$. Such a procedure is called *spin^c-quantization* ([4][7][19]) nowadays and considered as a quantization of spin^c-manifolds. When (X, ω) is a symplectic toric manifold with the action of a torus T we can choose an almost complex structure so that it is integrable and invariant under the action of the torus T. Then L has a structure of a holomorphic line bundle and the Riemann-Roch number is equal to the dimension of $H^0(X, L)$, the space of holomorphic sections of L. Moreover when we consider a lift of the torus action to the pre-quantizing line bundle, $RR_T(X, L) = H^0(X, L)$ becomes a representation of the torus T. Classical theorem of Danilov [5] says that the representation $RR_T(X, L)$ can be described in terms of the integral points in the Delzant polytope. Precisely we have

$$RR_T(X, L) = \bigoplus_{\xi \in \mu(M) \cap \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi)}, \tag{1.1}$$

where μ is the moment map, $\mathfrak{t}^*_{\mathbb{Z}}$ is the integral weight lattice in the dual of the Lie algebra of T and $\mathbb{C}_{(\xi)}$ is the representation of the torus associated with the integral weight $\xi \in \mathfrak{t}^*_{\mathbb{Z}}$. Though Danilov's original proof was based on an algebraic geometric setting, a proof in the symplectic geometric setting is also known. See [14] for example.

A *folded symplectic manifold* introduced by Cannas da Silva, Guillemin and Woodward in [3] is a pair consisting of an even-dimensional smooth manifold and a closed 2-form which may degenerate in a transverse way and it is called the *folded symplectic form*. When the degenerate locus (which becomes a hypersurface and called the *fold*) has a structure of a circle bundle whose vertical tangent bundle coincides with the degenerate direction of the folded symplectic form, the folded symplectic manifold is called an *origami manifold*. By definition a folded symplectic manifold (resp. origami manifold) is a generalization of a symplectic manifold, and several notions and studies in symplectic geometry are generalized to the folded symplectic (resp. origami) case, such as pre-quantizing line bundle, Hamiltonian group action, moment map, convexity property and so on. It is known that a folded symplectic manifold is not orientable in general, and hence it does not admit an almost complex structure, however, if it is orientable, then it admits a stable almost complex structure as shown in [3, Theorem 2]. Since the stable almost complex structure determines a spin^c-structure, we can define its spin^c-quantization by the index of spin^c-Dirac operator. If the folded symplectic manifold is equipped with a Hamiltonian group action, then it becomes a virtual representation and is also called the Riemann-Roch character. In particular the spin^c-quantization of a toric origami manifold is a virtual representation of the torus.

In this paper we give a proof of the following generalization of Danilov's formula (1.1) for spin^c-quantization of toric origami manifolds by making use of the localization theorem of index developed in [10, 11].

Theorem (Theorem 5.9)**.** *Let (M,ω) be an oriented toric origami manifold with the action of a torus T and a T-equivariant pre-quantizing line bundle (L,∇). Then we have*

$$RR_T(M,L) = \bigoplus_{\xi\in\mu(M^+)\cap\mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi)} - \bigoplus_{\xi\in\mu(M^-)\cap\mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi)}$$

as elements in the character ring of T.

Precise statement and notations are explained in the subsequent sections. The formula itself can be obtained as a consequence of the cobordism theorem [2, Theorem 4.1] and Danilov's formula (1.1) for symplectic toric manifolds. There is an another possible approach which uses the theory of *multi-fans* introduced by Hattori and Masuda [15]. Masuda and Park showed in [18] that one can associate a multi-fan for each oriented toric origami manifold. In view of the theory of multi-fans the above formula can be considered as a special case of the equivariant index formula [15, Theorem 11.1], which is based on the fixed point formula. In contrast to these proofs, our proof is direct and geometric, which detects the contribution of each lattice point directly. Once we construct a geometric structure which we call an *acyclic compatible system* on an open subset of the manifold, then the index of Dirac operator is localized at the complement of the open subset by the localization formula in [11]. In this paper we construct an acyclic compatible system on the complement of the inverse image of the lattice points and the fold for toric origami manifolds. It implies that the Riemann-Roch character is equal to the sum of contributions of the lattice points and the fold. We show that the contribution of the lattice point ξ is equal to $\mathbb{C}_{(\xi)}$ with sign determined by the orientation and the contribution of the fold is zero. Our proof does not rely on neither the original Danilov's formula nor the fixed point formula. In fact, as a special case, our proof gives a new direct proof of Danilov's formula for symplectic toric manifolds. Note that there is an another generalization of the formula (1.1) by Karshon and Tolman [17]. They gave a formula for toric manifolds with a torus invariant *presymplectic form*. Though their proof is based on the holomorphic structure of toric manifolds, our proof does not use such rigid structure and it is topological and flexible.

This paper is organized as follows. In Section 2 we summarize several known facts about folded symplectic manifolds, origami manifolds and toric origami manifolds, which we use in this paper. The convexity theorem for toric origami manifolds (Theorem 2.5) is essential for us. In Section 3 we discuss stable almost complex structures on folded symplectic manifolds. We construct a $\mathbb{Z}/2$-graded Clifford module bundle in terms of the stable almost complex structure. In Section 4 we construct a structure of *(good) compatible fibration* on toric origami manifolds, which is a family of torus fibrations (foliations) with specific compatibility condition introduced in [11]. The construction is based on an open covering of the convex polytope associated with the natural stratification of the polytope with respect to the dimension of the faces. Strictly speaking there exist *cracks* on which we can not extend the compatible fibration keeping the compatibility condition. Though the crack causes an extra contribution to the Riemann-Roch character, we show that it is equal to 0. In Section 5 we construct a *compatible system* on the compatible fibration of the toric origami manifolds, which is a family of Dirac-type operators along the fibers of the compatible fibration with specific anti-commutativity. In [11] the authors had already constructed compatible system for Hamiltonian torus manifolds, and our construction for the complement of the fold is based on that. On the other hand a neighbourhood of the fold has a structure of a quotient of the product of the fold and the cylinder with the standard folded symplectic structure by a natural S^1-action. We use this structure to define the Dirac-type operator along fibers near the fold. To discuss the localization it is essential to investigate the *acyclicity* of the compatible system. The fundamental property of the moment map says that it is acyclic outside the lattice points and the fold. In Section 5.2 we explain the localization formula of the Riemann-Roch character by making use of the acyclic compatible system. In Section 6 we compute the local contributions of the crack, lattice points and the fold. We first consider the symplectic toric case, i.e., origami manifolds with the empty fold, and compute the local contribution. We use a decomposition of a neighbourhood of the fiber, the inverse image of a lattice point, into the product of the cotangent bundle of the fiber and the normal direction of the symplectic submanifold containing the fiber. We apply the product formula ([11, Theorem 8.8]) to the neighbourhood of the fiber. The vanishing of the contribution from the fold follows from the product structure of a neighbourhood of the fold. The last three sections are appendixes. In Appendix A we give a brief summary of the theory of local index following [11, 12] and [9]. In Appendix B we show a useful formula of local indices of vector spaces, which will be essential in the proof of Lemma 6.2 and Lemma 6.3. In Appendix C we give a direct computation of the local index of the folded cylinder and show that it is equal to 0. We use this result to show that vanishing of the contribution from the fold.

2. Folded symplectic forms and toric origami manifolds

2.1. Folded symplectic forms and origami manifolds. In this section we recall basic definitions and facts on folded symplectic manifolds and origami manifolds. Details can be found in [2], [3], [16] and [18].

A folded symplectic form ω on a smooth $2n$-dimensional manifold M is a closed 2-form whose top power ω^n vanishes transversally on a submanifold Z and whose restriction to Z has maximal rank. In this case Z is a hypersurface in M and is called the *folding hypersurface* or *fold*. The pair (M, ω) is called a *folded symplectic manifold* and the 2-form ω is called a *folded symplectic form*. Let $i_Z : Z \hookrightarrow M$ be the inclusion of Z into M. The restriction $i_Z^*\omega$ determines a line field on Z, called the *null foliation*, whose fiber at $z \in Z$ is $\ker(i_Z^*\omega_z)$.

Suppose that (M, ω) is an oriented folded symplectic manifold with non-empty fold Z.

Then $M \setminus Z$ is not connected and has a decomposition $M \setminus Z = M_+ \sqcup M_-$, where M_+ (resp. M_-) is the union of connected components such that $\omega^n|_{M_+}$ agrees (resp. disagrees) with the given orientation of M.

Definition 2.1. A folded symplectic manifold (M, ω) is called an *origami manifold* if the null foliation $\ker(i_Z^*\omega)$ is the vertical tangent bundle of a principal S^1-bundle structure $\pi : Z \to B$ over Z with a compact base B.

Note that since B is compact the total space Z is also compact. As in the symplectic reduction procedure, there is the unique symplectic form ω_B on B satisfying $\pi^*\omega_B = i_Z^*\omega$. An analogue of Darboux's theorem for folded symplectic forms says that near any point $p \in Z$ there exists a coordinate chart centered at p where the folded symplectic form ω can be written as

$$x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \cdots + dx_n \wedge dy_n.$$

In this local description, the fold Z is given by the equation $x_1 = 0$ and the null foliation is the line field spanned by $\frac{\partial}{\partial y_1}$. This local description has a global variant.

Theorem 2.2 (Theorem 1 in [3]). *Let (M, ω) be an oriented origami manifold with fold $Z \to B$. Fix a connection 1-form α of $Z \to B$. Then there exists a neighbourhood $\mathcal{U}$ of Z and an orientation preserving diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \to \mathcal{U}$ such that*

$$\varphi \circ \iota_0 = \iota_Z$$

and

$$\varphi^*\omega = p_Z^*\iota_Z^*\omega + d(t^2 p_Z^*\alpha),$$

where $\iota_0 : Z \to Z \times (-\varepsilon, \varepsilon)$ is the inclusion $z \mapsto (z, 0)$ and $p_Z : Z \times (-\varepsilon, \varepsilon) \to Z$ is the natural projection.

Example 2.3. For a positive integer n let S^{2n} be the unit sphere in $\mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{C}^n \oplus \mathbb{R}$ with coordinates $x_1, y_1, \cdots, x_n, y_n, h$. Let ω be the restriction to S^{2n} of the 2-form $dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$ on $\mathbb{R}^{2n} \oplus \mathbb{R}$. Then ω is a folded symplectic form on S^{2n} with the fold S^{2n-1}, the equator sphere given by $h = 0$. The Hopf fibration $S^1 \hookrightarrow S^{2n-1} \to \mathbb{C}\mathrm{P}^{n-1}$ gives a structure of origami manifold on (S^{2n}, ω).

2.2. Hamiltonian torus actions and toric origami manifolds. The action of a compact Lie group G on an origami manifold (M, ω) is called *Hamiltonian* if it admits a moment map μ, that is, a map $\mu : M \to \mathfrak{g}^* = \mathrm{Lie}(G)^*$ satisfying the conditions :

• μ is equivariant with respect to the given action of G on M and the coadjoint action of G on $\mathfrak{g}^*$.

• for any $v \in \mathfrak{g}$ we have $d\langle \mu, v \rangle = \iota(v^M)\omega$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ and $\iota(v^M)\omega$ is the contraction of ω by the induced fundamental vector field v^M.

Definition 2.4. A Hamiltonian torus origami manifold (M, ω, T, μ) (or M for short) is a connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus T with a choice of a corresponding moment map μ. If the dimension of the torus T is half of that of M, then we call (M, ω, T, μ) a *toric origami manifold*.

If the fold Z is empty, a Hamiltonian torus origami manifold is a Hamiltonian torus manifold in the usual sense. The following is an origami analogue of the famous convexity theorem for Hamiltonian torus manifolds.

Theorem 2.5 (Theorem 3.2 in [2]). *Let (M, ω, T, μ) be a connected compact origami manifold with null fibration $\pi : Z \to B$ and a Hamiltonian torus action of a torus T with moment map μ. Then :*
(a) *The image $\mu(M)$ is the union of a finite number of convex polytopes $\Delta_1, \cdots, \Delta_N$ in the dual of the Lie algebra $\mathfrak{t}^*$, each of which is the image of the moment map restricted to the closure of a connected component of $M \smallsetminus Z$.*
(b) *Over each connected component Z' of Z, the null fibration is given by a subgroup of T if and only if $\mu(Z')$ is a facet of each of the one or two polytopes corresponding to the neighbourhood(s) of $M \smallsetminus Z$, and when those are two polytopes Δ_1 and Δ_2 there exists an open subset $\widetilde{\Delta}_{Z'}$ containing $\mu(Z')$ such that $\widetilde{\Delta}_{Z'} \cap \Delta_1 = \widetilde{\Delta}_{Z'} \cap \Delta_2$.*
We call such images $\mu(M)$ origami polytopes.

Example 2.6. Consider the origami manifold (S^{2n}, ω) given in Example 2.3. Let $T := (S^1)^n$ be the n-dimensional torus. Then the action of T on S^{2n} given by

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_1, h) := (t_1 z_1, \dots, t_n z_n, h)$$

for $(t_1, \dots, t_n) \in T$ and $(z_1, \dots, z_1, h) \in S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}$ is Hamiltonian (in fact, toric) action with the moment map $\mu : S^{2n} \to \mathbb{R}^n$,

$$\mu(z_1, \dots, z_n, h) := \frac{1}{2}(|z_1|^2, \dots, |z_n|^2).$$

The image of μ is the union of two copies of the n-simplex, $\xi_1, \cdots, \xi_n \geq 0, \xi_1 + \cdots + \xi_n \leq 1/2$, and the image of fold S^{2n-1} is the "hypotenuse", $\xi_1 + \cdots + \xi_n = 1/2$. See Figure 1 for the case of $n = 2$.

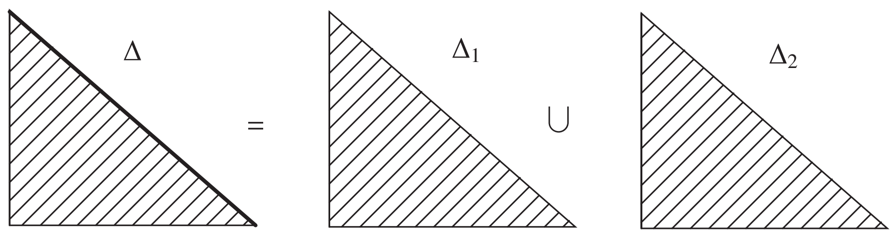

Fig. 1. An origami polytope for S^4

3. Stable almost complex structure and Clifford module bundle

Let (M, ω) be a $2n$-dimensional oriented folded symplectic manifold with fold Z and $\mathcal{U}$ an open neighbourhood of Z as in Theorem 2.2. Let M_+ (resp. M_-) be the union of connected components of $M \smallsetminus Z$ such that $\omega^n|_{M_+}$ agrees (resp. disagrees) with the given orientation of M. In [3], it was shown that M has a stable almost complex structure. More precisely the following holds.

Theorem 3.1 (Theorem 2 in [3]). *There exists an almost complex structure $\widetilde{J}$ on the real $(2n+2)$-dimensional vector bundle $TM \oplus \mathbb{R}^2$, and a $\mathbb{C}$-linear isomorphism*

$$(TM \oplus \mathbb{R}^2)|_{M \setminus \mathcal{U}} \cong T(M \setminus \mathcal{U}) \oplus \mathbb{C}.$$

Moreover, $TM \oplus \mathbb{R}^2$ has a symplectic structure $\widetilde{\omega}$ which is canonical up to homotopy, and the homotopy class of $\widetilde{J}$ is unique provided $\widetilde{J}$ is compatible with the natural symplectic structure on $TM \oplus \mathbb{R}^2$.

Remark 3.2. One can see in the proof of [3, Theorem 2] that the above $\widetilde{J}$ has the following properties.

(1) Let J be an almost complex structure on $M \setminus Z$ which is compatible with $\omega|_{M \setminus Z}$. Then one can construct $\widetilde{J}$ so that the following equality holds.

$$\begin{cases} \widetilde{J}|_{M_+ \setminus \mathcal{U}} = J|_{M_+ \setminus \mathcal{U}} \oplus (\sqrt{-1}) \\ \widetilde{J}|_{M_- \setminus \mathcal{U}} = J|_{M_- \setminus \mathcal{U}} \oplus (-\sqrt{-1}). \end{cases} \tag{3.1}$$

(2) By using a connection of the principal S^1-bundle $Z \to B$ we have the splitting of the tangent bundle $TZ \cong \pi^*TB \oplus T_\pi Z$, where $T_\pi Z$ is the tangent bundle along the fiber, which is a real line bundle over Z. Since $T\mathcal{U}$ is oriented, and hence, TZ is also oriented, the fact that B is a symplectic manifold implies that $T_\pi Z$ is an orientable. In particular, $T_\pi Z$ is trivial real line bundle. Under these identifications the almost complex structure $\widetilde{J}|_{\mathcal{U}}$ in Theorem 3.1 on $T\mathcal{U} \oplus \mathbb{R}^2 \cong \pi^*TB \oplus T_\pi Z \oplus \mathbb{R} \oplus \mathbb{R}^2$ can be taken as the direct sum of almost complex structures on the symplectic vector bundle π^*TB and the trivial bundle $T_\pi Z \oplus \mathbb{R} \oplus \mathbb{R}^2$ of real rank 4.

(3) If a compact Lie group G acts on (M, ω), then we can take $\widetilde{J}$ to be G-invariant. In fact we will use such an invariant $\widetilde{J}$ in the subsequent sections.

By using $\widetilde{J}$ and $\widetilde{\omega}$, we have a Riemannian metric on $TM \oplus \mathbb{R}^2$, and TM is equipped with the metric as a subbundle of $TM \oplus \mathbb{R}^2$. Moreover the stable almost complex structure induces a spin^c-structure on M. Now we construct a Clifford module bundle over TM in terms of this stable almost complex structure.

We first explain the construction for the vector space case. Let E be an even dimensional Euclidean vector space. Suppose that a complex structure $J_{\widetilde{E}}$ on $\widetilde{E} := E \oplus \mathbb{R}^e$ which preserves the metric on $\widetilde{E}$ is given for a non-negative (even) integer e. By using $J_{\widetilde{E}}$ we have a $\mathbb{Z}/2$-graded $Cl(\widetilde{E}) = Cl(E) \otimes Cl(\mathbb{R}^e)$-module $W_{\widetilde{E}} := \wedge^\bullet_{\mathbb{C}} \widetilde{E}$, the exterior product algebra of the Hermitian vector space $\widetilde{E}$. The Clifford action of $Cl(\widetilde{E})$ is defined by the wedge product and the interior product. We define W_E as the set of all linear maps from an irreducible representation W_e of the Clifford algebra $Cl_e := Cl(\mathbb{R}^e)$ to $W_{\widetilde{E}}$ which commute with the Clifford action of Cl_e,

$$W_E := \mathrm{Hom}_{Cl_e}(W_e, W_{\widetilde{E}}),$$

where Cl_e acts on $W_{\widetilde{E}}$ by using the inclusion $Cl_e \hookrightarrow Cl(E \oplus \mathbb{R}^e)$. Note that W_E is equipped with the Clifford action of $Cl(E)$ by

$$\alpha \cdot \phi : v \mapsto \alpha\phi(v)$$

for $\alpha \in Cl(E)$ and $v \in W_e$ using the inclusion $Cl(E) \hookrightarrow Cl(E \oplus \mathbb{R}^e)$.

Lemma 3.3. *W_E is an irreducible $\mathbb{Z}/2$-graded $Cl(E)$-module.*

Proof. Suppose that E is equipped with an almost complex structure J_E and $J_{\widetilde{E}}$ is the direct sum of J_E and the standard complex structure $\sqrt{-1}$ on $\mathbb{R}^e = \mathbb{C}^{e/2}$ (for a specific order of the basis of $\mathbb{R}^e$). In this case, one can see that $\wedge^\bullet_{\mathbb{C}}\widetilde{E} = \wedge^\bullet_{\mathbb{C}}E \otimes \wedge^\bullet_{\mathbb{C}}\mathbb{R}^e$ and

$$W_E = \mathrm{Hom}_{Cl_e}(W_e, \wedge^\bullet_{\mathbb{C}}\widetilde{E}) = \wedge^\bullet_{\mathbb{C}}E \otimes \mathrm{Hom}_{Cl_e}(W_e, \wedge^\bullet_{\mathbb{C}}\mathbb{R}^e) = \wedge^\bullet_{\mathbb{C}}E.$$

It implies that W_E is an irreducible $Cl(E)$-module. Since any complex structure on $\widetilde{E}$ is homotopic to the direct sum $J_E \oplus \sqrt{-1}$ and the irreducible representation of $Cl(E)$ is unique, we complete the proof. □

By applying the above construction for an almost complex structure on $TM \oplus \mathbb{R}^2$ we have the $\mathbb{Z}/2$-graded $Cl(TM)$-module bundle

$$W := \mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(TM \oplus \mathbb{R}^2)) \tag{3.2}$$

over M. Note that we have $W|_{M_\pm \setminus \mathcal{V}} \cong \wedge^\bullet_{\mathbb{C}}T(M_\pm \setminus \mathcal{V})$ by (3.1), which is the standard $Cl(T(M_\pm \setminus \mathcal{V}))$-module bundle of $M_\pm \setminus \mathcal{V}$. For any Hermitian line bundle L we have a twisted $\mathbb{Z}/2$-graded $Cl(E)$-module bundle $W_L := W \otimes L$.

DEFINITION 3.4. For a compact oriented origami manifold (M, ω) without boundary and a Hermitian line bundle L over M the *Riemann-Roch number $RR(M, L)$* is defined as the index of spin^c-Dirac operator which acts on the smooth sections of the Clifford module bundle W_L:

$$RR(M, L) := \mathrm{ind}(W_L).$$

REMARK 3.5. Since any two Dirac-type operators can be joined in the space of Dirac-type operators the index $RR(M, L) = \mathrm{ind}(W_L)$ does not depend on the choice of the Dirac-type operators by the homotopy invariance of the analytic index.

4. Compatible fibration on toric origami manifolds

In this section we construct a structure of *good compatible fibration* on toric origami manifolds. The notion of good compatible fibration is a family of torus fibrations (or more generally foliations) over an open covering of the manifold with some compatibility condition and is introduced in [11]. See also Definition A.2.

ASSUMPTION 4.1. In this section we consider a toric origami manifold (M, ω, T, μ) satisfying the following assumptions.

- M is connected, oriented, and compact without boundary.
- (M, ω, T, μ) satisfies the condition (b) in Theorem 2.5. Namely, the null foliation is given by a subgroup of T.

Suppose that $\dim M = 2n$. Let $\mu(M) = \bigcup_i \Delta_i$ be the union of convex polytopes associated with the moment map $\mu : M \to \mathfrak{t}^*$. For each i let $\Delta_i = \Delta_Z \cup \bigcup_{j=0}^n \bigcup_{k=1}^{m_j} \Delta_{i,k}^{(j)}$ be the stratification of Δ_i, where we put[1] $\Delta_Z := \mu(Z)$ and $\{\Delta_{i,1}^{(j)}, \cdots, \Delta_{i,m_j}^{(j)}\}$ is the set of all j-dimensional faces of Δ_i for each $j \in \{0, \cdots, n\}$. We take and fix a neighbourhood $\mathcal{V} := Z \times (-\varepsilon, \varepsilon)$ of Z in M as

[1]Strictly speaking we consider each connected component of Z.

in Theorem 2.2 for some small $\varepsilon > 0$, and we may assume that $\overline{\mathcal{U}} = Z \times [-\varepsilon, \varepsilon,]$ and an open neighbourhood $\widetilde{\Delta}_Z$ in Theorem 2.5(b) has the form $\widetilde{\Delta}_Z = \mu(\mathcal{U})$.

The construction of the good compatible fibration is divided into two parts, fibrations near the fold and fibrations outside the fold.

4.1. Torus actions near the fold. We set $\mathcal{U}' := Z \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. We take $\varepsilon > 0$ small enough so that the S^1-action on Z can be extended to a free S^1-action on $\mathcal{U}'$. By using this S^1-action we have an S^1-bundle structure on $\mathcal{U}'$ with the base space $B \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$.

4.2. Torus actions outside the fold. We construct a family of torus actions on $M \smallsetminus \overline{\mathcal{U}}$. We put $\Delta'_{i,Z} := \Delta_i \smallsetminus \mu(\overline{\mathcal{U}})$ and we first construct an open covering

$$\Delta'_{i,Z} = \left(\bigcup_{j=0}^{n} \bigcup_{k=1}^{m_j} \widetilde{\Delta}_{i,k}^{(j)} \right)$$

by the following procedure. See also Figure 2.

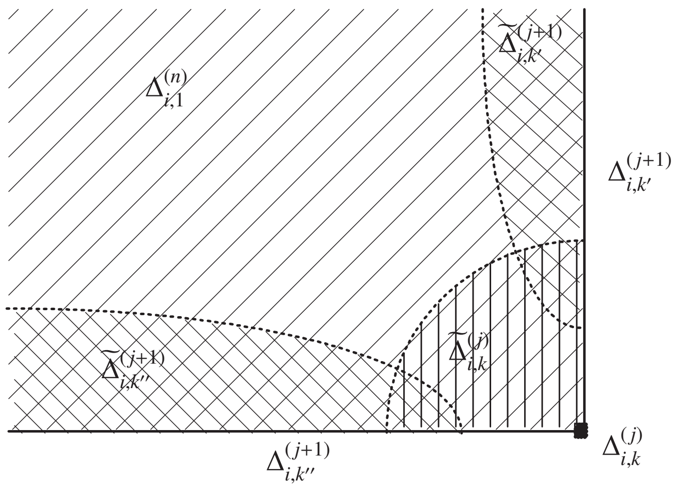

Fig. 2. Open covering outside the image of the fold

(0) For each $k \in \{1, \cdots, m_0\}$ take a small open neighbourhood $\widetilde{\Delta}_{i,k}^{(0)}$ of $\Delta_{i,k}^{(0)}$ in $\Delta'_{i,Z}$ so that $\widetilde{\Delta}_{i,k}^{(0)} \cap \widetilde{\Delta}_{i,k'}^{(0)} = \emptyset$ if $k \neq k'$.

(1) For each $k \in \{1, \cdots, m_1\}$ take a small open neighbourhood $\widetilde{\Delta}_{i,k}^{(1)}$ of

$$\left(\Delta'_{i,Z} \smallsetminus \bigcup_{k_0=1}^{m_0} \widetilde{\Delta}_{i,k_0}^{(0)} \right) \cap \Delta_{i,k}^{(1)}$$

in $\Delta'_{i,Z}$ so that $\widetilde{\Delta}_{i,k}^{(1)} \cap \widetilde{\Delta}_{i,k'}^{(1)} = \emptyset$ if $k \neq k'$.

(2) For each $k \in \{1, \cdots, m_2\}$ take a small open neighbourhood $\widetilde{\Delta}_{i,k}^{(2)}$ of

$$\left(\Delta'_{i,Z} \smallsetminus \bigcup_{j=0}^{1} \bigcup_{k_j=1}^{m_j} \widetilde{\Delta}_{i,k_j}^{(j)} \right) \cap \Delta_{i,k}^{(2)}$$

in $\Delta'_{i,Z}$ so that $\widetilde{\Delta}_{i,k}^{(2)} \cap \widetilde{\Delta}_{i,k'}^{(2)} = \emptyset$ if $k \neq k'$.

$\vdots$

(n-1) For each $k \in \{1, \cdots, m_{n-1}\}$ take a small open neighbourhood $\widetilde{\Delta}_{i,k}^{(n-1)}$ of

$$\left(\Delta'_{i,Z} \setminus \bigcup_{j=0}^{n-2} \bigcup_{k_j=0}^{m_j} \widetilde{\Delta}_{i,k_j}^{(j)} \right) \cap \Delta_{i,k}^{(n-1)}$$

in $\Delta'_{i,Z}$ so that $\widetilde{\Delta}_{i,k}^{(n-1)} \cap \widetilde{\Delta}_{i,k'}^{(n-1)} = \emptyset$ if $k \neq k'$.

(n) We set $\widetilde{\Delta}_{i,1}^{(n)} := \mathrm{int}(\Delta_i) = \mathrm{int}(\Delta_i^{(n)})$.

For a toric manifold $M \setminus Z$ it is well-known that for each i, j, k there exists a subtorus $T_{i,k}^{(j)}$ of T such that $\dim T_{i,k}^{(j)} = n - j$ and for any $x \in \mu^{-1}(\mathrm{int}(\Delta_{i,k}^{(j)})) \setminus Z$ the stabilizer subgroup at x is equal to $T_{i,k}^{(j)}$. Particularly we have $T_{i,1}^{(n)} = \{e\}$. We take and fix a rational metric of the Lie algebra $\mathfrak{t}$ so that for each subspace $\mathfrak{h}$ in $\mathfrak{t}$ spanned by rational vectors one can associate the orthogonal complement subgroup $\exp(\mathfrak{h}^{\perp})$ as a compact subgroup of T. Let $G_{i,k}^{(j)}$ be the orthogonal complement subgroup associated with (the Lie algebra of) the stabilizer subgroup $T_{i,k}^{(j)}$. Note that we have $G_{i,1}^{(n)} = T$. Define an open subset of M by $M_{i,k}^{(j)} := \mu^{-1}(\widetilde{\Delta}_{i,k}^{(j)})$, which has the natural $G_{i,k}^{(j)}$-action and the following properties.

- Each $G_{i,k}^{(j)}$ acts on $M_{i,k}^{(j)}$, and all orbits of $G_{i,k}^{(j)}$-action have the maximal dimension $\dim G_{i,k}^{(j)}$.
- If $\widetilde{\Delta}_{i,k}^{(j)} \cap \widetilde{\Delta}_{i,k'}^{(j')} \neq \emptyset$, then we have $G_{i,k}^{(j)} \subset G_{i,k'}^{(j')}$ or $G_{i,k}^{(j)} \supset G_{i,k'}^{(j')}$.

4.3. Good compatible fibration on toric origami manifolds. By taking each open subset small enough we may assume that $\mathcal{U}' \cap M_{i,k}^{(j)} = \emptyset$ for all i, j, k with $j \neq n$. The union $\mathcal{U}' \cup \bigcup_{i,j,k} M_{i,k}^{(j)}$ is not an open covering of the whole M. There exist a family of compact sets, which we call the *crack* $C_{i,k}^{Z}$ defined by

$$C_{i,k}^{Z} := \mu^{-1}\left(\mu(\overline{\mathcal{U}} \setminus \mathcal{U}') \cap \Delta_{i,k}^{(n-1)}\right).$$

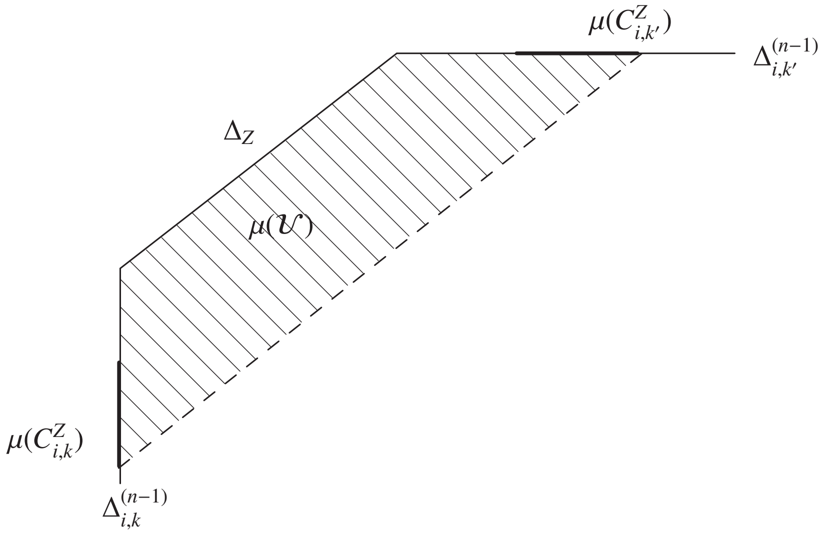

Fig. 3. Crack near the fold

Though we do not know the way to extend the good compatible fibration across the crack, we have the following.

Proposition 4.2. *A family of open subsets* $\{\mathcal{U}', M_{i,k}^{(j)}\}_{i,j,k}$ *defines a structure of good compatible fibration* (Definition A.2) *on the complement* $M \setminus \bigcup_{i,k} C_{i,k}^Z$.

Example 4.3. Consider the toric origami manifold S^4 with the moment map $\mu : S^4 \to \mathbb{R}^2$ whose origami polytope is the union of two copies of the triangle, $\mu(S^4) = \Delta = \Delta_1 \cup \Delta_2$. The open covering $\{\mathcal{U}', M_{i,k}^{(j)}\}_{i,j,k}$ consists of the inverse images of the following two copies of 5 open subsets of Δ_1 $(= \Delta_2)$ for any small $\varepsilon > 0$:

- $\widetilde{\Delta}_Z$: small open neighbourhood of the hypotenuse $\xi_1 + \xi_2 = 1/2$.
- $\widetilde{\Delta}_1^{(0)} = \widetilde{\Delta}_2^{(0)}$: small open ball of radius $\varepsilon > 0$ centered at $(0, 0)$.
- $\widetilde{\Delta}_{1,1}^{(1)} = \widetilde{\Delta}_{2,1}^{(1)}$: small open neighbourhood of the line segment, $0 \leq \xi_1 < \varepsilon, \varepsilon/2 \leq \xi_2 \leq 1 - \varepsilon$.
- $\widetilde{\Delta}_{1,2}^{(1)} = \widetilde{\Delta}_{2,2}^{(1)}$: small open neighbourhood of the line segment $0 \leq \xi_2 < \varepsilon, \varepsilon/2 \leq \xi_1 \leq 1 - \varepsilon$.
- $\mathrm{int}\Delta_1 = \mathrm{int}\Delta_2$.

In this case the cracks consist of the inverse images of two compact subsets $c_{1,1}^Z = c_{2,1}^Z$ and $c_{1,2}^Z = c_{2,2}^Z$ defined by

$$c_{1,1}^Z \left(= c_{2,1}^Z\right) \ : \ \xi_1 = 0, 1 - \varepsilon \leq \xi_2 \leq 1 - \varepsilon/2$$

and

$$c_{1,2}^Z \left(= c_{2,2}^Z\right) \ : \ \xi_2 = 0, 1 - \varepsilon \leq \xi_1 \leq 1 - \varepsilon/2.$$

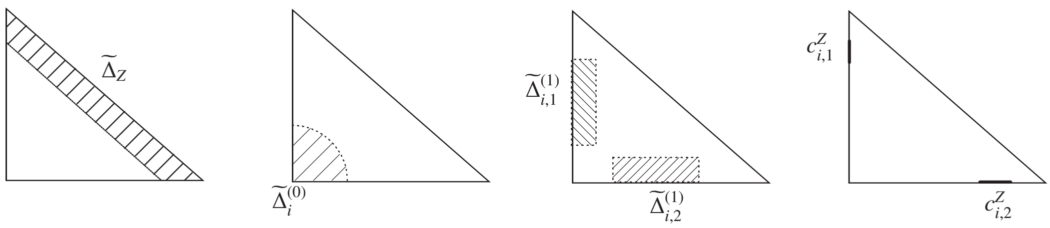

Fig. 4. Covering of the S^4

5. Compatible system on toric origami manifolds

In this section we construct a *compatible system (of Dirac-type operators)* on toric origami manifolds. The notion of compatible system is introduced in [11], which is a family of Dirac-type operators along leaves of compatible fibration and satisfies some anti-commutativity. See also Definition A.4.

Assumption 5.1. In this section we consider a toric origami manifold (M, ω, T, μ) satisfying the following assumption.

- (M, ω, T, μ) satisfies Assumption 4.1.
- The de Rham cohomology class $[\omega]$ has an integral lift in $H^2(M, \mathbb{Z})$.
- A T-equivariant pre-quantizing line bundle (L, ∇) is fixed. Namely, L is a T-equivariant Hermitian line bundle over M and ∇ is a T-invariant Hermitian connection whose curvature form is equal to $-\sqrt{-1}\omega$.

Together with the assumptions we may choose a stable almost complex structure $\widetilde{J}$ as in Theorem 3.1 so that the tangent bundle of each symplectic submanifold $\mu^{-1}(\mathrm{int}(\Delta^{(j)}_{i,k}))$ is preserved by $\widetilde{J}$ for all i, j and k. Under the above assumption we use the $\mathbb{Z}/2$-graded Clifford module bundle W_L as in the end of Section 3. As it is shown in Section 4, $M \smallsetminus \bigcup_{i,k} C^Z_{i,k}$ has a structure of good compatible fibration $\{\mathcal{V}', M^{(j)}_{i,k}\}_{i,j,k}$. Since $\{M^{(j)}_{i,k}\}_{i,j,k}$ is a good compatible fibration on an open toric manifold $M \smallsetminus \overline{\mathcal{V}}$, we have a compatible system $\{D^{(j)}_{i,k}\}_{i,j,k}$ on it as in [11, Theorem 5.1]. Namely for each i, j, k we have the following.

- $D^{(j)}_{i,k}$ is a first order formally self-adjoint differential operator of degree-one, which acts on the space of smooth sections of $W_L|_{M^{(j)}_{i,k}}$.
- $D^{(j)}_{i,k}$ contains only the differentials along the $G^{(j)}_{i,k}$-orbits.
- For each $x \in M^{(j)}_{i,k}$, the restriction of $D^{(j)}_{i,k}$ to the orbit $G^{(j)}_{i,k} \cdot x$ is a Dirac-type operator on the $\mathbb{Z}/2$-graded $Cl(T(G^{(j)}_{i,k} \cdot x))$-module bundle $W_L|_{G^{(j)}_{i,k}\cdot x}$.
- Let $\widetilde{u}$ be a $G^{(j)}_{i,k}$-invariant section of the normal bundle to the orbit $G^{(j)}_{i,k} \cdot x$. Then $D^{(j)}_{i,k}$ anti-commutes with the Clifford multiplication $c(\widetilde{u})$ of $\widetilde{u}$:

$$D^{(k)}_{i,k} c(\widetilde{u}) + c(\widetilde{u}) D^{(k)}_{i,k} = 0. \tag{5.1}$$

Now we construct a differential operator D_Z along the S^1-orbits on $\mathcal{V}$. We first study the product structure of $W|_{\mathcal{V}}$. Hereafter we use the identification $\mathcal{V} = Z \times (-\varepsilon, \varepsilon) \cong (Z \times S^1 \times (-\varepsilon, \varepsilon))/S^1$ with respect to the diagonal S^1-action. According to Remark 3.2(2) we may assume that the almost complex structure $\widetilde{J}|_{\mathcal{V}}$ in Theorem 3.1 on $T\mathcal{V} \oplus \mathbb{R}^2 \cong \pi^*TB \oplus T_\pi Z \oplus \mathbb{R} \oplus \mathbb{R}^2$ is the direct sum of almost complex structures on the symplectic vector bundle π^*TB and the trivial bundle $T_\pi Z \oplus \mathbb{R} \oplus \mathbb{R}^2$ of real rank 4. Then we have

$$W|_{\mathcal{V}} = \mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(T\mathcal{V} \oplus \mathbb{R}^2)) = \pi^*(\wedge^\bullet_{\mathbb{C}} TB) \otimes \mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(T_\pi Z \oplus \mathbb{R}^3)).$$

On the other hand we have the commutative diagram of bundle maps

$$\begin{tikzcd}
T_\pi Z \arrow[d] & q^*(T_\pi Z) \cong p^*(TS^1) \arrow[l] \arrow[r] \arrow[d] & TS^1 \arrow[d] \\
Z & Z \times S^1 \arrow[l, "q"'] \arrow[r, "p"'] & S^1,
\end{tikzcd}$$

where $p : Z \times S^1 \to S^1$ is the projection to the S^1-factor and $q : Z \times S^1 \to (Z \times S^1)/S^1 \cong Z$, $(z, t) \mapsto zt^{-1}$ is the quotient map with respect to the diagonal action of S^1. The isomorphism in the middle column is given by the differential of the map $S^1 \to Z$, $t \mapsto zt^{-1}$ for $z \in Z$. The commutative diagram implies that the vector bundle $T_\pi Z \oplus \mathbb{R}^3 \to \mathcal{V} \cong (Z \times S^1 \times (-\varepsilon, \varepsilon))/S^1$ can be obtained as a quotient bundle of $p^*(TS^1) \oplus \mathbb{R}^3 \to Z \times S^1 \times (-\varepsilon, \varepsilon)$. In particular $\mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(T_\pi Z \oplus \mathbb{R}^3)) \to \mathcal{V}$ can be obtained as a quotient bundle of $\mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(TS^1 \oplus \mathbb{R}^3)) \to Z^1 \times S^1 \times (-\varepsilon, \varepsilon)$, where the complex structure on $TS^1 \oplus \mathbb{R}^3$ is given by the same formula for B_t as in the proof of [3, Theorem 2] under a trivialization. Note that $\mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(TS^1 \oplus \mathbb{R}^3))$ has a structure of $\mathbb{Z}/2$-graded $Cl(TS^1 \oplus \mathbb{R})$-module bundle over $S^1 \times (-\varepsilon, \varepsilon)$.

Now we decompose the line bundle L over $\mathcal{V}$. Let $(L_0, \nabla) \to S^1 \times (-\varepsilon, \varepsilon)$ be the pre-quantizing line bundle over the folded cylinder as in Appendix C.

Proposition 5.2. *If we take ε small enough, then the diffeomorphism $\varphi : \mathcal{U} \xrightarrow{\cong} Z \times (-\varepsilon, \varepsilon)$ as in* Theorem 2.2 *can be lifted to an isomorphism between $L|_{\mathcal{U}} \to \mathcal{U}$ and $(L|_Z \boxtimes L_0)/S^1 \to (Z \times S^1 \times (-\varepsilon, \varepsilon))/S^1 = Z \times (-\varepsilon, \varepsilon)$.*

Proof. Note that there exists the canonical isomorphism $\widetilde{\varphi}_0$ between $\iota_Z^* L$ and $\iota_0^* \left((L|_Z \boxtimes L_0)/S^1\right)$. Fix a Hermitian connection of $(L|_Z \boxtimes L_0)/S^1$. Then the we have the required isomorphism by using $\widetilde{\varphi}_0$ and the parallel transport. □

Summarising we have the following.

Proposition 5.3. *Let $W_{B,L_B} := \wedge^\bullet_{\mathbb{C}} TB \otimes (L|_Z/S^1)$ be a $\mathbb{Z}/2$-graded $Cl(TB)$-module bundle over B. Let $W_{0,L_0} := \mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(TS^1 \oplus \mathbb{R}^3)) \otimes L_0$ be a $\mathbb{Z}/2$-graded $Cl(TS^1 \oplus \mathbb{R})$-module bundle over $S^1 \times (-\varepsilon, \varepsilon)$ as in the above construction. The $\mathbb{Z}/2$-graded Clifford module bundle $W_L|_{\mathcal{U}} \to \mathcal{U}$ is isomorphic to the quotient bundle of the tensor product $\pi^* W_{B,L_B} \otimes p^* W_{0,L_0} \to Z \times S^1 \times (-\varepsilon, \varepsilon)$ with respect to the diagonal S^1-action, where $\pi : Z \times S^1 \times (-\varepsilon, \varepsilon) \to B$ and $p : Z \times S^1 \times (-\varepsilon, \varepsilon) \to S^1 \times (-\varepsilon, \varepsilon)$ are natural projections.*

Let D_{S^1} be a Dirac-type operator along the S^1-orbits in $S^1 \times (-\varepsilon, \varepsilon)$, which acts on the space of smooth sections of W_{0,L_0}. See Appendix C for the explicit description of D_{S^1}. Let ϵ_B be the map representing the $\mathbb{Z}/2$-grading of W_{B,L_B}, i.e., $\epsilon_B(v) = (-1)^{\deg(v)}(v)$ for $v \in W_{B,L_B}$. The product of operators $\epsilon_B \otimes D_{S^1}$ is S^1-invariant, and it induces a differential operator D_Z acting on the smooth sections of $W|_{\mathcal{U}}$ through the isomorphism in Proposition 5.2. Since the S^1-action on Z is given by a subgroup of T, D_Z is a differential operator along the S^1-orbits and satisfies the anti-commutativity as in (5.1).

Proposition 5.4. *The family of differential operators $\{D_Z, D_{i,k}^{(j)}\}_{i,j,k}$ is a compatible system on the compatible fibration defined by the torus actions $\{S^1 \curvearrowright \mathcal{U}', G_{i,k}^{(j)} \curvearrowright M_{i,k}^{(j)}\}_{i,j,k}$.*

5.1. Acyclicity of the compatible system. In this section we determine the condition for the compatible system $\{D_Z, D_{i,k}^{(j)}\}_{i,j}$ to be *acyclic* ([11, Definition 6.10] or Definition A.5).

Let $\mathfrak{g}_{i,k}^{(j)*}$ be the dual of the Lie algebra of the subtorus $G_{i,k}^{(j)}$ and $(\mathfrak{g}_{i,k}^{(j)*})_{\mathbb{Z}}$ the integral weight lattice of $\mathfrak{g}_{i,k}^{(j)*}$. Let $\iota_{i,k}^{(j)} : \mathfrak{g}_{i,k}^{(j)} \to \mathfrak{g}$ be the inclusion of the Lie subalgebra. Note that the composition $\mu_{i,k}^{(j)} := (\iota_{i,k}^{(j)*}) \circ \mu : M_{i,k}^{(j)} \to \mathfrak{g}_{i,k}^{(j)*}$ is the moment map for the Hamiltonian $G_{i,k}^{(j)}$-action on $M_{i,k}^{(j)}$. We put $M_{i,k}^{(j)\circ} := M_{i,k}^{(j)} \smallsetminus (\mu_{i,k}^{(j)})^{-1}((\mathfrak{g}_{i,k}^{(j)*})_{\mathbb{Z}})$.

Proposition 5.5. *For each $x \in M_{i,k}^{(j)\circ}$, we have* $\ker(D_{i,k}^{(j)}|_{G_{i,k}^{(j)} \cdot x}) = 0$.

Proof. Note that for each $x \in M_{i,k}^{(j)}$ the kernel of $D_{i,k}^{(j)}|_{G_{i,k}^{(j)} \cdot x}$ vanishes if and only if there are no non-trivial global parallel sections of $L|_{G_{i,k}^{(j)} \cdot x}$. The proposition follows from the fact that if there exists a global parallel section, then we have $\mu_{i,k}^{(j)}(x) = \iota_{i,k}^{(j)*}(\mu(x))$ lies in the integral weight lattice $(\mathfrak{g}_{i,k}^{(j)*})_{\mathbb{Z}}$. □

We may take $\varepsilon > 0$ small enough so that $\mu(\mathcal{U}) = \mu(Z \times (-\varepsilon, \varepsilon))$ does not contain any integral lattice points outside $\mu(Z) = \Delta_Z$. Then we have the following by the same argument as that for Proposition 5.5.

Proposition 5.6. *For each $x \in \mathcal{U}' \smallsetminus Z$, we have* $\ker(D_Z|_{S^1 \cdot x}) = 0$.

We put $V := \left(\mathcal{U}' \cup \bigcup_{i,j,k} M_{i,k}^{(j)\circ}\right) \smallsetminus \left(Z \cup \bigcup_{i,k} C_{i,k}^{Z}\right)$. Then $M \smallsetminus V$ is compact. Since $\{S^1 \curvearrowright \mathcal{U}', G_{i,k}^{(j)} \curvearrowright M_{i,k}^{(j)}\}_{i,j,k}$ is a good compatible fibration one can see that the following two types of the anti-commutators on the intersections are non-negative.

- $D_{i,k}^{(j)} D_{i,k'}^{(j')} + D_{i,k'}^{(j')} D_{i,k}^{(j)}$ on $M_{i,k}^{(j)} \cap M_{i,k'}^{(j')}$, and
- $D_Z D_{i,1}^{(n)} + D_{i,1}^{(n)} D_Z$ on $\mathcal{U}' \cap M_{i,1}^{(n)}$.

See [11, Proposition 5.8, Lemma 5.9] for example. Together with Proposition 5.5 this fact implies the following.

Proposition 5.7. *The compatible system $\{D_Z, D_{i,k}^{(j)}\}_{i,j,k}$ is acyclic over V.*

5.2. Localization formula and Danilov-type formula. As in Definition 3.4, the Riemann-Roch number $RR(M, L)$ is defined for any origami manifold (M, ω) with pre-quantizing line bundle (L, ∇). If (M, ω) is a toric origami manifold with the action of a torus T, then the resulting index is an element of the character ring $R(T)$ of T. In this case we call the index the *equivariant Riemann-Roch number* or *Riemann-Roch character* and is denoted by $RR_T(M, L)$.

We use notations in the previous sections and assume Assumption 5.1. For each i, $j(\neq n)$, and k we may assume that

$$\widetilde{\Delta}_{i,k}^{(j)} \cap \mathrm{int}\Delta_i \cap \mathfrak{t}_{\mathbb{Z}}^* = \emptyset,$$

and we take and fix a T-invariant small open neighbourhood $V_{i,k}^{(j)}$ of $(\mu_{i,k}^{(j)})^{-1}((\mathfrak{g}_{i,k}^{(j)*})_{\mathbb{Z}})$ for each i, j and k. By the above assumption one has that if $j \neq n$, then $V_{i,k}^{(j)} \cap \mu^{-1}(\mathfrak{t}_{\mathbb{Z}}^*)$ consists of the inverse image of lattice points in the boundary $\partial\Delta_i = \Delta_i \smallsetminus \mathrm{int}\Delta_i$. We also take and fix a small open neighbourhood $V_{i,k}^{Z}$ of the crack $C_{i,k}^{Z}$ so that it does not contain any integral points for each i and k. Note that each open subset $V_{i,k}^{(j)} \cap V$ (resp. $V_{i,k}^{Z} \cap V$) with compact complement $V_{i,k}^{(j)} \smallsetminus V_{i,k}^{(j)} \cap V = (\mu_{i,k}^{(j)})^{-1}((\mathfrak{g}_{i,k}^{(j)*})_{\mathbb{Z}}) (\supset M_{i,k}^{(j)} \cap \mu^{-1}(\mathfrak{g}_{\mathbb{Z}}^*))$ (resp. $V_{i,k}^{Z} \smallsetminus V_{i,k}^{Z} \cap V = C_{i,k}^{Z}$) is equipped with an acyclic compatible system by Proposition 5.7, and hence, the T-equivariant local index $\mathrm{ind}_T(V_{i,k}^{(j)}, V_{i,k}^{(j)} \cap V)$ (resp. $\mathrm{ind}_T(V_{i,k}^{Z}, V_{i,k}^{Z} \cap V)$) is defined (Theorem A.7) . As in the same way one can define the T-equivariant local index for the fold, $\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \smallsetminus Z)$, is defined.

The localization formula (Theorem A.8) implies that the Riemann-Roch character is localized at $\mu^{-1}(\mathfrak{g}_{\mathbb{Z}}^*) \cup Z \cup \bigcup_{i,k} C_{i,k}^{Z} \subset M \smallsetminus V$ as follows.

Theorem 5.8. *Under* Assumption 5.1 *we have the localization formula of T-equivariant index*

$$RR_T(M, L) = \mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \smallsetminus Z) + \sum_{i,j,k} \mathrm{ind}_T(V_{i,k}^{(j)}, V_{i,k}^{(j)} \cap V) + \sum_{i,k} \mathrm{ind}_T(V_{i,k}^{Z}, V_{i,k}^{Z} \cap V).$$

By computing the contributions $\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \smallsetminus Z)$ (Theorem 6.9), $\mathrm{ind}_T(V_{i,k}^{(j)}, V_{i,k}^{(j)} \cap V)$ (Theorem 6.10, Theorem 6.11) and $\mathrm{ind}_T(V_{i,k}^{Z}, V_{i,k}^{Z} \cap V)$ (Theorem 6.12) in the subsequent section, we have the following Danilov-type formula.

Theorem 5.9. *Under* Assumption 5.1 *we have the following equality as elements in the character ring $R(T)$.*

$$RR_T(M, L) = \sum_{\xi_+ \in \mu(M^+) \cap \mathfrak{t}_{\mathbb{Z}}^*} \mathbb{C}_{(\xi_+)} - \sum_{\xi_- \in \mu(M^-) \cap \mathfrak{t}_{\mathbb{Z}}^*} \mathbb{C}_{(\xi_-)}, \tag{5.2}$$

where for each $\xi \in \mathfrak{t}^*_{\mathbb{Z}}$ *we denote by* $\mathbb{C}_{(\xi)}$ *the irreducible representation of* T *whose weight is given by* ξ.

To compute the local contributions in the subsequent sections, we will use the following notations. We divide the collection of Delzant polytopes $\{\Delta_i\}_{i=1,\cdots,N}$ into two subsets,

$$\{\Delta_i\}_{i=1,\cdots,N} = \{\Delta_i^+\}_{i=1,\cdots,N_+} \cup \{\Delta_i^-\}_{i=1,\cdots,N_-},$$

where $N_+ + N_- = N$ and the sign is determined by the condition $\mu(M^\pm) = \bigcup_{i=1}^{N_\pm} \Delta_i^\pm$. In a similar way we also use notations $\Delta_{i,k}^{(j)\pm}$, $V_{i,k}^{(j)\pm}$, $\mu_{i,k}^{(j)\pm}$ and $\mathfrak{g}_{i,k}^{(j)\pm}$.

In terms of this notations the formula (5.2) can be rewritten as

$$RR_T(M,L) = \sum_{i,j,k} \left(\sum_{\xi_+ \in \mathrm{int}\Delta_{i,k}^{(j)+} \cap \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi_+)} - \sum_{\xi_- \in \mathrm{int}\Delta_{i,k}^{(j)-} \cap \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi_-)} \right)$$

Example 5.10. Consider the toric origami manifold (S^{2n}, ω), the unit sphere, with the moment map $\mu : S^{2n} \to \mathbb{R}^n$ as in Example 2.6, whose origami polytope is the union of two copies of the n-simplex, $\mu(S^{2n}) = \Delta = \Delta_1 \cup \Delta_2$. Since $\mu((S^{2n})^+) \cap \mathfrak{t}^*_{\mathbb{Z}} = \mu((S^{2n})^-) \cap \mathfrak{t}^*_{\mathbb{Z}}$, one has $RR_T(S^{2n}, L) = 0$ for any T-equivariant pre-quantizing line bundle L.

Note that if we use the folded symplectic form $k\omega$ for any positive constant k, then the origami polytope for $(S^{2n}, k\omega)$ is the similar extension with ratio k of the original origami polytope. In this case one also has $RR_T(S^{2n}, L_k) = 0$ for any T-equivariant pre-quantizing line bundle L_k.

5.3. Comments on another possible approaches. The formula (5.2) in Theorem 5.9 itself can be obtained as a consequence of the cobordism theorem [2, Theorem 4.1] and Danilov's theorem for symplectic toric manifolds.

There is an another possible approach which uses the theory of *multi-fans* introduced in [15]. The equivariant index formula [15, Theorem 11.1], which is based on the fixed point formula, would be available to the left hand side of (5.2). In fact as it is shown in [18] one can associate a multi-fan for each oriented toric origami manifold.

It would be possible to show the formula (5.2) by using the theory of *transverse index* in [1][20]. In [1] it was shown that the Riemann-Roch character $RR_T(M,L)$ can be realized as a perturbation of Dirac operator by the Clliford multiplication of the Kirwan vector field of the moment map. By considering the perturbation $RR_T(M,L)$ is localized at the zero locus of the Kirwan vector field, i.e., the fixed point set M^T. Under Assumption 4.1, the fold has a free S^1-action, and hence, there are no contributions of the fold to $RR_T(M,L)$. In particular $RR_T(M,L)$ is the sum of contributions of the vertices of the image of the moment map $\mu(M \setminus Z)$. As in [21, Example 13] the contribution from a fixed point is infinite sum of one dimensional representations of T in general. It implies that $RR_T(M,L)$ is expressed as a cancellation of infinite sum of one dimensional representations. See also [8] for the infinite dimensional nature of the transverse index and the finite dimensional nature of the index theory in [10, 11].

In contrast to these approaches our proof is direct and geometric, which detects the contribution of each lattice point directly and contains a new proof of original Danilov's theorem as a special case.

6. Computation of local contributions

6.1. Toric case. In this subsection we consider the symplectic toric case, i.e., toric origami manifolds with empty fold. We first summarize the set-up and notations.

Let X be a $2n$-dimensional symplectic manifold equipped with a Hamiltonian torus action of an n-dimensional torus G. We assume that there exists a G-equivariant pre-quantizing line bundle $L_X \to X$. Let $\mu_X : X \to \mathfrak{g}^* = \mathrm{Lie}(G)^*$ and $\Delta_X = \mu_X(X)$ be the corresponding moment map and the Delzant polytope. We take and fix an m-dimensional face Δ' of Δ_X and a point ξ in the relative interior $\mathrm{int}(\Delta')$. Let $F := \mu_X^{-1}(\xi)$ be the m-dimensional isotropic torus in X and $X' := \mu_X^{-1}(\Delta')$ be the $2m$-dimensional symplectic submanifold of X. We take and fix a point $x \in F \subset X'$. Let H be the stabilizer subgroup at x with respect to G-action and $H^\perp$ the complementary orthogonal subtorus of H in G with respect to a rational metric of $\mathfrak{g}$. Note that H (resp. $H^\perp$) is an $n-m$-dimensional (resp. m-dimensional) subtorus of G. We denote the inclusion map of Lie-algebra and its dual by $\iota_H : \mathrm{Lie}(H) = \mathfrak{h} \to \mathfrak{g}$ and $\iota_H^* : \mathfrak{g}^* \to \mathfrak{h}^*$ respectively.

We first give following comments.

- Since the computation is purely local, we do not need the compactness of Δ_X. In fact we only use a part of the Delzant condition near ξ.
- We fix a G-invariant ω-compatible almost complex structure on X so that it also induces a G-invariant ω-compatible almost complex structure on the inverse image of each face of Δ_X.
- F is a Lagrangian torus in the symplectic submanifold X'.
- F can be described as the orbit $F = G \cdot x = H^\perp \cdot x$.
- The intersection $H \cap H^\perp$ is a finite Abelian group.
- Since x is a fixed point with respect the H-action, the moment map image $(\iota_H^* \circ \mu)(x) = \iota_H^*(\xi)$ of x with respect to the H-action is an element in the weight lattice $\mathfrak{h}_{\mathbb{Z}}^*$.
- The argument below still holds when there exists a finite subgroup of G which acts trivially on X. In fact in the proof of Lemma 6.2 we deal with the symplectic toric manifold X_1 for which such a subgroup $H \cap H_1 \cap H_1^\perp$ may exist.

If Y is a smooth manifold and Y' is its smooth submanifold, then we denote the normal bundle of Y' in Y by $\nu_Y(Y')$. We also denote the fiber at $y \in Y'$ by $\nu_Y(Y')_y$. There exists a G-invariant tubular neighbourhood N_F of F and G-equivariant diffeomorphism

$$N_F \cong (\nu_X(F)_x \times G)/H = (\nu_X(F)_x \times H^\perp)/H \cap H^\perp,$$

where we use the G-action on the right hand side through the identification $G = H \cdot H^\perp = (H \times H^\perp)/H \cap H^\perp$ arising from the exact sequence

$$\begin{gathered} H \cap H^\perp \to H \times H^\perp \to H \cdot H^\perp = G \\ h \mapsto (h, h^{-1}), (h_1, h_2) \mapsto h_1 h_2. \end{gathered}$$

Since F is a Lagrangian torus in X' we have

$$\nu_X(F)_x \times H^\perp = \nu_X(X')_x \times \nu_{X'}(F)_x \times H^\perp = \nu_X(X')_x \times T_x^*(H^\perp \cdot x) \times H^\perp = \nu_X(X')_x \times T^* H^\perp,$$

and hence, we have a G-equivariant isomorphism

$$N_F \cong (\nu_X(X')_x \times T^*H^\perp)/H \cap H^\perp. \tag{6.1}$$

Now we describe the restriction $L_X|_{N_F}$. We first define an H-equivariant line bundle $L_1 := \nu_X(X')_x \times L_X|_x \to \nu_X(X')_x$, where we regarded $L_X|_x$ as a representation of H. Note that $\nu_X(X')_x$ has a natural symplectic structure and L_1 is equipped with a structure of pre-quantizing line bundle with respect to the symplectic structure. Let L_2 be the pull-back of $L_X|_{N_F}$ with respect to the natural map $T^*H^\perp \to (\nu_X(X')_x \times T^*H^\perp)/H \cap H^\perp$, which is an $H \times H^\perp$-equivariant line bundle over $T^*H^\perp$. Note that though H-action on $T^*H^\perp$ is trivial, the action on L_2 is non-trivial in general. We define an $H^\perp$-equivariant line bundle $\hat{L}_2 \to T^*H^\perp$ by $\hat{L}_2 := \mathrm{Hom}(L_X|_x, L_2)$. Then $\hat{L}_2$ is isomorphic to L_2 as $H^\perp$-equivariant line bundle and the induced H-action on $\hat{L}_2$ is trivial. We have two line bundles with connection $(L_1 \boxtimes \hat{L}_2)/H \cap H^\perp$ and $L_X|_{N_F}$ over $(\nu_X(X')_x \times T^*H^\perp)/H \cap H^\perp = N_F$. The restrictions of these two line bundles to the zero-section F in N_F are isomorphic to each other as line bundles with connection. The Darboux type theorem ([12, Proposition 7.11]) implies that the G-equivariant isomorphism can be extended to a G-invariant neighbourhood of F.

Remark 6.1. Strictly speaking we have to consider the data on sufficiently small neighbourhoods of the origin in $\nu_X(X')_x$ and the zero section $H^\perp$ in $T^*H^\perp$ as a Lagrangian torus to consider the above isomorphisms and the local indices in the subsequent argument, though, we use the same notations $\nu_X(X')$ and $T^*H^\perp$ to simplify the notations.

Let $\Delta_1, \ldots, \Delta_{n-m}$ be codimension one faces of Δ_X such that Δ' is the intersection of them, $\Delta' = \Delta_1 \cap \cdots \cap \Delta_{n-m}$. For each $l = 1, 2, \ldots, n-m$, let H_l be the circle subgroup of H which acts trivially on the symplectic submanifold $X_l := \mu_X^{-1}(\Delta_l)$ and $H_l^\perp$ the orthogonal complement of H_l. If we choose any members $\Delta_{l_1}, \ldots, \Delta_{l_\alpha}$, then we have a locally free action of the intersection $H_{l_1}^\perp \cap \cdots \cap H_{l_\alpha}^\perp = (H_{l_1} \cdot \cdots \cdot H_{l\alpha})^\perp$ on a small neighbourhood of the inverse image of the complement of a neighbourhood of the boundary $\partial(\Delta_{l_1} \cap \cdots \cap \Delta_{l_\alpha})$ in $\Delta_{l_1} \cap \cdots \cap \Delta_{l_\alpha}$. Such a family of torus actions determines a good compatible fibration as in Section 4.3. For each H_l we have the decomposition $H_l^\perp = (H \cap H_l^\perp) \cdot H^\perp$. On the other hand there exists a natural action of the product $H \times H^\perp$ on N_F under the identification (6.1). Then the above good compatible fibration is induced from the action of the subgroup $(H \cap H_l^\perp) \times H^\perp$ in $H \times H^\perp$.

The G-equivariant local index $\mathrm{ind}_G(N_F, N_F \smallsetminus F)$ is defined by using these structures and it is equal to the $H \cap H^\perp$-invariant part of the $H \times H^\perp$-equivariant local index $\mathrm{ind}_{H\times H^\perp}(\nu_X(X')_x \times T^*H^\perp, \nu_X(X')_x \times T^*H^\perp \smallsetminus \{0\} \times H^\perp)$. For simplicity we use the following type of notations for the equivariant local indices:

$$RR_H(\nu_X(X')_x) := \mathrm{ind}_H(\nu_X(X')_x, \nu_X(X')_x \smallsetminus \{0\})$$

and

$$RR_{H^\perp}(T^*H^\perp) := \mathrm{ind}_{H^\perp}(T^*H^\perp, T^*H^\perp \smallsetminus H^\perp).$$

Lemma 6.2. *$RR_H(\nu_X(X')_x) = \mathbb{C}_{(\iota_H^*(\xi))} = L_X|_x \in R(H)$.*

Proof. The second equality follows from the property of the moment map and the Kostant formula. We show the first equality by induction on $n - m = \dim(\nu_X(X')_x)/2$. If $n - m = 1$, then the equality follows from the direct computation. See [22, Example 2.3] for example. Suppose that $n - m$ is grater than 1 and the statement holds for any situation with codimension $n - m - 1$. We consider the decomposition $\nu_X(X') = \nu_X(X_1) \oplus \nu_{X_1}(X')$ and $H = H_1 \cdot (H \cap H_1^\perp)$. According to the decomposition the H-action on $\nu_X(X')$ factors the action of the product of H_1-action on $\nu_X(X_1)$ and $H \cap H_1^\perp$-action on $\nu_{X_1}(X')$. By Proposition B.2 we have that $RR_H(\nu_X(X')_x)$ is equal to the $H_1 \cap (H \cap H_1^\perp)$-invariant part of the product $RR_{H_1}(\nu_X(X_1)_x) \otimes RR_{H \cap H_1^\perp}(\nu_{X_1}(X')_x)$. Note that $H_1^\perp$-action on X_1 gives a structure of a symplectic toric manifold whose momentum polytope is $\iota^*_{H_1^\perp}(\Delta_1)$. By the assumption of the induction we have $RR_{H_1^\perp}(\nu_{X_1}(X')_x) = \mathbb{C}_{(\iota^*_{H_1^\perp}(\xi))}$. By considering the subgroup $H \cap H_1^\perp$ we have $RR_{H \cap H_1^\perp}(\nu_{X_1}(X')_x) = \mathbb{C}_{(\iota^*_{H \cap H_1^\perp}(\xi))}$, and hence,

$$RR_{H_1}(\nu_X(X_1)_x) \otimes RR_{H \cap H_1^\perp}(\nu_{X_1}(X')_x) = \mathbb{C}_{(\iota^*_{H_1}(\xi))} \otimes \mathbb{C}_{(\iota^*_{H \cap H_1^\perp}(\xi))} = \mathbb{C}_{(\iota^*_{H_1}(\xi) \oplus \iota^*_{H \cap H_1^\perp}(\xi))}.$$

Note that under the natural isomorphism $\mathrm{Lie}(H_1)^* \oplus \mathrm{Lie}(H \cap H_1^\perp)^* \cong \mathfrak{h}^*$ we have $\iota^*_{H_1}(\xi) \oplus \iota^*_{H \cap H_1^\perp}(\xi) = \iota^*_H(\xi)$. As we noted in the beginning of this section $\iota^*_H(\xi)$ is an element of the weight lattice $\mathfrak{h}^*_{\mathbb{Z}}$, the $H_1 \times (H \cap H_1^\perp)$-representation $\mathbb{C}_{\iota^*_{H_1}(\xi) \oplus \iota^*_{H \cap H_1^\perp}(\xi)}$ induces an H-representation $\mathbb{C}_{(\iota^*_H(\xi))}$, and hence, it implies that $RR_{H_1}(\nu_X(X_1)_x) \otimes RR_{H \cap H_1^\perp}(\nu_{X_1}(X')_x)$ decsends to an H-representation. In particular the index $RR_{H_1}(\nu_X(X_1)_x) \otimes RR_{H \cap H_1^\perp}(\nu_{X_1}(X')_x)$ is $H_1 \cap H \cap H_1^\perp$-invariant, and we complete the proof. □

Let $\iota_{H^\perp} : \mathfrak{h}^\perp \to \mathfrak{g}$ be the inclusion and $\iota^*_{H^\perp}$ its dual. We may assume that the moment map image $(\iota^*_{H^\perp} \circ \mu)(x) = \iota^*_{H^\perp}(\xi)$ of x with respect to the $H^\perp$-action is an element in the weight lattice $(\mathfrak{h}^\perp)^*_{\mathbb{Z}}$. Otherwise the compatible system on T^*H is acyclic, and hence, the local index $RR_{H^\perp}(T^*H^\perp)$ is zero.

Lemma 6.3. $RR_{H^\perp}(T^*H^\perp) = \mathbb{C}_{(\iota^*_{H^\perp}(\xi))} \in R(H^\perp)$.

Proof. Since the $H^\perp$-action on $T^*H^\perp$ is free, the induced good compatible fibration (system) on $TH^\perp$ consists of two open subsets, a small open neighbourhood of the zero-section $H^\perp$ and its complement. On the other hand by fixing a decomposition $H^\perp = (S^1)^m$, we have a product structure of compatible fibration and compatible system, where the S^1-equivariant data is determined by the inclusion $\iota_i : S^1 \hookrightarrow (S^1)^m = H^\perp$ to the ith factor for $i = 1, \cdots, m$. By applying Proposition B.2 the local index $RR_{H^\perp}(T^*H^\perp)$ is equal to the product of $RR_{S^1}(T^*S^1)$ defined the structure induced form ι_i's. Then the lemma follows from the computation of $RR_{S^1}(T^*S^1)$ (See [12, Proposition 5.3] for example.). □

Together with the product formula, Lemma 6.2 and Lemma 6.3 imply the following.

Proposition 6.4. *We have the equality*

$$RR_{H \times H^\perp}(\nu_X(X')_x \times T^*H^\perp) = \mathbb{C}_{(\iota^*_H(\xi) \oplus \iota^*_{H^\perp}(\xi))} \in R(H \times H^\perp).$$

Theorem 6.5. $\mathrm{ind}_G(N_F, N_F \setminus F) \neq 0$ *if and only if* $\xi \in \mathfrak{g}^*_{\mathbb{Z}}$, *and if* $\xi \in \mathfrak{g}^*_{\mathbb{Z}}$, *then we have* $\mathrm{ind}_G(N_F, N_F \setminus F) = \mathbb{C}_{(\xi)}$.

Proof. As we explained, $\mathrm{ind}_G(N_F, N_F \setminus F)$ is equal to the $H \cap H^\perp$-invariant part of $RR_{H \times H^\perp}(\nu_X(X')_x \times T^*H^\perp)$ which is represented by a one-dimensional representation of $H \times H^\perp$. Suppose that the invariant part is non-zero. Then the one-dimensional representation $RR_{H \times H^\perp}(\nu_X(X')_x \times T^*H^\perp)$ descends to a representation of G. Since $\mathfrak{h}^* \oplus \mathfrak{h}^{\perp *}$ is isomorphic to $\mathfrak{g}^*$

by $\iota_H^* \oplus \iota_{H^\perp}^*$, the invariant part is equal to the point $\iota_H^*(\xi) \oplus \iota_{H^\perp}^*(\xi) = \xi \in \mathfrak{g}_{\mathbb{Z}}^*$ by Proposition 6.4. Conversely if $\xi \in \mathfrak{g}_{\mathbb{Z}}^*$, then $RR_{H \times H^\perp}(\nu_X(X')_x \times T^*H^\perp)$ represents a point in $\mathfrak{g}_{\mathbb{Z}}^*$, and hence, a representation of G. In particular we have $\mathrm{ind}_G(N_F, N_F \setminus F) \neq 0$ as the $H \cap H^\perp$-invariant part. $\square$

Definition 6.6. A G-orbit F is called a *Bohr-Sommerfeld orbit* (*BS-orbit* for short) if there exists a non-trivial global parallel section on the restriction $(L_X, \nabla)|_F$.

Proposition 6.7. *A G-orbit F is BS-orbit if and only if* $\mathrm{ind}_G(N_F, N_F \setminus F) \neq 0$.

Proof. We fix the decomposition $H^\perp = (S^1)^m$ as in the proof of Lemma 6.3. The computation in [10, Remark 6.10] says that $RR_{S^1}(T^*S^1)$ is isomorphic to the space of parallel sections $\Gamma^{\mathrm{par}}(S^1, \iota_i^* \hat{L}_2|_{S^1})$ for each $i = 1, \cdots, m$. By the product structure of $(TH^\perp, \hat{L}_2)$ near the zero section, the space of parallel sections $\Gamma^{\mathrm{par}}(H^\perp, \hat{L}_2|_{H^\perp})$ is generated by a constant section and isomorphic to the product of $\Gamma^{\mathrm{par}}(S^1, \iota_i^* \hat{L}_2|_{S^1}) \cong RR_{S^1}(T^*S^1)$. It implies that $\Gamma^{\mathrm{par}}(H^\perp, \hat{L}_2|_{H^\perp})$ is isomorphic to $RR_{H^\perp}(T^*H^\perp)$ as $H^\perp$-representation. Similarly by considering the restriction to the origin, we have that the one-dimensional representation $RR_H(\nu_X(X')_x)$ is isomorphic to $L_X|_x$ as H-representation. Then we have that $RR_H(\nu_X(X')_x) \otimes RR_{H^\perp}(T^*H^\perp)$ is isomorphic to $\Gamma^{\mathrm{par}}(\{0\} \times H^\perp, L_X|_x \otimes \hat{L}_2|_{H^\perp})$ as $H \times H^\perp$-representation.

If F is a BS-orbit, then there exists a non-trivial global parallel section $s_F : F \to L|_F$. By considering the pull-back we have a non-trivial global parallel section $\widetilde{s_F} : H^\perp \to L_X|_x \otimes \hat{L}_2|_{H^\perp}$, which is $H \cap H^\perp$-invariant, and hence, it implies $\mathrm{ind}_G(N_F, N_F \setminus F) \neq 0$.

Conversely suppose that $\mathrm{ind}_G(N_F, N_F \setminus F) \neq 0$. Then by the isomorphism $RR_H(\nu_X(X')_x) \otimes RR_{H^\perp}(T^*H^\perp) \cong \Gamma^{\mathrm{par}}(\{0\} \times H^\perp, L_X|_x \otimes \hat{L}_2|_{H^\perp})$ there exists an $H \cap H^\perp$-invariant non-trivial global parallel sections of $L_X|_x \otimes \hat{L}_2|_{H^\perp}$. It induces a non-trivial global parallel section s_F of $L_X|_F$ by the natural map $H^\perp \to H^\perp \cdot x = F$. $\square$

When we consider the situation in Section 5.2 we have

$$\sum_{\xi \in \Delta_i} \mathrm{ind}_T(N_F, N_F \setminus F) = \sum_{j,k} \mathrm{ind}_T(V_{i,k}^{(j)+}, V_{i,k}^{(j)+} \cap V).$$

As a particular case we have a proof of Danilov's theorem for symplectic toric manifolds.

Theorem 6.8. *If X is a closed symplectic toric manifold with pre-quantizing line bundle L, then we have the following equality of the G-equivariant Riemann-Roch number.*

$$RR_G(X, L) = \sum_{\xi \in (\Delta_X)_{\mathbb{Z}}} \mathbb{C}_{(\xi)}.$$

6.2. Contribution from the fold. In the subsequent subsections we consider the toric origami case as in Theorem 5.9. In this subsection we compute the contribution from the folded part, $\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \setminus Z)$.

Theorem 6.9. *We have*

$$\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \setminus Z) = 0$$

as a T-equivariant index.

Proof. As it is showed in Proposition 5.3 and by definition of D_Z, the acyclic compatible system on $\mathcal{U}' \smallsetminus Z$ has a natural product structure between them on B and $S^1 \times (-\varepsilon/2, \varepsilon/2)$, and hence, its local index $\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \smallsetminus Z)$ is equal to the product of them in the sense of the product formula [11, Theorem 8.8]. On the other hand the compatible system on $S^1 \times (-\varepsilon/2, \varepsilon/2)$ is the one associated with the natural folded structure on it, and it will be shown in Appendix C that its local index is equal to 0. See Proposition C.2. These facts imply $\mathrm{ind}_T(\mathcal{U}', \mathcal{U}' \smallsetminus Z) = 0$. □

6.3. Contribution from the positive unfolded part. We compute the contribution from the unfolded part of the positive orientation, $\mathrm{ind}_T(V^{(j)+}_{i,k}, V^{(j)+}_{i,k} \smallsetminus (\mu^{(j)+}_{i,k})^{-1}((\mathfrak{g}^{(j)*}_{i,k})_{\mathbb{Z}}))$. Since $V^{(j)+}_{i,k}$ is away from the fold Z, the local situation is same as that for the genuine toric case, and hence, we can apply Theorem 6.5.

Theorem 6.10. *We may choose $\widetilde{\Delta}^{(j)+}_{i,k}$ small enough so that $\widetilde{\Delta}^{(j)+}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}} = \mathrm{int}\Delta^{(j)+}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}}$. Then we have*

$$\mathrm{ind}_T(V^{(j)+}_{i,k}, V^{(j)+}_{i,k} \smallsetminus (\mu^{(j)+}_{i,k})^{-1}((\mathfrak{g}^{(j)*}_{i,k})_{\mathbb{Z}})) = \sum_{\xi \in \mathrm{int}\Delta^{(j)+}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi)}.$$

Proof. Since the compatible system $\{D_Z, D^{(j)}_{i,k}\}_{i,j,k}$ is acyclic on V, the complement of the inverse images of lattice points, the excision formula implies that the T-equivariant local index $\mathrm{ind}_T(V^{(j)+}_{i,k}, V^{(j)+}_{i,k} \smallsetminus (\mu^{(j)+}_{i,k})^{-1}((\mathfrak{g}^{(j)*}_{i,k})_{\mathbb{Z}}))$ is equal to the sum of contributions of the inverse image of the lattice point which is contained in $V^{(j)+}_{i,k}$. Each inverse image has a neighbourhood of the form N_F as in Subsection 6.1, and hence, the contribution of the lattice point ξ is the representation corresponding to the lattice point $\mathbb{C}_{(\xi)}$. □

6.4. Contribution from the negative unfolded part. We compute the contribution from the unfolded part of the negative orientation, $\mathrm{ind}_T(V^{(j)-}_{i,k}, V^{(j)-}_{i,k} \smallsetminus (\mu^{(j)-}_{i,k})^{-1}((\mathfrak{g}^{(j)*}_{i,k})_{\mathbb{Z}}))$. The situation is same as that for the positive unfolded part up to the orientation. The difference appears only in the $\mathbb{Z}/2$-grading of the Clifford module bundle. Namely the $\mathbb{Z}/2$-grading in the negative case is opposite to the positive case, and hence, the resulting index has the opposite sign. The proof of the following theorem can be shown by the similar way for the proof of Theorem 6.10.

Theorem 6.11. *We may choose $\widetilde{\Delta}^{(j)-}_{i,k}$ small enough so that $\widetilde{\Delta}^{(j)-}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}} = \mathrm{int}\Delta^{(j)-}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}}$. Then we have*

$$\mathrm{ind}_T(V^{(j)-}_{i,k}, V^{(j)-}_{i,k} \smallsetminus (\mu^{(j)-}_{i,k})^{-1}((\mathfrak{g}^{(j)*}_{i,k})_{\mathbb{Z}})) = -\sum_{\xi \in \mathrm{int}\Delta^{(j)-}_{i,k} \cap \mathfrak{t}^*_{\mathbb{Z}}} \mathbb{C}_{(\xi)}.$$

6.5. Contribution from the crack. We compute the contribution from the crack, $\mathrm{ind}_T(V^Z_{i,k}, V^Z_{i,k} \cap V) = \mathrm{ind}_T(V^Z_{i,k}, V^Z_{i,k} \smallsetminus C^Z_{i,k})$, and show that it is equal to 0. Note that each $V^Z_{i,k}$ has two components $V^Z_{i,k} \cap M^+$ and $V^Z_{i,k} \cap M^-$. Then the open subsets $V^Z_{i,k} \cap M^+$ and $V^Z_{i,k} \cap M^-$ are isomorphic to each other as open symplectic toric manifolds up to their orientations.

Theorem 6.12. *We have the equality*

$$\mathrm{ind}_T(V^Z_{i,k}, V^Z_{i,k} \smallsetminus C^Z_{i,k}) = 0$$

as T-equivariant indices for each i and k.

Proof. Since $V_{i,k}^Z \cap M^+$ and $V_{i,k}^Z \cap M^-$ are isomorphic up to their orientations we have

$$\mathrm{ind}_T(V_{i,k}^Z, V_{i,k}^Z \smallsetminus C_{i,k}^Z) = \mathrm{ind}_T(V_{i,k}^Z \cap M^+, V_{i,k}^Z \cap M^+ \smallsetminus C_{i,k}^Z) + \mathrm{ind}_T(V_{i',k'}^Z \cap M^-, V_{i,k}^Z \cap M^- \smallsetminus C_{i,k}^Z) = 0.$$

□

Appendix A. Acyclic compatible systems and their local indices

In this appendix we give a brief summary of some definitions of compatible fibration, acyclic compatible system and their local indices following [11, 12] and [9]. We adopt combinations of definitions in [11] and [12]. Let V be a smooth manifold.

Definition A.1. A *compatible fibration on V* is a collection of the data $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ consisting of an open covering $\{V_\alpha\}_{\alpha\in A}$ of V and a foliation $\mathscr{F}_\alpha$ on V_α with compact leaves which satisfies the following properties.

(1) The holonomy group of each leaf of $\mathscr{F}_\alpha$ is finite.
(2) For each α and β, if a leaf $L \in \mathscr{F}_\alpha$ has non-empty intersection $L \cap V_\beta \neq \emptyset$, then, $L \subset V_\beta$.

Definition A.2. A compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ on V is called *good* if for all α and β with $V_\alpha \cap V_\beta \neq \emptyset$ the following condition (i) or (ii) holds.

(i) For each leaf $L_\alpha \in \mathscr{F}_\alpha$, there exists a leaf $L_\beta \in \mathscr{F}_\beta$ such that $L_\alpha \subset L_\beta$.
(ii) For each leaf $L_\beta \in \mathscr{F}_\beta$, there exists a leaf $L_\alpha \in \mathscr{F}_\alpha$ such that $L_\beta \subset L_\alpha$.

Let (V, g) be a Riemannian manifold, W a $Cl(TV)$-module bundle over V. Suppose that V is equipped with a compatible fibration $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$. We impose the following conditions on the Riemannian metric g.

Assumption A.3. Let $\nu_\alpha = \{u \in TV_\alpha \mid g(u, v) = 0 \text{ for all } v \in T\mathscr{F}_\alpha\}$ be the normal bundle of $\mathscr{F}_\alpha$. Then, $g|_{\nu_\alpha}$ is invariant under holonomy, and gives a transverse invariant metric on ν_α.

Definition A.4. A *compatible system* on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ is a data $\{D_\alpha\}_{\alpha\in A}$ satisfying the following properties.

(1) $D_\alpha\colon \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is an order-one formally self-adjoint differential operator.
(2) D_α contains only the derivatives along leaves of $\mathscr{F}_\alpha$.
(3) D_α is a Dirac-type operator along leaves. Namely the principal symbol of D_α is given by the composition of the dual of the natural inclusion $\iota_\alpha\colon T\mathscr{F}_\alpha \to TV_\alpha$ and the Clifford multiplication $c\colon T^*\mathscr{F}_\alpha \cong T\mathscr{F}_\alpha \subset TV_\alpha \to \mathrm{End}(W|_{V_\alpha})$.
(4) For a leaf $L \in \mathscr{F}_\alpha$ let $\widetilde{u} \in \Gamma(\nu_\alpha|_L)$ be a section of $\nu_\alpha|_L$ parallel along L. $\widetilde{u}$ acts on $W|_L$ by the Clifford multiplication $c(\widetilde{u})$. Then D_α and $c(\widetilde{u})$ anti-commute each other, i.e.

$$0 = \{D_\alpha, c(\widetilde{u})\} := D_\alpha \circ c(\widetilde{u}) + c(\widetilde{u}) \circ D_\alpha$$

as an operator on $W|_L$.

As in [11, Lemma 3.4] for each leaf $L \in \mathscr{F}_\alpha$ we have a small open tubular neighbourhood

V_L of L and the finite covering $q_L : \widetilde{V}_L \to V_L$ such that the induced foliation on $\widetilde{V}_L$ is a bundle foliation with the projection $\pi_L : \widetilde{V}_L \to \widetilde{U}_L$.

Definition A.5. A compatible system $\{D_\alpha\}_{\alpha\in A}$ on $(\{V_\alpha, \mathscr{F}_\alpha\}, W)$ is said to be *acyclic* if it satisfies the following conditions.

(1) The Dirac-type operator $q_L^* D_\alpha|_{\pi_L^{-1}(\widetilde{b})}$ has zero kernel for each $\alpha \in A$, leaf $L \in \mathscr{F}_\alpha$ and $\widetilde{b} \in \widetilde{U}_L$.

(2) If $V_\alpha \cap V_\beta \neq \emptyset$, then the anti-commutator $\{D_\alpha, D_\beta\} := D_\alpha D_\beta + D_\beta D_\alpha$ is a non-negative operator on $V_\alpha \cap V_\beta$.

As in [11, Section 5] we can construct such structures, good compatible fibration and compatible system, on Hamiltonian torus manifolds. Though the good compatible fibrations form a nice class, we have to generalize it to treat the product of such structures.

Definition A.6. Suppose that a compact Lie group G acts on a Riemannian manifold V in an isometric way. Let $\{V_\alpha, \mathscr{F}_\alpha\}_{\alpha\in A}$ be a compatible fibration on V. If the following conditions are satisfied, then we call the compatible fibration a *G-tangential compatible fibration* (or *tangential compatible fibration* for short).

- $\{V_\alpha\}_{\alpha\in A}$ is a G-invariant open covering of V.
- Each leaf L of $\mathscr{F}_\alpha$ has positive dimension for all $\alpha \in A$.
- For each leaf L of $\mathscr{F}_\alpha$ there exists some $x \in V_\alpha$ such that L is contained in the G-orbit $G \cdot x$.

A compatible system on a G-tangential compatible fibration is called *G-tangential compatible system* (or *tangential compatible system* for short).

Any non-trivial torus action induces a good compatible fibration, which is a tangential compatible fibration. Moreover the product of two such good compatible fibrations is a tangential compatible fibration which is not good in general.

Theorem A.7 (Theorem 7.2 and Proposition 7.3 in [11],Theorem 3.7 in [9])**.** *Suppose that V is an open subset of M whose complement is compact. If V is equipped with a G-tangential acyclic compatible system $\{V_\alpha, \mathscr{F}_\alpha, D_\alpha\}_{\alpha\in A}$, then we can define the local index*

$$\operatorname{ind}(M, \{V_\alpha, \mathscr{F}_\alpha, D_\alpha\}_{\alpha\in A}, W) = \operatorname{ind}(M, V, W) = \operatorname{ind}(M, V) \in \mathbb{Z},$$

which satisfies the excision formula, sum formula and product formula.

Let us briefly recall the definition of the local index $\operatorname{ind}(M, V, W)$. Let $D : \Gamma(W) \to \Gamma(W)$ be a Dirac-type operator. We consider the perturbation $D_t := D + t\sum_{\alpha\in A} \rho_\alpha D_\alpha \rho_\alpha$ for $t \gg 1$, where $\{\rho_\alpha\}_{\alpha\in A}$ is a family of smooth cut-off functions which is constant along leaves of $\mathscr{F}_\alpha$ and satisfies some estimates as in [11, Subection 4.1]. Such a perturbation D_t gives a Fredholm operator on the space of L^2-sections of W. The local index $\operatorname{ind}(M, V)$ is defined as the analytic index of D_t for $t \gg 1$. The excision formula implies the following localization formula of Dirac-type operator.

Theorem A.8. *Suppose that M is compact without boundary and an open subset V of M is equipped with a G-tangential acyclic compatible system. Then the index of any Dirac-*

type operator $\mathrm{ind}(W)$ *is localized at the complement* $M \smallsetminus V$. *Namely we have*

$$\mathrm{ind}(W) = \mathrm{ind}(M, V).$$

Appendix B. A formula of local indices of vector spaces

In this appendix we give a formula of equivariant local indices of vector spaces. For $l = 1, 2$ let G_l be an m_l-dimensional torus and R_l an m_l-dimensional Hermitian vector space on which G_l acts unitary and effective way. We put the following assumptions for $l = 1, 2$.

Assumption B.1. (1) A G_l-tangential equivariant compatible fibration (Definition A.6) on $R_l^\times := R_l \smallsetminus \{0\}$ is given.
(2) For the compatible fibration in (1), a G_l-tangential equivariant acyclic compatible system on $R_l^\times$ is given.

By the assumption we have two equivariant local indices $\mathrm{ind}_{G_1}(R_1, R_1^\times)$ and $\mathrm{ind}_{G_2}(R_2, R_2^\times)$. Now we fix $\varepsilon > 0$ small enough and define two compatible fibrations and acyclic compatible systems on the product $R := R_1 \times R_2$.

Define two subsets R' and R'' of R by

$$R' := \{(v_1, v_2) \in R \mid |v_1| > \varepsilon,\ |v_2| < \varepsilon\},$$

and

$$R'' := \{(v_1, v_2) \in R \mid |v_1| < \varepsilon,\ |v_2| > \varepsilon\}.$$

We consider a structure of G_1-tangential (resp. G_2-tangential) compatible fibration on R' (resp. R'') induced from the first (resp. second) factor. We also define a subset R_∞ of R by

$$R_\infty := \{(v_1, v_2) \in R \mid |v_1| > \varepsilon/2,\ |v_2| > \varepsilon/2\},$$

which is also equipped with a compatible fibration and compatible system arising from the product structure. Then the union $\widetilde{R}_\infty := R' \cup R'' \cup R_\infty$ gives an open covering of the complement of a compact neighbourhood of the origin of R. Note that the above compatible fibration and compatible system define a $G_1 \times G_2$-tangential equivariant compatible fibration

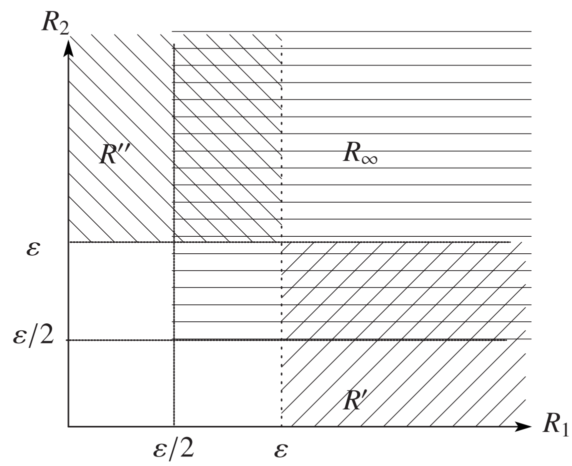

Fig. 5. Open covering $\widetilde{R}_\infty$.

and acyclic compatible system on $\widetilde{R}_\infty$, and hence, we have the equivariant local index

$$\mathrm{ind}_{G_1\times G_2}(R,\widetilde{R}_\infty).$$

For $l=1,2$ define open subsets $R_{l,0}$ and $R_{l,\infty}$ of R_l by

$$R_{l,0}:=\{v\in R_l \mid |v|<\varepsilon\},$$

and

$$R_{l,\infty}:=\{v\in R_l \mid |v|>\varepsilon/2|\}.$$

We set $R_\infty^{\mathrm{prod}}:=(R_{1,\infty}\times R_{2,0})\cup(R_{1,0}\times R_{2,\infty})\cup(R_{1,\infty}\times R_{2,\infty})$, which gives an open covering of a complement of a compact neighbourhood of the origin of R. We consider the trivial fibration on $R_{l,0}$ and the G_l-tangential compatible fibration on $R_{l,\infty}$. The product of these structures induces a $G_1\times G_2$-tangential equivariant compatible fibration and acyclic compatible system on R_∞^{prod}, and hence, we have the equivariant local index

$$\mathrm{ind}_{G_1\times G_2}(R,R_\infty^{\mathrm{prod}}).$$

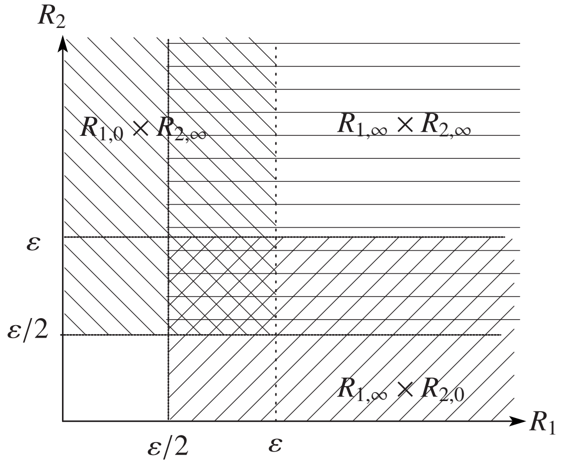

Fig. 6. Open covering R_∞^{prod}.

Proposition B.2. *We have the following equality among equivariant local indices.*

$$\mathrm{ind}_{G_1\times G_2}(R,\widetilde{R}_\infty)=\mathrm{ind}_{G_1\times G_2}(R,R_\infty^{\mathrm{prod}})=\mathrm{ind}_{G_1}(R_1,R_1^\times)\otimes\mathrm{ind}_{G_2}(R_2,R_2^\times)\in R(G_1\times G_2).$$

Proof. The first equality follows from the cobordism invariance of local index ([9, Theorem 7.1]). In fact the union of these two acyclic compatible systems on $\widetilde{R}_\infty$ and R_∞^{prod} is also $G_1\times G_2$-tangential acyclic compatible system. The second equality follows from the product formula ([11, Theorem 8.8]). □

Appendix C. Local index of folded cylinder

In this appendix we consider a natural folded symplectic structure on the cylinder and several geometric structures on it, which plays important role in the study of local property of the neighbourhood of the fold in a folded symplectic manifold. We consider a perturbation of the Dirac operator and give the direct computation of the L^2-kernel of the perturbed Dirac operator. We show that the L^2-kernel is trivial, in particular, the local index is equal to 0.

For any $\varepsilon > 0$, a folded symplectic structure on a cylinder (of finite length) $M_\varepsilon := (-\varepsilon, \varepsilon)\times S^1$ is given by a closed 2-form $2rdr \wedge d\theta$, where (r, θ) is a coordinate function on M_ε. Here we use the opposite orientation of the cylinder as that in Section 5 and subsequent argument for conventional reason. The standard S^1-action on the S^1-factor is Hamiltonian (in fact it is toric origami) with the moment map $(r, \theta) \mapsto r^2$. Moreover the trivial line bundle L_0 with connection $d - 2\pi\sqrt{-1}r^2 d\theta$ and the trivial lift of the S^1-action to the fiber direction gives an S^1-equivariant pre-quantizing line bundle over M_ε. To give a computation of the local index of this toric origami manifold, we need a Clifford module bundle, Dirac-type operator along the S^1-orbits over a completion of M_ε as a Riemannian manifold. We summarize the set-up as follows.

SET-UP.

- $M := \mathbb{R} \times S^1$: cylinder of infinite length
- (r, θ) : coordinate function on M
- $g := dr^2 + d\theta^2$: Riemannian metric on M
- $\rho : \mathbb{R} \to \mathbb{R}$: smooth function with

$$\rho(r) = \begin{cases} r^2 & (|r| < 1/4)) \\ 1/2 & (|r| > 1/2) \end{cases}$$

- $\omega := \rho'(r)dr \wedge d\theta$: closed 2-form on M
- $J : \partial_r \mapsto \partial_\theta,\ \partial_\theta \mapsto -\partial_r$: almost complex structure on M
- $TM_{\mathbb{C}} = (TM, J)$: complex tangent bundle with frame ∂_θ
- $W^+ := M \times \mathbb{C},\ W^- := TM_{\mathbb{C}},\ W := W^+ \oplus W^-$: $\mathbb{Z}/2$-graded vector bundle
- $c : T^*M \to \mathrm{End}(W)$: Clifford action on W defined by

$$c(dr) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}, \quad c(d\theta) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- $\nabla^W = d - 2\pi\rho(r)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} d\theta$: Clifford connection of W
- $D = D^+ + D^- : \Gamma(W) \to \Gamma(W)$: Dirac operator,

$$\begin{aligned} D &= c(\partial_r)\nabla^W_{\partial_r} + c(\partial_\theta)\nabla^W_{\partial_\theta} = D^+ + D^- \\ &= \begin{pmatrix} 0 & -\partial_\theta - \sqrt{-1}\partial_r + 2\pi\sqrt{-1}\rho \\ \partial_\theta - \sqrt{-1}\partial_r - 2\pi\sqrt{-1}\rho & 0 \end{pmatrix} \end{aligned}$$

- Let S^1 acts on M in the standard way, and we take a lift of the S^1-action on W so that the action on the fiber direction is trivial.
- $D_{S^1} = D^+_{S^1} + D^-_{S^1} : \Gamma(W) \to \Gamma(W)$: Dirac operator along the S^1-orbits,

$$D_{S^1} = c(\partial_\theta)\nabla^W_{\partial_\theta} = \begin{pmatrix} 0 & -\partial_\theta + 2\pi\sqrt{-1}\rho \\ \partial_\theta - 2\pi\sqrt{-1}\rho & 0 \end{pmatrix}$$

REMARK C.1. When we consider the restriction to an S^1-invariant small open neighbourhood M_ε of $\{0\} \times S^1 = S^1$ in M, the closed 2-form ω is the folded symplectic form on M_ε

and the $\mathbb{Z}/2$-graded Clifford module bundle W is the one associated with the pre-quantizing line bundle L_0, the trivial line bundle with the connection $d - 2\pi\sqrt{-1}\rho d\theta$. Note that M_ε has a unique spin^c-structure and the Clifford module bundle W which is isomorphic to $W_{0,L_0} := \mathrm{Hom}_{Cl_2}(W_2, \wedge^\bullet_{\mathbb{C}}(TS^1 \oplus \mathbb{R}^3)) \otimes L_0$ as in Proposition 5.3.

By using this data we have a compatible system on M_ε and can define the S^1-equivariant local index $\mathrm{ind}_{S^1}(S^1 \times (-\varepsilon, \varepsilon), S^1 \times (-\varepsilon, \varepsilon) \smallsetminus S^1)$. The index is defined by the following perturbation of the Dirac operator:

$$D_t = D_t^+ + D_t^-,\ D_t^+ := D^+ + tD^+_{S^1},\ D_t^- := D^- + tD^-_{S^1},$$

$$D_t^+ = (1+t)(\partial_\theta - 2\pi\sqrt{-1}\rho) - \sqrt{-1}\partial_r,$$

and

$$D_t^- = -(1+t)(\partial_\theta - 2\pi\sqrt{-1}\rho) - \sqrt{-1}\partial_r.$$

Proposition C.2. *We have* $\ker_{L^2}(D_t^+) = \ker_{L^2}(D_t^-) = 0$ *for any* $t \geq 0$. *In particular we have* $\mathrm{ind}_{S^1}(S^1 \times (-\varepsilon, \varepsilon), S^1 \times (-\varepsilon, \varepsilon) \smallsetminus S^1) = 0$, *for any* $\varepsilon > 0$.

Proof. By using the Fourier expansion $\phi(r, \theta) = \sum_{m\in\mathbb{Z}} a_m(r)e^{2\pi\sqrt{-1}m\theta}$ for smooth section ϕ of W^+, the equation $D_t^+\phi = 0$ can be rewritten as a series of differential equations

$$a'_m(r) = 2\pi(1+t)(m - \rho(r))a_m(r) \quad (m \in \mathbb{Z}).$$

Each of these equations has solutions

$$a_m(r) = \alpha_m \exp\left(2\pi(1+t)\int_0^r (m - \rho(r))dr\right),$$

where $\alpha_m \in \mathbb{C}$ is constant. Suppose that the solution ϕ is an L^2-section. Since $\rho \equiv 1/2$ on $\pm r \gg 0$ we have $\alpha_m = 0$ for all $m \in \mathbb{Z}$. In particular there are no non-trivial L^2-solutions of $D_t^+\phi = 0$ for any t. As in the same way the equation $D_t^-\phi = 0$ for $\phi(r, \theta) = \sum_{m\in\mathbb{Z}} b_m(r)e^{2\pi\sqrt{-1}m\theta}$ has solutions

$$b_m(r) = \beta_m \exp\left(-2\pi(1+t)\int_0^r (m - \rho(r))dr\right) \quad (m \in \mathbb{Z}),$$

for any constant $\beta_m \in \mathbb{C}$ and one can see that there are no non-trivial L^2-solutions. □

REMARK C.3. The vanishing of the index can be deduced from the existence of an orientation reversing isomorphism of $S^1 \times (-\varepsilon, \varepsilon)$ defined by $(\theta, t) \mapsto (\theta, -t)$.

ACKNOWLEDGEMENTS. The author would like to thank Mikio Furuta and Takahiko Yoshida for stimulating conversations. Especially the argument in Section 6.1 is based on the discussion with them. The author is grateful to the anonymous referee for his/her comments and pointing out some mistakes on the previous version. Due to referee's comments on the proof of Theorem 6.12, the author could simplify it.

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