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ON KOHNEN PLUS-SPACE OF JACOBI FORMS OF HALF INTEGRAL WEIGHT OF MATRIX INDEX

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Abstract

We introduce a plus-space of Jacobi forms, which is a certain subspace of Jacobi forms of half-integral weight of matrix index. This is an analogue to the Kohnen plus-space in the framework of Jacobi forms. We shall show a linear isomorphism between the plus-space of Jacobi forms and the space of Jacobi forms of integral weight of certain matrix index. Moreover, we shall show that this linear isomorphism is compatible with the action of Hecke operators of both spaces. This result is a kind of generalization of Eichler-Zagier-Ibukiyama correspondence, which is an isomorphism between the generalized plus-space of Siegel modular forms of general degree and Jacobi forms of index 1 of general degree.

1. Introduction

The Kohnen plus-space plays an important role in the theory of modular forms of half-integral weight. In the present article we will introduce an analogue of Kohnen plus-space for Jacobi forms of half-integral weight of certain matrix indices.

Let $\mathcal{M}_1 \in L_r^*$ be a half-integral symmetric matrix of size r and $L \in M_{r,1}(\mathbb{Z})$ be a column vector of size r such that the matrix

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}L \\ \frac{1}{2}{}^tL & 1 \end{pmatrix} \in L_{r+1}^*$$

is positive-definite. We put

$$\mathfrak{M} := 4\mathcal{M}_1 - L\,{}^tL \in L_r^*.$$

Remark that $\mathfrak{M}$ is positive-definite.

Let k and n be non-negative integers. Let $J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+}$ be the plus-space of Jacobi forms (see §3 for the definitinon) which is a certain subspace of $J_{k-\frac{1}{2},\mathfrak{M}}^{(n)}$, where $J_{k-\frac{1}{2},\mathfrak{M}}^{(n)}$ denotes the space of Jacobi forms of weight $k-\frac{1}{2}$ and index $\mathfrak{M}$ on $\Gamma_0^{(n)}(4)$, and where $\Gamma_0^{(n)}(4)$ consists of matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$ such that $C \equiv 0_n \mod 4$. Let $J_{k,\mathcal{M}}^{(n)}$ be the space of Jacobi forms of weight k and index $\mathcal{M}$ on Γ_n. We denote by $J_{k-\frac{1}{2},\mathfrak{M}}^{cusp\,(n)+}$ (resp. $J_{k,\mathcal{M}}^{cusp\,(n)}$) the subspace of Jacobi cusp forms in $J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+}$ (resp. $J_{k,\mathcal{M}}^{(n)}$). The main result is

Theorem 1.1. *Let k be an even integer. We obtain linear isomorphisms*

$$J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+} \cong J_{k,\mathcal{M}}^{(n)}$$

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and

$$J^{cusp\,(n)+}_{k-\frac{1}{2},\mathfrak{M}} \cong J^{cusp\,(n)}_{k,\mathcal{M}}$$

as Hecke algebra modules.

Remark that we can regard Jacobi forms as vector valued modular forms with certain (projective) representations of $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$. Therefore, the above isomorphisms are certain isomorphisms between certain vector valued modular forms with respect to $\mathrm{Mp}(n,\mathbb{Z})$, which is the metaplectic cover of Γ_n.

In this paper we shall also give a necessary and sufficient condition of Jacobi cusp forms with a bound of the absolute value of the Jacobi forms (Lemma 2.1). The half of the statement of Lemma 2.1 has already been shown by Murase [4, Lemma 1.4].

We also remark that Theorem 1.1 gives a generalization of the result given by Ibukiyama [3, Theorem 1] (it is the case $r = 0$). The most part of the proof of Theorem 1.1 is an analogue to the one of [3, Theorem 1]. However, the calculations are more complicate, because we have to treat the Jacobi group instead of the Siegel modular group.

The importance of the plus space of Jacobi forms will be recognized, when one considers the Fourier-Jacobi expansion of Siegel modular forms of half-integral weight which belong to the plus space of Siegel modular forms. In particular, a weak version of Theorem 1.1 has been used in [2] to prove a certain lifting from pairs of two elliptic modular forms to Siegel modular forms of half-integral weight of even degrees. Here a weak version means that we have treated in [2] a certain subspace of $J^{+(n)}_{k-\frac{1}{2},\mathfrak{M}}$ ($\mathfrak{M} \in L_2^*$, $r = 1$), which is obtained through the Fourier-Jacobi expansion of Siegel modular forms belonging to the plus space of Siegel modular forms (see [2, Lemma 4.2] for the detail).

The paper is organized as follows. In Section 2 we give a notation and definitions what we need later to state our result precisely, and we shall show a lemma which states a necessary and sufficient condition of Jacobi cusp forms (Lemma 2.1). In Section 3 we shall introduce the plus-space of Jacobi forms. In Section 4 we shall prove the bijection of the linear map appeared in Theorem 1.1, while in Section 5 we shall prove the compatibility between this linear map and the action of Hecke operators of both spaces.

2. Notation

We denote by $\mathbb{R}^+$ the set of all positive real numbers. The symbol $M_{n,m}(R)$ denotes the set of $n \times m$ matrices with entries in a ring R, and we put $M_n(R) := M_{n,n}(R)$. The letter $\mathrm{Sym}_n(R)$ denotes the set of symmetric matrices of size n with entries in a ring R. We denote by L_n^* the set of all semi positive-definite, half-integral symmetric matrices of size n, and we denote by L_n^+ all positive definite matrices in L_n^*. We write ${}^tB$ for the transpose of a matrix B. For two matrices $A \in M_n(R)$ and $B \in M_{n,m}(R)$ we write $A[B] := {}^tBAB$. The identity matrix (resp. zero matrix) of size n is denoted by 1_n (resp. 0_n). The symbol $\mathrm{tr}(S)$ denotes the trace of a square matrix S and we put $e(S) := e^{2\pi\sqrt{-1}\,\mathrm{tr}(S)}$ for a square matrix S. For square matrices $a_1, ..., a_n$, we write the diagonal matrix $\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ as $\mathrm{diag}(a_1, ..., a_n)$. For any odd prime p the symbol $\left(\frac{*}{p}\right)$ denotes the Legendre symbol.

The symbol $\mathfrak{H}_n$ denotes the Siegel upper half space of degree n, and $\mathrm{Sp}(n,\mathbb{R})$ denotes the real symplectic group of size $2n$. We put $\Gamma_n := \mathrm{Sp}(n,\mathbb{Z})$. We denote by $M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ the vector space of Siegel modular forms of weight $k-\frac{1}{2}$ of degree n. The plus-space of $M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ is denoted by $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$, which is a certain subspace of $M_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ and it is a generalization of Kohnen plus-space for general degree (cf. Ibukiyama [3]). The symbol $S^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ denotes the vector space of all Siegel cusp forms in $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$.

2.1. Jacobi group. We put

$$\mathrm{GSp}^+(n,\mathbb{R}) := \left\{ M \in \mathrm{GL}(2n,\mathbb{R}) \;\middle|\; M\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}{}^tM = \gamma\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \text{ with some } \gamma \in \mathbb{R}^+ \right\}.$$

The number γ in the above set depends on the choice of M and is called the similitude of M. For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}^+(n,\mathbb{R})$ and for any $\tau \in \mathfrak{H}_n$, the linear fractional transformation is defined by $M\cdot\tau := (A\tau+B)(C\tau+D)^{-1}$.

We set

$$G^J_{n,r} \;:=\; \{(M,[(\lambda,\mu),\kappa]) \mid M \in \mathrm{GSp}^+(n,\mathbb{R}), \lambda,\mu \in M_{n,r}(\mathbb{R}), \kappa \in Sym_r(\mathbb{R})\}.$$

By the embedding

$$\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, [(\lambda,\mu),\kappa]\right) \to \begin{pmatrix} A & 0 & B & 0 \\ 0 & \gamma 1_r & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_r \end{pmatrix}\begin{pmatrix} 1_n & 0 & 0 & \mu \\ {}^t\lambda & 1_r & {}^t\mu & {}^t\lambda\mu+\kappa \\ 0 & 0 & 1_n & -\lambda \\ 0 & 0 & 0 & 1_r \end{pmatrix},$$

we can regard $G^J_{n,r}$ as a subgroup of $\mathrm{GSp}^+(n+r,\mathbb{R})$, where γ is the similitude of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. In particular, $G^J_{n,r}$ is viewed as a semi direct product: $G^J_{n,r} = GSp^+(n,\mathbb{R}) \ltimes H_{n,r}(\mathbb{R})$, where

$$H_{n,r}(\mathbb{R}) := \{[(\lambda,\mu),\kappa] \,|\, \lambda,\mu \in M_{n,r}(\mathbb{R}), \kappa \in Sym_r(\mathbb{R})\}.$$

For any $M \in \mathrm{GSp}^+(n,\mathbb{R})$, $h = [(\lambda,\mu),\kappa]$, $h' = [(\lambda',\mu'),\kappa'] \in H_{n,r}(\mathbb{R})$ the composition rule is given by

$$\begin{aligned} hh' &= [(\lambda+\lambda',\mu+\mu'),\kappa+\kappa'-{}^t\lambda'\mu-{}^t\mu\lambda'], \\ M^{-1}hM &= [(\lambda^*,\mu^*),\gamma^{-1}(\kappa+{}^t\lambda\mu)-{}^t\lambda^*\mu^*], \end{aligned}$$

where $\begin{pmatrix} \lambda^* \\ \mu^* \end{pmatrix} = \gamma^{-1t}M\begin{pmatrix} \lambda \\ \mu \end{pmatrix}$ and where γ is the similitude of M. We call $G^J_{n,r}$ the Jacobi group. By abuse of language, for $g = (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), [(\lambda,\mu),\kappa]) \in G^J_{n,r}$ we call γ the similitude of g, if γ is the similitude of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. For the sake of simplicity, if there is no confusion, we write M for $(M,[(0,0),0]) \in G^J_{n,r}$ and write $[(\lambda,\mu),\kappa]$ for $(1_{2n},[(\lambda,\mu),\kappa]) \in G^J_{n,r}$.

We set $H_{n,r}(\mathbb{Z}) := \{[(\lambda,\mu),\kappa] \,|\, \lambda,\mu \in M_{n,r}(\mathbb{Z}), \kappa \in Sym_r(\mathbb{Z})\}$.

We define the action of $G^J_{n,r}$ on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$: for any $g = (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), [(\lambda,\mu),\kappa]) \in G^J_{n,r}$ and for any $(\tau,z) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$, we define

$$g\cdot(\tau,z) \;:=\; (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\cdot\tau, \gamma{}^t(C\tau+D)^{-1}(z+\tau\lambda+\mu)),$$

where $\gamma \in \mathbb{R}^+$ is the similitude of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$.

Let $S \in \mathrm{Sym}_r(\mathbb{R})$ and let $l \in \mathbb{R}$. For any $g = (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), [(\lambda,\mu),\kappa]) \in G^J_{n,r}$ and for any $(\tau,z) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$ we set

$$\begin{aligned} & J_{l,S}(g,(\tau,z)) \\ := \quad & \det(C\tau+D)^l e(\gamma S((C\tau+D)^{-1}C)[z+\tau\lambda+\mu])e(-\gamma S(\tau[\lambda]+2{}^t\lambda z+2{}^t\lambda\mu+\kappa)), \end{aligned}$$

where we take the principal branch $-\pi < \arg(\det(C\tau+D)) \leq \pi$, and where γ is the similitude of g.

If $l \in \mathbb{Z}$, the factor $J_{l,S}(*,*)$ satisfies the cocycle condition:

$$J_{l,S}(g_1g_2,(\tau,z)) = J_{l,S}(g_1, g_2\cdot(\tau,z))J_{l,\gamma_1 S}(g_2,(\tau,z)),$$

where $g_1, g_2 \in G^J_{n,r}$ and where $\gamma_1 \in \mathbb{R}^+$ is the similitude of g_1. We define

$$(\phi|_{l,S}\, g)(\tau,z) \quad := \quad J_{l,S}(g,(\tau,z))^{-1}\phi(g\cdot(\tau,z))$$

for a function ϕ on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$ and for $g \in G^J_{n,r}$.

2.2. Covering group. The group $\mathfrak{G}$ consists of (M,φ), such that $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{GSp}^+(n,\mathbb{R})$ and φ is a holomorphic function on $\mathfrak{H}_n$ which satisfy the relation $|\varphi(\tau)|^2 = \det M^{-\frac{1}{2}}|\det(C\tau+D)|$. The group action of $\mathfrak{G}$ is defined by $(M_1,\varphi_1(\tau))\cdot(M_2,\varphi_2(\tau)) := (M_1M_2, \varphi_1(M_1\cdot\tau)\varphi_2(\tau))$ for $(M_i,\varphi_i) \in \mathfrak{G}$ $(i = 1,2)$.

We put $\theta(\tau) := \theta_n(\tau) = \sum_{p\in M_{n,1}(\mathbb{Z})} e(\frac{1}{4}{}^t p\tau p)$ for $\tau \in \mathfrak{H}_n$. We denote by $\Gamma_0^{(n)}(4)^*$ the subgroup of $\mathfrak{G}$ which consists of $(M, \theta(M\cdot\tau)\theta(\tau)^{-1})$ with $M \in \Gamma_0^{(n)}(4)$, where $\Gamma_0^{(n)}(4)$ consists of matrix $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_n$ such that $C \equiv 0_n \mod 4$.

2.3. Jacobi forms of half-integral weight. For the definition of Jacobi forms of integral weight, the reader is referred to Ziegler [6]. In this subsection we review a definition of Jacobi forms of half-integral weight.

Let n, $r \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. A holomorphic function $\phi : \mathfrak{H}_n \times M_{n,r}(\mathbb{C}) \to \mathbb{C}$ is called a Jacobi form of weight $k-\frac{1}{2}$ and index $S \in L_r^+$ on $\Gamma_0^{(n)}(4)$, if ϕ satisfies the following three conditions:

(i) For any $M^* \in \Gamma_0^{(n)}(4)^*$,

$$\phi|_{k-\frac{1}{2},S}\, M^* = \phi.$$

Here we defined

$$\left(\phi|_{k-\frac{1}{2},S}(M,\varphi)\right)(\tau,z) \quad := \quad \varphi(\tau)^{-2k+1}e(-S\,{}^t z(C\tau+D)^{-1}Cz)\phi(M\cdot(\tau,z))$$

for $(M,\varphi) \in \mathfrak{G}$ $(M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right))$.

(ii) For any $\lambda, \mu \in M_{n,r}(\mathbb{Z})$,

$$e(\lambda S\,{}^t\lambda\tau + 2\lambda S\,{}^t z)\phi(\tau, z+\tau\lambda+\mu) \quad = \quad \phi(\tau,z).$$

(iii) For any $M \in \Gamma_n$, the form $\phi^2|_{2k-1,2S}\, M$ has the Fourier expansion:

$$(\phi^2|_{2k-1,2S}\, M)(\tau,z) \quad = \quad \sum_{N,R} C_M(N,R)\, e\!\left(\frac{1}{h}N\tau + R^t z\right)$$

with a positive number h. Here, in the summation, $N \in L_n^*$ and $R \in M_{n,r}(\mathbb{Z})$ run over all matrices such that $4h^{-1}N - R(2S)^{-1}{}^tR \geq 0$.

For $n \geq 2$, it is known that the condition (iii) follows from the other conditions by the Koecher principle (cf. [6]). For a semi positive-definite matrix $S \in L_r^*$, the Jacobi forms of index S is defined likely as in the case of integral weight (cf. [6, Def. 1.3.]).

For any half-integer $k-\frac{1}{2}$ and for any $S \in L_r^*$ we denote by $J^{(n)}_{k-\frac{1}{2},S}$ the vector space of Jacobi forms of weight $k-\frac{1}{2}$ and index S on $\Gamma_0^{(n)}(4)$. For any integer l and for any $S \in L_r^*$ we denote by $J^{(n)}_{l,S}$ the vector space of Jacobi forms of weight l and index S on Γ_n.

Let $\phi \in J^{(n)}_{l,S}$ be a Jacobi form for $l \in \frac{1}{2}\mathbb{Z}$. We call ϕ a *Jacobi cusp form*, if ϕ satisfies the condition that for any $M \in \Gamma_n$ the Fourier coefficients $C_M(N,R)$ satisfy $C_M(N,R) = 0$ unless $4h^{-1}N - R(2S)^{-1}{}^tR > 0$.

Lemma 2.1. *Let $l \in \frac{1}{2}\mathbb{Z}$ be an integer or a half-integer. Let ϕ be a Jacobi form of weight l of index $S \in L_r^*$ of degree n. Then ϕ is a Jacobi cusp form, if and only if*

$$|\det Y^{l/2} e^{-2\pi\, tr(Y^{-1}S[{}^t\beta])}\,\phi(\tau,z)|$$

is bounded on the domain $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. Here $(\tau,z) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$, $Y = Im(\tau)$ and $\beta = Im(z)$.

Proof. We shall show only the case $S \in L_r^+$, since the proof for $S \in L_r^*$ is similar.

We put $g(\tau,z) := \det Y^{l/2} e^{-2\pi\, tr(Y^{-1}S[{}^t\beta])}\,\phi(\tau,z)$. When $l \in \mathbb{Z}$ it is shown in [4, Lemma 1.4] that if $|g(\tau,z)|$ is bounded on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$, then ϕ is a Jacobi cusp form. When $l \in \frac{1}{2}\mathbb{Z}$, ϕ^2 has weight $2l \in \mathbb{Z}$ and index $2S$. Thus, if $|g(\tau,z)|$ is bounded, then $|g(\tau,z)|^2$ is bounded and ϕ is a Jacobi cusp form due to the definition. Hence, we only need to show the opposite direction.

We assume that ϕ is a Jacobi cusp form. We write $Z_0 := (i1_n, 0) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. We remark that there exists a $\xi \in G^J_{n,r}$ such that $\xi \cdot Z_0 = (\tau,z)$ for a fixed $(\tau,z) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. With this ξ we have

$$g(\tau,z) = \left|J_{l,S}(\xi,Z_0)^{-1}\right| \phi(\xi \cdot Z_0).$$

For any $M \in \Gamma_n$ we put $g_M(\tau,z) := g(M\cdot(\tau,z)) = g(M\xi\cdot Z_0)$. For any $(\lambda,\mu) \in M_{n,r}(\mathbb{Z}) \times M_{n,r}(\mathbb{Z})$, we have

$$\begin{aligned} (2.1) \qquad |g_M(\tau, z+\tau\lambda+\mu)| &= |g((M[(\lambda,\mu),0]\xi)\cdot Z_0)| \\ &= |g(([(\lambda',\mu'),\kappa']M\xi)\cdot Z_0)| \\ &= |g_M(\tau,z)|, \end{aligned}$$

where $[(\lambda',\mu'),\kappa'] = M[(\lambda,\mu),0]M^{-1} \in H_{n,r}(\mathbb{Z})$. Here we used the condition (ii) in the definition of Jacobi forms. Let

$$(\phi^2|_{2l,2S}M)(\tau,z) = \sum_{N,R} C_M(N,R)\, e\!\left(\frac{1}{h}N\tau + R^t z\right)$$

be the Fourier expansion of $\phi^2|_{2l,2S}M$. Then we have

$$\begin{aligned} |g_M(\tau,z)|^2 &= |g(M\cdot(\tau,z))|^2 = |g(M\xi\cdot Z_0)|^2 \\ &= |J_{2l,2S}(M\xi,Z_0)^{-1}\phi(M\xi\cdot Z_0)^2| \\ &= |J_{2l,2S}(M,\xi\cdot Z_0)^{-1}J_{2l,2S}(\xi,Z_0)^{-1}\phi(M\xi\cdot Z_0)^2| \\ &= |J_{2l,2S}(\xi,Z_0)^{-1}(\phi^2|_{2l,2S}M)(\tau,z)| \\ &= \left|\det Y^l e^{-2\pi\, tr(Y^{-1}(2S)[\beta])}\sum_{N,R} C_M(N,R)\, e\!\left(\frac{1}{h}N\tau + R^t z\right)\right|. \end{aligned}$$

Thus,

$$|g_M(\tau + hB, z)| = |g_M(\tau, z)| \tag{2.2}$$

for any $B \in Sym_n(\mathbb{Z})$. For $Y = \mathrm{Im}(\tau)$ we put

$$P(Y) \;\; := \;\; \left\{ Y\lambda \in M_{n,r}(\mathbb{R}) \;\middle|\; \lambda = (\lambda_{i,j}) \in M_{n,r}(\mathbb{R}),\ |\lambda_{i,j}| \le \frac{1}{2} \right\}.$$

For a positive number c, we set

$$D_c := \left\{ (\tau, z) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C}) \;\middle|\; |x_{i,j}| \le \frac{1}{2}h,\ |\alpha_{i,j}| \le \frac{1}{2},\ Y > c1_n,\ \beta \in P(Y) \right\},$$

where $x_{i,j}$ is the (i, j)-th component of $X = Re(\tau)$ and $\alpha_{i,j}$ is the (i, j)-th component of $\alpha = Re(z)$.

To show that $|g(\tau, z)|$ is bounded on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$, it is enough to show that for any $M \in \Gamma_n$, we have $|g_M(\tau, z)| \to 0$ for $\mathrm{tr}(Y) \to +\infty$ or for $|\beta_{i,j}| \to +\infty$ with a (i, j). Here $\beta_{i,j}$ denotes the (i, j)-th component of β. In particular, due to the identities (2.1) and (2.2), we can take such limit in the domain D_c. We have the fact that if $(\tau, z) \in D_c$ and if $|\beta_{i,j}| \to +\infty$ with a (i, j), then $\mathrm{tr}(Y) \to +\infty$. Therefore we only need to show $|g_M(\tau, z)| \to 0$ for $\mathrm{tr}(Y) \to +\infty$.

As for the Fourier expansion of $\phi^2|_{2l,2S} M$ we remark that $C_M(N, R) = 0$ unless $\frac{1}{h}N - \frac{1}{4}R(2S)^{-1\,t}R > 0$. Thus, we assume the condition $\frac{1}{h}N - \frac{1}{4}R(2S)^{-1}\left[{}^tR\right] > 0$.

We write $T = 2S$. We now have

$$\begin{aligned} & -Y^{-1}T[\beta] - \frac{1}{h}NY - \frac{1}{2}R\left[{}^t\beta\right] - \frac{1}{2}Y^{-1}\beta\left[{}^tR\right]Y \\ = \;& -\left(\frac{1}{h}N - \frac{1}{4}RT^{-1}\left[{}^tR\right]\right)Y - Y^{-1}T\left[{}^t\left(\beta + \frac{1}{2}YRT^{-1}\right)\right]. \end{aligned}$$

Hence

$$\begin{aligned} & |g_M(\tau, z)|^2 \\ = \;& \left| \sum_{N,R} C_M(N, R) \det Y^l \exp\left[-2\pi\, \mathrm{tr}\left\{\left(\frac{1}{h}N - \frac{1}{4}RT^{-1\,t}R\right)Y + Y^{-1}T\left[{}^t\left(\beta + \frac{1}{2}YRT^{-1}\right)\right]\right\}\right]\right|. \end{aligned}$$

If $\mathrm{tr}(Y) \to +\infty$, then $\mathrm{tr}\left\{\left(\frac{1}{h}N - \frac{1}{4}RT^{-1}\left[{}^tR\right]\right)Y\right\} \to +\infty$ and

$$\det Y^l e^{-2\pi\, \mathrm{tr}\left\{\left(\frac{1}{h}N - \frac{1}{4}RT^{-1\,t}R\right)Y + Y^{-1}T\left[{}^t\left(\beta + \frac{1}{2}YRT^{-1}\right)\right]\right\}} \to 0.$$

To show that $g_M(\tau, z)$ converges uniformly on the domain D_c, we write $\beta = Y\lambda \in P(Y)$ with $\lambda \in M_{n,r}(\mathbb{R})$. Remark that the absolute value of any component of λ is less than or equals to $\frac{1}{2}$. We have

$$\begin{aligned} & \left(\frac{1}{h}N - \frac{1}{4}RT^{-1}\left[{}^tR\right]\right)Y + Y^{-1}T\left[{}^t\left(\beta + \frac{1}{2}YRT^{-1}\right)\right] \\ = \;& \left(\frac{1}{h}N - \frac{1}{4}T^{-1}\left[{}^tR\right] + T\left[{}^t\left(\lambda + \frac{1}{2}RT^{-1}\right)\right]\right)Y \end{aligned}$$

and

$$
\begin{aligned}
&|g_M(\tau,z)|^2\\
= &\left|\sum_{N,R} C_M(N,R)\,\det Y^l\,\exp\left[-2\pi\,\mathrm{tr}\left\{\left(\frac{1}{h}N-\frac{1}{4}T^{-1}\left[{}^tR\right]+T\left[{}^t\left(\lambda+\frac{1}{2}RT^{-1}\right)\right]\right)Y\right\}\right]\right|.
\end{aligned}
$$

In particular, any

$$\frac{1}{h}N-\frac{1}{4}T^{-1}\left[{}^tR\right]+T\left[{}^t\left(\lambda+\frac{1}{2}RT^{-1}\right)\right]$$

is positive definite. Since the set $\{\lambda=(\lambda_{i,j})\in M_{n,r}(\mathbb{R})\,|\,|\lambda_{i,j}|\le\frac{1}{2}\}$ is compact, there exists a $(iY_0, iY_0\lambda_0)\in\mathfrak{H}_n\times M_{n,r}(\mathbb{C})$ and a constant c' such that

$$
\begin{aligned}
&\det Y^l\,\exp\left[-2\pi\,\mathrm{tr}\left\{\left(\frac{1}{h}N-\frac{1}{4}T^{-1}\left[{}^tR\right]+T\left[{}^t\left(\lambda+\frac{1}{2}RT^{-1}\right)\right]\right)Y\right\}\right]\\
\le\ & c'\,\det Y_0^l\,\exp\left[-2\pi\,\mathrm{tr}\left\{\left(\frac{1}{h}N-\frac{1}{4}T^{-1}\left[{}^tR\right]+T\left[{}^t\left(\lambda_0+\frac{1}{2}RT^{-1}\right)\right]\right)Y_0\right\}\right]
\end{aligned}
$$

for any $(\tau,z)\in D_c$ and for any (N,R). Since $g_M(iY_0, iY_0\lambda_0)$ converges absolutely, we conclude that $g_M(\tau,z)$ converges uniformaly on the domain D_c,

Thus, if $\mathrm{tr}(Y)\to+\infty$, then $|g_M(\tau,z)|\to 0$. Hence we conclude that $|g(\tau,z)|$ is bounded on $\mathfrak{H}_n\times M_{n,r}(\mathbb{C})$. □

3. Plus-space of Jacobi forms

For k, $r\in\mathbb{N}$ we put

$$P_r^k \ := \ \left\{\mathfrak{M}\in L_r^*\,|\,\mathfrak{M}\equiv(-1)^{k+1}L\,{}^tL \mod 4Sym_n(\mathbb{Z}) \text{ with some } L\in M_{r,1}(\mathbb{Z})\right\}.$$

We remark that P_r^k depends only on the choice of r and of the parity of k.

We shall define the plus-space of Jacobi forms $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$, which is a subspace of $J^{(n)}_{k-\frac{1}{2},\mathfrak{M}}$.

Definition 1 (Plus-space of Jacobi forms). Let $\mathfrak{M}\in P_r^k$. Let $\phi\in J^{(n)}_{k-\frac{1}{2},\mathfrak{M}}$. We take the Fourier expansion

$$\phi(\tau,z) \ = \ \sum_{\substack{M\in L_n^*, S\in M_{n,r}(\mathbb{Z})\\ 4M-S\mathfrak{M}^{-1}\,{}^tS\ge 0}} C(M,S)\,e\left(M\tau+S\,{}^tz\right).$$

The subspace $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}\subset J^{(n)}_{k-\frac{1}{2},\mathfrak{M}}$ consists of all ϕ which satisfies the condition that $C(M,S)=0$ unless $\begin{pmatrix} M & \frac{1}{2}S\\ \frac{1}{2}{}^tS & \mathfrak{M}\end{pmatrix}\in P^k_{n+r}$.

In this article we call $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ the plus-space of Jacobi forms of weight $k-\frac{1}{2}$ and index $\mathfrak{M}$ on $\Gamma_0^{(n)}(4)$.

Remark:

1. The plus-space of Jacobi forms is defined not for all matrix indices, but for matrix indices in P_r^k. For example, if $r=1$, then the index $\mathfrak{M}\in\mathbb{N}$ has to satisfy the condition $\mathfrak{M}\equiv 0,(-1)^{k+1} \mod 4$.
2. If $r=0$, we regard $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ as the generalized plus-space $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ which is

introduced in [3].

(3) If $\phi(\tau,z) \in J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$, then it is not difficult to check that $\phi(\tau,0)$ belongs to $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ as a function of $\tau \in \mathfrak{H}_n$.

In this article we consider the case $k \in 2\mathbb{Z}$.

By the Fourier-Jacobi expansion we have the map

$$M^+_{k-\frac{1}{2}}(\Gamma_0^{(n+r)}(4)) \to \bigoplus_{\mathfrak{M}\in P_r^k} J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}. \tag{3.1}$$

On the other hand, by the Fourier-Jacobi expansion we have the map

$$J^{(n+r)}_{k,1} \to \bigoplus_{\mathcal{M}\in L^*_{r,1}} J^{(n)}_{k,\mathcal{M}}, \tag{3.2}$$

where we put

$$L^*_{r,1} \;\; := \;\; \left\{ \mathcal{M} \in L^*_{r+1} \,|\, \mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}L \\ \frac{1}{2}{}^tL & 1 \end{pmatrix} \text{ with } \mathcal{M}_1 \in L^*_r, L \in M_{r,1}(\mathbb{Z}) \right\}.$$

Here the Fourier-Jacobi expansion of $\phi(\tau,z) \in J^{(n+r)}_{k,1}$ means that if we take the expansion

$$\phi(\tau,z)e(\omega) \;\; = \sum_{\mathcal{M}\in L^*_{r,1}} \phi_{\mathcal{M}}(\tau',z')e(\mathcal{M}\omega')$$

for $\begin{pmatrix} \tau & z \\ {}^tz & \omega \end{pmatrix} = \begin{pmatrix} \tau' & z' \\ {}^tz' & \omega' \end{pmatrix}$, $\tau \in \mathfrak{H}_{n+r}$, $\omega \in \mathfrak{H}_1$, $z \in M_{n+r,1}(\mathbb{C})$, $\tau' \in \mathfrak{H}_n$, $\omega' \in \mathfrak{H}_{r+1}$ and $z' \in M_{n,r+1}(\mathbb{C})$, then $\phi_{\mathcal{M}} \in J^{(n)}_{k,\mathcal{M}}$. It means that $\phi_{\mathcal{M}}$ is the $\mathcal{M}$-th Fourier-Jacobi coefficient of ϕ.

If $k \in 2\mathbb{Z}$, due to the results given by Eichler-Zagier [1] (for $n = 1$) and by Ibukiyama [3] (for $n > 1$) it is known that $M^+_{k-\frac{1}{2}}(\Gamma_0^{(n)}(4))$ and $J^{(n)}_{k,1}$ are isomorphic as Hecke algebra modules. Therefore we can expect that $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ and $J^{(n)}_{k,\mathcal{M}}$ are isomorphic. However we do not know that the maps (3.1), (3.2) of Fourier-Jacobi expansions are surjective, hence the isomorphism between $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ and $J^{(n)}_{k,\mathcal{M}}$ is not obvious.

4. Isomorphism map

Let $\mathcal{M}_1 \in L^*_r$ be a half-integral symmetric matrix and $L \in M_{r,1}(\mathbb{Z})$ be a column vector such that the matrix

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}L \\ \frac{1}{2}{}^tL & 1 \end{pmatrix} \in L^*_{r+1},$$

is positive-definite. We recall $\mathfrak{M} := 4\mathcal{M}_1 - L\,{}^tL \in L^*_r$. In the following sections we use these symbols $\mathcal{M}$, $\mathcal{M}_1$, L and $\mathfrak{M}$. We assume $k \in 2\mathbb{Z}$.

4.1. Definition of the map. Let $\psi \in J^{(n)}_{k,\mathcal{M}}$. We take the Fourier expansion

$$\psi(\tau,z) \;\; = \sum_{\substack{N\in L^*_n, R\in M_{n,r+1}(\mathbb{Z}) \\ 4N - R\mathcal{M}^{-1}\,{}^tR \geq 0}} A(N,R)\, e\!\left(N\tau + R\,{}^tz\right).$$

For $R_2 \in M_{n,1}(\mathbb{Z})$ and for $(\tau,z') \in \mathfrak{H}_n \times M_{n,1}(\mathbb{C})$, we define

$$\vartheta_{2,R_2}(\tau, z') \quad := \sum_{\substack{p\in M_{n,1}(\mathbb{Z}) \\ p\equiv R_2 \mod 2M_{n,1}(\mathbb{Z})}} e\left(\frac{1}{4}{}^t p\tau p + {}^t p z'\right).$$

For $\tau \in \mathfrak{H}_n$ and for $z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C})$ ($z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C})$), we define

$$\vartheta_{2,R_2,L}(\tau, z_1, z_2) \quad := \quad \vartheta_{2,R_2}\left(\tau, \frac{1}{2}z_1 L + z_2\right).$$

The following lemma plays an important role in this article.

Lemma 4.1 (Theta decomposition)**.** *For $\tau \in \mathfrak{H}_n$, $z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C})$ ($z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C})$), we have*

$$\psi(\tau, z) \quad = \sum_{R_2\in\mathbb{Z}^n/(2\mathbb{Z})^n} f_{R_2}(\tau, z_1)\vartheta_{2,R_2,L}(\tau, z_1, z_2),$$

where

$$f_{R_2}(\tau, z_1) \quad = \sum_{N\in L_n^*, R_1\in M_{n,r}(\mathbb{Z})} A(N, (R_1, R_2)) \times e\left(\frac{1}{4}\left(4N - R_2{}^tR_2\right)\tau + \frac{1}{4}(4R_1 - 2R_2{}^tL)^t z_1\right).$$

Here in the above summation N and R_1 run over all matrices which satisfy

$$4N - R_2{}^tR_2 - \mathfrak{M}^{-1}[{}^t\left(2R_1 - R_2{}^tL\right)] \geq 0.$$

Proof. For $\lambda \in M_{n,1}(\mathbb{Z})$ we put

$$\begin{pmatrix} N' & \frac{1}{2}R' \\ \frac{1}{2}{}^tR' & \mathcal{M} \end{pmatrix} \quad := \quad \begin{pmatrix} N & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix}\left[\begin{pmatrix} 1_n & 0 \\ {}^t\lambda' & 1_{r+1} \end{pmatrix}\right].$$

Here $\lambda' = \begin{pmatrix} 0 & \lambda \end{pmatrix} \in M_{n,r+1}(\mathbb{Z})$ ($0 \in M_{n,r}(\mathbb{Z})$). Due to the definition of Jacobi forms we have $A(N', R') = A(N, R)$. We write $R = (R_1, R_2) \in M_{n,r}(\mathbb{Z}) \times M_{n,1}(\mathbb{Z})$. Since

$$A\left(N + \frac{1}{2}R_2\left[{}^t\lambda\right] + \frac{1}{2}\lambda{}^tR_2 + \lambda\left[{}^t\lambda\right], R + \lambda\begin{pmatrix}{}^tL & 2\end{pmatrix}\right) \quad = \quad A(N', R') \;=\; A(N, R),$$

we obtain

$$\begin{aligned}
\psi(\tau, z) \quad &= \sum_{\substack{N\in L_n^*, R\in M_{n,r+1}(\mathbb{Z}) \\ 4N - R\mathcal{M}^{-1}{}^tR \geq 0}} A(N, R)\, e\left(N\tau + R^t z\right) \\
&= \sum_{R_2 \mod 2M_{n,1}(\mathbb{Z})} \sum_{N\in L_n^*} \sum_{R_1\in M_{n,r}(\mathbb{Z})} A(N, (R_1, R_2)) \\
&\quad \times \quad e\left(\frac{1}{4}\left(4N - R_2{}^tR_2\right)\tau + \frac{1}{4}\left(4R_1 - 2R_2{}^tL\right){}^t z_1\right) \\
&\quad \times \sum_{\lambda\in M_{n,1}(\mathbb{Z})} e\left(\frac{1}{4}(R_2 + 2\lambda)^t(R_2 + 2\lambda)\tau + (R_2 + 2\lambda)^t\left(\frac{1}{2}z_1 L + z_2\right)\right).
\end{aligned}$$

The condition $4N - R_2{}^tR_2 - \mathfrak{M}^{-1}[{}^t(2R_1 - R_2{}^tL)] \geq 0$ follows from $4N - R\mathcal{M}^{-1}{}^tR \geq 0$ and $\mathcal{M}^{-1} = \begin{pmatrix} 4\mathfrak{M}^{-1} & 0 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} 1_r & -\frac{1}{2}L \\ 0 & 1 \end{pmatrix}\right]$. □

We now define the map $\iota_{\mathcal{M}} : J^{(n)}_{k,\mathcal{M}} \to J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$. Let $\psi \in J^{(n)}_{k,\mathcal{M}}$. We use the same symbols $A(N,R)$ and $f_{R_2}(\tau, z_1)$ which are obtained by ψ as before. We define $\phi = \iota_{\mathcal{M}}(\psi)$ by

$$\phi(\tau, z_1) := \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(4\tau, 4z_1) \tag{4.1}$$

for $(\tau, z_1) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. We have the Fourier expansion

$$\phi(\tau, z_1) = \sum_{\substack{M \in L_n^*, S \in M_{n,r}(\mathbb{Z}) \\ 4M - S\mathfrak{M}^{-1}{}^tS \geq 0}} C(M,S)\, e(M\tau + S\,{}^tz_1),$$

where $C(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL) = A(N,R)$ for $N \in L_n^*$ and for $R = (R_1, R_2) \in M_{n,r+1}(\mathbb{Z})$ $(R_1 \in M_{n,r}(\mathbb{Z}), R_2 \in M_{n,1}(\mathbb{Z}))$.

In §4.2 we shall show that the above ϕ belongs to $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$. In §4.3 we shall show that the linear map $\iota_{\mathcal{M}}$ is bijective.

4.2. Half-integral weight. Let $\psi \in J^{(n)}_{k,\mathcal{M}}$ and $\phi = \iota_{\mathcal{M}}(\psi)$ be the form constructed by (4.1) in §4.1. In this subsection we shall show that the form ϕ belongs to $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$.

4.2.1. Heisenberg group part.

Lemma 4.2. *Let $\phi = \iota_M(\psi)$ be as above. For any $\lambda_1, \mu_1 \in M_{n,r}(\mathbb{Z})$ we have*

$$\phi(\tau, z_1 + \tau\lambda_1 + \mu_1) = e(-\lambda_1\mathfrak{M}{}^t\lambda_1\tau - 2\lambda_1\mathfrak{M}{}^tz_1)\phi(\tau, z_1).$$

Proof. We recall $\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & \frac{1}{2}L \\ \frac{1}{2}{}^tL & 1 \end{pmatrix}$. For $\lambda_1, \mu_1 \in M_{n,r}(\mathbb{Z})$ and for $z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C})$ $(z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C}))$, we have

$$\begin{aligned} \psi(\tau, z + \tau(\lambda_1, 0) + (\mu_1, 0)) &= e(-(\lambda_1, 0)\mathcal{M}{}^t(\lambda_1, 0)\tau - 2(\lambda_1, 0)\mathcal{M}{}^t(z_1, z_2))\psi(\tau, z) \\ &= e(-\lambda_1\mathcal{M}_1{}^t\lambda_1\tau - 2\lambda_1\mathcal{M}_1{}^tz_1 - \lambda_1L{}^tz_2)\psi(\tau, z), \end{aligned} \tag{4.2}$$

where $(\lambda_1, 0), (\mu_1, 0) \in M_{n,r+1}(\mathbb{Z})$ $(0 \in M_{n,1}(\mathbb{Z}))$. On the other hand

$$\begin{aligned} &\psi(\tau, z + \tau(\lambda_1, 0) + (\mu_1, 0)) \\ &= \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(\tau, z_1 + \tau\lambda_1 + \mu_1)\vartheta_{2,R_2,L}(\tau, z_1 + \tau\lambda_1 + \mu_1, z_2) \\ &= \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(\tau, z_1 + \tau\lambda_1 + \mu_1)\vartheta_{2,R_2+\lambda_1L,L}(\tau, z_1, z_2) \\ &\qquad \times e\left(\frac{1}{2}{}^tR_2\mu_1L - \frac{1}{4}\tau[\lambda_1L] - \frac{1}{2}{}^tL{}^t\lambda_1z_1L - {}^tL{}^t\lambda_1z_2\right). \end{aligned} \tag{4.3}$$

Due to (4.2), (4.3) and because of the linear independence of $\{\vartheta_{2,R_2,L}\}_{R_2}$, we obtain

$$\begin{aligned} &f_{R_2+\lambda_1L}(\tau, z_1) \\ &= f_{R_2}(\tau, z_1 + \tau\lambda_1 + \mu_1)\, e\left(\frac{1}{4}\lambda_1\mathfrak{M}{}^t\lambda_1\tau + \frac{1}{2}\lambda_1\mathfrak{M}{}^tz_1 + \frac{1}{2}\mu_1L{}^tR_2\right). \end{aligned} \tag{4.4}$$

Hence

$$\phi(\tau, z_1 + \tau\lambda_1 + \mu_1) = \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(4\tau, 4z_1 + 4\tau\lambda_1 + 4\mu_1)$$

$$= \sum_{R_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2+\lambda_1 L}(4\tau, 4z_1)$$
$$\times e\left(-\frac{1}{4}\lambda_1\mathfrak{M}^t\lambda_1(4\tau) - \frac{1}{2}\lambda_1\mathfrak{M}(4^t z_1) - \frac{1}{2}(4\mu_1)L^t R_2\right)$$
$$= e(-\lambda_1\mathfrak{M}^t\lambda_1\tau - 2\lambda_1\mathfrak{M}^t z_1)\,\phi(\tau, z_1).$$

□

4.2.2. Symplectic group part. Let $\Gamma_0^{(n)}(4)^*$ be the subgroup of $\mathfrak{G}$ which is denoted in §2.2.

Lemma 4.3. *Let $\phi = \iota_{\mathcal{M}}(\psi)$ be as before. For any $M \in \Gamma_0^{(n)}(4)^*$ we have $\phi|_{k-\frac{1}{2},\mathfrak{M}}M = \phi$.*

Proof. Due to the transformation formula of $\psi \in J_{k,\mathcal{M}}^{(n)}$ we have

$$\psi(-\tau^{-1}, \tau^{-1}z) = e\left(\mathcal{M}^t z\tau^{-1}z\right)\det(\tau)^k\psi(\tau, z). \tag{4.5}$$

On the other hand

$$\psi(-\tau^{-1}, \tau^{-1}z) = \sum_{R_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(-\tau^{-1}, \tau^{-1}z_1)\,\vartheta_{2,R_2,L}(-\tau^{-1}, \tau^{-1}z_1, \tau^{-1}z_2) \tag{4.6}$$

for $z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C})$ ($z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C})$). It is known the identity

$$\begin{aligned}\vartheta_{2,R_2}&\left(-\tau^{-1}, \tau^{-1}z\right) \\ &= 2^{-\frac{n}{2}}\det(-i\tau)^{\frac{1}{2}}\, e({}^t z\tau^{-1}z) \sum_{T_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e\left(-\frac{1}{2}{}^t R_2 T_2\right)\vartheta_{2,T_2}(\tau, z).\end{aligned} \tag{4.7}$$

Hence from (4.5), (4.6) and (4.7), we have

$$\begin{aligned} e&\left(\mathcal{M}^t z\tau^{-1}z\right)\det(\tau)^k\psi(\tau, z) \\ &= 2^{-n/2}\det(-i\tau)^{1/2} e\left({}^t\left(\frac{1}{2}z_1 L + z_2\right)\tau^{-1}\left(\frac{1}{2}z_1 L + z_2\right)\right) \\ &\quad\times \sum_{T_2}\left\{\sum_{R_2} f_{R_2}\left(-\tau^{-1}, \tau^{-1}z_1\right) e\left(-\frac{1}{2}{}^t R_2 T_2\right)\right\}\vartheta_{2,T_2,L}(\tau, z_1, z_2).\end{aligned} \tag{4.8}$$

Here R_2 and T_2 run over a complete set of representatives of $M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))$. By comparing the term of $\vartheta_{2,0,L}$ in (4.8) we have

$$e\left(\tfrac{1}{4}\mathfrak{M}^t z_1\tau^{-1}z_1\right)\det(\tau)^k f_0(\tau, z_1) \quad = 2^{-n/2}\det(-i\tau)^{1/2}\sum_{R_2} f_{R_2}\left(-\tau^{-1}, \tau^{-1}z_1\right). \tag{4.9}$$

Since $\psi(\tau, -z) = \psi(\tau, z)$, we have $A(N, -R) = A(N, R)$. Hence $f_0(\tau, -z_1) = f_0(\tau, z_1)$. By replacing (τ, z_1) by $(-(4\tau)^{-1}, \tau^{-1}z_1)$ in the identity (4.9), we obtain

$$\begin{aligned}\phi(\tau, z_1) &= 2^{\frac{n}{2}}\det(-(4\tau)^{-1})^k\det((4\tau)^{-1}i)^{-\frac{1}{2}} e(-\mathfrak{M}^t z_1\tau^{-1}z_1) f_0(-(4\tau)^{-1}, \tau^{-1}z_1) \\ &= 2^{\frac{n}{2}}(-1)^{\frac{nk}{2}}\det((4\tau)^{-1}i)^{k-\frac{1}{2}} e(-\mathfrak{M}^t z_1\tau^{-1}z_1) f_0(-(4\tau)^{-1}, \tau^{-1}z_1).\end{aligned} \tag{4.10}$$

We calculate $\phi|_{k-\frac{1}{2},\mathfrak{M}}v(4s)^*$ for symmetric matrix $s \in \mathrm{Sym}_n(\mathbb{Z})$, where $v(4s) := \begin{pmatrix}1_n & 0\\ 4s & 1_n\end{pmatrix}$ and $v(4s)^* := (v(4s), \theta(v(4s)\tau)/\theta(\tau))$.

We put $t(s) := \begin{pmatrix} 1_n & s \\ 0 & 1_n \end{pmatrix}$ and $J_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Then, $\upsilon(4s) = J_n\, t(-4s) J_n^{-1}$. By using the transformation formula of $\vartheta_{2,0}$ in (4.7) we have

$$\begin{aligned}
&\theta(J_n\, t(-4s)J_n^{-1}\cdot\tau)\\
&\quad = \vartheta_{2,0}(J_n\, t(-4s)J_n^{-1}\cdot\tau, 0)\\
&\quad = 2^{-\frac{n}{2}}\det\Bigl(-i\left(t(-4s)J_n^{-1}\cdot\tau\right)\Bigr)^{\frac12}\sum_{T_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))}\vartheta_{2,T_2}\Bigl(t(-4s)J_n^{-1}\cdot\tau, 0\Bigr)\\
&\quad = 2^{-\frac{n}{2}}\det\Bigl(i\left(\tau^{-1}+4s\right)\Bigr)^{\frac12}\sum_{T_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))}\vartheta_{2,T_2}\Bigl(J_n^{-1}\cdot\tau, 0\Bigr)\\
&\quad = 2^{-n}\det\Bigl(i\left(\tau^{-1}+4s\right)\Bigr)^{\frac12}\det(-i\tau)^{\frac12}\sum_{T_2,V_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e\Bigl(-\frac{1}{2}{}^tT_2V_2\Bigr)\vartheta_{2,V_2}(\tau,0)\\
&\quad = \det\Bigl(i\left(\tau^{-1}+4s\right)\Bigr)^{\frac12}\det(-i\tau)^{\frac12}\,\theta(\tau).
\end{aligned}$$

Thus, we have

$$\frac{\theta(\upsilon(4s)\cdot\tau)}{\theta(\tau)} = \det\left(i\left(\tau^{-1}+4s\right)\right)^{\frac12}\det(-i\tau)^{\frac12}.$$

Since $\upsilon(4s)\cdot(\tau,z_1) = (\tau(4s\tau+1_n)^{-1}, {}^t(4s\tau+1_n)^{-1}z_1)$, we obtain

$$\begin{aligned}
&(\phi|_{k-\frac12,\mathfrak{M}}\upsilon(4s))(\tau,z_1)\\
&\quad = \left(\frac{\theta(\upsilon(4s)\cdot\tau)}{\theta(\tau)}\right)^{-2k+1} e(-\mathfrak{M}{}^tz_1(4s\tau+1_n)^{-1}(4s)z_1)\\
&\qquad \times\ \phi(\tau(4s\tau+1_n)^{-1}, {}^t(4s\tau+1_n)^{-1}z_1)\\
&\quad = \left(\det\left(i\left(\tau^{-1}+4s\right)\right)^{\frac12}\det(-i\tau)^{\frac12}\right)^{-2k+1} e(-\mathfrak{M}{}^tz_1(4s\tau+1_n)^{-1}(4s)z_1)\\
&\qquad \times\ 2^{\frac{n}{2}}(-1)^{\frac{nk}{2}}\det((4\tau(4s\tau+1_n)^{-1})^{-1}i)^{k-\frac12} e(-\mathfrak{M}{}^tz_1\tau^{-1}{}^t(4s\tau+1_n)^{-1}z_1)\\
&\qquad \times\ f_0(-(4\tau(4s\tau+1_n)^{-1})^{-1}, (4s\tau+1_n)\tau^{-1}{}^t(4s\tau+1_n)^{-1}z_1).
\end{aligned}$$

By using the identities $\tau(4s\tau+1_n)^{-1} = {}^t(4s\tau+1_n)^{-1}\tau$ and $f_0(\tau-s,z_1) = f_0(\tau,z_1)$ and due to the identity (4.10) we have

$$\begin{aligned}
&(\phi|_{k-\frac12,\mathfrak{M}}\upsilon(4s))(\tau,z_1)\\
&\quad = 2^{\frac{n}{2}}(-1)^{\frac{nk}{2}}\det(-4i\tau)^{-k+\frac12} e(-\mathfrak{M}{}^tz_1\tau^{-1}z_1) f_0(-(4\tau)^{-1},\tau^{-1}z_1)\\
&\quad = \phi(\tau,z_1).
\end{aligned}$$

The transformation formulas $\phi|_{k-\frac12,\mathfrak{M}}(t(s),1) = \phi$ and $\phi|_{k-\frac12,\mathfrak{M}}(d(A),1) = \phi$ are obvious, where we put $d(A) := \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$ for $A\in GL_n(\mathbb{Z})$. It is known that $\Gamma_0^{(n)}(4)$ is generated by three types of elements $\upsilon(4s)$, $t(s)$ and $d(A)$ (cf. [3, Lemma 2.1]). Therefore $\Gamma_0^{(n)}(4)^*$ is generated by three types of elements $\upsilon(4s)^*$, $(t(s),1)$ and $(d(A),1)$. Hence $\phi|_{k-\frac12,\mathfrak{M}}M = \phi$ for any $M\in\Gamma_0^{(n)}(4)^*$. □

Proposition 4.4. *Let k be an even integer and $\psi \in J^{(n)}_{k,\mathcal{M}}$. Let $\phi = \iota_{\mathcal{M}}(\psi)$ be as in* §4.1. *Then ϕ belongs to $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$. Moreover, if ψ is a Jacobi cusp form, then ϕ is also a Jacobi cusp form.*

Proof. If $\psi \in J^{(n)}_{k,\mathcal{M}}$, then due to Lemma 4.2 and 4.3, we conclude that ϕ belongs to $J^{(n)}_{k-\frac{1}{2},\mathfrak{M}}$. Because of the construction of ϕ it is not difficult to see that ϕ belongs to the plus-space of Jacobi forms $J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$.

We now assume that $\psi \in J^{(n)}_{k,\mathcal{M}}$ is a Jacobi cusp form. We shall show that $\phi = \iota_{\mathcal{M}}(\psi) \in J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ is a Jacobi cusp form. We take the theta decomposition

$$\psi(\tau, z) \quad = \sum_{R_2 \in \mathbb{Z}^n/(2\mathbb{Z})^n} f_{R_2}(\tau, z_1)\vartheta_{2,R_2,L}(\tau, z_1, z_2),$$

where $z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C})$ $(z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C}))$. Since ψ is a Jacobi form and due to the transformation formula of $\vartheta_{2,R_2,L}$, we have the fact that for any $M \in \Gamma_n$, the form

$$J_{k-\frac{1}{2},\frac{1}{4}\mathfrak{M}}(M, (\tau, z_1))^{-1} f_{R_2}(M \cdot (\tau, z_1))$$

is a linear combination of $\{f_{T_2}(\tau, z_1)\}_{T_2}$. Here the definition of $J_{k-\frac{1}{2},\frac{1}{4}\mathfrak{M}}(M, (\tau, z_1))$ has been given in the section 2.1. We write $Y = \mathrm{Im}(\tau)$ and $\beta_1 = \mathrm{Im}(z_1)$. Since

$$f_{T_2}(\tau, z_1 + 2\tau\lambda_1 + 2\mu_1) \quad = \quad f_{T_2}(\tau, z_1)\, e\left(-\frac{1}{4}\mathfrak{M}[{}^t(2\lambda_1)]\tau - \frac{1}{2}\mathfrak{M}\,{}^t(2\lambda_1)z_1\right)$$

for any $\lambda_1, \mu_1 \in M_{n,r}(\mathbb{Z})$ (cf. the identity (4.4)), we have

$$|\det Y^{\frac{2k-1}{4}} e^{-2\pi\, tr(\frac{1}{4}Y^{-1}\mathfrak{M}[{}^t\beta_1])} f_{T_2}(\tau, z_1)| \to 0$$

for $\mathrm{tr}(Y) \to \infty$ (cf. the proof of Lemma 2.1). We take a $\xi \in G^J_{n,r}$ such that $\xi \cdot Z_0 = (\tau, z_1)$, where $Z_0 = (i1_n, 0) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. Then

$$\begin{aligned} & |J_{k-\frac{1}{2},\frac{1}{4}\mathfrak{M}}(M\xi, Z_0)^{-1} f_{R_2}(M \cdot (\tau, z_1))| \\ = \;& |\det Y^{\frac{2k-1}{4}} e^{-2\pi\, tr(\frac{1}{4}Y^{-1}\mathfrak{M}[{}^t\beta_1])} J_{k-\frac{1}{2},\frac{1}{4}\mathfrak{M}}(M, (\tau, z_1))^{-1} f_{R_2}(M \cdot (\tau, z_1))| \to 0 \end{aligned}$$

for $\mathrm{tr}(Y) \to \infty$. Therefore, the form

$$|\det Y^{\frac{2k-1}{4}} e^{-2\pi\, tr(\frac{1}{4}Y^{-1}\mathfrak{M}[{}^t\beta_1])} f_{R_2}(\tau, z_1)|$$

is bounded on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. Thus,

$$|\det Y^{\frac{2k-1}{4}} e^{-2\pi\, tr(Y^{-1}\mathfrak{M}[{}^t\beta_1])} \phi(\tau, z_1)|$$

is bounded on $\mathfrak{H}_n \times M_{n,r}(\mathbb{C})$. Due to Lemma 2.1, we conclude that ϕ is a Jacobi cusp form. □

4.3. Inverse map. In this subsection we will show that the map $\iota_{\mathcal{M}} : J^{(n)}_{k,\mathcal{M}} \to J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$ defined in §4.1 is bijective. Let $\phi \in J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$. We take the Fourier expansion of ϕ:

$$\phi(\tau, z_1) \quad = \sum_{\substack{M \in L_n^*, S \in M_{n,r}(\mathbb{Z}) \\ 4M - S\mathfrak{M}^{-1}{}^tS \geq 0}} C(M, S)\, e(M\tau + S\,{}^tz_1).$$

If $C(M,S) \neq 0$, then there exist $N \in L_n^*$, $R_1 \in M_{n,r}(\mathbb{Z})$ and $R_2 \in M_{n,1}(\mathbb{Z})$, which satisfy $M = 4N - R_2{}^tR_2$ and $S = 4R_1 - 2R_2{}^tL$. This R_2 is uniquely determined up to modulo $2M_{n,1}(\mathbb{Z})$, because if $4N - R_2{}^tR_2 = 4N' - R_2'{}^tR_2'$, then $R_2{}^tR_2 \equiv R_2'{}^tR_2' \mod 4$. Therefore there exists $\{f_{R_2}\}_{R_2}$ which satisfies

$$\phi(\tau, z_1) = \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(4\tau, 4z_1),$$

where

$$f_{R_2}(\tau, z_1) = \sum_{N,R_1} C(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL)\ e\left(\frac{1}{4}\left(4N - R_2{}^tR_2\right)\tau + \frac{1}{4}(4R_1 - 2R_2{}^tL)^t z_1\right).$$

Here (N, R_1) runs over all elements in $L_n^* \times M_{n,r}(\mathbb{Z})$, such that $\begin{pmatrix} N & R_1 & R_2 \\ {}^tR_1 & M_1 & \frac{1}{2}L \\ {}^tR_2 & \frac{1}{2}{}^tL & 1 \end{pmatrix} \in L_{n+r+1}^*$

By using these f_{R_2} the inverse image $\psi = \iota_{\mathcal{M}}^{-1}(\phi)$ of ϕ is given by

$$\psi(\tau, (z_1, z_2)) = \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(\tau, z_1)\vartheta_{2,R_2}(\tau, (z_1, z_2)), \tag{4.11}$$

where $\tau \in \mathfrak{H}_n$, $z_1 \in M_{n,r}(\mathbb{C})$ and $z_2 \in M_{n,1}(\mathbb{C})$.

In this subsection we shall show that this ψ belongs to $J_{k,\mathcal{M}}^{(n)}$.

4.3.1. Heisenberg group part.

Lemma 4.5. *Let ϕ and ψ be as above in §4.3. For any $\lambda = (\lambda_1, \lambda_2) \in M_{n,r+1}(\mathbb{Z})$ ($\lambda_1 \in M_{n,r}(\mathbb{Z}), \lambda_2 \in M_{n,1}(\mathbb{Z})$) and for any $\mu = (\mu_1, \mu_2) \in M_{n,r+1}(\mathbb{Z})$ ($\mu_1 \in M_{n,r}(\mathbb{Z}), \mu_2 \in M_{n,1}(\mathbb{Z})$), we have*

$$\psi(\tau, z + \tau\lambda + \mu) = e\left(-\mathcal{M}\left({}^t\lambda\tau\lambda + 2{}^t\lambda z\right)\right)\psi(\tau, z).$$

Proof. For $(\tau, z_1) \in \mathfrak{H}_n \times M_{n,r}(\mathbb{C})$ we have

$$f_{R_2}(\tau, z_1) = 2^{-n}\sum_{s} e\left(\frac{1}{2}{}^tR_2 s R_2\right)\phi\left(\frac{1}{4}\tau + \frac{1}{2}s, \frac{1}{4}z_1\right), \tag{4.12}$$

where s runs over all diagonal matrices of size n with entries in $\{0, 1\}$.

For $(\tau, z) \in \mathfrak{H}_n \times M_{n,r+1}(\mathbb{C})$ ($z = (z_1, z_2) \in M_{n,r+1}(\mathbb{C}), z_1 \in M_{n,r}(\mathbb{C}), z_2 \in M_{n,1}(\mathbb{C})$), due to the definition of $\vartheta_{2,R_2,L}$, we obtain

$$\begin{aligned}
&\vartheta_{2,R_2,L}(\tau, z_1 + \tau\lambda_1 + \mu_1, z_2 + \tau\lambda_2 + \mu_2) \\
&= \sum_{\substack{p \in M_{n,1}(\mathbb{Z}) \\ p \equiv R_2 \mod 2M_{n,1}(\mathbb{Z})}} e\left(\frac{1}{4}\tau[p + \lambda_1 L + 2\lambda_2] + {}^t(p + \lambda_1 L + 2\lambda_2)\left(\frac{1}{2}z_1 L + z_2\right)\right) \\
&\qquad \times e\left(-\frac{1}{4}\tau[\lambda_1 L] - \tau[\lambda_2] - \frac{1}{2}{}^t\lambda_2\tau\lambda_1 L - \frac{1}{2}{}^t(\lambda_1 L)\tau\lambda_2\right) \\
&\qquad \times e\left(-\frac{1}{2}{}^t(\lambda_1 L)z_1 L - {}^t\lambda_2 z_1 L - {}^t(\lambda_1 L)z_2 - 2{}^t\lambda_2 z_2 + \frac{1}{2}{}^tp\mu_1 L\right) \\
&= e\left(-\begin{pmatrix} \frac{1}{4}L{}^tL & \frac{1}{2}L \\ \frac{1}{2}{}^tL & 1 \end{pmatrix}\left({}^t\lambda\tau\lambda + 2{}^t\lambda z\right)\right) e\left(\frac{1}{2}{}^tR_2\mu_1 L\right)\vartheta_{2,R_2+\lambda_1 L,L}(\tau, z_1, z_2).
\end{aligned}$$

On the other hand, by using the transformation formula

$$\phi(\tau, z_1 + \tau\lambda_1 + \mu_1) = e(-\mathfrak{M}(\tau[\lambda_1] + 2\lambda_1{}^t z_1))\,\phi(\tau, z_1),$$

we have

$$\begin{aligned}
&\phi\left(\frac{1}{4}\tau + \frac{1}{2}s, \frac{1}{4}(z_1 + \tau\lambda_1 + \mu_1)\right)\\
&= e\left(-\mathfrak{M}\left\{\left(\frac{1}{4}\tau + \frac{1}{2}s\right)[\lambda_1] + \frac{1}{2}{}^t\lambda_1(z_1 - 2s\lambda_1 + \mu_1)\right\}\right)\\
&\qquad \times \phi\left(\frac{1}{4}\tau + \frac{1}{2}s, \frac{1}{4}(z_1 - 2s\lambda_1 + \mu_1)\right).
\end{aligned}$$

We obtain

$$\begin{aligned}
\phi\left(\frac{1}{4}\tau + \frac{1}{2}s, \frac{1}{4}(z_1 - 2s\lambda_1 + \mu_1)\right) &= \sum_{T_2} f_{T_2}(\tau + 2s, z_1 - 2s\lambda_1 + \mu_1)\\
&= \sum_{T_2} e\left(-\frac{1}{2}T_2{}^t T_2 s\right) e\left(-\frac{1}{2}T_2{}^t L^t \mu_1\right) f_{T_2}(\tau, z_1).
\end{aligned}$$

Therefore, by using the identity (4.12) we get

$$\begin{aligned}
&f_{R_2}(\tau, z_1 + \tau\lambda_1 + \mu_1)\\
&= 2^{-n}\sum_{s} e\left(\frac{1}{2}R_2{}^t R_2 s\right)\phi\left(\frac{1}{4}\tau + \frac{1}{2}s, \frac{1}{4}(z_1 + \tau\lambda_1 + \mu_1)\right)\\
&= 2^{-n}\sum_{s} e\left(\frac{1}{2}R_2{}^t R_2 s\right) e\left(-\mathfrak{M}\left\{\left(\frac{1}{4}\tau + \frac{1}{2}s\right)[\lambda_1] + \frac{1}{2}{}^t\lambda_1(z_1 - 2s\lambda_1 + \mu_1)\right\}\right)\\
&\qquad \times \sum_{T_2} e\left(-\frac{1}{2}T_2{}^t T_2 s\right) e\left(-\frac{1}{2}T_2{}^t L^t \mu_1\right) f_{T_2}(\tau, z_1)\\
&= 2^{-n}\sum_{T_2} e\left(-\frac{1}{2}T_2{}^t L^t \mu_1\right) e\left(-\mathfrak{M}\left\{\frac{1}{4}\tau[\lambda_1] + \frac{1}{2}{}^t\lambda_1(z_1 + \mu_1)\right\}\right) f_{T_2}(\tau, z_1)\\
&\qquad \times \sum_{s} e\left(\frac{1}{2}R_2{}^t R_2 s\right) e\left(-\frac{1}{2}T_2{}^t T_2 s\right) e\left(\frac{1}{2}\lambda_1 L^t L^t \lambda_1 s\right)\\
&= e\left(-\frac{1}{2}R_2{}^t L^t \mu_1\right) e\left(-\frac{1}{4}\mathfrak{M}\left(\tau[\lambda_1] + 2^t\lambda_1 z_1\right)\right) f_{R_2+\lambda_1 L}(\tau, z_1).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\psi(\tau, z + \tau\lambda + \mu)\\
&= \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}(\tau, z_1 + \tau\lambda_1 + \mu_1)\,\vartheta_{2,R_2,L}(\tau, z_1 + \tau\lambda_1 + \mu_1, z_2 + \tau\lambda_2 + \mu_2)\\
&= e\left(-\mathcal{M}\left({}^t\lambda\tau\lambda + 2^t\lambda z\right)\right)\psi(\tau, z).
\end{aligned}$$

□

4.3.2. Symplectic group part.

Lemma 4.6. *Let ϕ and ψ be as in the beginning of* §4.3. *Then ψ satisfies*

$$\det(C\tau + D)^{-k} e(-\mathcal{M}\,{}^t z (C\tau + D)^{-1} C z) \psi(M \cdot (\tau, z)) \quad = \quad \psi(\tau, z)$$

for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n$.

Proof. It is enough to show the transformation formula of ψ for three types of matrices $v(s) = \left(\begin{smallmatrix} 1_n & 0 \\ s & 1_n \end{smallmatrix}\right)$, $t(s) = \left(\begin{smallmatrix} 1_n & s \\ 0 & 1_n \end{smallmatrix}\right)$ and $d(A) = \left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$, where $s \in \mathrm{Sym}_n(\mathbb{Z})$ and $A \in \mathrm{GL}(n, \mathbb{Z})$.

The transformation formula of ψ for $t(s)$ and for $d(A)$ are obvious. Thus, we shall show the transformation formula of ψ for $v(s)$.

We recall $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$. Since $v(s) = J_n t(-s) J_n^{-1}$, we have

$$\begin{aligned}
&\vartheta_{2,R_2,L}(v(s) \cdot (\tau, z_1, z_2)) \\
= \;& \vartheta_{2,R_2}\left(J_n t(-s) J_n^{-1} \cdot \left(\tau, \frac{1}{2} z_1 L + z_2\right)\right) \\
= \;& 2^{-\frac{n}{2}} \det\left(i(\tau^{-1} + s)\right)^{\frac{1}{2}} e((-\tau^{-1} - s)^{-1}[-\tau^{-1}(2^{-1} z_1 L + z_2)]) \\
& \quad \times \sum_{T_2} e(-2^{-1}\,{}^t R_2 T_2)\, \vartheta_{2,T_2}(t(-s) J_n^{-1} (\tau, 2^{-1} z_1 L + z_2)) \\
= \;& 2^{-\frac{n}{2}} \det\left(i(\tau^{-1} + s)\right)^{\frac{1}{2}} e(((-1_n - s\tau)^{-1} \tau^{-1})[2^{-1} z_1 L + z_2]) \\
& \times \sum_{T_2} e(-2^{-1}\,{}^t R_2 T_2)\, e(-4^{-1} s[T_2])\, \vartheta_{2,T_2}(-\tau^{-1}, -\tau^{-1}(2^{-1} z_1 L + z_2))) \\
= \;& 2^{-\frac{n}{2}} \det\left(i(\tau^{-1} + s)\right)^{\frac{1}{2}} e(((-1_n - s\tau)^{-1} \tau^{-1})[2^{-1} z_1 L + z_2]) \\
& \times \sum_{T_2} e(-2^{-1}\,{}^t R_2 T_2)\, e(-4^{-1} s[T_2])\, 2^{-\frac{n}{2}} \det(-i\tau)^{\frac{1}{2}} e(\tau^{-1}[2^{-1} z_1 L + z_2]) \\
& \times \sum_{V_2} e(-2^{-1}\,{}^t T_2 V_2)\, \vartheta_{2,V_2}(\tau, -(2^{-1} z_1 L + z_2)).
\end{aligned}$$

Here, in the above summations, T_2, and V_2 run over a complete set of the representatives of $M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))$. Since $\vartheta_{2,V_2,L}(\tau, -z_1, -z_2) = \vartheta_{2,V_2,L}(\tau, z_1, z_2)$, we have

$$\begin{aligned}
&\vartheta_{2,R_2,L}(v(s) \cdot (\tau, z_1, z_2)) \\
= \;& 2^{-n} \det\left(i(\tau^{-1} + s)\right)^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} e(((s\tau + 1_n)^{-1} s)[2^{-1} z_1 L + z_2]) \\
& \times \sum_{T_2, V_2} e(-2^{-1}\,{}^t R_2 T_2)\, e(-4^{-1} s[T_2])\, e(-2^{-1}\,{}^t T_2 V_2)\, \vartheta_{2,V_2,L}(\tau, z_1, z_2).
\end{aligned}$$

On the other hand, from (4.12) we have $f_{V_2}(\tau, z_1) = 2^{-n} \sum_{s_1} e(2^{-1} s_1[V_2])\, \phi\left(\frac{1}{4}\tau + \frac{1}{2}s_1, \frac{1}{4}z_1\right)$, where s_1 runs over all diagonal matrices of size n with entries in $\{0, 1\}$. Similarly to the case of Siegel modular forms (cf. [3, p.119]), we put $\gamma_s(s_1) := \begin{pmatrix} 1_n + 2s_1 s & -s_1 s s_1 \\ 4s & 1_n - 2ss_1 \end{pmatrix} \in \Gamma_0^{(n)}(4)$. Then $\gamma_s(s_1) \cdot (\frac{1}{4}\tau + \frac{1}{2}s_1, \frac{1}{4}z_1) = (\frac{1}{4}v(s)\tau + \frac{1}{2}s_1, \frac{1}{4}{}^t(s\tau + 1_n)z_1)$. Due to the transformation formula of ϕ for $\gamma_s(s_1)$, we have

$$
\begin{aligned}
&f_{V_2}(\tau, z_1)\\
&\quad = 2^{-n} \sum_{s_1} e(2^{-1} s_1[V_2])\, e(-4^{-1}\mathfrak{M}^t z_1{}^t(s\tau + 1_n)^{-1} s z_1)\\
&\qquad \times \left(\theta\left(\frac{1}{4}v(s)\tau + \frac{1}{2}s_1\right)\theta\left(\frac{1}{4}\tau + \frac{1}{2}s_1\right)^{-1}\right)^{-2k+1} \phi\left(\frac{1}{4}v(s)\tau + \frac{1}{2}s_1, \frac{1}{4}{}^t(s\tau + 1_n)^{-1} z_1\right).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\vartheta_{2,q}(v(s)\tau, 0)\\
&\quad = \vartheta_{2,q}(J_n t(-s) J_n^{-1}\tau, 0)\\
&\quad = 2^{-\frac{n}{2}} \det(-i(t(-s)J_n^{-1}\tau))^{\frac{1}{2}} \sum_{\nu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1\,t}\nu q)\, \vartheta_{2,\nu}(t(-s)J_n^{-1}\tau, 0)\\
&\quad = 2^{-\frac{n}{2}} \det(i(\tau^{-1} + s))^{\frac{1}{2}} \sum_{\nu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1\,t}\nu q)\, e(-4^{-1}s[\nu])\, \vartheta_{2,\nu}(-\tau^{-1}, 0)\\
&\quad = 2^{-n} \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} \sum_{\nu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1\,t}\nu q)\, e(-4^{-1}s[\nu])\\
&\qquad\qquad \times \sum_{\mu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1\,t}\nu\mu)\vartheta_{2,\mu}(\tau, 0),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\theta\left(\frac{1}{4}v(s)\tau + \frac{1}{2}s_1\right)\\
&\quad = \sum_{q \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1}s_1[q])\, \vartheta_{2,q}(v(s)\tau, 0)\\
&\quad = 2^{-n} \sum_{q \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} \sum_{\nu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} \sum_{\mu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))}\\
&\qquad \times e(-2^{-1}s_1[q])\, e(-2^{-1\,t}\nu q)\, e(-4^{-1}s[\nu])\, e(-2^{-1\,t}\nu\mu)\\
&\qquad \times \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} \vartheta_{2,\mu}(\tau, 0)\\
&\quad = 2^{-n} \sum_{\nu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} \sum_{\mu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-4^{-1}s[\nu])\, e(-2^{-1\,t}\nu\mu)\\
&\qquad \times \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} \vartheta_{2,\mu}(\tau, 0) \sum_{q \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1}(s_1[q] + {}^t\nu q))\\
&\quad = \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} e(-4^{-1}s[s_1]) \sum_{\mu \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z})} e(-2^{-1\,t}s_1\mu)\, \vartheta_{2,\mu}(\tau, 0)\\
&\quad = \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}}\, e(-4^{-1}s[s_1])\, \theta\left(\frac{1}{4}\tau + \frac{1}{2}s_1\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&f_{V_2}(\tau, z_1)\\
&\quad = 2^{-n} \det(s\tau + 1_n)^{-k} \det(i(\tau^{-1} + s))^{\frac{1}{2}} \det(-i\tau)^{\frac{1}{2}} e(-4^{-1}\mathfrak{M}^t z_1{}^t(s\tau + 1_n)^{-1} s z_1)\\
&\qquad \times \sum_{s_1} e(2^{-1}s_1[V_2])\, e(-4^{-1}s[s_1]) \sum_{R_2 \in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} f_{R_2}\left(v(s)\tau + 2s_1, {}^t(s\tau + 1_n)^{-1} z_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2^{-n}\det(s\tau+1_n)^{-k}\det(i(\tau^{-1}+s))^{\frac{1}{2}}\det(-i\tau)^{\frac{1}{2}}e(-4^{-1}\mathfrak{M}{}^tz_1{}^t(s\tau+1_n)^{-1}sz_1)\\
&\quad\times\sum_{s_1}\sum_{R_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(2^{-1}s_1[V_2])\,e(-4^{-1}s[s_1])\,e(-2^{-1}s_1[R_2])\\
&\quad\times f_{R_2}\Big(v(s)\tau,{}^t(s\tau+1_n)^{-1}z_1\Big)\\
&= 2^{-n}\det(s\tau+1_n)^{-k}\det(i(\tau^{-1}+s))^{\frac{1}{2}}\det(-i\tau)^{\frac{1}{2}}e(-4^{-1}\mathfrak{M}{}^tz_1{}^t(s\tau+1_n)^{-1}sz_1)\\
&\quad\times\sum_{T_2,R_2\in M_{n,1}(\mathbb{Z})/(2M_{n,1}(\mathbb{Z}))} e(-2^{-1}{}^tT_2V_2)\,e(-4^{-1}s[T_2])\,e(-2^{-1}{}^tR_2T_2)f_{R_2}(v(s)\cdot(\tau,z_1))\,.
\end{aligned}
$$

By using the above identities we obtain

$$
\begin{aligned}
&\psi(v(s)\cdot(\tau,z))\\
&\quad=\sum_{R_2} f_{R_2}(v(s)\cdot(\tau,z_1))\,\vartheta_{2,R_2,L}(v(s)\cdot(\tau,z_1,z_2))\\
&\quad= 2^{-n}\det\Big(i(\tau^{-1}+s)\Big)^{\frac{1}{2}}\det(-i\tau)^{\frac{1}{2}}\,e\Big(((s\tau+1_n)^{-1}s)\Big[2^{-1}z_1L+z_2\Big]\Big)\\
&\qquad\times\sum_{V_2}\sum_{R_2,T_2} e\Big(-2^{-1}{}^tR_2T_2\Big)e\Big(-4^{-1}s[T_2]\Big)e\Big(-2^{-1}{}^tT_2V_2\Big)f_{R_2}(v(s)\cdot(\tau,z_1))\\
&\qquad\times\vartheta_{2,V_2,L}(\tau,z_1,z_2)\\
&\quad=\det(s\tau+1_n)^ke(4^{-1}\mathfrak{M}({}^t(s\tau+1_n)^{-1}s)[z_1]+((s\tau+1_n)^{-1}s)\Big[2^{-1}z_1L+z_2\Big])\\
&\qquad\times\sum_{V_2} f_{V_2}(\tau,z_1)\vartheta_{2,V_2,L}(\tau,z_1,z_2)\\
&\quad=\det(s\tau+1_n)^ke(\mathcal{M}{}^tz(s\tau+1_n)^{-1}sz)\,\psi(\tau,z).
\end{aligned}
$$

Thus, we obtain the transformation formula of ψ for $v(s)$. □

Proposition 4.7. *The linear map $\iota_{\mathcal{M}}: J_{k,\mathcal{M}}^{(n)}\to J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+}$ is bijective. Moreover, the linear map $\iota_{\mathcal{M}}$ induces the bijection between the Jacobi cusp forms of both spaces.*

Proof. By the virtue of Proposition 4.4, Lemma 4.5 and Lemma 4.6, the linear map $\iota_{\mathcal{M}}$ is bijective. Moreover, it is shown in Proposition 4.4 that if $\psi\in J_{k,\mathcal{M}}^{(n)}$ is a Jacobi cusp form, then $\iota_{\mathcal{M}}(\psi)\in J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+}$ is a Jacobi cusp form. It is not difficult to see that if $\phi\in J_{k-\frac{1}{2},\mathfrak{M}}^{(n)+}$ is a Jacobi cusp form, then $\psi=\iota_{\mathcal{M}}^{-1}(\phi)\in J_{k,\mathcal{M}}^{(n)}$ is a Jacobi cusp form. (For the inverse image ψ of ϕ, see (4.11) in §4.3). Thus, we conclude the proposition. □

5. Compatibility of the linear map with the action of Hecke operators

In this section we shall show the compatibility between the map $\iota_{\mathcal{M}}$ and Hecke operators acting on both spaces.

5.1. Hecke operators. Let p be an odd prime. We set

$$K_\alpha := \mathrm{diag}(1_\alpha, p1_{n-\alpha}, p^21_\alpha, p1_{n-\alpha}).$$

Let $S\in L_{r+1}^*$. We shall review the Hecke operators acting on $J_{k,S}^{(n)}$. For $0\le\alpha\le n$ and for $\psi\in J_{k,S}^{(n)}$, we define,

$$\psi|T_{\alpha,n-\alpha}(p^2) \quad := \quad \sum_{\lambda,\mu} \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)} \psi|_{k,S} \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) |_{k,S} [\lambda, \mu],$$

and where λ and μ run over a complete set of representatives of $M_{n,r+1}(\mathbb{Z})/(pM_{n,r+1}(\mathbb{Z}))$, and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ runs over a complete set of representatives of $\Gamma_n \backslash \Gamma_n K_\alpha \Gamma_n$, and where we defined

$$\begin{aligned} (\psi|_{k,S} \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right))(\tau, z) \quad &:= \quad p^{-(r+1)k} \det(C\tau + D)^{-k} e(S\,{}^t z(C\tau + D)^{-1} Cz) \\ &\quad \times \phi((A\tau + B)(C\tau + D)^{-1}, p^t(C\tau + D)^{-1} z). \end{aligned}$$

One can check $\psi|T_{\alpha,n-\alpha}(p^2) \in J^{(n)}_{k,S}$.

We shall review the Hecke operators acting on $J^{(n)}_{k-\frac{1}{2},S}$. For the definition of $\Gamma_0^{(n)}(4)^* \subset \mathfrak{G}$, see §2.2, and for the definition of "$|_{k-\frac{1}{2},S}$", see §2.3. For $\phi \in J^{(n)}_{k-\frac{1}{2},S}$ we define

$$\phi|\widetilde{T}_{\alpha,n-\alpha}(p^2) \quad := \quad \sum_{\lambda,\mu} \sum_{M^*} \phi|_{k-\frac{1}{2},S} M^*|_{k-\frac{1}{2},S} [\lambda, \mu],$$

where λ and μ run over a complete set of representatives of $M_{n,r}(\mathbb{Z})/(pM_{n,r}(\mathbb{Z}))$, and M^* runs over a complete set of representatives of

$$\Gamma_0^{(n)}(4)^* \backslash \Gamma_0^{(n)}(4)^* \left(K_\alpha, p^{\alpha/2}\right) \Gamma_0^{(n)}(4)^*.$$

One can check $\phi|\widetilde{T}_{\alpha,n-\alpha}(p^2) \in J^{(n)}_{k-\frac{1}{2},S}$ for any α, such that $0 \le \alpha \le n$.

To describe a complete set of representatives of $\Gamma_0^{(n)}(4)^* \backslash \Gamma_0^{(n)}(4)^* \left(K_\alpha, p^{\alpha/2}\right) \Gamma_0^{(n)}(4)^*$ we prepare the following symbols. We put $d_{i,j} := \mathrm{diag}(1_i, p1_j, p^2 1_{n-i-j})$ and put

$$B_0 \quad := \quad \left\{ b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & {}^t b_1 & b_2 \end{pmatrix} \in M_n(\mathbb{Z}) \;\middle|\; \begin{matrix} a_1 = {}^t a_1 \in M_j(p),\ rank_p(a_1) = j - n + \alpha \\ b_1 \in M_{j,n-i-j}(p),\ b_2 \in M_{n-i-j}(p^2) \end{matrix} \right\},$$

and where $M_{l,m}(p^\delta)$ is a complete set of representatives of $M_{l,m}(\mathbb{Z})$ modulo p^δ, and we set $M_l(p^\delta) := M_{l,l}(p^\delta)$, and $\mathrm{rank}_p(a_1)$ is the rank of a_1 over the finite field $\mathbb{Z}/p\mathbb{Z}$.

The symbol U_0 denotes a complete set of representatives of

$$(\mathrm{SL}(n, \mathbb{Z}) \cap d_{i,j}^{-1} \mathrm{SL}(n, \mathbb{Z}) d_{i,j}) \backslash \mathrm{SL}(n, \mathbb{Z}).$$

For $b \in B_0$ and for $u \in U_0$, we put $\kappa(bu) := \varepsilon(a_1)$, where $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & {}^t b_1 & b_2 \end{pmatrix}$, $a_1 \in M_j(p)$, and where $\varepsilon(a_1) := \sqrt{\left(\frac{\det a_1'}{p}\right)}$ with $a_1' \in M_{j-n+\alpha}(\mathbb{Z})$ which satisfies $a_1 \equiv v \left(\begin{smallmatrix} a_1' & 0 \\ 0 & 0 \end{smallmatrix}\right) v^{-1} \mod p$ with some $v \in \mathrm{SL}(j, \mathbb{Z})$.

We quote the following lemma from Zhuravlev [5, p.173].

Lemma 5.1. *A complete set of representatives of* $\Gamma_0^{(n)}(4)^* \backslash \Gamma_0^{(n)}(4)^* \left(K_\alpha, p^{\alpha/2}\right) \Gamma_0^{(n)}(4)^*$ *is given by*

$$\left\{ \left(\begin{pmatrix} p^2 d_{i,j}^{-1} & b \\ 0 & d_{i,j} \end{pmatrix} \begin{pmatrix} {}^t u^{-1} & 0 \\ 0 & u \end{pmatrix}, \kappa(bu) p^{(n-2i-j)/2} \right) \;\middle|\; i + j \le n, b \in B_0, u \in U_0 \right\}.$$

Let $\mathcal{M} := \left(\begin{smallmatrix} \mathcal{M}_1 & \frac{1}{2}L \\ \frac{1}{2}{}^t L & 1 \end{smallmatrix}\right) \in L^+_{r+1}$ and $\mathfrak{M} := 4\mathcal{M}_1 - L\,{}^t L \in L^+_r$ be the same symbols in the beginning of §4.

Proposition 5.2. *Let $\psi \in J^{(n)}_{k,\mathcal{M}}$ and $\phi = \iota_{\mathcal{M}}(\psi) \in J^{(n)+}_{k-\frac{1}{2},\mathfrak{M}}$. For any odd prime p and for any $0 \le \alpha \le n$, we have*

$$\iota_{\mathcal{M}}\left(\psi|T_{\alpha,n-\alpha}(p^2)\right) = p^{-(r+1)(k-n)+(n-\alpha)/2}\, \phi|\tilde{T}_{\alpha,n-\alpha}(p^2).$$

Proof. Similarly to the proof of [3, Theorem 2], we will conclude this proposition by comparing the Fourier coefficients of $\iota_{\mathcal{M}}\left(\psi|T_{\alpha,n-\alpha}(p^2)\right)$ and of $\phi|\tilde{T}_{\alpha,n-\alpha}(p^2)$.

We write the Fourier expansions:

$$\begin{aligned}
\psi(\tau, z) &= \sum_{\substack{N\in L_n^*, R\in M_{n,2}(\mathbb{Z})\\ 4N-R\mathcal{M}^{-1}\,{}^tR\ge 0}} A_1(N,R)\, e(N\tau + R^tZ),\\
(\psi|T_{\alpha,n-\alpha}(p^2))(\tau, z) &= \sum_{\substack{N\in L_n^*, R\in M_{n,2}(\mathbb{Z})\\ 4N-R\mathcal{M}^{-1}\,{}^tR\ge 0}} A_2(N,R)\, e(N\tau + R^tZ),\\
\phi(\tau, z_1) &= \sum_{\substack{M\in L_n^*, S\in M_{n,1}(\mathbb{Z})\\ 4M-S\mathfrak{M}^{-1}\,{}^tS\ge 0}} C_1(M,S)\, e(M\tau + S\,{}^tZ),\\
\phi|\widetilde{T}_{\alpha,n-\alpha}(p^2))(\tau, z_1) &= \sum_{\substack{M\in L_n^*, S\in M_{n,1}(\mathbb{Z})\\ 4M-S\mathfrak{M}^{-1}\,{}^tS\ge 0}} C_2(M,S)\, e(M\tau + S\,{}^tZ).
\end{aligned}$$

Since $\phi = \iota_{\mathcal{M}}(\psi)$, we have $C_1(4N-R_2{}^tR_2, 4R_1-2R_2{}^tL) = A_1(N,R)$ for $(N,R) \in L_n^*\times M_{n,r+1}(\mathbb{Z})$ $(R = (R_1,R_2) \in M_{n,r+1}(\mathbb{Z}), R_1 \in M_{n,r}(\mathbb{Z}), R_2 \in M_{n,1}(\mathbb{Z}))$.

We now calculate $A_2(N,R)$. We have

$$\begin{aligned}
&e(N\tau + {}^tRz)|_{k,\mathcal{M}} \begin{pmatrix} p^{2t}D^{-1} & B\\ 0 & D\end{pmatrix} |_{k,\mathcal{M}}[\lambda,\mu]\\
&\quad= p^{-(r+1)k}\det D^{-k} e(N(p^{2t}D^{-1}\tau + B)D^{-1} + p^tR^tD^{-1}z)|_{k,\mathcal{M}}[\lambda,\mu]\\
&\quad= p^{-(r+1)k}\det D^{-k} e(\mathcal{M}(\tau[\lambda] + 2^t\lambda z))e(N(p^{2t}D^{-1}\tau + B)D^{-1} + p^tR^tD^{-1}(z+\tau\lambda+\mu))\\
&\quad= p^{-(r+1)k}\det D^{-k} e(\hat{N}\tau + {}^t\hat{R}z + NBD^{-1} + p^tR^tD^{-1}\mu),
\end{aligned}$$

where

$$(5.1)\qquad \begin{cases} \hat{N} = p^2N[{}^tD^{-1}] + \frac{1}{2}p\lambda^tR^tD^{-1} + \frac{1}{2}pD^{-1}R^t\lambda + \mathcal{M}[{}^t\lambda],\\ \hat{R} = pD^{-1}R + 2\lambda\mathcal{M}. \end{cases}$$

If $\hat{R} \in M_{n,r+1}(\mathbb{Z})$, then $pD^{-1}R = \hat{R} - 2\lambda\mathcal{M} \in M_{n,r+1}(\mathbb{Z})$. Therefore, for $(\hat{N},\hat{R}) \in L_n^*\times M_{n,r+1}(\mathbb{Z})$ we have

$$\begin{aligned}
A_2(\hat{N},\hat{R}) &= p^{-(r+1)k}\sum_{D,B}\det D^{-k}\sum_{\lambda,\mu\in M_{n,r+1}(\mathbb{Z})} A_1(N,R)e(NBD^{-1} + p^tR^tD^{-1}\mu)\\
&= p^{-(r+1)(k-n)}\sum_{D,B}\det D^{-k}\sum_{\lambda\in M_{n,r+1}(\mathbb{Z})} A_1(N,R)e(NBD^{-1}),
\end{aligned}$$

where, in the above summations, we take matrices D and B such that $\begin{pmatrix} p^{2t}D^{-1} & B\\ 0 & D\end{pmatrix}$ runs over a complete set of representatives of $\Gamma_n\backslash\Gamma_nK_\alpha\Gamma_n$, and where

$$(5.2)\qquad \begin{cases} N = & \frac{1}{p^2}D\left(\left(\hat{N} - \frac{1}{4}\hat{R}\mathcal{M}^{-1\,t}\hat{R}\right) + \frac{1}{4}\mathcal{M}^{-1}\left[{}^t\left(\hat{R} - 2\lambda\mathcal{M}\right)\right]\right){}^tD, \\ R = & \frac{1}{p}D\hat{R} - \frac{2}{p}D\lambda\mathcal{M}. \end{cases}$$

Here the condition (5.1) is equivalent to (5.2). We write $R = (R_1, R_2)$, $\hat{R} = (\hat{R}_1, \hat{R}_2)$, $\lambda = (\lambda_1, \lambda_2) \in M_{n,r+1}(\mathbb{Z})$ ($\hat{R}_1$, $\lambda_1 \in M_{n,r}(\mathbb{Z})$, $R_1 \in M_{n,r}(\mathbb{Q})$, $\hat{R}_2$, $\lambda_2 \in M_{n,1}(\mathbb{Z})$, $R_2 \in M_{n,1}(\mathbb{Q})$), and we assume the condition (5.2). If $(N, R) \notin L_n^* \times M_{n,r}(\mathbb{Z})$, then $A_1(N, R) = 0$. Hence, we assume $(N, R) \in L_n^* \times M_{n,r}(\mathbb{Z})$. In particular, we assume $R_1 \in M_{n,r}(\mathbb{Z})$ and $R_2 \in M_{n,1}(\mathbb{Z})$. By a straightforward calculation we have

$$\begin{aligned} 4N - R_2{}^tR_2 &= \frac{1}{p^2}D(\hat{N} - \frac{1}{4}\hat{R}_2{}^t\hat{R}_2){}^tD \\ &\quad + \frac{1}{p^2}D\left\{-\frac{1}{2}\lambda_1{}^t(4\hat{R}_1 - 2\hat{R}_2{}^tL) - \frac{1}{2}(4\hat{R}_1 - 2\hat{R}_2{}^tL){}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tD, \\ 4R_1 - 2R_2{}^tL &= \frac{1}{p}D(4\hat{R}_1 - 2\hat{R}_2{}^tL) - \frac{2}{p}D\lambda_1\mathfrak{M}. \end{aligned}$$

We remark that $A_1(N, R)$ depends on the choice of $(\hat{N}, \hat{R})$, D and λ_1, and $A_1(N, R)$ is independent of the choice of λ_2. Since $R_2 = \frac{1}{p}D(\hat{R}_2 - \lambda_1 L - 2\lambda_2)$ and $R_2 \in M_{n,1}(\mathbb{Z})$, we have

$$(5.3)\qquad \hat{R}_2 - \lambda_1 L - 2\lambda_2 \in pD^{-1}M_{n,1}(\mathbb{Z}).$$

Let $d_{i,j}$, B_0 and U_0 be symbols defined in Lemma 5.1. We put $D = d_{i,j}u$ and $B = bu$ with $u \in U_0$ and with $b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_1 & pb_1 \\ 0 & {}^tb_1 & b_2 \end{pmatrix} \in B_0$. Then the condition (5.3) is equivalent to

$$(5.4)\qquad u(\hat{R}_2 - \lambda_1 L - 2\lambda_2) \in \begin{pmatrix} p1_i & 0 \\ 0 & 1_{n-i} \end{pmatrix} M_{n,1}(\mathbb{Z}).$$

Taking into account the condition (5.4) on $\lambda_2 \in M_{n,1}(\mathbb{Z})$ we have

$$\begin{aligned} &\sum_{\lambda_2} e\left(\frac{1}{4p^2}\mathcal{M}^{-1}\left[{}^t(\hat{R} - 2\lambda\mathcal{M})\right]{}^tDB\right) \\ &= \sum_{\lambda_2} e\left(\frac{1}{4p^2}\left\{\begin{pmatrix} \mathcal{M}_1 - \frac{1}{4}L{}^tL & 0 \\ 0 & 1 \end{pmatrix}^{-1}\left[\begin{pmatrix} 1_r & -\frac{1}{2}L \\ 0 & 1 \end{pmatrix}{}^t(\hat{R} - 2\lambda\mathcal{M})\right]\right\}{}^tDB\right) \\ &= e\left(\frac{1}{4p^2}\left\{4\mathfrak{M}^{-1}\left[{}^t\hat{R}_1 - \frac{1}{2}L{}^t\hat{R}_2\right] + \mathfrak{M}[{}^t\lambda_1] - 2\hat{R}\begin{pmatrix} 1_r \\ \frac{1}{2}{}^tL \end{pmatrix}{}^t\lambda_1 - 2\lambda_1\begin{pmatrix} 1_r & \frac{1}{2}L \end{pmatrix}{}^t\hat{R}\right\}{}^tDB\right) \\ &\quad \times \sum_{\lambda_2} e\left(\frac{1}{4p^2}u(\hat{R}_2 - \lambda_1 L - 2\lambda_2){}^t(\hat{R}_2 - \lambda_1 L - 2\lambda_2){}^tud_{i,j}b\right) \\ &= e\left(\frac{1}{4p^2}\left\{4\mathfrak{M}^{-1}\left[{}^t\hat{R}_1 - \frac{1}{2}L{}^t\hat{R}_2\right] + \mathfrak{M}[{}^t\lambda_1] - 2\hat{R}\begin{pmatrix} 1_r \\ \frac{1}{2}{}^tL \end{pmatrix}{}^t\lambda_1 - 2\lambda_1\begin{pmatrix} 1_r & \frac{1}{2}L \end{pmatrix}{}^t\hat{R}\right\}{}^tDB\right) \\ &\quad \times\ p^{n-i-j} \sum_{\lambda' \in M_{n,1}(\mathbb{Z})} e\left(\frac{1}{p}a_1[\lambda']\right) \\ &= e\left(\frac{1}{4p^2}\left\{4\mathfrak{M}^{-1}\left[{}^t\hat{R}_1 - \frac{1}{2}L{}^t\hat{R}_2\right] + \mathfrak{M}[{}^t\lambda_1] - 2\hat{R}\begin{pmatrix} 1_r \\ \frac{1}{2}{}^tL \end{pmatrix}{}^t\lambda_1 - 2\lambda_1\begin{pmatrix} 1_r & \frac{1}{2}L \end{pmatrix}{}^t\hat{R}\right\}{}^tDB\right) \\ &\quad \times\ p^{n-i+(n-j-\alpha)/2}\varepsilon(a_1), \end{aligned}$$

where $\varepsilon(a_1)$ is the symbol defined before Lemma 5.1. We recall $A_1(N,R) = C_1(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL)$. Therefore,

$$\begin{aligned}
&A_2(\hat{N},\hat{R}) \\
&= p^{-(r+1)(k-n)} \sum_{i,j,b,u} \det D^{-k} \sum_{\lambda=(\lambda_1,\lambda_2)\in M_{n,r+1}(\mathbb{Z})} A_1(N,R)e(NBD^{-1}) \\
&= p^{-(r+1)(k-n)} \sum_{i,j,b,u} \det D^{-k} \sum_{\lambda=(\lambda_1,\lambda_2)\in M_{n,r+1}(\mathbb{Z})} A_1(N,R) \\
&\quad \times e\left(\frac{1}{p^2}D\left(\left(\hat{N} - \frac{1}{4}\hat{R}\mathcal{M}^{-1}\left[{}^t\hat{R}\right]\right) + \frac{1}{4}\mathcal{M}^{-1}\left[{}^t\left(\hat{R} - 2\lambda\mathcal{M}\right)\right]\right){}^tDBD^{-1}\right) \\
&= p^{-(r+1)(k-n)} \sum_{i,j,b,u} p^{-k(2n-2i-j)+n-i+(n-j-\alpha)/2}\varepsilon(a_1)e\left(\frac{1}{p^2}\left(\hat{N} - \frac{1}{4}\hat{R}\mathcal{M}^{-1}\left[{}^t\hat{R}\right]\right){}^tDB\right) \\
&\quad \times e\left(\frac{1}{p^2}\left\{\mathfrak{M}^{-1}\left[{}^t\hat{R}_1 - \frac{1}{2}L\left[{}^t\hat{R}_2\right]\right]\right\}{}^tDB\right) \sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL) \\
&\quad \times e\left(\frac{1}{4p^2}\left\{\mathfrak{M}[{}^t\lambda_1] - 2\hat{R}\begin{pmatrix}1_r \\ \frac{1}{2}{}^tL\end{pmatrix}{}^t\lambda_1 - 2\lambda_1\begin{pmatrix}1_r & \frac{1}{2}L\end{pmatrix}{}^t\hat{R}\right\}{}^tDB\right),
\end{aligned}$$

where, in the above summations, i, j run over $i + j \le n$, $j \ge n - \alpha$, and u runs over U_0, and b runs over B_0, and where $D = d_{i,j}u$ and $B = bu$, and (N,R) is determined by the identities (5.2). Remark that if $(N,R) \notin M_{n,r+1}(\mathbb{Z})$, then $C_1(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL) = A_1(N,R) = 0$.

Thus, we conclude

$$\begin{aligned}
A_2(\hat{N},\hat{R}) &= p^{-(r+1)(k-n)} \sum_{i,j,b,u} p^{-k(2n-2i-j)+n-i+(n-j-\alpha)/2}\varepsilon(a_1)e\left(\frac{1}{p^2}\left(\hat{N} - \frac{1}{4}\hat{R}_1{}^t\hat{R}_1\right){}^tDB\right) \\
&\quad \times \sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(4N - R_2{}^tR_2, 4R_1 - 2R_2{}^tL) \\
&\quad \times e\left(\frac{1}{4p^2}\left\{\mathfrak{M}[{}^t\lambda_1] - 2\hat{R}\begin{pmatrix}1_r \\ \frac{1}{2}{}^tL\end{pmatrix}{}^t\lambda_1 - 2\lambda_1\begin{pmatrix}1_r & \frac{1}{2}L\end{pmatrix}{}^t\hat{R}\right\}{}^tDB\right),
\end{aligned}$$

where the summations are the same as the above.

We now calculate $C_2(4\hat{N} - \hat{R}_2{}^t\hat{R}_2, 4\hat{R}_1 - 2\hat{R}_2{}^tL)$ and will show that $C_2(4\hat{N} - \hat{R}_2{}^t\hat{R}_2, 4\hat{R}_1 - 2\hat{R}_2{}^tL)$ coincides with $A_2(\hat{N},\hat{R})$ up to constant as functions of $(\hat{N},\hat{R})$.

For $M \in L_n^*$, $S \in M_{n,r}(\mathbb{Z})$, $\left(\begin{pmatrix} p^{2t}D^{-1} & B \\ 0 & D\end{pmatrix}, \kappa(B)\det D^{\frac{1}{2}}\right) \in \Gamma_0^{(n)}(4)^*\left(K_\alpha, p^{\alpha/2}\right)\Gamma_0^{(n)}(4)^*$, we have

$$\begin{aligned}
&e(M\tau + S{}^tz_1)|_{k-\frac{1}{2},\mathfrak{M}}\left(\begin{pmatrix} p^{2t}D^{-1} & B \\ 0 & D\end{pmatrix}, \kappa(B)\det D^{\frac{1}{2}}\right)|_{k-\frac{1}{2},\mathfrak{M}}[\lambda_1,\mu_1] \\
&\quad = \kappa(B)^{-2k+1}\det(D)^{-k+\frac{1}{2}}e(M(p^{2t}D^{-1}\tau + B)D^{-1} + pS{}^tz_1D^{-1})|_{k-\frac{1}{2},\mathfrak{M}}[\lambda_1,\mu_1] \\
&\quad = \kappa(B)\det(D)^{-k+\frac{1}{2}}e(\mathfrak{M}\left(\tau[\lambda_1] + 2\lambda_1{}^tz\right)) \\
&\qquad \times e(M(p^{2t}D^{-1}\tau + B)D^{-1} + pS^t(z_1 + \tau\lambda_1 + \mu_1)D^{-1}) \\
&\quad = \kappa(B)\det(D)^{-k+\frac{1}{2}}e(\hat{M}\tau + \hat{S}{}^tz + MBD^{-1} + pS{}^t\mu_1D^{-1}),
\end{aligned}$$

where we put

$$\text{(5.5)} \qquad \begin{cases} \hat{M} &= p^2M[{}^tD^{-1}] + \mathfrak{M}[{}^t\lambda_1] + pD^{-1}S{}^t\lambda_1, \\ \hat{S} &= 2\lambda_1\mathfrak{M} + pD^{-1}S, \end{cases}$$

and where $\kappa(B)$ is the symbol defined before Lemma 5.1. We assume $(\hat{M},\hat{S}) \in L_n^* \times M_{n,r}(\mathbb{Z})$. Then, we have $pD^{-1}S = \hat{S} - 2\lambda_1\mathfrak{M} \in M_{n,r}(\mathbb{Z})$. Since

$$MBD^{-1} = \frac{1}{p^2}D\left\{\hat{M} - \frac{1}{2}\lambda_1{}^t\hat{S} - \frac{1}{2}\hat{S}\,{}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tDBD^{-1},$$

and due to Lemma 5.1, we have

$$\begin{aligned} C_2(\hat{M},\hat{S}) &= \sum_{D,B}\kappa(B)\det(D)^{-k+\frac{1}{2}}\sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(M,S)\,e(MBD^{-1}) \\ &= \sum_{D,B}\kappa(B)\det(D)^{-k+\frac{1}{2}}\sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(M,S) \\ &\quad \times e\left(\frac{1}{p^2}\left\{\hat{M} - \frac{1}{2}\lambda_1{}^t\hat{S} - \frac{1}{2}\hat{S}\,{}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tDB\right), \end{aligned}$$

where D and B runs over matrices such that

$$\left\{(D,B)\ \middle|\ i+j\le n,\ j\ge n-\alpha,\ D=d_{i,j}u,\ B=bu,\ b\in B_0,\ u\in U_0\right\},$$

and where $(M,S)\in L_n^*\times M_{n,r}(\mathbb{Z})$ is determined by $(\hat{M},\hat{S})$, D and λ_1 through the identity (5.5). We choose a $(\hat{N},\hat{R})\in L_n^*\times M_{n,r+1}(\mathbb{Z})$ ($\hat{R}=(\hat{R}_1,\hat{R}_2), \hat{R}_1\in M_{n,r}(\mathbb{Z}), \hat{R}_2\in M_{n,1}(\mathbb{Z})$) which satisfies $\hat{M} = 4\hat{N} - \hat{R}_2{}^t\hat{R}_2$ and $\hat{S} = 4\hat{R}_1 - 2\hat{R}_2{}^tL$. Then (M,S) in the above summations is

$$\begin{aligned} M &= \frac{1}{p^2}D\left\{\hat{M} - \frac{1}{2}\lambda_1{}^t\hat{S} - \frac{1}{2}\hat{S}\,{}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tD = 4N - R_2{}^tR_2, \\ S &= \frac{1}{p}D\left(\hat{S} - 2\lambda_1\mathfrak{M}\right) = 4R_1 - 2R_2{}^tL, \end{aligned}$$

where $(N,R)\in L_n^*\times M_{n,r+1}(\mathbb{Z})$ is determined by $(\hat{N},\hat{R})$ through the identity (5.2). Therefore

$$\begin{aligned} &C_2(4\hat{N} - \hat{R}_2{}^t\hat{R}_2, 4\hat{R}_1 - 2\hat{R}_2{}^tL) \\ &= \sum_{D,B}\kappa(B)\det(D)^{-k+\frac{1}{2}}\sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(4N - R_2{}^tR_2, 4\hat{R}_1 - 2\hat{R}_2{}^tL) \\ &\quad \times e\left(\frac{1}{p^2}\left\{4\hat{N} - \hat{R}_2{}^t\hat{R}_2 - \lambda_1{}^t\left(2\hat{R}_1 - \hat{R}_2{}^tL\right) - \left(2\hat{R}_1 - \hat{R}_2{}^tL\right){}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tDB\right) \\ &= \sum_{i,j,b,u}\varepsilon(a_1)p^{(-k+\frac{1}{2})(2n-2i-j)}\sum_{\lambda_1\in M_{n,r}(\mathbb{Z})} C_1(4N - R_2{}^tR_2, 4\hat{R}_1 - 2\hat{R}_2{}^tL) \\ &\quad \times e\left(\frac{1}{p^2}\left\{4\hat{N} - \hat{R}_2{}^t\hat{R}_2 - \lambda_1{}^t\left(2\hat{R}_1 - \hat{R}_2{}^tL\right) - \left(2\hat{R}_1 - \hat{R}_2{}^tL\right){}^t\lambda_1 + \mathfrak{M}[{}^t\lambda_1]\right\}{}^tDB\right) \\ &= p^{(r+1)(k-n)-(n-\alpha)/2}A_2(\hat{N},\hat{R}), \end{aligned}$$

where, in the above summation, i, j run over $i+j\le n$, $j\ge n-\alpha$, and u runs over U_0, and b runs over B_0, We conclude

$$\phi|\tilde{T}_{\alpha,n-\alpha}(p^2) = p^{(r+1)(k-n)-(n-\alpha)/2}\,\iota_{\mathcal{M}}\left(\psi|T_{\alpha,n-\alpha}(p^2)\right).$$

□

Theorem 1.1 follows from Proposition 4.7 and Proposition 5.2.

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