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A RIGIDITY OF EQUIVARIANT HOLOMORPHIC MAPS INTO A COMPLEX GRASSMANNIAN INDUCED FROM ORTHOGONAL DIRECT SUMS OF HOLOMORPHIC LINE BUNDLES

Isami KOGA

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Abstract

In the present paper, we study holomorphic maps induced from orthogonal direct sums of holomorphic line bundles over a compact simply connected homogeneous Kähler manifold into a complex Grassmannian. Then we show if such maps are equivariant, then they are unique up to complex isometry.

1. Introduction

Holomorphic maps into a complex projective space have been studied for a long time and there are many results.

E. Calabi studied holomorphic isometric embeddings into a complex projective space in [1]. He proved local and global rigidity theorems of them. M. Takeuchi notice that those maps are also equivariant with respect to complex isometry group and constructed all holomorphic isometric embeddings of compact homogeneous Kähler manifolds into complex projective spaces in [8].

Recently, some mathematicians study holomorphic or harmonic maps into a complex or real oriented Grassmannian, which are a kind of generalization of complex projective spaces or spheres. For example, in [3] J. Fei, X. Jiao, L. Xiao and X. Xu studied $SU(2)$-equivariant harmonic maps of complex projective line into complex Grassmannians. In [4] L.He, Jiao and X. Zhou studied a rigidity of holmorphic maps of complex projective line into $Gr_2(\mathbb{C}^5)$. They used a method of moving frame and harmonic sequence, which are constructed by S. S. Chern and J. G. Wolfson in [2].

In the present paper, we study holomorphic maps into a complex Grassmannian by using another method. We focus on the relation of holomorphic maps into a complex Grassmannian and holomorphic vector bundles. This is a theory to study harmonic maps into a real oriented or complex Grassmannian considered from Nagatomo in [7]. He proved the following theorem in the same paper.

Theorem 1 ([7], Theorem 5.20). *Let $M := G/K_0$ be a compact reductive Riemannian homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Fix a homogeneous vector bundle $V = G \times_{K_0} V_0 \to G/K_0$ of rank q.*

Let $f : M \to Gr_p(\mathbb{K}^n)$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, be a full harmonic map satisfying following two conditions:

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(i) *The pull-back bundle $f^*Q \to M$ with the pull-back metric and connection is gauge equivalent to $V \to M$ with the invariant metric and the canonical connection. (Hence, $q = n - p$.)*

(ii) *The mean curvature operator $A \in \Gamma(\mathrm{End}V)$ of a map f is expressed as $-\mu\mathrm{Id}_V$ for some positive real number μ.*

Then there exists an eigenspace $W \subset \Gamma(V)$ of the Laplacian of an eigenvalue μ equipped with L_2-scalar product $(\cdot,\cdot)_W$ and a semi-positive symmetric or Hermitian endomorphism $T \in \mathrm{End}(W)$. Regard W as $\mathfrak{g}$-representation (ρ, W). The pair (W, T) satisfies the following conditions.

(I) *The vector space $\mathbb{K}^n$ is a subspace of W with the inclusion $\iota : \mathbb{K}^n \to W$ and $V \to M$ is globally generated by $\mathbb{K}^n$.*

(II) *As a subspace, $\mathbb{K}^n = (\mathrm{Ker}T)^\perp$, and the restriction of T is positive endomorphism of $\mathbb{K}^n$.*

(III) *The endomorphism T satisfies*

$$(T^2 - \mathrm{Id}_W, GH(V_0, V_0))_H = 0, \quad (T^2, GH(\rho(\mathfrak{m})V_0, V_0))_H = 0, \tag{1.1}$$

where V_0 is regarded as a subspace of W.

(IV) *The endomorphism T gives an embedding of $Gr_p(\mathbb{K}^n)$ into $Gr_{p'}(W)$, where $p' = n + \dim \mathrm{Ker}T$ and also gives a bundle isomorphism $\phi : V \to f^*Q$.*

Then, $f : M \to Gr_p(\mathbb{K}^n)$ can be expressed as

$$f(x) = (\iota^* T \iota)^{-1}\left(f_0(x) \cap (\mathrm{Ker}T)^\perp\right), \tag{1.2}$$

where ι^* denotes the adjoint operator of ι under the induced scalar product on $\mathbb{K}^n$ from $(\cdot,\cdot)_W$ on W and f_0 the standard map induced by W.

The pairs (f_1, ϕ_1) and (f_2, ϕ_2) are gauge equivalent if and only if

$$\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2, \tag{1.3}$$

where (T_i, ι_i) correspond to f_i under the expression in (1.2) *respectively.*

Conversely, suppose that a vector space $\mathbb{K}^n$, an eigenspace $W \subset \Gamma(V)$ with eigenvalue μ and a semi-positive symmetric or Hermitian endomorphism $T \in \mathrm{End}(W)$ satisfying condition (I), (II) *and* (III) *are given. Then there exists a unique embedding of $Gr_p(\mathbb{K}^n)$ into $Gr_{p'}(W)$ and the map $f : M \to Gr_p(\mathbb{K}^n)$ defined in* (1.2) *is a full harmonic map into $Gr_p(\mathbb{K}^n)$ satisfying condition* (i) *and* (ii) *with bundle isomorphism $V \cong f^*Q$.*

This theorem can be applied to holomorphic maps into a complex Grassmannian.

In [6] the author studied holomorphic isometric immersions of Hermitian symmetric spaces of compact type into a complex Grassmannian. He defined a *projectively flat map*, which is a holomorphic map whose pull-back of the universal quotient bundle $Q \to Gr_p(\mathbb{C}^n)$ becomes projectively flat by pull-back connection. This is a kind of generalization of holomorphic maps into a complex projective space.

He also showed that holomorphic isometric projectively flat immersions of Hermitian symmetric spaces of compact type have a certain rigidity. This result can be considered as a partially extension of a theorem of Calabi in [1]. In order to show this results, he essentially used the decomposability of the pull-back of the universal quotient bundle.

Inspiring this fact and results of Calabi and Takeuchi, we study holomorphic maps which are equivariant and the pull-back of the universal quotient bundle is decomposed to the direct sum of holomorphic line bundles. Here we define G-equivariance of a holomorphic map $f : G/K \to Gr_p(\mathbb{C}^n)$ by the following:

Definition 2. Let $M = G/K$ be a complex homogeneous space of a compact semi-simple Lie group G. A holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ is called *G-equivariant* if there exists a Lie group homomorphism $\rho : G \to SU(n)$ such that

$$f(gx) = \rho(g)f(x), \qquad \text{for } g \in G,\ x \in M. \tag{1.4}$$

When a holomorphic map $f : G/K \to Gr_p(\mathbb{C}^n)$ is G-equivariant, it is not always uniquely determined the Lie group homomorphism $\rho : G \to SU(n)$ which satisfies (1.4). This is because there may exist more than one G-actions to a holomorphic Hermitian vector bundle which preserve holomorphic structure and Hermitian metric.

In section 2 we provide facts of geometry of complex Grassmannian and in section 3 we study a holomorphic maps induced by a holomorphic vector bundle, especially having homogeneous structure.

In section 4 we introduce and prove main theorem in the present paper (Theorem 9).

As an application, we study equivariant holomorphic maps into a complex projective space in the last section.

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2. Preliminaries 1: vector bundles on a complex Grassmannian

For a detail of the argument of this section, see [7]. Let $\mathbb{C}^n$ be an n-dimensional complex vector space with a Hermitian inner product $(\cdot,\cdot)_n$ and $Gr_p(\mathbb{C}^n)$ be the complex Grassmannian manifold of complex p-planes in $\mathbb{C}^n$. We denote by $S \to Gr_p(\mathbb{C}^n)$ the tautological bundle and by $\underline{\mathbb{C}^n} := Gr_p(\mathbb{C}^n) \times \mathbb{C}^n \to Gr_p(\mathbb{C}^n)$ the trivial bundle of rank n. They are holomorphic vector bundles. The trivial bundle $\underline{\mathbb{C}^n} \to Gr_p(\mathbb{C}^n)$ has the Hermitian fibre metric induced by $(\cdot,\cdot)_n$, which is denoted by the same notation. Since $S \to Gr_p(\mathbb{C}^n)$ is a subbundle of $\underline{\mathbb{C}^n} \to Gr_p(\mathbb{C}^n)$, the bundle $S \to Gr_p(\mathbb{C}^n)$ has a Hermitian fibre metric h_S induced from $(\cdot,\cdot)_n$ and we obtain a holomorphic vector bundle $Q \to Gr_p(\mathbb{C}^n)$ satisfying the following short exact sequence:

$$0 \longrightarrow S \longrightarrow \underline{\mathbb{C}^n} \longrightarrow Q \longrightarrow 0. \tag{2.1}$$

This is called the *universal quotient bundle* over $Gr_p(\mathbb{C}^n)$. When we denote by $S^\perp \to Gr_p(\mathbb{C}^n)$ the orthogonal complement bundle of $S \to Gr_p(\mathbb{C}^n)$ in $\underline{\mathbb{C}^n} \to Gr_p(\mathbb{C}^n)$, $Q \to Gr_p(\mathbb{C}^n)$ is isomorphic to $S^\perp \to Gr_p(\mathbb{C}^n)$ as a C^∞-complex vector bundle. Thus $Q \to Gr_p(\mathbb{C}^n)$ has the Hermitian fibre metric h_Q induced by the Hermitian fibre metric of $S^\perp \to Gr_p(\mathbb{C}^n)$.

These vector bundles are all homogeneous vector bundles. We set $\tilde{G} := SU(n)$ and $\tilde{K} := S(U(p) \times U(q))$. Then $Gr_p(\mathbb{C}^n) \cong \tilde{G}/\tilde{K}$. Let $\mathbb{C}^p$ be a p-dimensional complex subspace of $\mathbb{C}^n$ such that $\mathbb{C}^p$ is an irreducible representation space of $\tilde{K}$ and $\mathbb{C}^q$ the orthogonal complement of $\mathbb{C}^p$. Then $S \to Gr_p(\mathbb{C}^n)$, $S^\perp \to Gr_p(\mathbb{C}^n)$ and $Q \to Gr_p(\mathbb{C}^n)$ are expressed as the following:

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p, \qquad S^{\perp} = \tilde{G} \times_{\tilde{K}} \mathbb{C}^q, \qquad Q = \tilde{G} \otimes_{\tilde{K}} (\mathbb{C}^n/\mathbb{C}^p).$$

For the exact sequence (2.1), the inclusion $i_S : S \to \underline{\mathbb{C}^n}$ is expressed as

$$S = \tilde{G} \times_{\tilde{K}} \mathbb{C}^p \ni [g, v] \longmapsto ([g], gv) \in \tilde{G}/\tilde{K} \times \mathbb{C}^n = \underline{\mathbb{C}^n},$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^p$. Similarly we define a inclusion $i_Q : Q \to \underline{\mathbb{C}^n}$:

$$Q \cong S^{\perp} \ni [g, v] \longmapsto ([g], gv) \in \underline{\mathbb{C}^n},$$

for $g \in \tilde{G}$ and $v \in \mathbb{C}^q$. When we regard $Q \to Gr_p(\mathbb{C}^n)$ a subbundle of $\underline{\mathbb{C}^n} \to Gr_p(\mathbb{C}^n)$ as above, the $\tilde{G}$-action to $Q \to Gr_p(\mathbb{C}^n)$ is expressed as the following:

$$g \cdot ([\tilde{g}], \tilde{g}v) = (g \cdot [\tilde{g}], g\tilde{g}v), \qquad \text{for } g, \tilde{g} \in \tilde{G},\ v \in \mathbb{C}^q.$$

Since the holomorphic tangent bundle $T_{1,0}Gr \to Gr_p(\mathbb{C}^n)$ is identified with $S^* \otimes Q \to Gr_p(\mathbb{C}^n)$, where $S^* \to Gr$ is the dual bundle of $S \to Gr$, complex manifold $Gr_p(\mathbb{C}^n)$ has a homogeneous Hermitian metric $h_{Gr} := h_{S^*} \otimes h_Q$. This is called the Hermitian metric of Fubini-Study type of $Gr_p(\mathbb{C}^n)$ induced from $(\cdot, \cdot)_n$.

We denote by $\pi_p : \mathbb{C}^n \to \mathbb{C}^p$ and $\pi_q : \mathbb{C}^n \to \mathbb{C}^q$ the orthogonal projection. Then the ajoint map $\pi_S : \underline{\mathbb{C}^n} \to S$ and $\pi_Q : \underline{\mathbb{C}^n} \to Q$ of i_S and i_Q is expressed as the following respectively:

$$\pi_S : \underline{\mathbb{C}^n} \ni ([g], w) \longmapsto [g, \pi_p(g^{-1}w)] \in S,$$
$$\pi_Q : \underline{\mathbb{C}^n} \ni ([g], w) \longmapsto [g, \pi_q(g^{-1}w)] \in Q.$$

By using the bundle projection π_S and π_Q, for each vector $w \in \mathbb{C}^n$ we obtain a section of $S \to Gr_p(\mathbb{C}^n)$ and $Q \to Gr_p(\mathbb{C}^n)$ respectively:

$$\pi_S : \mathbb{C}^n \longrightarrow \Gamma(S) : w \longmapsto \pi_S(\cdot, w),$$
$$\pi_Q : \mathbb{C}^n \longrightarrow \Gamma(Q) : w \longmapsto \pi_Q(\cdot, w).$$

It is well-known that π_Q generates holomorphic sections. It follows from a Borel-Weil theory that $\mathbb{C}^n$ is identified with the space $H^0(Q)$ of global holomorphic sections of $Q \to Gr_p(\mathbb{C}^n)$ by π_Q. Since the bundle projection π_Q is the third arrow in (2.1), $Q \to Gr_p(\mathbb{C}^n)$ is globally generated by $\mathbb{C}^n$.

Remark 2.1. When we consider the case that $p = n - 1$ [1], $(Gr_{n-1}(\mathbb{C}^n), h_{Gr})$ is the complex projective space with Fubini-Study metric of constant holomophic sectional curvature 2. (See [6].)

3. Preliminaries 2: standard maps and gauge condition

In this section, we study relations between holomorphic maps into a complex Grassmannian and holomrophic vector bundles over a base manifold.

Let M be a compact Kähler manifold and $V \to M$ a holomorphic vector bundle of rank q equipped with a Hermitian metric h_V and the Hermitian connection ∇^V. We denote by $W = H^0(V)$ the space of holomorphic sections of $V \to M$ and by N the dimension of W. Suppose that $V \to M$ is *globally generated* by W. This means that the following evaluation

[1] In this paper, the complex projective space means the complex Grassmannian manifold $Gr_{n-1}(\mathbb{C}^n)$, not $Gr_1(\mathbb{C}^n)$.

homomorphism

$$ev : \underline{W} := M \times W \longrightarrow V, \quad (x, t) \longmapsto ev_x(t) = t(x)$$

is surjective. For each $x \in M$, we have the kernel Ker ev_x of the linear map $ev_x : W \to V_x$. Dimensions of Ker ev_x is independent of $x \to M$. Therefore we obtain a map

$$f_0 : M \longrightarrow Gr_p(W), \quad x \longmapsto \text{Ker } ev_x,$$

where $p = N - p$. Since $V \to M$ is holomorphic and W is the space of holomorphic sections, f is a holomorphic map.

DEFINITION 3 ([7]). Let M be a compact Kähler manifold, $V \to M$ a holomorphic Hermitian vector bundle and $W = H^0(V)$ the space of holomorphic sections of $V \to M$. We set $(\cdot, \cdot)_W$ an L_2-Hermitian inner product of W, which induces an invariant Kähler metric of $Gr_p(W)$. If $V \to M$ is globally generated by W, the holomorphic map

$$f_0 : M \longrightarrow Gr_p(W), \quad x \longmapsto \text{Ker } ev_x$$

is called the *standard map* induced from $V \to M$, where $q = \text{Rank } V$, $N = \dim W$, $p = N - q$.

Conversely we construct a holomorphic vector bundle and a space of holomorphic sections induced from a holomorphic map.

Let $(\mathbb{C}^n, (\cdot, \cdot)_n)$ be an n-dimensional complex vector space with a Hermitian inner product and $f : M \to Gr_p(\mathbb{C}^n)$ a holomorphic map. Pulling the universal quotient bundle $Q \to Gr_p(\mathbb{C}^n)$ back, we obtain a holomorphic vector bundle $f^*Q \to Gr_p(\mathbb{C}^n)$ with induced metric h_Q and connection ∇^Q. Since $\mathbb{C}^n$ is identified with the space of holomorphic sections of $Q \to Gr_p(\mathbb{C}^n)$, we have a linear map $i : \mathbb{C}^n \to H^0(f^*Q)$ by restricting holomorphic sections of $Q \to Gr_p(\mathbb{C}^n)$ to M, where $H^0(f^*Q)$ is the space of holomorphic sections of $f^*Q \to M$. By using i we have an evaluation homomorphism:

$$ev_{\mathbb{C}} : \underline{\mathbb{C}^n} := M \times \mathbb{C}^n \longrightarrow f^*Q, \quad (x, v) \longmapsto ev_{\mathbb{C}_x}(v) = i(v)(x).$$

By definition of $Q \to Gr_p(\mathbb{C}^n)$, we have $f(x) = \text{Ker } ev_{\mathbb{C}_x}$.

In the present paper, we study holomorphic maps which have a relation to a fixed holomorphic vector bundle, which is called *gauge condition*.

DEFINITION 4 ([7]). Let M be a compact Kähler manifold. We fix a holomorphic vector bundle $V \to M$ equipped with a Hermitian metric h_V and Hermitian connection ∇^V. A holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ is called *satisfying the gauge condition with* $V \to M$ if there exists a holomorphic isomorphism $\phi : V \to f^*Q$ preserving metrics and connections.

We dentote by W the space of holomorphic sections of $V \to M$. Suppose that a holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ satisfies the gauge condition with W. Then we have a linear map $i : \mathbb{C}^n \to W \cong H^0(f^*Q)$.

DEFINITION 5 ([7]). A holomoprhic map $f : M \to Gr_p(\mathbb{C}^n)$ is called *full* if the corresponding linear map $i : \mathbb{C}^n \to H^0(f^*Q)$ is injective.

Remark 3.1. When $p = n - 1$, Definition 5 is the same as the well-known definition.

Let $f : M \to Gr_{n-1}(\mathbb{C}^n)$ be a holomorphic map. In submanifold theory f is called NOT full if there exists a linear subspace U of $\mathbb{C}^n$ which is contained in $f(x) \subset \mathbb{C}^n$ for any x. Let $\iota : \mathbb{C}^n \to H^0(f^*Q)$ be a linear map obtained by restricting each sections of $Q \to Gr_{n-1}(\mathbb{C}^n)$ to M, then $\mathrm{Ker}\, \iota = U$.

Suppose that $f : M \to Gr_p(\mathbb{C}^n)$ is full. By definition $\mathbb{C}^n$ is regarded as a subspace of W by i. We denote by $ev : \underline{W} \to V$ the evaluation homomorphism. Restricting ev to $M \times \mathbb{C}^n$, we obtain a bundle homomorphism

$$ev_{\mathbb{C}} : M \times \mathbb{C}^n \longrightarrow V, \quad (x, v) \longmapsto ev_{\mathbb{C}_x}(v) = i(v)(x).$$

Then we have

$$f(x) = \mathrm{Ker}\, ev_{\mathbb{C}_x} = \mathrm{Ker}\, ev_x \cap \mathbb{C}^n \subset \mathbb{C}^n.$$

We notice that the Hermitian inner product $(\cdot, \cdot)_n$ is not always coincide with $(\cdot, \cdot)_W$. We set $\underline{T}$ the positive Hermitian endomorphism of $\mathbb{C}^n$ satisfying that

$$(\underline{T}u, \underline{T}v)_n = (u, v)_W, \qquad u, v \in \mathbb{C}^n. \tag{3.1}$$

Then we have an complex isometry

$$\underline{T}^{-1} : (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_n) \longrightarrow (Gr_p(\mathbb{C}^n), (\cdot, \cdot)_W), \quad U \longmapsto \underline{T}^{-1}U. \tag{3.2}$$

Let $\pi : W \to \mathbb{C}^n$ be the orthogonal projection with respect to $(\cdot, \cdot)_W$ and we denote by $T := \underline{T} \circ \pi$ an endomorphism of W, which is semi-positive Hermitian.

Consequently, a holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ satisfying the gauge condition with $V \to M$ is expressed as a semi-positive Hermitian endomorphism of W:

$$f : M \longrightarrow (Gr_p(\mathrm{Ker}\, T^{\perp}), (\cdot, \cdot)_W), \quad x \longmapsto T|_{\mathrm{Ker}\, T^{\perp}}^{-1}(f_0(x) \cap \mathrm{Ker}\, T^{\perp}). \tag{3.3}$$

In the remainder of this section we study holomorphic maps satisfying the gauge condition with holomorphic homogeneous Hermitian vecetor bundles.

Let $M = G/K$ be a compact homogeneous Kähler manifold of a compact semi-simple Lie group G and a closed subgroup K. Let $V \to M$ be a holomorphic Hermitian vector bundle.

Definition 6. Let $V \to M$ be a holomorphic vector bundle over a complex homogeneous space. We denote by $pr : V \to M$ the bundle projection. We say that $V \to M$ has a G-action when each g corresponds to a bundle holomorphic isomorphism $g : V \to V$ and satisfies that $pr \circ g = g$, where the right hand side is the natural action of G to M.

When $V \to M$ has a G-action, there exists a K-module V_0 such that $V = G \times_K V_0$. We assume that the G-action preserves Hermitian metric h_V and Hermitian connection ∇^V. The invariant Hermitian metric h_V is obtained by a Hermitian inner product of V_0, which is invariant by K.

For a section t of $V \to M$ and $g \in G$, we obtain a new section gt of V:

$$(gt)(x) = g\left(t(g^{-1}x)\right).$$

It is well known that the space W of holomorphic sections of $V \to M$ is G-module by the above action.

Proposition 7. *The evaluation homomorphism $ev : \underline{W} \to V$ is G-equivariant.*

Proof. We define G-action to $\underline{W} \to M$ by the following:

$$g \cdot (x, t) := (gx, gt), \qquad g \in G,\ x \in M,\ t \in W.$$

For any $g \in G$, $x \in M$ and $t \in W$, we have

$$ev(g \cdot (x, t)) = ev(gx, gt) = (gt)(gx) = g(tg^{-1}(gx)) = g(t(x)) = g \cdot ev(t, x).$$

Therefore ev is G-equivariant. □

We set $e \in G$ the identity element and $o = [e] \in G/K$. Suppose that $V \to M$ is globally generated by W. Then the fibre V_o at o of $V \to M$ is identified with V_0 as a K-module. It follows from the hypothesis that the evaluation homomorphism induces a surjective linear map:

$$ev_o : W \longrightarrow V_0 \ :\ t \longmapsto t(o).$$

We denote by $U_0 = \text{Ker } ev_o$ the kernel of ev_o, which is a K-submodule of W. Since ev is G-equivariant, we have

$$\text{Ker } ev_{[g]} = g\text{Ker } ev_o = gU_0, \qquad g \in G. \tag{3.4}$$

It follows that the standard map from $V \to M$ is G-equivariant.

We identify V_0 with $U_0^{\perp}$ and denote by

$$\pi_0 := ev_o : W \longrightarrow V_0.$$

Then for any $g \in G$ and $t \in W$, we compute that

$$\begin{aligned} ev([g], t) =& t([g]) = t(go) = gg^{-1} \cdot t(go) = g\left((g^{-1}t)(o)\right) \\ =& g \cdot ev(o, g^{-1}t) = g[e, \pi_0(g^{-1}t)] = [g, \pi_0(g^{-1}t)]. \end{aligned}$$

And the adjoint homomorphism $ev^* : V \to \underline{W}$ of ev is expressed as

$$ev^*([g, v]) = ([g], gv), \qquad g \in G,\ v \in V_0. \tag{3.5}$$

4. Holomorphic maps satisfying the gauge condition with the orthogonal direct sum of a holomorphic homogeneous line bundle

Let $M = G/K$ be a compact simply connected homogeneous Kähler manifold of a compact semi-simple Lie group G and a closed subgroup K. We fix a holomorphic homogeneous line bundle $L = G \times_K L_0 \to M$ with an invariant metric h_L and the Hermitian connection ∇^L. We denote by $W = H^0(L)$ the complex vector space of global holomorphic sections of $L \to M$ with L_2-Hermitian inner product $(\cdot, \cdot)_W$ and by N the dimension of W. Let $\tilde{L} \to M$ be the orthogonal direct sum of q-copies of $L \to M$.

We assume that the Hermitian metric, connection and G-action of $\tilde{L} \to M$ are induced from each $L \to M$. Then $\tilde{L} \to M$ has the following expression:

$$\tilde{L} = \bigoplus^{q} L = \bigoplus (G \times_K L_0) = G \times_K \bigoplus L_0.$$

Thus the space $\tilde{W}$ of global holomorphic sections of $\tilde{L} \to M$ is regarded as a orthogonal direct sum of q-copies of W as a G-module. We denote by $(\cdot,\cdot)_{\tilde{W}}$ the L_2-Hermitian inner product of $\tilde{W}$. We denote by $ev_0 : W \to L$ and $ev_1 : \tilde{W} \to \tilde{L}$ the evaluation maps respectively. For $x \in M$ and $t_1 \oplus \cdots \oplus t_q \in \tilde{W}$, we have

$$\begin{aligned} ev_1([g], t_1 \oplus \cdots \oplus t_q) &= t_1(x) \oplus \cdots \oplus t_q(x) \\ &= ev_0([g], t_1) \oplus \cdots \oplus ev_0([g], t_q) \in \tilde{L}. \end{aligned}$$

Then we have $ev_1 = ev_0 \oplus \cdots \oplus ev_0$. Consequently the standard map f_1 induced from $\tilde{L} \to M$ is expressed as the following:

$$\begin{aligned} f_1 : &M \longrightarrow Gr_{N-1}(W) \times \cdots \times Gr_{N-1}(W) \longrightarrow Gr_{q(N-1)}(\mathbb{C}^n), \\ &x \longmapsto (f_0(x), \cdots, f_0(x)) \longmapsto f_0(x) \oplus \cdots \oplus f_0(x), \end{aligned} \tag{4.1}$$

where f_0 is the standard map by $L \to M$. Since f_0 is G-equivariant, f_1 is also G-equivariant.

Proposition 8. *Let $M = G/K$ be a compact complex homogeneous Kähler manifold of a compact semi-simple Lie group G and $f : M \to Gr_p(\mathbb{C}^n)$ a full holomorphic G-equivariant map. Then the pull-back bundle $f^*Q \to M$ is homogeneous and $\mathbb{C}^n$ is regarded as a G-submodule of $H^0(f^*Q)$.*

Proof. Since f is G-equivariant, there exists a Lie group homomorphism $\rho : G \to SU(n)$ and G acts on $\mathbb{C}^n$ by ρ. The definition of the pull-back bundle $f^*Q \to M$ is that

$$f^*Q = \{([g], v) \in M \times Q | f([g]) = \pi(v)\},$$

where $\pi : Q \to Gr_p(\mathbb{C}^n)$ is the natural projection. For any $([\tilde{g}], v) \in f^*Q$ and $g \in G$, we have the action of G to $f^*Q \to M$ by

$$g \cdot ([\tilde{g}], v) = (g[\tilde{g}], \rho(g)v).$$

Since G acts on M transitively, $f^*Q \to M$ is homogeneous.

It follows that the space $H^0(f^*Q)$ of global holomorphic sections has G-action defined by the following:

$$(g \cdot t)(x) := g\left(t(g^{-1}x)\right), \qquad g \in G,\ t \in H^0(f^*Q),\ x \in M.$$

For $v \in \mathbb{C}^n$, we have a holomorphic section $t_v \in H^0(f^*Q)$ by

$$t_v(x) := (x, \pi_Q(v)(f(x))), \qquad x \in M.$$

Thus we compute that

$$\begin{aligned} (g \cdot t_v)(x) &= g(g^{-1}x, \pi_Q(v)(f(g^{-1}x))) = g(g^{-1}x, \pi_Q(v)(\rho(g^{-1})f(x))) \\ &= g(g^{-1}x, \rho(g^{-1}\pi_Q(\rho(g)v)f(x)) = (x, \pi_Q(\rho(g)v)(f(x))) = t_{\rho(g)v}(x). \end{aligned}$$

Therefore $\mathbb{C}^n$ is a G-submodule of $H^0(f^*Q)$. □

In the present paper, our main purpose is to prove the following theorem.

Theorem 9. *Let $M = G/K$ be a compact simply connected homogeneous Kähler manifold and we fix a holomorphic homogeneous line bundle $L = G \times_K L_0 \to M$ equipped with an invariant Hermitian metric h_L and the invariant Hermitian connection ∇^L. We denote by $\tilde{L} = L \oplus \cdots \oplus L$ the orthogonal direct sum of q-copies of $L \to M$. We consider that Hermitian metric, connection and G-action of $\tilde{L} \to M$ are induced from those of each $L \to M$. We also denote by W and $\tilde{W}$ the complex vector space of holomorphc sections of $L \to M$ and $\tilde{L} \to M$ respectively.*

Let $f : M \to Gr_p(\mathbb{C}^n)$ be a full holomorphic map satisfying the gauge condition with $\tilde{L} \to M$. If f is G-equivariant and there exists a G-equivariant bundle isomorphism $\phi : \tilde{L} \to f^*Q$ preserving Hermitian metrics, then f is congruent to the standard map induced from $\tilde{L} \to M$.

Proof. Since $f : M \to Gr_p(\mathbb{C}^n)$ is full, $\mathbb{C}^n$ is a subspace of $\tilde{W}$. It follows from the previous section that there exists a semi-positive Hermitian endomorphism $T : \tilde{W} \to \tilde{W}$ such that the holomorphic map $f : M \to Gr_p(\mathbb{C}^n)$ and the bundle isomorphism $\phi : \tilde{L} \to f^*Q$ are expressed as

$$f([g]) = T^{-1}\left(\tilde{f}_0([g]) \cap (\mathrm{Ker}T)^{\perp}\right), \tag{4.2}$$

$$\phi([g, v]) = ([g], Tgv), \tag{4.3}$$

for $g \in G$ and $v \in \tilde{L}_0$. In order to prove this theorem, we show that the Hermitian endomorphism T is the identity map of $\tilde{W}$.

Since f is G-equivariant, there exists a Lie group homomorphism $\rho : G \to SU(n)$ which satisfies the following equation:

$$f(g[\tilde{g}]) = \rho(g)f([\tilde{g}]), \qquad g, \tilde{g} \in G.$$

Thus $\mathbb{C}^n$ has G-action by ρ. It follows from Proposition 8 that $\mathbb{C}^n$ is a G-submodule of $\tilde{W}$.

Lemma 10. *The semi-positive Hermitian endomorphism $T : \tilde{W} \to \tilde{W}$ is G-equivariant.*

Proof. By definition T is a composed endomorphism of an orthogonal projection $\pi : \tilde{W} \to \mathrm{Ker}T^{\perp}$ and a positive Hermitian endomorphism $\underline{T} : \mathrm{Ker}T^{\perp} \to \mathrm{Ker}T^{\perp}$.

Since $\mathbb{C}^n = \mathrm{Ker}T^{\perp}$ is a G-submodule of $\tilde{W}$ by the natural way, π is G-equivariant.

The positive Hermitian endomorphism $\underline{T}$ satisfies the equality

$$(\underline{T}u, \underline{T}v)_n = (u, v)_W, \qquad u, v \in \mathrm{Ker}T^{\perp}.$$

Therefore for any $g \in G$ and $u, v \in \mathrm{Ker}T^{\perp}$, we obtain

$$(g\underline{T}^2 u, v)_n = (\underline{T}^2 gu, v)_n.$$

It follows from the positivity of $\underline{T}$ that $\underline{T}^2$ (and also $\underline{T}$) is G-equivariant. Consequently $T = \underline{T} \circ \pi$ is G-equivariant. □

Lemma 11.

$$T(\tilde{L}_0) \subset \tilde{L}_0.$$

Proof. Since T is G-equivariant, this is also K-equivariant. Since the orthogonal projection $\pi_j : \tilde{W} \to W$ is K-equivariant for each $j = 1, \cdots, q$, $\pi_j \circ T : \tilde{W} \to W$ is a K-equivariant endomorphism. Thus $\pi_j \circ T(\tilde{L}_0) \subset W$ is a K-submodule of W. It follows from Schur's lemma and Borel-Weil theory and highest weight theory that $\pi_j \circ T(\tilde{L}_0) \subset L_0$. Concequently $T(\tilde{L}_0) \subset (\tilde{L}_0)$. □

We denote by the same notation $T : \tilde{L}_0 \to \tilde{L}_0$ the restriction of $T : \tilde{W} \to \tilde{W}$ to $\tilde{L}_0$.

Lemma 12. *The semi-positive Hermitian endomorphism $T : \tilde{W} \to \tilde{W}$ is the identity map.*

Proof. Since the bundle isomorphism $\phi : \tilde{L} \to f^*Q$ preserves fibre metrics and T is Hermitian, we have

$$(v_1, v_2)_{\tilde{L}_0} = ([e, v_1], [e, v_2])_{\tilde{L}} = ([e, Tv_1], [e, Tv_2])_{\tilde{L}} = (Tv_1, Tv_2)_{\tilde{L}_0} = (T^2 v_1, v_2)_{\tilde{L}_0},$$

for any $v_1, v_2 \in \tilde{L}_0$. Therefore $T^2 : \tilde{L}_0 \to \tilde{L}_0$ is the identity map.

Since W is G-irreducible and T is G-equivariant, $T^2 : \tilde{W} \longrightarrow \tilde{W}$ is the identity map by Schur's lemma and so is T because T is semi-positive Hermitian. □

Consequently we finish the proof of Theorem 9 □

In Theorem 9, we can take some holomorphic homogeneous line bundles $L_i \to M$ which is not always isomorphic to each other.

Theorem 13. *Let $M = G/K$ be a compact simply connected homogeneous Kähler manifold of semi-simple Lie group G. Let $f : M \to Gr_p(\mathbb{C}^n)$ be a full holomorphic G-equivariant map. Assume that $f^*Q \to M$ is decomposed to orthogonal direct sum of holomorphic line bundles $f^*Q = L_1 \oplus \cdots \oplus L_q$ as a holomorphic homogeneous Hermitian vector bundle. Then f is the standard map induced from $L_1 \oplus \cdots \oplus L_q$.*

Proof. Rearranging $L_1 \oplus \cdots \oplus L_q$ we obtain

$$f^*Q = L_{1,1} \oplus \cdots \oplus L_{1,l_1} \oplus \cdots \oplus L_{s,1} \oplus \cdots \oplus L_{s,l_s},$$

where $L_{i,j} \cong L_{k,l}$ if and only if $i = k$. Similarly we obtain the decomposition of the complex vector space $H^0(f^*Q)$:

$$H^0(f^*Q) = W_{1,1} \oplus \cdots \oplus W_{1,l_1} \oplus \cdots \oplus W_{s,1} \oplus \cdots \oplus W_{s,l_s},$$

where $W_{i,j}$ is the complex vector space of holomorphic sections of $L_{i,j} \to M$ and $W_{i,j} \cong W_{k,l}$ if and only if $i = k$.

Then there exists a semi-positive Hermitian inner product

$$T : \bigoplus_{i,j} W_{i,j} \longrightarrow \bigoplus_{i,j} W_{i,j}$$

which satisfies (4.2) and (4.3).

In the same manner as in the proof of Lemma 10 we can prove that T is G-equivariant. It follows from Schur's lemma that the image of T restricting to $W_{i,1} \oplus \cdots \oplus W_{i,l_i}$ is included in $W_{i,1} \oplus \cdots \oplus W_{i,l_i}$. Since $W_{i,j}$ and $W_{i,k}$ are isomorphic as a G-module, by Lemma 11 and Lemma 12 we conclude that the restriction of T to $W_{i,1} \oplus \cdots \oplus W_{i,l_i}$ is the identity map for $i = 1, \cdots, s$. Therefore T is the identity map. □

5. Existence

In the previous section we show the uniqueness of holomorphic maps of compact simply connected homogeneous Kähler manifold satisfying the gauge condition with the orthogonal direct sum of holomorphic line bundles. In this section we show the existence of such maps.

Theorem 14. *The standard map induced from a holomorphic homogeneous line bundle equipped with an invariant Hermitian metric and invariant Hermitian connection over a compact simply connected homogeneous Kähler manifold satisfies the gauge condition.*

Proof. Let M be a compact simply connected homogeneous Kähler manifold and $L \to M$ a holomorphic homogeneous line bundle equipped with an invariant metric h_L and invariant Hermitian connection ∇^L. Let $f_0 : M \to Gr_{n-1}(W)$ be the standard map induced from $L \to M$, where W is the space of holomorphic sections of $L \to M$ and $n = \dim W$. It is known that the pull-back $f^*Q \to M$ of the universal quotient bundle is holomorphically isomorphic to $L \to M$. Since f_0 is G-equivariant, f^*Q has invariant metric h_Q and invariant connection ∇^Q which is compatible with the holomorphic structure of $f^*Q \to M$. Since the invariant connection in a line bundle is Einstein-Hermitian and the Einstein-Hermitian connection is unique up to gauge equivalence, $(L \to M, h_L, \nabla^L)$ is gauge equivalent to (f^*Q, h_Q, ∇^Q). □

6. Application

In this section we consider equivariant holomorphic maps into a complex projective space.

Theorem 15. *Let $M = G/K$ be a compact simply connected homogeneous Kähler manifold of semi-simple Lie group G. We denote by $Gr_{n-1}(\mathbb{C}^n)$ a complex projective space of dimension $n-1$ equipped with Kähler metric having constant holomorphic sectional curvature 2. If a full holomorphic map $f : M \to Gr_{n-1}(\mathbb{C}^n)$ is G-equivariant, then there eixists a holomorphic line bundle $L \to M$ with an invariant Hermitian metric and connection such that f is congruent to the standard map induced by $L \to M$*

Proof. In this case, the pull-back bundle $f^*Q \to M$ is of rank 1. Thus $f^*Q \to M$ is expressed as a homogeneous line bundle $L = G \times_K L_0 \to M$, where L_0 is a 1-dimensional K-module.

It follows from the G-equivariance of f that the pull-back metric and connection are invariant. Therefore by Theorem 9 f is the standard map by $L \to M$. □

Remark 6.1. At a proof of Theorem 15 we do not require the positivity of $L \to M$. This means that if f is not immersed, this theorem holds.

For example, we set $M = F_{1,2} = SU(3)/U(1) \times U(1)$, which is a full flag manifold. Then $SU(3)$-equivariant fibrations $F_{1,2} \to \mathbb{C}P^2$ are determined by semi-positive line bundles $\mathcal{O}(1,0) \to F_{1,2}$ or $\mathcal{O}(0,1) \to F_{1,2}$.

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Graduate school of Mathematics
Kyushu university
744 Motooka, Nishi-ku, Fukuoka, 819–0395
Japan
e-mail: i-koga@math.kyushu-u.ac.jp