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ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS FOR THE LAPLACE EQUATION WITH A LARGE SPECTRAL PARAMETER AND THE INHOMOGENEOUS ROBIN TYPE CONDITIONS

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Abstract

Reduced problems are elliptic problems with a large parameter (as the spectral parameter) given by the Laplace transform of time dependent problems. In this paper, asymptotic behavior of the solutions of the reduced problem for the classical heat equation in bounded domains with the inhomogeneous Robin type conditions is discussed. The boundary of the domain consists of two disjoint surfaces, outside one and inside one. When there are inhomogeneous Robin type data at both boundaries, it is shown that asymptotics of the value of the solution with respect to the large parameter at a given point inside the domain is closely connected to the distance from the point to the both boundaries. It is also shown that if the inside boundary is strictly convex and the data therein vanish, then the asymptotics is different from the previous one.

The method for the proof employs a representation of the solution via single layer potentials. It is based on some non trivial estimates on the integral kernels of related integral equations which are previously established and used in studying an inverse problem for the heat equation via the enclosure method.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^3$ with C^2 boundary. Let D be an open subset of Ω with C^2 boundary and such that $\overline{D} \subset \Omega$; $\Omega \setminus \overline{D}$ is connected. We denote by ν_ξ, ν_y the unit outward normal vectors at $\xi \in \partial D$ and $y \in \partial\Omega$ on ∂D and $\partial\Omega$ respectively. For $\rho_1 \in C(\partial\Omega)$ and $\rho_2 \in C(\partial D)$, we consider the following problem:

$$\tag{1.1} \begin{cases} (\triangle - \lambda^2) w(x;\lambda) = 0 & \text{in } \Omega \setminus \overline{D}, \\ (\partial_\nu + \rho_1(x)) w(x;\lambda) = g_1(x) & \text{on } \partial\Omega, \\ (\partial_\nu + \rho_2(x)) w(x;\lambda) = g_2(x) & \text{on } \partial D, \end{cases}$$

where $\partial_\nu = \sum_{j=1}^{3} (\nu_x)_j \partial_{x_j}$. Note that in this paper, as written in $w(x;\lambda)$ of (1.1), x expresses a variable of functions in subsets of $\mathbb{R}^3$. On the other hand, to avoid confusion, for describing points on ∂D, ξ, η and ζ are used, and, for points on $\partial\Omega$, notations y and z are used.

If $\lambda \in \mathbb{C}$, $-\pi/2 < \arg\lambda < \pi/2$, for any pair $(g_1, g_2) \in L^2(\partial\Omega) \times L^2(\partial D)$, there exists the unique L^2-solution $w(x;\lambda)$ of (1.1). The purpose of the present paper is to study for asymptotic behavior of $w(x;\lambda)$ ($\mathrm{Re}\,\lambda > 0$) as $\mathrm{Re}\,\lambda \to \infty$.

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It seems that the theme treated in this paper studied from 1960's. In the case of the Dirichlet condition, for any $q \in \overline{\Omega} \setminus D$, Varadhan [6] obtained

$$\lim_{\mu\to\infty} \frac{\log|\phi(q;\mu)|}{\mu} = -\min\{\mathrm{dist}(q,\partial\Omega),\mathrm{dist}(q,\partial D)\}, \tag{1.2}$$

where $\mathrm{dist}(q,\partial\Omega) = \inf_{y\in\partial\Omega}|q-y|$, $\mathrm{dist}(q,\partial D) = \inf_{\xi\in\partial D}|q-\xi|$, and $\phi(x;\mu)$ is the solution of

$$\begin{cases} (\triangle-\mu^2)\phi(x;\mu) = 0 & \text{in } \Omega\setminus\overline{D}, \\ \phi(x;\mu) = 1 & \text{on } \partial\Omega\cup\partial D. \end{cases}$$

In [6], Varadhan used (1.2) to give the short time asymptotics of the heat kernel. See also [5] and references therein for the subject itself.

Throughout this paper, we use the following notations:

$$\begin{cases} \mathcal{M}_{\partial\Omega}(q) = \{y\in\partial\Omega \,|\, |q-y| = \mathrm{dist}(q,\partial\Omega)\}, \\ \mathcal{M}_{\partial D}(q) = \{\xi\in\partial D \,|\, |q-\xi| = \mathrm{dist}(q,\partial D)\}. \end{cases} \tag{1.3}$$

As is in the following theorem, for the solution $w(x;\lambda)$, a similar asymptotic formula to $\phi(x;\mu)$ holds.

Theorem 1.1. *Let Ω, D and ρ_j $(j=1,2)$ be as above and take $q\in\overline{\Omega}\setminus D$. Assume that $g_1\in C(\partial\Omega)$ and $g_2\in C(\partial D)$, and there exists a constant $C_0>0$ such that*

$$g_1(y)\geq C_0 \quad (y\in\mathcal{M}_{\partial\Omega}(q)) \quad \textit{and} \quad g_2(\xi)\geq C_0 \quad (\xi\in\mathcal{M}_{\partial D}(q)). \tag{1.4}$$

Then there exists a constant $\delta_0>0$ such that

$$\lim_{|\lambda|\to\infty} \frac{\log|w(q;\lambda)|}{\lambda} = -d(q) \quad \textit{uniformly in } \lambda\in\Lambda_{\delta_0},$$

where $d(q) = \min\{\mathrm{dist}(q,\partial\Omega),\mathrm{dist}(q,\partial D)\}$ and

$$\Lambda_{\delta_0} = \{\lambda\in\mathbb{C}, |\mathrm{Im}\,\lambda| \leq \delta_0\frac{\mathrm{Re}\,\lambda}{\log\mathrm{Re}\,\lambda}, \mathrm{Re}\,\lambda\geq e\}.$$

Thus, when both g_1 and g_2 have the same sign, the distance from $q\in\overline{\Omega}\setminus D$ to the boundary $\partial\Omega\cup\partial D$ determines the asymptotic behavior of the solution $w(q;\lambda)$ as $\mu = \mathrm{Re}\,\lambda\to\infty$. Theorem 1.1 shows that the value of the boundary data at all points of the boundary $\partial\Omega\cup\partial D$ does not contribute to the asymptotic behavior. Only the values on the set $\mathcal{M}_{\partial\Omega}(p)\cup\mathcal{M}_{\partial D}(p)$ are important. Note that Theorem 1.1 holds without any geometrical condition on ∂D like convexity.

Remark 1.2. In the case of $q\in\Omega\setminus\overline{D}$, if all points $y\in\mathcal{M}_{\partial\Omega}(q)$ and $\xi\in\mathcal{M}_{\partial D}(q)$ are non-degenerate critical points of the functions $y\mapsto|y-q|$ and $\xi\mapsto|\xi-q|$, the set Λ_{δ_0} in Theorem 1.1 can be replaced to $\mathbb{C}_{\delta_0} = \{\lambda\in\mathbb{C} \,|\, |\mathrm{Im}\,\lambda|\leq\delta_0\mathrm{Re}\,\lambda\}$ for any fixed $\delta_0>0$ (cf. Proposition 5.2).

For the proof of Theorem 1.1, we use a representation of the solution $w(x;\lambda)$ via single layer potentials (cf. Proposition 2.1). Note that the solution consists of the direct parts derived from the data on each boundary and the "reflected parts" being the parts of reflected solutions at the opposite boundary for each direct part. The solution is constructed in Section 2. In the case of Theorem 1.1, from the form of the solution giving in Section 2, the shortest distance from q to the both boundaries $\partial\Omega$ and ∂D is simply dominant for the asymptotics. There is no contribution of "reflected parts" since the shortest length of any broken path connecting q, a point of one boundary and a point of the other boundary is larger than $d(q)$ (cf. Proposition 5.1).

Now, we turn to consider the case that $g_2(x) = 0$ in (1.1), i.e. the following problem:

$$\begin{cases} (\triangle - \lambda^2)w(x;\lambda) = 0 & \text{in } \Omega \setminus \overline{D}, \\ (\partial_\nu + \rho_1(x))w(x;\lambda) = g_1(x) & \text{on } \partial\Omega, \\ (\partial_\nu + \rho_2(x))w(x;\lambda) = 0 & \text{on } \partial D. \end{cases} \tag{1.5}$$

In this case, we can expect that different phenomena from one for (1.1) may occur since (1.4) does not hold, and there are only signals from the direct part emanating from g_1 on $\partial\Omega$ and its reflected part by the other boundary ∂D. For $q \in \Omega \setminus \overline{D}$, the shorter distance between the direct part and reflected part is given by the former, i.e. $\mathrm{dist}(q, \partial\Omega)$, since the length of any reflected broken path is larger than $\mathrm{dist}(q, \partial\Omega)$. But, when $q \in \partial D$, the direct paths and reflected paths coincide, which means that both the effects coming from the direct path and the reflected one should be counted. Hence, further study is needed when we consider the case $q \in \partial D$. In what follows, we write the reference point q as $p \in \partial D$ when the case $g_2 = 0$ is treated.

For any fixed $p \in \partial D$, we divide $\mathcal{M}_{\partial\Omega}(p)$ into the following three sets:

$$\mathcal{M}^{\pm}_{\partial\Omega}(p) = \{y \in \mathcal{M}_{\partial\Omega}(p) \mid \pm \nu_p \cdot (y - p) > 0\},$$
$$\mathcal{M}^{g}_{\partial\Omega}(p) = \{y \in \mathcal{M}_{\partial\Omega}(p) \mid \nu_p \cdot (y - p) = 0\}.$$

In what follows, for fixed $q \in \overline{\Omega} \setminus D$, we say that $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is a degenerate critical point of finite order for the function $\partial\Omega \ni y \mapsto |y - q| \in \mathbb{R}$ if there exist constants $l_0 > 0$, $r_1 > 0$ and $C' > C > 0$ such that

$$\mathrm{dist}\,(q, \partial\Omega) + C|y - y_0|^{2+l_0} \le |y - q| \le \mathrm{dist}\,(q, \partial\Omega) + C'|y - y_0|^{2+l_0} \tag{1.6}$$
$$(y \in \partial\Omega \cap B(y_0, r_1)),$$

where $B(y_0, r_1) = \{\, y \in \partial\Omega \mid |y - y_0| < r_1 \,\}$. Note that a point $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is a non-degenerate critical point of the function $\partial\Omega \ni y \mapsto |y - q| \in \mathbb{R}$ if we can take $l_0 = 0$ in estimate (1.6) of $|y - q|$. As is in Section 4, $q \notin \partial\Omega$ if (1.6) holds for some $l_0 \ge 0$ (cf. Proposition 4.4).

Theorem 1.3. *Let Ω, D and ρ_j $(j = 1, 2)$ be as above and take $p \in \partial D$. Assume that*
1) *the domain D is strictly convex;*
2) *$\mathcal{M}^{+}_{\partial\Omega}(p) \neq \emptyset$ and $\mathcal{M}^{g}_{\partial\Omega}(p) = \emptyset$ hold;*
3) *$g_1 \in C(\partial\Omega)$ and there exists a constant $C_0 > 0$ such that $g_1(y) \ge C_0$ $(y \in \mathcal{M}^{+}_{\partial\Omega}(p))$;*
4) *Every point $y_0 \in \mathcal{M}^{-}_{\partial\Omega}(p)$ is a non-degenerate or a degenerate critical point of finite order for the function $\partial\Omega \ni y \mapsto |y - p| \in \mathbb{R}$.*

Then there exists a constant $\delta_0 > 0$ such that

$$\lim_{|\lambda|\to\infty} \frac{\log|w(p;\lambda)|}{\lambda} = -\mathrm{dist}(p, \partial\Omega) \quad \textit{uniformly in } \lambda \in \Lambda_{\delta_0}.$$

Note that if $g_2 \neq 0$ and (1.4) hold, then for any $p \in \partial D$, Theorem 1.1 yields

$$\lim_{|\lambda|\to\infty} \frac{\log|w(p;\lambda)|}{\lambda} = -\min\{\mathrm{dist}(p, \partial\Omega), \mathrm{dist}(p, \partial D)\} = 0.$$

Comparing this result with Theorem 1.3, the asymptotic behavior of the solution $w(p;\lambda)$ of (1.5) as $|\lambda| \to \infty$ is completely different from that of the solution of (1.1) with $g_2 \neq 0$.

Remark 1.4. When there exists a degenerate point $y_0 \in \mathcal{M}^+_{\partial\Omega}(p)$, assumption 4) can be relaxed according to the degeneracy of y_0. Assumption 4) in Theorem 1.3 is needed only in the case that every point in $\mathcal{M}^+_{\partial\Omega}(p)$ is a non-degenerate critical point of the function $\partial\Omega \ni y \mapsto |y-p|$.

In Section 5, a proof of Theorem 1.1 is given. In this case, it has simple structure since only the direct parts from both boundaries are dominant. The direct parts of the solution $w(x;\lambda)$ can be simply reduced to some Laplace integrals. In particular, lower bound estimates for them are essential. To get them, we need much more argument than usual, which is given in Section 4.

Sections 6 and 7 are devoted to show Theorem 1.3. Even in this case, we need the representation of the solution $w(x;\lambda)$ used in the proof of Theorem 1.1. Since $g_2 = 0$, as is in (6.2), the formula of $w(p;\lambda)$ consists of the direct part from the outside boundary and the reflected part corresponding to this direct part. The both parts contribute as the main part. For the direct part, it is the same as for Theorem 1.1. Hence, in this case, the problem is to count for the contribution of this "reflection effects".

For this purpose, we need to prepare non trivial estimates on the integral kernels of some operators appearing in the reflected part. These estimates are previously established in [1] and used in studying an inverse problem for the heat equation via the enclosure method (cf. [2]). In the inverse problem of [2], the original problems are reduced to giving lower bounds for some Laplace integrals. In this reduction procedure, we need the same type estimates of the integral kernels as for Theorem 1.3. In Section 3, the necessary estimates for the integral kernels are given.

In [1], the key estimate for the integral kernels are obtained if the boundary has C^{2,α_0} regularity for some $0 < \alpha_0 \leq 1$ and is strictly convex. This regularity assumption is needed to apply the inverse problem developed in [2]. The estimates of the kernels themselves can be given for C^2 boundaries, however, additional argument is needed. This argument is condensed into the proof of Lemma 3.4 although the estimates given in Lemma 3.4 is just the same as those in Proposition 2.1 of [1].

As is given (6.2), $w(p;\lambda)$ is given by

$$w(p;\lambda) = \frac{1}{2\pi}\int_{\partial\Omega} e^{-\lambda|p-y|}\varphi_1(y;\lambda)\Big\{\frac{1}{|y-p|} + A(y,p;\lambda)\Big\}dS_y,$$

where $\varphi_1(y;\lambda)$ is a continuous function on $\partial\Omega$ uniformly positive near $\mathcal{M}^+_{\partial\Omega}(p)$ for λ, and $A(y,p;\lambda)$ is the amplitude function for the reflected part (see (6.3) for the details). Thus, the reflected part is also written by the similar form to the direct part. As is Proposition 6.4, the point is that for $y \in \partial\Omega$ near $\mathcal{M}^+_{\partial\Omega}(p)$ and $\mathcal{M}^-_{\partial\Omega}(p)$, the amplitude function $A(y,p;\lambda)$ for the reflected part given in (6.3) has different asymptotic behavior. Near $\mathcal{M}^+_{\partial\Omega}(p)$, the main term of $A(y,p;\lambda)$ is the same as the amplitude for the direct part, however, near $\mathcal{M}^-_{\partial\Omega}(p)$, the main term cancels out one for the direct part. Thus, in this approach, to find the asymptotic behavior of $A(y,p;\lambda)$ is important, which is given in Section 7.

In the case of Theorem 1.3, the problem is finally reduced to investigating similar Laplace integrals to ones appeared in the case of Theorem 1.1. Thus, the arguments in Section 4 for Laplace integrals are also important to obtain Theorem 1.3. Our approach seems to be simple, but, it gives a direct dependance for what contributes to the main part of the asymptotics and how, which is the advantage of this method.

2. Construction of the solution $w(x;\lambda)$

In this section, the solution of $w(x;\lambda)$ is constructed by using the single layer potentials on $\partial\Omega$ and ∂D. For the function

$$E_\lambda(x,\tilde{x}) = \frac{e^{-\lambda|x-\tilde{x}|}}{2\pi|x-\tilde{x}|},\ x\neq\tilde{x},\ |\arg\lambda| < \frac{\pi}{4},$$

which satisfies the equation $(\triangle - \lambda^2)E(x) + 2\delta(x-\tilde{x}) = 0$, we put

$$V_\Omega(\lambda)g(x) = \int_{\partial\Omega} E_\lambda(x,y)g(y)dS_y,\ \ x\in\mathbb{R}^3\setminus\partial\Omega,$$

and

$$V_D(\lambda)h(x) = \int_{\partial D} E_\lambda(x,\zeta)h(\zeta)dS_\zeta,\ \ x\in\mathbb{R}^3\setminus\partial D.$$

We construct $w(x;\lambda)$ in the form

$$w(x;\lambda) = V_\Omega(\lambda)\psi_1(x;\lambda) + V_D(\lambda)\psi_2(x;\lambda) \tag{2.1}$$

where $\psi_1(\,\cdot\,;\lambda)\in C(\partial\Omega)$ and $\psi_2(\,\cdot\,;\lambda)\in C(\partial D)$ are unknown functions to be determined. In what follows, for Frechét spaces X and Y, $B(X,Y)$ denotes the set of continuous linear operators from X to Y. If X and Y are Banach spaces $B(X,Y)$ is the set of bounded linear operators from X to Y. We also write $B(X) = B(X,X)$.

As is in [4], for example, $V_\Omega(\lambda)$ and $V_D(\lambda)$ satisfy the following properties:

- $V_\Omega(\lambda)g$ satisfies $(\triangle-\lambda^2)V_\Omega(\lambda)g = 0$ in $\mathbb{R}^3\setminus\partial\Omega$.
- $V_D(\lambda)h$ satisfies $(\triangle-\lambda^2)V_D(\lambda)h = 0$ in $\mathbb{R}^3\setminus\partial D$.

These yield that w having the form (2.1) satisfies the equation $(\triangle-\lambda^2)w = 0$ in $\Omega\setminus\overline{D}$.

- $V_\Omega(\lambda)\in B(C(\partial\Omega), C^\infty(\mathbb{R}^3\setminus\partial\Omega)\cap C(\mathbb{R}^3))$ and the Neumann derivative for $V_\Omega(\lambda)g$ at $y\in\partial\Omega$

$$\frac{\partial}{\partial\nu_y}V_\Omega(\lambda)g|_{\partial\Omega}(y) = \lim_{\epsilon\downarrow 0}\sum_{j=1}^{3}(\nu_y)_j\left(\frac{\partial}{\partial x_j}V_\Omega(\lambda)g\right)(y-\epsilon\nu_y)$$

exists and is given by the formula

$$\frac{\partial}{\partial\nu_y}V_\Omega(\lambda)g|_{\partial\Omega}(y) = g(y) + \int_{\partial\Omega}\frac{\partial}{\partial\nu_y}E_\lambda(y,z)g(z)dS_z,\ \ y\in\partial\Omega.$$

- $V_D(\lambda)\in B(C(\partial D), C^\infty(\mathbb{R}^3\setminus\partial D)\cap C(\mathbb{R}^3))$ and the Neumann derivative for $V_D(\lambda)h$ at $\xi\in\partial D$

$$\frac{\partial}{\partial\nu_\xi}V_D(\lambda)h|_{\partial D}(\xi) = \lim_{\epsilon\downarrow 0}\sum_{j=1}^{3}(\nu_\xi)_j\left(\frac{\partial}{\partial x_j}V_D(\lambda)h\right)(\xi+\epsilon\nu_\xi)$$

exists and is given by the formula

$$\frac{\partial}{\partial\nu_\xi}V_D(\lambda)h|_{\partial D}(\xi) = -h(\xi) + \int_{\partial D}\frac{\partial}{\partial\nu_\xi}E_\lambda(\xi,\zeta)h(\zeta)dS_\zeta,\ \ \xi\in\partial D.$$

We define $Y_{ij}(\lambda)$ $(i, j = 1, 2)$ by

$$Y_{11}(\lambda)\psi_1(y;\lambda) = -\int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_y}E_\lambda(y,z) + \rho_1(y)E_\lambda(y,z)\Big)\psi_1(z;\lambda)dS_z \quad (y\in\partial\Omega),$$
$$Y_{12}(\lambda)\psi_2(y;\lambda) = -\int_{\partial D}\Big(\frac{\partial}{\partial\nu_y}E_\lambda(y,\zeta) + \rho_1(y)E_\lambda(y,\zeta)\Big)\psi_2(\zeta;\lambda)dS_\zeta \quad (y\in\partial\Omega),$$
$$Y_{21}(\lambda)\psi_1(\xi;\lambda) = \int_{\partial\Omega}\Big(\frac{\partial}{\partial\nu_\xi}E_\lambda(\xi,z) + \rho_2(\xi)E_\lambda(\xi,z)\Big)\psi_1(z;\lambda)dS_z \quad (\xi\in\partial D)$$

and

$$Y_{22}(\lambda)\psi_2(\xi;\lambda) = \int_{\partial D}\Big(\frac{\partial}{\partial\nu_\xi}E_\lambda(\xi,\zeta) + \rho_2(\xi)E_\lambda(\xi,\zeta)\Big)\psi_2(\zeta;\lambda)dS_\zeta \quad (\xi\in\partial D).$$

From the form of $Y_{ij}(\lambda)$, it follows that $Y_{11}(\lambda)\in B(C(\partial\Omega))$, $Y_{22}(\lambda)\in B(C(\partial D))$, $Y_{12}(\lambda)\in B(C(\partial D), C(\partial\Omega))$ and $Y_{21}(\lambda)\in B(C(\partial\Omega), C(\partial D))$, and

$$\begin{aligned}\|Y_{11}(\lambda)\|_{B(C(\partial\Omega))} &+ \|Y_{22}(\lambda)\|_{B(C(\partial D))} + \|Y_{12}(\lambda)\|_{B(C(\partial D),C(\partial\Omega))}\\ &+ \|Y_{21}(\lambda)\|_{B(C(\partial\Omega),C(\partial D))} \le C(\operatorname{Re}\lambda)^{-1} \qquad (\lambda\in\mathbb{C}, \operatorname{Re}\lambda>0).\end{aligned} \tag{2.2}$$

From the properties of the single layer potentials stated above, we can reduce original problem (1.1) to a system of integral equations for the densities $\psi_1(\,\cdot\,;\lambda)\in C(\partial\Omega)$ and $\psi_2(\,\cdot\,;\lambda)\in C(\partial D)$ given by

$$\begin{cases}\psi_1(x;\lambda) - Y_{11}(\lambda)\psi_1(x;\lambda) - Y_{12}(\lambda)\psi_2(x;\lambda) = g_1(x) & \text{on } \partial\Omega,\\ \psi_2(x;\lambda) - Y_{21}(\lambda)\psi_1(x;\lambda) - Y_{22}(\lambda)\psi_2(x;\lambda) = -g_2(x) & \text{on } \partial D.\end{cases} \tag{2.3}$$

If we choose a constant $\mu_0 > 0$ sufficiently large, from (2.2), for any λ with $\operatorname{Re}\lambda \ge \mu_0$, the inverse $(I - Y_{11}(\lambda))^{-1}$ and $(I - Y_{22}(\lambda))^{-1}$ are constructed by the Neumann series. To solve (2.3), we put

$$\begin{aligned}Z_1(\lambda) &= Y_{12}(\lambda)(I - Y_{22}(\lambda))^{-1}Y_{21}(\lambda)(I - Y_{11}(\lambda))^{-1} \qquad (\lambda\in\mathbb{C}, \operatorname{Re}\lambda\ge\mu_0),\\ Z_2(\lambda) &= Y_{21}(\lambda)(I - Y_{11}(\lambda))^{-1}Y_{12}(\lambda)(I - Y_{22}(\lambda))^{-1} \qquad (\lambda\in\mathbb{C}, \operatorname{Re}\lambda\ge\mu_0)\end{aligned}$$

and define $\varphi_j(x;\lambda)$ $(j = 1, 2)$, $\varphi_{12}(x;\lambda)$ and $\varphi_{21}(x;\lambda)$ by

$$\begin{aligned}\varphi_1(x;\lambda) &= (I - Y_{11}(\lambda))^{-1}(I - Z_1(\lambda))^{-1}g_1(x),\\ \varphi_2(x;\lambda) &= -(I - Y_{22}(\lambda))^{-1}(I - Z_2(\lambda))^{-1}g_2(x),\\ \varphi_{12}(x;\lambda) &= (I - Y_{11}(\lambda))^{-1}Y_{12}(\lambda)\varphi_2(x;\lambda),\\ \varphi_{21}(x;\lambda) &= (I - Y_{22}(\lambda))^{-1}Y_{21}(\lambda)\varphi_1(x;\lambda).\end{aligned} \tag{2.4}$$

From (2.2), it also follows that $(I - Z_j(\lambda))^{-1}$ $(j = 1, 2)$ exists for $\operatorname{Re}\lambda\ge\mu_0$ if we choose a constant $\mu_0 > 0$ sufficiently large. Noting that uniqueness of the solutions of (2.3) holds for $\lambda\in\mathbb{C}$ with large $\operatorname{Re}\lambda$, we get the solution $(\psi_1(x;\lambda), \psi_2(x;\lambda))$ of (2.3) can be expressed as

$$\psi_1(y;\lambda) = \varphi_1(y;\lambda) + \varphi_{12}(y;\lambda), \quad \psi_2(\xi;\lambda) = \varphi_{21}(\xi;\lambda) + \varphi_2(\xi;\lambda) \qquad (\lambda\in\mathbb{C}, \operatorname{Re}\lambda\ge\mu_0)$$

for some fixed $\mu_0 > 0$ sufficiently large. This fact and (2.1) imply the following representation of the solution $w(q;\lambda)$:

Proposition 2.1. *For any $\delta_0 > 0$, there exists a constant $\mu_0 \geq 1$ such that the solution $w(q;\lambda)$ of (1.1) with $\lambda \in \mathbb{C}$, $|\mathrm{Im}\,\lambda| \leq \delta_0^{-1}\mathrm{Re}\,\lambda$ is represented as*

$$w(q;\lambda) = V_\Omega(\lambda)\varphi_1(q;\lambda) + V_\Omega(\lambda)\varphi_{12}(q;\lambda) + V_D(\lambda)\varphi_2(q;\lambda) + V_D(\lambda)\varphi_{21}(q;\lambda) \qquad (q \in \Omega \setminus \overline{D}). \tag{2.5}$$

To show Theorem 1.1, the above formula (2.5) is enough, however, for Theorem 1.3, i.e. the case $g_2 = 0$, which gives $\varphi_2 = 0$ and $\varphi_{12} = 0$, we need to decompose the term $V_D(\lambda)\varphi_{21}(q;\lambda)$ to pick up its main part. By using the transposed operator ${}^tY_{22}(\lambda)$ and ${}^tY_{21}(\lambda)$ of $Y_{22}(\lambda)$ and $Y_{12}(\lambda)$ defined by

$$\int_{\partial D} ({}^tY_{22}(\lambda)g)(\xi)h(\xi)dS_\xi = \int_{\partial D} g(\xi)(Y_{22}(\lambda)h)(\xi)dS_\xi \qquad (g, h \in C(\partial D))$$

and

$$\int_{\partial D} ({}^tY_{12}(\lambda)g)(\xi)h(\xi)dS_\xi = \int_{\partial\Omega} g(y)(Y_{12}(\lambda)h)(y)dS_y \qquad (g \in C(\partial\Omega) \text{ and } h \in C(\partial D)),$$

respectively, we obtain

$$\begin{aligned} V_D(\lambda)\varphi_{21}(q;\lambda) &= \int_{\partial D} E_\lambda(q,\xi)(I - Y_{22}(\lambda))^{-1}Y_{21}(\lambda)\varphi_1(\xi;\lambda)dS_\xi \\ &= \int_{\partial\Omega} {}^tY_{21}(\lambda)({}^t((I - Y_{22}(\lambda))^{-1})E_\lambda(q,\cdot))(y)\varphi_1(y;\lambda)dS_y. \end{aligned} \tag{2.6}$$

Hence, to give the decomposition, the integral kernel representation of ${}^tY_{21}(\lambda){}^t((I-Y_{22}(\lambda))^{-1})$ is needed. Since this term contains the integral operator $(I - Y_{22}(\lambda))^{-1}$ which is constructed by the Neumann series $\sum_{n=0}^{\infty}(Y_{22}(\lambda))^n$, we need estimates of the repeated kernel. This is given in the next section after preparing estimates for boundary integrals.

3. Estimates of boundary integrals and integral kernels

In the proof of Theorems 1.1 and 1.3, many boundary integrals appear. Many of them are treated in [1] in the case that ∂D is strictly convex and C^{2,α_0} for some $0 < \alpha_0 \leq 1$, however, this regularity assumption can be relaxed to C^2. In this section, we give the outline of this procedure. In addition, we also prepare basic estimates for treating these integrals.

We begin by recalling the following well known facts on compact and C^2 surfaces. We denote $\mathcal{B}^2(\mathbb{R}^2)$ by the set of C^2 functions f in $\mathbb{R}^2$ such that the norm $\|f\|_{\mathcal{B}^2(\mathbb{R}^2)}$ is finite, where the norm is defined by $\|f\|_{\mathcal{B}^2(\mathbb{R}^2)} = \max_{|\alpha|\leq 2}\sup_{x\in\mathbb{R}^2}|\partial_x^\alpha f(x)|$.

Lemma 3.1. *Assume that ∂D is of class C^2. (i) There exists a positive constant C such that, for all ξ and $\zeta \in \partial D$*

$$|\nu_\xi - \nu_\zeta| \leq C|\xi - \zeta|, \;\; |\nu_\xi \cdot (\xi - \zeta)| \leq C|\xi - \zeta|^2.$$

(ii) There exists $0 < r_0$ such that, for all $\xi \in \partial D$, $\partial D \cap B(\xi, 2r_0)$ can be represented as a graph of a function on the tangent plane of ∂D at ξ, that is, there exist an open neighborhood U_ξ of $(0,0)$ in $\mathbb{R}^2$ and a function $g \in \mathcal{B}^2(\mathbb{R}^2)$ with $g(0,0) = 0$ and $\nabla g(0,0) = 0$ such that the

map

$$U_\xi \ni \sigma = (\sigma_1, \sigma_2) \mapsto \xi + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma_1, \sigma_2)\nu_\xi \in \partial D \cap B(\xi, 2r_0)$$

gives a system of local coordinates around ξ, *where* $\{e_1, e_2\}$ *is an orthogonal basis for* $T_\xi(\partial D)$. *Moreover the norm* $\|g\|_{\mathcal{B}^2(\mathbb{R}^2)}$ *has an upper bound independent of* $\xi \in \partial D$.

In this paper we call this system of coordinates the standard system of local coordinates around ξ.

First, estimates of the boundary integrals needed to show Theorem 1.1 is given.

Lemma 3.2. *Assume that* $\partial\Omega$ *is of class* C^2. *Then it follows that*

$$\int_{\partial\Omega} \frac{dS_y}{|q-y|} \le C \qquad (q \in \overline{\Omega} \setminus D). \tag{3.1}$$

Proof. Since $\partial\Omega$ is compact, from (ii) of Lemma 3.1, there exist constant $\delta > 0$ and $r_0' > 0$ such that for any $y_0 \in \partial\Omega$, $B(y_0, 2\delta) \cap \partial\Omega$ has the standard local coordinate

$$U \ni \sigma = {}^t(\sigma_1, \sigma_2) \mapsto s(\sigma) = y_0 + \sigma_1 e_1 + \sigma_2 e_2 + g(\sigma)(-\nu_{y_0}) \in B(y_0, 2\delta) \cap \partial\Omega,$$
$$|g(\sigma)| \le C|\sigma|^2 \qquad (\sigma \in U \subset \{\, \sigma \in \mathbb{R}^2 \mid |\sigma| < r_0' \},\ \text{uniformly in } y_0 \in \partial\Omega),$$

and $B(y_0, 2\delta) \cap \Omega \subset \{\, x \in \mathbb{R}^3 \mid \nu_{y_0} \cdot (x - y_0) < 0 \,\}$. Note that $|q-y| \ge \delta$ $(y \in \partial\Omega)$ if $\mathrm{dist}(q, \partial\Omega) \ge \delta$. Hence, in this case, (3.1) holds. When $q \in \overline{\Omega} \setminus D$ satisfies $\mathrm{dist}(q, \partial\Omega) \le \delta$, we can choose a point $y_0 \in \partial\Omega$ in the above coordinate as it satisfies $|q - y_0| = \mathrm{dist}(q, \partial\Omega)$. For this y_0, it follows that $(q - y_0) \cdot e_j = 0$ $(j = 1, 2)$ since

$$|q - y_0|^2 \le |q - s(\sigma)|^2 = |q - y_0|^2 - 2\sum_{j=1}^{2}(q - y_0) \cdot e_j \sigma_j + O(|\sigma|^2) \qquad \text{near } \sigma = 0.$$

Noting that $q \in \overline{\Omega} \cap B(y_0, 2\delta)$, we have $q - y_0 = -|q - y_0|\nu_{y_0}$, which implies that

$$|q - s(\sigma)|^2 = (|q - y_0| - g(\sigma))^2 + |\sigma|^2 \ge |\sigma|^2 \qquad (\sigma \in U). \tag{3.2}$$

Note also that $y \in \partial\Omega$ with $y \notin B(y_0, 2\delta) \cap \partial\Omega$ satisfies $|q-y| \ge |y_0 - y| - |y_0 - q| \ge 2\delta - \delta = \delta$. Hence it follows that

$$\int_{\partial\Omega} \frac{1}{|q-y|} dS_y \le \delta^{-1} \int_{\partial\Omega} 1 dS_y + C \int_U \frac{1}{|\sigma|} d\sigma \le C_{\delta, r_0'} < \infty,$$

which shows Lemma 3.2. □

REMARK 3.3. Similaly to showing (3.1), it follows that for any constant $c_0 > 0$, there exists a constant $C > 0$ such that

$$\int_{\partial D} \frac{e^{-c_0\mu|\xi-\zeta|}}{|\xi-\zeta|^k} dS_\zeta \le \frac{C}{2-k}\mu^{-(2-k)} \quad (0 \le k < 2, \mu \ge 1, \xi \in \partial D).$$

It also holds that

$$\int_{\partial D} \frac{dS_\xi}{|q-\xi|^k} + \int_{\partial\Omega} \frac{dS_y}{|q-y|^k} \le \frac{C}{2-k} \qquad (q \in \overline{\Omega} \setminus D, 0 \le k < 2).$$

These are used to show Theorem 1.3 frequently.

The estimates introduced above are enough to show Theorem 1.1. Thus, strict convexity for ∂D does not need to obtain Theorem 1.1. This is used to get Theorem 1.3. We also need to care for uniformity in g of the standard local coordinate around $\xi \in \partial D$, which may depend on ξ. From (ii) of Lemma 3.1, there exists a constant $R_1 > 0$ independent of $\xi \in \partial D$ such that

$$|g(\sigma)| \leq R_1 |\sigma|^2 \tag{3.3}$$

holds for any $\sigma \in U_\xi$. Note that (3.3) is also given by taking $r_0 \leq 1/(4C)$, where $C > 0$ is the constant given in (i) of Lemma 3.1. To check it, for $\sigma \in U_\xi$, take $\zeta = \xi + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_\xi \in \partial D$. Since $g(\sigma) = -(\zeta - \xi) \cdot \nu_\xi$, Lemma 3.1 implies that $|g(\sigma)| \leq C|\zeta - \xi|^2 < |\zeta - \xi|/2$ for $|\zeta - \xi| < 2r_0 \leq 1/(2C)$. Noting $|\zeta - \xi|^2 = |\sigma|^2 + |g(\sigma)|^2 \leq |\sigma|^2 + |\zeta - \xi|^2/4$, we obtain $|\zeta - \xi|^2 < 4|\sigma|^2/3$, which yields $|g(\sigma)| \leq C|\zeta - \xi|^2 \leq 4C|\sigma|^2/3$. Thus, (3.3) holds if we take $R_1 = 4C/3$ and $r_0 \leq 1/(4C)$.

As above, the constant r_0 in (ii) of Lemma 3.1 can be chosen as small as possible if necessary. In what follows, to show main theorems, we need to take r_0 sufficiently small with finite many times. The same notation r_0 is used even when r_0 is changed to smaller one.

Next we show U_ξ contains a ball with a radius independent of ξ. For $\zeta = \xi + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_\xi \in \partial D \cap B(\xi, 2r_0)$, $|\sigma| \leq |\zeta - \xi| = \sqrt{|\sigma|^2 + g(\sigma)^2} \leq |\sigma|\sqrt{1 + R_1^2(2r_0)^2}$ holds, which yields $\sigma \in U_\xi$ for σ with $|\sigma| < r_1$ if we take

$$r_1 = \frac{2r_0}{\sqrt{1 + R_1^2(2r_0)^2}} < 2r_0. \tag{3.4}$$

Note that this $r_1 > 0$ does not depend on $\xi \in \partial D$.

To obtain the same estimates as in [1] for C^2 boundary case, we have to go back to give the estimates of the length of the broken lines $|\xi - \zeta| + |\zeta - p|$ ($\xi, \zeta, p \in \partial D$) in Proposition 3.1 in [1]. Here, strict convexity is needed. Strict convexity is described by the principal curvatures, i.e. the inverse of the eigenvalues of the Weingartain map $\mathcal{A}_\xi$ defined by

$$(\mathcal{A}_\xi v, v) = -\sum_{j,k=1}^{2} \nu_\xi \cdot \frac{\partial^2 \zeta}{\partial \sigma_j \partial \sigma_k}(0) a_j a_k, \qquad \left(v = \sum_{j=1}^{2} a_j \frac{\partial \zeta}{\partial \sigma_j}(0), a_1, a_2 \in \mathbb{R}\right).$$

Geometrically, ∂D is strictly convex if and only if all of the principal curvatures of ∂D at any point $\xi \in \partial D$ are positive. Since ∂D is C^2 and compact, from strict convexity of ∂D it follows that there exists a constant $R_2 > 0$ such that

$$-\sum_{j,k=1}^{2} \nu_\xi \cdot \frac{\partial^2 \zeta}{\partial \sigma_j \partial \sigma_k}(0) a_j a_k, \geq R_2 |v|^2 \qquad \left(v = \sum_{j=1}^{2} a_j \frac{\partial \zeta}{\partial \sigma_j}(0), a_1, a_2 \in \mathbb{R}\right). \tag{3.5}$$

Note also that strict convexity of ∂D is equivalent to $(\zeta(\sigma) - \xi) \cdot \nu_\xi < 0$ ($\sigma \neq 0$) for any standard local coordinate $\zeta = \zeta(\sigma)$. This fact and the compactness of ∂D imply

$$R_3 = \inf_{\xi, \zeta \in \partial D, \xi \neq \zeta} \frac{-(\zeta - \xi) \cdot \nu_\xi}{|\zeta - \xi|^2} > 0,$$

which yields

$$-(\zeta-\xi)\cdot\nu_\xi \geq R_3|\zeta-\xi|^2 \quad (\xi,\zeta\in\partial D). \tag{3.6}$$

Note that estimate (3.6) implies $g(\sigma) \geq R_3|\sigma|^2$ ($\sigma \in U_\xi, \xi\in\partial D$) for any standard local coordinate.

Let r_0 be the same constant as (ii) of Lemma 3.1. Take any points $\xi\in\partial D$ and $\eta\in\partial D\cap B(\xi,2r_0)$ with $\eta\neq\xi$. Note that $(\eta-\xi)\cdot\nu_\xi<0$ since ∂D is strictly convex. For these ξ and η, we put $e_1=(\eta-\xi-((\eta-\xi)\cdot\nu_\xi)\nu_\xi)/|\eta-\xi-((\eta-\xi)\cdot\nu_\xi)\nu_\xi|$ and take $e_2\in\mathbb{R}^3$ so that the pair $\{e_1,e_2,\nu_\xi\}$ consists of an orthonormal basis of $\mathbb{R}^3$. This pair can be determined since $\eta-\xi-((\eta-\xi)\cdot\nu_\xi)\nu_\xi\neq 0$ if r_0 is chosen small if necessary. From the definition, $(\eta-\xi)\cdot e_1>0$ holds. Since $\{e_1,e_2\}$ is an orthonormal basis of $T_\xi(\partial D)$, we can choose a standard system of local coordinates around ξ satisfying $\eta=\xi+\sigma_1^0e_1-g(\sigma_1^0,0)\nu_\xi\in\partial D\cap B(x,2r_0)$, $(\sigma_1^0)^2+g(\sigma_1^0,0)^2<(2r_0)^2$ and $\sigma_1^0>0$. Note that any point $\zeta\in\partial D\cap B(\xi,2r_0)$ is represented by $\zeta=\xi+\sigma_1e_1+\sigma_2e_2-g(\sigma)\nu_\xi$ with $\sigma\in\mathbb{R}^2$ satisfying $\sigma_1^2+\sigma_2^2+g(\sigma)^2<(2r_0)^2$.

The standard local coordinate mentioned above is used to show the following lemma:

Lemma 3.4. *Assume that ∂D is of class C^2 and strictly convex.*
(i) It follows that

$$|\xi-\zeta|+|\zeta-\eta|\geq|\xi-\eta|+\frac{1}{2}\frac{\sigma_2^2}{|\zeta-\xi|} \qquad (\zeta\in\partial D\cap B(\xi,2r_0)).$$

(ii) If r_0 is chosen small enough, it follows that

$$|\xi-\zeta|+|\zeta-\eta|\geq|\xi-\eta|+\frac{c_0}{|\zeta-\xi|}((\sigma_1^0)^2\sigma_1^2+\sigma_2^2)$$

for all $\sigma=(\sigma_1,\sigma_2)$ and $\sigma^0=(\sigma_1^0,0)$ with $\sigma_1<2\sigma_1^0/3$, $|\sigma|<r_1$ and $|\sigma^0|<r_1$, where $r_1=2r_0/\sqrt{1+R_1^2(2r_0)^2}$ is given in (3.4)*, and c_0 is a positive constant depending only on ∂D.*

Remark 3.5. When ∂D is C^{2,α_0}, Lemma 3.4 is the same as Proposition 2.1 in [1]. This is the only part which is required C^{2,α_0} regularities for ∂D in [1]. Hence from Lemma 3.4, all estimates in [1] can be obtained in the case that ∂D is strictly convex with C^2 regularity.

Proof of Lemma 3.4. For $\zeta=\xi+\sigma_1e_1+\sigma_2e_2-g(\sigma)\nu_\xi\in\partial D\cap B(\xi,2r_0)$, we put $\zeta'=\xi+\sigma_1e_1-g(\sigma)\nu_\xi$. Since $(\eta-\xi)\cdot e_2=0$, i.e. $(\zeta-\xi)\cdot(\eta-\xi)=(\zeta'-\xi)\cdot(\eta-\xi)$, it follows that

$$|\eta-\xi|=(\eta-\zeta)\cdot\frac{\eta-\xi}{|\eta-\xi|}+(\zeta-\xi)\cdot\frac{\eta-\xi}{|\eta-\xi|}\leq|\eta-\zeta|+\frac{(\zeta'-\xi)\cdot(\eta-\xi)}{|\eta-\xi|}$$

which yields

$$\begin{aligned}|\xi-\zeta|+|\zeta-\eta|-|\eta-\xi|&\geq|\zeta-\xi|-\frac{(\zeta'-\xi)\cdot(\eta-\xi)}{|\eta-\xi|}\\&\geq|\zeta-\xi|-|\zeta'-\xi|=\frac{|\zeta-\xi|^2-|\zeta'-\xi|^2}{|\zeta-\xi|+|\zeta'-\xi|}\geq\frac{\sigma_2^2}{2|\zeta-\xi|}.\end{aligned} \tag{3.7}$$

This shows (i) of Lemma 3.4. Note that this proof is just the same as in [1].

Next we show (ii) for the C^2 surface ∂D of strictly convex. The estimate (ii) is given by the fact that there exists a constant $c_0>0$ depending only on ∂D such that

$$1 - \frac{(\zeta' - \xi) \cdot (\eta - \xi)}{|\eta - \xi||\zeta - \xi|} \geq c_0(\sigma_1^0)^2 \quad (\sigma_1 < 2\sigma_1^0/3, |\sigma_1| < r_1, |\sigma_1^0| < r_1) \tag{3.8}$$

for sufficiently small r_0. From (3.7) and (3.8), it follows that

$$|\xi - \zeta| + |\zeta - \eta| - |\eta - \xi| \geq |\zeta - \xi|\Big(1 - \frac{(\zeta' - \xi) \cdot (\eta - \xi)}{|\eta - \xi||\zeta - \xi|}\Big) \geq \frac{c_0(\sigma_1^0)^2|\zeta - \xi|^2}{|\zeta - \xi|}.$$

This implies (ii) of Lemma 3.4 since $|\zeta - \xi| \geq |\sigma| \geq |\sigma_1|$. Thus, it suffices to show (3.8).

We put $\sigma^0 = (\sigma_1^0, 0)$. By the standard local coordinate, the right side of (3.8) is expressed as

$$1 - \frac{\sigma_1^0\sigma_1 + g(\sigma^0)g(\sigma)}{\sqrt{((\sigma_1^0)^2 + g(\sigma^0)^2)(|\sigma|^2 + g(\sigma)^2)}} = 1 - \frac{\sigma_1/|\sigma| + (g(\sigma^0)/\sigma_1^0)(g(\sigma)/|\sigma|)}{\sqrt{(1 + (g(\sigma^0)/\sigma_1^0)^2)(1 + (g(\sigma)/|\sigma|)^2)}}.$$

Since $1/\sqrt{1 + X} \leq 1 - X/2 + 3X^2/8$ $(X \geq 0)$ and $g \geq 0$, from (3.3), there exists a constant $C > 0$ such that for all σ and σ_0 with $|\sigma| < r_1$ and $|\sigma_0| < r_1$

$$\begin{aligned} 1 - \frac{(\zeta' - \xi) \cdot (\eta - \xi)}{|\eta - \xi||\zeta - \xi|} &\geq 1 - \frac{\sigma_1}{|\sigma|} - \frac{g(\sigma^0)}{\sigma_1^0} \cdot \frac{g(\sigma)}{|\sigma|} + \frac{1}{2}\frac{\sigma_1}{|\sigma|}\Big(\frac{g(\sigma^0)^2}{(\sigma_1^0)^2} + \frac{g(\sigma)^2}{|\sigma|^2}\Big) \\ &\qquad - C(|\sigma|^4 + |\sigma_1^0|^4) \\ &= \Big(1 - \frac{\sigma_1}{|\sigma|}\Big)\Big(1 - \frac{1}{4}\Big(\frac{g(\sigma^0)}{\sigma_1^0} + \frac{g(\sigma)}{|\sigma|}\Big)^2\Big) \\ &\qquad + \frac{1}{4}\Big(1 + \frac{\sigma_1}{|\sigma|}\Big)\Big(\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|}\Big)^2 - C(|\sigma|^4 + |\sigma_1^0|^4) \\ &\geq \Big(1 - \frac{\sigma_1}{|\sigma|}\Big)(1 - R_1^2 r_1^2) + \frac{1}{4}\Big(1 + \frac{\sigma_1}{|\sigma|}\Big)\Big(\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|}\Big)^2 - C(|\sigma|^4 + |\sigma_1^0|^4), \end{aligned} \tag{3.9}$$

where the constant C does not depend on r_1 given by (3.4) with $0 < r_0 \leq 1$.

From (ii) of Lemma 3.1, we can take a constant $M > 0$ independent of $\xi \in \partial D$ with $\sup_{\sigma \in U_\xi} |\partial_{\sigma_i}\partial_{\sigma_j} g(\sigma)| \leq M$ for $i, j = 1, 2$. For this M and R_3 given in (3.6), we choose $\varepsilon_0 = \min\{1/4, (R_3/(240(M + R_1))^2\} > 0$, and divide the case $1 - \varepsilon_0 \leq \sigma_1/|\sigma|$ and $1 - \varepsilon_0 \geq \sigma_1/|\sigma|$. In the latter case, (3.9) implies

$$1 - \frac{(\zeta' - \xi) \cdot (\eta - \xi)}{|\eta - \xi||\zeta - \xi|} \geq \varepsilon_0(1 - R_1^2 r_1^2) - C r_1^4,$$

which yields

$$1 - \frac{(\zeta' - \xi) \cdot (\eta - \xi)}{|\eta - \xi||\zeta - \xi|} \geq \frac{\varepsilon_0}{2} \quad (|\sigma| < r_1, |\sigma^0| < r_1 \text{ and } 1 - \varepsilon_0 \geq \sigma_1/|\sigma|) \tag{3.10}$$

if we take $r_1 < \min\{1/(2R_1), \varepsilon_0^{1/4}/(4C)^{1/4}\}$.

Next we consider the case $1 - \varepsilon_0 \leq \sigma_1/|\sigma|$. In this case, $\sigma_1 > 0$. Since $1 = (\sigma_1/|\sigma|)^2 + (\sigma_2/|\sigma|)^2 \geq (1 - \varepsilon_0)^2 + (\sigma_2/|\sigma|)^2 > 1 - 2\varepsilon_0 + (\sigma_2/|\sigma|)^2$, it follows that

$$(\sigma_2/|\sigma|)^2 < 2\varepsilon_0, \qquad \Big(\frac{\sigma_2}{\sigma_1}\Big)^2 < \frac{2\varepsilon_0}{(1 - \varepsilon_0)^2} \leq \frac{32\varepsilon_0}{9} < 4\varepsilon_0 \leq 1. \tag{3.11}$$

This estimate implies $|\sigma|^2 = \sigma_1^2(1 + (\sigma_2/\sigma_1)^2) < 2\sigma_1^2 \leq 8(\sigma_1^0)^2/9$ for $\sigma_1 < 2\sigma_1^0/3$. This estimate and (3.9) yield

$$1 - \frac{(\zeta' - \xi)\cdot(\eta - \xi)}{|\eta - \xi||\zeta - \xi|} \geq \frac{1}{4}\Big(\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|}\Big)^2 - 2C|\sigma_1^0|^4 \tag{3.12}$$

for any σ and σ^0 with $|\sigma| < r_1, |\sigma^0| < r_1, \sigma_1 < 2\sigma_1^0/3$ by taking $r_0 > 0$ sufficiently small to be $r_1 R_1 \leq 1$.

For treating the main term of the right side of (3.12), we divide the inside of the square term as

$$\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|} = \frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma_1, 0)}{|\sigma_1|} + \frac{|\sigma|(g(\sigma_1, 0) - g(\sigma)) + (|\sigma| - |\sigma_1|)g(\sigma)}{|\sigma_1||\sigma|}. \tag{3.13}$$

From $(\partial_{\sigma_2} g)(0, 0) = 0$, it follows that

$$\begin{aligned} g(\sigma) - g(\sigma_1, 0) = \sigma_1\sigma_2 &\int_0^1 \int_0^1 (\partial_{\sigma_1}\partial_{\sigma_2} g)(s\sigma_1, s\theta\sigma_2)\,ds\,d\theta \\ &+ \sigma_2^2 \int_0^1 \int_0^1 \theta(\partial_{\sigma_2}^2 g)(s\sigma_1, s\theta\sigma_2)\,ds\,d\theta, \end{aligned}$$

which yields

$$|g(\sigma) - g(\sigma_1, 0)| \leq M|\sigma_2|(|\sigma_1| + |\sigma_2|),$$

where $M > 0$ can be chosen independent of $\xi \in \partial D$. The above estimate and $|\sigma| - |\sigma_1| \leq |\sigma_2|$, (3.11), (3.3) and (3.13) imply

$$\begin{aligned} \frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|} &\geq \frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma_1, 0)}{|\sigma_1|} - \frac{(M + R_1)|\sigma_2|(|\sigma_1| + |\sigma_2|)}{|\sigma_1|} \\ &\geq \frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma_1, 0)}{|\sigma_1|} - \frac{8\sqrt{\varepsilon_0}(M + R_1)}{3}\sigma_1^0 \end{aligned}$$

for any σ and σ^0 with $|\sigma| < r_1, |\sigma^0| < r_1, \sigma_1 < 2\sigma_1^0/3$ and $1 - \varepsilon_0 \leq \sigma_1/|\sigma|$.

Now we show

$$\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma_1, 0)}{|\sigma_1|} \geq \frac{R_3}{9}\sigma_1^0 \quad (|\sigma| < r_1, |\sigma^0| < r_1, 0 < \sigma_1 < 2\sigma_1^0/3). \tag{3.14}$$

Once we obtain (3.14), from the above estimates, it follows that

$$\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma)}{|\sigma|} \geq \frac{R_3}{10}\sigma_1^0 \qquad (|\sigma| < r_1, |\sigma^0| < r_1, 0 < \sigma_1 < 2\sigma_1^0/3)$$

since $\varepsilon_0 \leq (R_3/(240(M + R_1)))^2$. Hence (3.12) imples

$$1 - \frac{(\zeta' - \xi)\cdot(\eta - \xi)}{|\eta - \xi||\zeta - \xi|} \geq \frac{R_3^2}{400}(\sigma_1^0)^2 - 2C|\sigma_1^0|^4 \geq \frac{R_3^2}{800}(\sigma_1^0)^2$$

for σ and σ^0 with $|\sigma| < r_1, |\sigma^0| < r_1, \sigma_1 < 2\sigma_1^0/3$ if we choose r_0 defined in (3.4) satisfying $r_1 < R_3/(40\sqrt{C})$. From the above estimate and (3.10), we obtain (3.8). Thus, it suffices to show (3.14) to finish the proof of (ii) of Lemma 3.4.

Let π_ξ be the plane defined by $\pi_\xi = \{z \in \mathbb{R}^3 | (z - \xi)\cdot e_2 = 0\}$. We put $\tilde{\zeta} = \xi + \sigma_1 e_1 - g(\sigma_1, 0)\nu_\xi$ and $\eta_0 = \xi + \sigma_1^0 e_1$. Take the curve γ on ∂D defined by $\gamma(t) = \xi + te_1 - g(t, 0)\nu_\xi$ $(-r_1 < t < r_1)$. Note that this curve is the section of ∂D with respect to the plane π_ξ.

The point q crossing the line $\xi\tilde{\zeta}$ and $\eta\eta_0$ is given by $q = \xi + \sigma_1^0 e_1 - (\sigma_1^0 g(\sigma_1,0)/\sigma_1)\nu_\xi$, and the point q' crossing the tangent line $l(t) = \gamma(\sigma_1) + t\gamma'(\sigma_1)$ $(t \in \mathbb{R})$ of γ at $\tilde{\zeta}$ and the line $\eta\eta_0$ is given by $q' = \xi + \sigma_1^0 e_1 - (g(\sigma_1,0) + (\sigma_1^0 - \sigma_1)\partial_{\sigma_1} g(\sigma_1,0))\nu_\xi$. Hence we obtain $|\eta - q'| = |g(\sigma_1^0,0) - (g(\sigma_1,0) + (\sigma_1^0 - \sigma_1)\partial_{\sigma_1} g(\sigma_1,0))|$ and $|\eta - q| = |g(\sigma_1^0,0) - (\sigma_1^0 g(\sigma_1,0)/\sigma_1)|$. Since ∂D is strictly convex, the curve γ is also strictly convex as the curve on the plane π_ξ. Hence, it follows that

$$\frac{g(\sigma_1,0)}{\sigma_1} < \partial_{\sigma_1} g(\sigma_1,0) < \frac{g(\sigma_1^0,0) - g(\sigma_1,0)}{\sigma_1^0 - \sigma_1},$$

which yields

$$|\eta - q'| = g(\sigma_1^0,0) - (g(\sigma_1,0) + (\sigma_1^0 - \sigma_1)\partial_{\sigma_1} g(\sigma_1,0)) < g(\sigma_1^0,0) - (\sigma_1^0 g(\sigma_1,0)/\sigma_1) = |\eta - q|.$$

From the above estimate, $\sigma_1 > 0$, (3.6) and $|\eta - \tilde{\zeta}| \geq \sigma_1^0 - \sigma_1 \geq \sigma_1^0/3$, it follows that

$$\begin{aligned}\frac{g(\sigma^0)}{\sigma_1^0} - \frac{g(\sigma_1,0)}{|\sigma_1|} &> \frac{|\eta - q'|}{\sigma_1^0} \geq \frac{\inf_{t\in\mathbb{R}} |\eta - l(t)|}{\sigma_1^0} \geq \frac{\inf_{x\in\mathbb{R}^3,(x-\tilde{\zeta})\cdot\nu_{\tilde{\zeta}}=0} |\eta - x|}{\sigma_1^0} \\ &= \frac{(\eta - \tilde{\zeta})\cdot(-\nu_{\tilde{\zeta}})}{\sigma_1^0} \geq \frac{R_3|\eta - \tilde{\zeta}|^2}{\sigma_1^0} \geq \frac{R_3\sigma_1^0}{9}.\end{aligned}$$

Thus we obtain (3.14), which completes the proof of Lemma 3.4. □

Next, we recall various estimates given in [1] for integrals on strictly convex boundary ∂D. For $\xi \in \partial D$ and $r > 0$, we put $S_r(\xi) = \{\zeta \in \partial D \,|\, |\zeta - \xi| < r\}$ and $S_r^-(\xi) = \{\zeta \in \partial D \,|\, |\zeta - \xi| \geq r\}$. The following estimates are given in Ikehata and Kawashita [1].

Lemma 3.6. *Assume that ∂D is of class C^2 and strictly convex. Then it follows that there exist constant $C > 0$ and (small) constant $r_0 > 0$ such that*

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \leq Ce^{-\mu\rho_0} \min\Big\{\frac{\rho_0^{3/2}}{\sqrt{\mu}}, \frac{1}{\mu^2\rho_0^3}\Big\} \quad (\xi,\eta \in \partial D, \rho_0 \leq r_0, \mu \geq 1), \tag{3.15}$$

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \leq Ce^{-\mu\rho_0}\mu^{-2} \qquad (\xi,\eta \in \partial D, \rho_0 \geq r_0, \mu \geq 1), \tag{3.16}$$

where $\rho_0 = |\xi - \eta|$. Further, it also follows that

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta - \eta|} dS_\zeta \leq Ce^{-\mu\rho_0} \min\Big\{\frac{1}{\mu\rho_0}, \mu^{-2/3}\Big\} \quad (\xi,\eta \in \partial D, \mu \geq 1), \tag{3.17}$$

$$\int_{S_{\rho_0}^-(\eta)\cup(S_{\rho_0}(\eta)\cap S_{\rho_0}^-(\xi))} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta - \eta|} dS_\zeta \leq \tilde{C}\mu^{-1}e^{-\mu\rho_0} \quad (\xi,\eta \in \partial D, \mu \geq 1). \tag{3.18}$$

Proofs of (3.15) and (3.16) are given in Proposition 3.1 of [1]. The estimates (3.17) and (3.18) are also shown by the same argument as for (5.6), (5.8) and (5.10) in Lemma 5.1 of [1]. Note that (3.15) and (3.16) yield

$$\mu^2 \int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \leq Ce^{-\mu\rho_0}\Big(1 + \min\Big\{\mu(\mu\rho_0^3)^{1/2}, \frac{1}{\rho_0^3}\Big\}\Big) \tag{3.19}$$

since $\partial D = (S_{\rho_0}(\eta) \cap S_{\rho_0}(\xi)) \cup S^-_{\rho_0}(\xi) \cup S^-_{\rho_0}(\eta)$ implies

$$\int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \leq e^{-\mu\rho_0} \int_{S^-_{\rho_0}(\xi)} e^{-\mu|\xi-\zeta|} dS_\zeta + e^{-\mu\rho_0} \int_{S^-_{\rho_0}(\eta)} e^{-\mu|\zeta-\eta|} dS_\zeta + \int_{S_{\rho_0}(\xi)\cap S_{\rho_0}(\eta)} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta.$$

We also need the following lemma:

Lemma 3.7. *Assume that ∂D is of class C^2 and strictly convex. Then, there exist constants $\mu_0 > 0$ and $C > 0$ such that*

$$\int_{\partial D} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \leq Ce^{-\mu|\xi-\eta|} \min\Big\{\frac{1}{\mu|\xi-\eta|}, \mu^{-2/3}, \mu^{-1} + |\xi-\eta|, \mu^{-1} + \sqrt{\frac{|\xi-\eta|}{\mu}}\Big\}$$
$$(\xi, \eta \in \partial D, \xi \neq \eta, \mu \geq \mu_0).$$

Lemma 3.7 can be shown by using the similar argument to the proof of Lemma 3.6 given in [1]. Here we only give a proof of Lemma 3.7, and omit showing Lemma 3.6.

Proof of Lemma 3.7. For $\xi, \eta \in \partial D$, we put $\rho_0 = |\xi - \eta|$. First we show

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \leq Ce^{-\mu\rho_0} \min\{\rho_0, \mu^{-1/2}\rho_0^{1/2}\} \tag{3.20}$$
$$(\xi, \eta \in \partial D, \xi \neq \eta, \mu \geq \mu_0).$$

For $r_0 > 0$ in Lemma 3.6, we put $\rho_0' = \min\{\rho_0, r_0\} > 0$. Since it follows that

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \leq \int_{S_{\rho_0'/2}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta + \int_{(S_{\rho_0}(\eta)\setminus S_{\rho_0'/2}(\eta))\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \equiv I_1 + I_2,$$

we give estimates for I_1 and I_2.

For I_1, we use the similar argument to showing (3.15) as in the proof of Proposition 3.1 in [1]. Take a standard system of local coordinates $\zeta = \eta + \sigma_1 e_1 + \sigma_2 e_2 - g(\sigma)\nu_\eta$ around $\eta \in \partial D$ satisfying $\xi = \eta + \sigma_1^0 e_1 - g(\sigma_1^0, 0)\nu_\eta$. Note that $g \geq 0$ holds since ∂D is strictly convex. For $\zeta \in S_{\rho_0'/2}(\eta) \cap S_{\rho_0}(\xi)$, it follows that $(\zeta - \eta) \cdot (\xi - \eta)/|\xi - \eta| \leq \rho_0'/2$, which yields $\sigma_1\sigma_1^0/\rho_0 \leq (\zeta - \eta) \cdot (\xi - \eta)/|\xi - \eta| \leq \rho_0'/2 \leq \rho_0/2$. This and (3.3) imply

$$\sigma_1 \leq \frac{\rho_0^2}{2\sigma_1^0} = \frac{\sigma_1^0}{2}(1 + (g(\sigma_1^0, 0)/\sigma_1^0)^2) \leq \sigma_1^0 \frac{1 + R_1^2(2r_0)^2}{2}.$$

Note that this is the same as (3.8) of [1]. Thus, choosing r_0 small enough to be $R_1^2(2r_0)^2 < 1/3$, we obtain $\sigma = (r\cos\theta, r\sin\theta) \in B_x'(0, 2\sigma_1^0/3) = \{\sigma \in \mathbb{R}^2 \,|\, |\sigma| < \rho_0, \sigma_1 < 2\sigma_1^0/3\}$. This fact and (ii) of Lemma 3.4 yield

$$|\xi - \zeta| + |\zeta - \eta| \geq \rho_0 + r(c_0(\rho_0)^2 + c_1 f(\theta)^2),$$

where $c_0 > 0$ and $c_1 > 0$ are constants, and $f(\theta)$ is given by

$$f(\theta) = \begin{cases} |\theta|, & \text{if } |\theta| \le \pi/2, \\ |\pi - \theta|, & \text{if } \pi/2 \le \theta \le \pi, \\ |\pi + \theta|, & \text{if } -\pi \le \theta \le -\pi/2. \end{cases}$$

As is in (3.10) of [1], using this coordinate, we obtain

$$\begin{aligned} I_1 &\le Ce^{-\mu\rho_0} \int_0^{\rho_0'} \int_{-\pi/2}^{\pi/2} \frac{e^{-\mu r(c_0(\rho_0')^2 + c_1\theta^2)}}{r} r dr d\theta \\ &\le Ce^{-\mu\rho_0} \int_0^{\rho_0'} e^{-\mu c_0 r(\rho_0')^2} \int_0^{\pi\sqrt{\mu r}/2} e^{-c_1\theta^2} \frac{d\theta}{\sqrt{\mu r}} dr \le Ce^{-\mu\rho_0}\mu^{-1/2} \int_0^{\rho_0'} r^{-1/2} dr, \end{aligned}$$

which yields $I_1 \le \tilde{C}e^{-\mu\rho_0}(\rho_0'/\mu)^{1/2}$. Note that this is the idea to show Proposition 3.1 of [1] (i.e. (3.15) in Lemma 3.6). Thus, (3.15) can be shown similarly. For the detail, see the proof of Proposition 3.1 of [1].

For I_2, if $\rho_0 \le r_0$, (3.15) implies

$$I_2 \le \frac{2}{\rho_0'} \int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \le \frac{2}{\rho_0} \frac{C\rho_0^{3/2}}{\sqrt{\mu}} \le \frac{C\rho_0^{1/2}}{\sqrt{\mu}}.$$

If $\rho_0 > r_0$, (3.16) yields

$$I_2 \le \frac{2}{\rho_0'} \int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)} dS_\zeta \le \frac{2}{r_0} C\mu^{-2} \le C\mu^{-2}.$$

The above estimates of I_1 and I_2 implies that

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \le Ce^{-\mu\rho_0}\mu^{-1/2}\rho_0^{1/2},$$

which yields the half of (3.20).

For the rest, the simple triangle inequality $|\xi - \zeta| + |\zeta - \eta| \ge |\xi - \eta|$ implies

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \le e^{-\mu|\xi-\eta|} \int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{1}{|\zeta-\eta|} dS_\zeta.$$

Since

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{1}{|\zeta-\eta|} dS_\zeta \le C\min\{\rho_0, r_0\},$$

and $\min\{\rho_0, r_0\} \le r_0 \le \rho_0$ when $\rho_0 \ge r_0$, it follows that

$$\int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(\xi)} \frac{e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}}{|\zeta-\eta|} dS_\zeta \le C\rho_0 e^{-\mu|\xi-\eta|}.$$

Thus, we obtain (3.20).

Since we can divide ∂D as $\partial D = S^-_{\rho_0}(\xi) \cup (S_{\rho_0}(\xi) \cap S^-_{\rho_0}(\eta)) \cup (S_{\rho_0}(\xi) \cap S_{\rho_0}(\eta))$, from (3.17), (3.18) and (3.20), we obtain Lemma 3.7. □

Next we turn to give estimates for the integral kernel of $Y_{22}(\lambda)(I - Y_{22}(\lambda))^{-1} = (I - Y_{22}(\lambda))^{-1}Y_{22}(\lambda)$. The integral kernel of the integral operator $Y_{22}(\lambda)$ is denoted by $Y_{22}(\xi, \zeta; \lambda)$

$(\xi, \zeta \in \partial D, \xi \neq \zeta)$. From the definition of $Y_{22}(\lambda)$, the kernel is given by

$$Y_{22}(\xi, \zeta; \lambda) = \frac{\partial}{\partial \nu_\xi} E_\lambda(\xi, \zeta) + \rho_2(\xi) E_\lambda(\xi, \zeta).$$

The operator $(I - Y_{22}(\lambda))^{-1}$ is constructed by the Neumann series $\sum_{n=0}^{\infty}(Y_{22}(\lambda))^n$. The n-th power $(Y_{22}(\lambda))^n$ of the operator $Y_{22}(\lambda)$ is also an integral operator with the kernel $Y_{22}^{(n)}(\xi, \zeta; \lambda)$. This kernel is called the repeated kernel, which is inductively given by

$$Y_{22}^{(n)}(\xi, \eta; \lambda) = \int_{\partial D} Y_{22}(\xi, \zeta; \lambda) Y_{22}^{(n-1)}(\zeta, \eta; \lambda) dS_\zeta \qquad (n = 2, 3, \ldots)$$

and $Y_{22}^{(1)}(\xi, \eta; \lambda) = Y_{22}(\xi, \eta; \lambda)$.

For $x \in \mathbb{R}^3$, $\xi \in \partial D$ and $x \neq \xi$, we put

$$K_\lambda(\xi, x) = \frac{\lambda}{2\pi} e^{-\lambda|\xi - x|} \frac{\nu_\xi \cdot (x - \xi)}{|\xi - x|^2}, \ \tilde{K}_\lambda(\xi, x) = \frac{e^{-\lambda|\xi - x|}}{2\pi} \Big(\frac{\nu_\xi \cdot (x - \xi)}{|\xi - x|^3} + \frac{\rho_2(\xi)}{|\xi - x|} \Big). \tag{3.21}$$

From the definition of $Y_{22}(\lambda)$ and (i) of Lemma 3.1, it follows that $Y_{22}(\xi, \eta; \lambda) = K_\lambda(\xi, \eta) + \tilde{K}_\lambda(\xi, \eta)$ for $\xi, \eta \in \partial D$ with $\xi \neq \eta$, and there exist constants C_0, $C > 0$ such that

$$|K_\lambda(\xi, \eta)| \leq C_0 \, \mu \, e^{-\mu|\xi - \eta|} \ (\xi, \eta \in \partial D, \ \xi \neq \eta, \ \lambda \in \mathbb{C}_{\delta_0}), \tag{3.22}$$

$$|\tilde{K}_\lambda(\xi, \eta)| \leq C_1 \, \frac{e^{-\mu|\xi - \eta|}}{|\xi - \eta|} \ (\xi, \eta \in \partial D, \ \xi \neq \eta, \ \lambda \in \mathbb{C}_{\delta_0}). \tag{3.23}$$

Note that from (3.22) and (3.23), it follows that

$$|Y_{22}(\xi, \eta; \lambda)| \leq C \frac{e^{-(\mu/2)|\xi - \eta|}}{|\xi - \eta|} \qquad (\xi, \eta \in \partial D, \ \xi \neq \eta, \ \lambda \in \mathbb{C}_{\delta_0}).$$

Hence, using Remark 3.3 inductively, we can see that there exists a constant $C > 0$ such that $|Y_{22}^{(n)}(\xi, \eta; \lambda)| \leq C^n \mu^{-n}$ $(\xi, \eta \in \partial D, \lambda \in \mathbb{C}_{\delta_0}, n = 2, 3, \ldots)$. This implies that $\sum_{n=2}^{\infty} Y_{22}^{(n)}(\xi, \eta; \lambda)$ is uniformly convergent for $\xi, \eta \in \partial D$, $\xi \neq \eta$, $\lambda \in \mathbb{C}_{\delta_0}$ with $\mathrm{Re}\,\lambda \geq \mu_0$ for some constant $\mu_0 \geq 1/(2C)$. Thus, the integral kernel of $Y_{22}(\lambda)(I - Y_{22}(\lambda))^{-1}$ is given by $Y_\lambda^\infty(\xi, \eta) = Y_{22}(\xi, \eta; \lambda) + \sum_{n=2}^{\infty} Y_{22}^{(n)}(\xi, \eta; \lambda)$ with the estimate

$$|Y_\lambda^\infty(\xi, \eta)| \leq C \Big\{ 1 + e^{-\mu|\xi - \eta|} \Big(\mu + \frac{1}{|\xi - \eta|} \Big) \Big\} \qquad (\xi, \eta \in \partial D, \xi \neq \eta, \lambda \in \mathbb{C}_{\delta_0}, \mathrm{Re}\,\lambda \geq \mu_0). \tag{3.24}$$

This is a simple and primary approach for getting the integral kernels, however, the obtained estimate (3.24) is too weak to find the points on $\partial\Omega$ and ∂D contributing the asymptotic behavior of $w(p; \lambda)$ for fixed $p \in \partial D$. Hence, we need to use more accurate estimates of the repeated kernels, which is established by [1].

Theorem 3.8. *Assume that ∂D is of class C^2 and strictly convex. Then there exist positive constants C and $\mu_0 \geq 1$ such that for all $\lambda \in \mathbb{C}_{\delta_0}$ with $\mu = Re\,\lambda \geq \mu_0$, the kernel $Y_\lambda(\xi, \eta)$ is measurable for $(\xi, \eta) \in \partial D \times \partial D$, continuous for $\xi \neq \eta$ and has the estimate*

$$|Y_\lambda^\infty(\xi, \eta)| \leq C \left(\mu + \frac{1}{|\xi - \eta|} \right) e^{-\mu|\xi - \eta|} \ (\xi, \eta \in \partial D, \ \xi \neq \eta).$$

Note that the estimate for $Y_\lambda^\infty(\xi, \eta)$ obtained in Theorem 3.8 has the exactly same exponential factor $e^{-\mu|\xi-\eta|}$ as is in estimates (3.22) and (3.23) for the original kernel $Y_{22}(\xi,\eta;\lambda)$. This is the advantage of Theorem 3.8, and is useful for applying problems which never allow losing exponential factors in each steps for estimating integral kernels. This type of problems originally arises in some inverse problem for the heat equation with the enclosure method developed by [2]. Note that Theorem 1.3 has the same structure. Theorem 3.8 is the same as Theorem 1.1 in [1]. In [1], for the boundary ∂D, C^{2,α_0} regularity with some $0<\alpha_0\le 1$ is assumed, however, this restriction put in [1] comes from getting the estimates stated in Lemma 3.4. Thus, regularity assumption can be reduced to C^2 since Lemma 3.4 is given for ∂D with C^2 regularity.

To show Theorem 1.3, we need to estimate the integral kernel of ${}^tY_{21}(\lambda)({}^t(I-Y_{22}(\lambda)))^{-1}$, which is given by

$$ {}^tY_{21}(\lambda)({}^t(I-Y_{22}(\lambda)))^{-1}f(y)=\int_{\partial D}M_\lambda(\xi,y)f(\xi)d\xi \qquad (f\in C(\partial D)) $$

by using the integral kernel $M_\lambda(\xi,y)$ ($y\in\partial\Omega$, $\xi\in\partial D$) of $(I-Y_{22}(\lambda))^{-1}Y_{21}(\lambda)$. Since $(I-Y_{22}(\lambda))^{-1}Y_{21}(\lambda)=Y_{21}(\lambda)+Y_{22}(\lambda)(I-Y_{22}(\lambda))^{-1}Y_{21}(\lambda)$, $M_\lambda(\xi,y)$ is of the form

$$ \begin{aligned} M_\lambda(\xi,y) &= Y_{21}(\xi,y;\lambda)+\int_{\partial D}Y_\lambda^\infty(\xi,\eta)Y_{21}(\eta,y;\lambda)dS_\eta \\ &\qquad (y\in\partial\Omega,\xi\in\partial D,\lambda\in\mathbb{C}_{\delta_0},\mathrm{Re}\,\lambda\ge\mu_0), \end{aligned} \tag{3.25} $$

where

$$ \begin{aligned} Y_{21}(\xi,y;\lambda) &= \frac{\partial}{\partial\nu_\xi}E_\lambda(\xi,y)+\rho_2(\xi)E_\lambda(\xi,y) \\ &= K_\lambda(\xi,y)+\tilde{K}_\lambda(\xi,y) \qquad (\xi\in\partial D,y\in\partial\Omega,\lambda\in\mathbb{C}_{\delta_0}) \end{aligned} \tag{3.26} $$

is the integral kernel of the integral operator $Y_{21}(\lambda)$.

From (3.26) and (3.21), it follows that

$$ |Y_{21}(\xi,y;\lambda)|\le C\mu e^{-\mu|\xi-y|} \quad (\xi\in\partial D,\ y\in\partial\Omega,\ \lambda\in\mathbb{C}_{\delta_0},\ \mu\ge\mu_0), \tag{3.27} $$

which yields

$$ |M_\lambda(\xi,y)|\le C\mu^2e^{-\mu|\xi-y|} \qquad (\xi\in\partial D,y\in\partial\Omega,\lambda\in\mathbb{C}_{\delta_0},\mu\ge\mu_0) \tag{3.28} $$

by (3.25), Theorem 3.8, Remark 3.3 and triangle inequality $|\xi-\zeta|+|\zeta-y|\ge|\xi-y|$.

The right side of (3.28) contains the term of order 2 in μ. The term providing this order surely contribute to the main term. To pick this up, we need the following estimate:

Proposition 3.9. *Assume that ∂D is of class C^2 and strictly convex. Then there exist positive constants C and $\mu_0\ge 1$ such that for all $\lambda\in\mathbb{C}_{\delta_0}$ with $\mu=Re\,\lambda\ge\mu_0$, the kernel $Y_\lambda^\infty(\xi,\eta)-K_\lambda(\xi,\eta)$ satisfies*

$$ |Y_\lambda^\infty(\xi,\eta)-K_\lambda(\xi,\eta)|\le Ce^{-\mu|\xi-\eta|}\left(1+\frac{1}{|\xi-\eta|}+\min\left\{\mu(\mu|\xi-\eta|^3)^{1/2},\ \frac{1}{|\xi-\eta|^3}\right\}\right) $$

for ξ, $\eta\in\partial D$ with $\xi\ne\eta$.

Proof. As is in Lemma 3.6, we put $\rho_0=|\xi-\eta|$. From $Y_\lambda^\infty(\xi,\eta)=Y_{22}(\xi,\eta;\lambda)+\sum_{n=2}^\infty Y_{22}^{(n)}(\xi,\eta;\lambda)$, and $Y_{22}(\xi,\eta;\lambda)=K_\lambda(\xi,\eta)+\tilde{K}_\lambda(\xi,\eta)$, it follows that

$$Y_\lambda^\infty(\xi,\eta) - K_\lambda(\xi,\eta) = \tilde{K}_\lambda(\xi,\eta) + \int_{\partial D} Y_\lambda^\infty(\xi,\zeta)Y_{22}(\zeta,\eta;\lambda)dS_\zeta.$$

Hence, it suffices to show

$$\Big|\int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}\left(\mu+\frac{1}{|\zeta-\xi|}\right)\left(\mu+\frac{1}{|\zeta-\eta|}\right)dS_\zeta\Big| \le Ce^{-\mu\rho_0}\left(1+\frac{1}{\rho_0}+\min\left\{\mu(\mu\rho_0^3)^{1/2},\ \frac{1}{\rho_0^3}\right\}\right), \tag{3.29}$$

since (3.23), (3.22) and Theorem 3.8 hold. Noting $|\zeta-\xi| \ge \rho_0/2$ for $|\zeta-\eta| \le \rho_0/2$, and $|\zeta-\eta| \ge \rho_0/2$ for $|\zeta-\xi| \le \rho_0/2$, we have

$$\Big|\int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}\left(\mu+\frac{1}{|\zeta-\xi|}\right)\left(\mu+\frac{1}{|\zeta-\eta|}\right)dS_\zeta\Big| \le \mu^2\int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}dS_\zeta + \left(\mu+\frac{2}{|\eta-\xi|}\right)\int_{\partial D} e^{-\mu(|\xi-\zeta|+|\zeta-\eta|)}\left(\frac{1}{|\zeta-\eta|}+\frac{1}{|\zeta-\xi|}\right)dS_\zeta.$$

This estimate, (3.19), (3.17) and (3.18) give (3.29), which completes the proof of Proposition 3.9. □

Note that Proposition 3.9 implies that

$$|Y_\lambda^\infty(\xi,\eta) - K_\lambda(\xi,\eta)| \le C\left(\mu+\frac{1}{|\xi-\eta|}\right)e^{-\mu|\xi-\eta|} \tag{3.30}$$

since $\min\{\sqrt{a}, a^{-1}\} \le 1$ for all $a > 0$.

Lemma 3.10. *Assume that ∂D is of class C^2 and strictly convex. Then there exist constants $C > 0$ and $\mu_0 > 0$ such that*

(3.31)
$$\int_{\partial D}(|Y_\lambda^\infty(\xi,\eta)| + |K_\lambda(\xi,\eta)| + |\tilde{K}_\lambda(\xi,\eta)|)\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi \le Ce^{-\mu|\eta-p|}\Big(1+\frac{1}{\mu|\eta-p|}+\min\big\{(\mu|\eta-p|)^{1/2}, \frac{1}{|\eta-p|}\big\}\Big)$$
$$(\eta, p\in\partial D, \eta\ne p, \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0)$$

and

(3.32)
$$\int_{\partial D}|\lambda||Y_\lambda^\infty(\xi,\eta) - K_\lambda(\xi,\eta)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi \le Ce^{-\mu|\eta-p|}\frac{1}{|\eta-p|}\Big(1+\frac{1}{|\eta-p|^3}\Big)$$
$$(\eta, p\in\partial D, \eta\ne p, \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0).$$

Proof. From Theorem 3.8, (3.22) and (3.23), the left side of (3.31) is estimated by

$$C\int_{\partial D} e^{-\mu(|\xi-\eta|+|\xi-p|)}\Big(\mu+\frac{1}{|\xi-\eta|}\Big)\frac{1}{|\xi-p|}dS_\xi$$

for some constant $C > 0$. We denote by I_1 the above integral. We put $\rho_0 = |\eta-p|$. Note that $|\eta-\xi| \ge \rho_0/2$ for $|\xi-p| < \rho_0/2$, and $|\xi-p| \ge \rho_0/2$ if $|\xi-\eta| < \rho_0/2$. Thus, either $|\xi-p| \ge \rho_0/2$ or $|\xi-\eta| \ge \rho_0/2$ hold. This and Lemma 3.7 imply

$$I_1 \le \int_{\partial D} e^{-\mu(|\eta-\xi|+|\xi-p|)} \left(\frac{\mu}{|\xi - p|} + \frac{2}{\rho_0|\xi-\eta|} + \frac{2}{\rho_0|\xi-p|} \right) dS_\xi$$
$$\le Ce^{-\mu\rho_0} \left\{ \mu \cdot (\mu^{-1} + \mu^{-1/2}\rho_0^{1/2}) + \frac{4}{\rho_0}(\mu^{-1}+\rho_0) \right\} \le Ce^{-\mu\rho_0} \left(1 + \mu^{1/2}\rho_0^{1/2} + \frac{1}{\mu\rho_0} \right).$$

Using Lemma 3.7 again, we also obtain

$$\int_{\partial D} e^{-\mu(|\eta-\xi|+|\xi-p|)} \frac{\mu}{|\xi-p|} dS_\xi \le Ce^{-\mu|\eta-p|}\mu \frac{1}{\mu|\eta-p|} = Ce^{-\mu|\eta-p|}\frac{1}{|\eta-p|}.$$

These estimates yield (3.31).

Next we show (3.32). From Proposition 3.9, for $|\xi-\eta| \ge \rho_0/2$, it follows that

$$|Y_\lambda^\infty(\eta,\xi) - K_\lambda(\eta,\xi)| \le Ce^{-\mu|\eta-\xi|} \left(1 + \frac{1}{|\eta-\xi|} + \frac{1}{|\eta-\xi|^3} \right) \le Ce^{-\mu|\eta-\xi|}\left(1+\rho_0^{-3}\right).$$

We put the left side of (3.32) I_2. The integral I_2 is divided into two integrals on $S_{\rho_0/2}(\eta)$ and $S^-_{\rho_0/2}(\eta)$. Since $\xi \in S_{\rho_0/2}(\eta)$ implies $|\xi - p| \ge \rho_0/2$, (3.30) and the above estimates yield

$$I_2 \le C\mu \int_{S_{\rho_0/2}(\eta)} e^{-\mu(|\eta-\xi|+|\xi-p|)} \left(\mu + \frac{1}{|\eta-\xi|} \right) \frac{2}{\rho_0} dS_\xi$$
$$+ C\mu \int_{S^-_{\rho_0/2}(\eta)} \frac{e^{-\mu(|\eta-\xi|+|\xi-p|)}}{|\xi-p|} \left(1+\rho_0^{-3}\right) dS_\xi$$
$$\le C\Big\{ \frac{\mu^2}{\rho_0} \int_{S_{\rho_0/2}(\eta)} e^{-\mu(|\eta-\xi|+|\xi-p|)} dS_\xi$$
$$+ \mu\left(1+\rho_0^{-3}\right) \int_{\partial D} \frac{e^{-\mu(|\eta-\xi|+|\xi-p|)}}{|\xi-p|} + \frac{e^{-\mu(|\eta-\xi|+|\xi-p|)}}{|\xi-\eta|} dS_\xi \Big\},$$

where we used $\rho_0^{-1} \le 1 + \rho_0^{-3}$. Here we note that (3.15), (3.16) and Remark 3.3 imply

$$\int_{S_{\rho_0/2}(\eta)} e^{-\mu(|\eta-\xi|+|\xi-p|)} dS_\xi \le \int_{S_{\rho_0}(\eta)\cap S_{\rho_0}(p)} e^{-\mu(|\eta-\xi|+|\xi-p|)} dS_\eta + \int_{S^-_{\rho_0}(p)} e^{-\mu(|\eta-\xi|+|\xi-p|)} dS_\xi$$
$$\le Ce^{-\mu\rho_0}\mu^{-2}\rho_0^{-3} + e^{-\mu\rho_0} \int_{S^-_{\rho_0}(p)} e^{-\mu|\eta-\xi|} dS_\xi \le Ce^{-\mu\rho_0}\mu^{-2}(1+\rho_0^{-3}).$$

Combining the above estimates with Lemma 3.7, we obtain

$$I_2 \le C\Big\{ \frac{\mu^2}{\rho_0} e^{-\mu\rho_0}\mu^{-2}(1+\rho_0^{-3}) + \mu\left(1+\rho_0^{-3}\right) e^{-\mu\rho_0}\frac{1}{\mu\rho_0} \Big\} \le C\frac{e^{-\mu\rho_0}}{\rho_0}(1+\rho_0^{-3}).$$

This completes the proof of Lemma 3.10. □

4. Estimate of Laplace integrals

The proof of Theorems 1.1 and 1.3 is reduced to estimating asymptotic behavior of integrals of Laplace types. In this section, we treat these integrals.

For $q \in \overline{\Omega} \setminus D$, $y_0 \in \mathcal{M}_{\partial\Omega}(q)$, which is defined in (1.3), and sufficiently small $r_0 > 0$, we choose a standard local coordinate $U \ni \sigma \mapsto s(\sigma) \in \partial\Omega \cap B(y_0, 2r_0)$ around a point $y_0 \in \mathcal{M}_{\partial\Omega}(q)$. Let $h(\sigma, \lambda)$ be a continuous function in σ of the form:

$$h(\sigma,\lambda) = h_1(\sigma) + \lambda^{-1}\tilde{h}_1(\sigma,\lambda) \quad (\sigma \in U, \lambda \in \mathbb{C}_{\delta_0}, \mu = \operatorname{Re}\lambda \geq \mu_0)$$

for some $\mu_0 > 0$. We put $S(\sigma) = |q - s(\sigma)|$ and take a function $\eta \in C_0^2(U)$ with $0 \leq \eta \leq 1$ on U and $\eta(0) > 0$. For these functions, we consider the following integral of Laplace type:

$$I(\lambda) = \int_U e^{-\lambda S(\sigma)} \frac{\eta(\sigma)}{S(\sigma)} h(\sigma,\lambda) d\sigma. \tag{4.1}$$

Proposition 4.1. *For Laplace integral* (4.1)*, assume that there exist constants $C_1 > 0$ and $C_1' > 0$ such that*

$$h_1(\sigma) \geq C_1, \quad |\tilde{h}_1(\sigma,\lambda)| \leq C_1' \qquad (\sigma \in U, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0). \tag{4.2}$$

Then, there exist constants $\delta_0 > 0$, $C_1 > 0$ and $\mu_0 > 0$ such that

$$\operatorname{Re}\{e^{\lambda \operatorname{dist}(p,\partial\Omega)} I(\lambda)\} \geq C_1 \mu^{-1} \quad (\lambda \in \Lambda_{\delta_0}, \mu = \operatorname{Re}\lambda \geq \mu_0).$$

Remark 4.2. For oscillatory integrals given by changing $e^{-\lambda S(\sigma)}$ to $e^{i\mu S(\sigma)}$ ($\mu >> 1$) in the definition of $I(\lambda)$, as is in Section 7 of [3], asymptotic behavior as $\mu \to \infty$ is studied even in degenerate cases (i.e. the case that the Hessian matrix of $S(\sigma)$ does not regular at some critical point of $S(\sigma)$). To obtain Proposition 4.1, the basic idea given in [3] can be used for treating degenerate cases although it does not directly work for the integrals $I(\lambda)$ of Laplace type.

Proof of Proposition 4.1. We put $\tau_{-\infty} = \inf_{\sigma \in U} S(\sigma)$ and $\tau_\infty = \sup_{\sigma \in U} S(\sigma)$, and $E_\tau = \{\sigma \in U \,|\, S(\sigma) \leq \tau\}$ for $\tau \in \mathbb{R}$. Note that $\tau_{-\infty} = |q - y_0| = \operatorname{dist}(q,\partial\Omega)$. We introduce the function $\beta_\lambda(\tau)$ defined by

$$\beta_\lambda(\tau) = \int_{E_\tau} \frac{\eta(\sigma)}{S(\sigma)} h(\sigma,\lambda) d\sigma \qquad (\tau \in \mathbb{R}).$$

From (3.2), $\eta(\sigma)/S(\sigma) \geq 0$ is integrable in U, which yields that $\beta_\lambda(\tau)$ is a function of bounded variation, $\beta_\lambda(\tau) = 0$ for $\tau < \tau_{-\infty}$ and $\beta_\lambda(\tau) = \beta_\lambda(\tau_\infty)$ for $\tau \geq \tau_\infty$. Note also that β_λ is a right continuous function in $\tau \in \mathbb{R}$. Indeed, for any $\tau_0 \in \mathbb{R}$ and $\sigma \in U$, $\sigma \notin E_\tau \setminus E_{\tau_0} = \{\sigma \in U \,|\, \tau_0 < S(\sigma) \leq \tau\}$ holds if $0 \leq \tau - \tau_0$ is small enough. This means that $\lim_{\tau \to \tau_0 + 0} \chi_{E_\tau}(\sigma) = \chi_{E_{\tau_0}}(\sigma)$, where $\chi_{E_\tau}(\sigma)$ is the characteristic function of the set E_τ. From this fact and Lebesgue's convergence theorem implies that β_λ is a right continuous.

Thus, using Stieltjes integral with respect to β_λ, for any $\tilde{\tau}_{-\infty} < \tau_{-\infty}$, we obtain

$$I(\lambda) = \int_U e^{-\lambda S(\sigma)} \frac{\eta(\sigma)}{S(\sigma)} h(\sigma,\lambda) d\sigma = \int_{\tilde{\tau}_{-\infty}}^{\tau_\infty} e^{-\lambda\tau} d\beta_\lambda(\tau),$$

which implies that

$$I(\lambda) = e^{-\lambda\tau_\infty}\beta_\lambda(\tau_\infty) + \lambda \int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\lambda}\beta_\lambda(\tau) d\tau. \tag{4.3}$$

We put

$$\beta_0(\tau) = \int_{E_\tau} \frac{\eta(\sigma)}{S(\sigma)} h_1(\sigma) d\sigma \qquad (\tau \in \mathbb{R}).$$

Since $\eta(\sigma)/S(\sigma) \geq 0$, from (4.2), it follows that

$$|\beta_\lambda(\tau)-\beta_0(\tau)| \le \int_{E_\tau} \frac{\eta(\sigma)}{S(\sigma)}|\lambda|^{-1}|\tilde{h}_1(\sigma,\lambda)|d\sigma \le C_1' C_1^{-1}|\lambda|^{-1}\int_{E_\tau}\frac{\eta(\sigma)}{S(\sigma)}h_1(\sigma)d\sigma$$

which means that

$$|\beta_\lambda(\tau)-\beta_0(\tau)| \le C_2|\lambda|^{-1}\beta_0(\tau) \qquad (\tau\in\mathbb{R}, \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0). \tag{4.4}$$

where $C_2 = C_1' C_1^{-1} > 0$.

From now on, we divide the proof into the following three cases: Case 1. $\tau_{-\infty}=\tau_\infty$, Case 2. $\tau_{-\infty}<\tau_\infty$ and $\beta_0(\tau_{-\infty})>0$, and Case 3. $\tau_{-\infty}<\tau_\infty$ and $\beta_0(\tau_{-\infty})=0$.

Case 1: In this case, note that $\beta_\lambda(\tau_{-\infty})=\beta_\lambda(\tau_\infty)>0$ since $E_{\tau_{-\infty}}=E_{\tau_\infty}=U$. Hence, from (4.3) it holds that $I(\lambda)=e^{-\lambda\tau_\infty}\beta_\lambda(\tau_\infty)=e^{-\lambda\tau_{-\infty}}\beta_\lambda(\tau_{-\infty})$, and from (4.4) it follows that

$$\mathrm{Re}\,\beta_\lambda(\tau_{-\infty}) \ge (1-C_2|\lambda|^{-1})\beta_0(\tau_{-\infty}) \ge 2^{-1}\beta_0(\tau_{-\infty}) \quad (\lambda\in\mathbb{C}_\delta, \mu\ge\mu_1),$$

where $\mu_1=\max\{\mu_0, 2C_2\}>0$. Hence we obtain $\mathrm{Re}\,\{e^{\lambda\tau_{-\infty}}I(\lambda)\}\ge 2^{-1}\beta_0(\tau_{-\infty})>0$ ($\lambda\in\mathbb{C}_\delta, \mu=\mathrm{Re}\,\lambda\ge\mu_1$).

Case 2: Since $\tau_{-\infty}=|q-y_0|=S(0)$, it follows that $e\cdot(q-y_0)=0$ for any $e\in\mathbb{R}^3$ perpendicular to ν_{y_0}, which yields $q-y_0=-|q-y_0|\nu_{y_0}$. Hence similarly to (3.2), we obtain

$$(S(\sigma))^2 = (|q-y_0|-g(\sigma))^2+|\sigma|^2 \ge |\sigma|^2 \qquad (\sigma\in U). \tag{4.5}$$

Since (4.5) implies

$$0\le\beta_0(\tau)\le\int_U \eta(\sigma)|\sigma|^{-1}h_1(\sigma)d\sigma<\infty \quad (\tau\in\mathbb{R}),$$

from (4.4), one gets $|\beta_\lambda(\tau)-\beta_0(\tau)|\le C\mu^{-1}$ for $\lambda\in\mathbb{C}_{\delta_0}$ with $\mu\ge\mu_0$.

Take any $\delta_0>0$ fixed. In this case, there exists $\delta'>0$ such that

$$|\beta_0(\tau)-\beta_0(\tau_{-\infty})| < \beta_0(\tau_{-\infty})/2(1+\delta_0) \qquad (0\le\tau-\tau_{-\infty}<\delta')$$

since β_0 is also right continuous and $\beta_0(\tau_{-\infty})>0$. Hence it follows that

$$|\beta_\lambda(\tau)-\beta_0(\tau_{-\infty})| \le |\beta_\lambda(\tau)-\beta_0(\tau)|+|\beta_0(\tau)-\beta_0(\tau_{-\infty})| \le C\mu^{-1}+\frac{\beta_0(\tau_{-\infty})}{2(1+\delta_0)}$$

for $0\le\tau-\tau_{-\infty}<\delta'$ and $\lambda\in\mathbb{C}_{\delta_0}$ with $\mu\ge\mu_0$. Combining this with the fact that

$$|\beta_\lambda(\tau)-\beta_0(\tau_{-\infty})|\le C \quad (\tau\in\mathbb{R}, \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0)$$

for some constant $C>0$, we obtain

$$\begin{aligned}\Big|\int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\lambda}\beta_\lambda(\tau)d\tau - \int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\lambda}\beta_0(\tau_{-\infty})d\tau\Big| &\le \int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\mu}|\beta_\lambda(\tau)-\beta_0(\tau_{-\infty})|d\tau \\ &\le \Big(\frac{\beta_0(\tau_{-\infty})}{2(1+\delta_0)}+\frac{C}{\mu}\Big)\int_{\tau_{-\infty}}^{\tau_{-\infty}+\delta'} e^{-\tau\mu}d\tau + C\int_{\tau_{-\infty}+\delta'}^{\tau_\infty} e^{-\tau\mu}d\tau \\ &\le \frac{\beta_0(\tau_{-\infty})}{2(1+\delta_0)\mu}e^{-\mu\tau_{-\infty}} + C(\mu^{-2}e^{-\mu\tau_{-\infty}}+\mu^{-1}e^{-\mu(\tau_{-\infty}+\delta')}).\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\mathrm{Re}\Big(e^{\lambda\tau_{-\infty}}\lambda\int_{\tau_{-\infty}}^{\tau_\infty}e^{-\tau\lambda}\beta_\lambda(\tau)d\tau\Big) &\ge \mathrm{Re}\Big(e^{\lambda\tau_{-\infty}}\beta_0(\tau_{-\infty})\lambda\int_{\tau_{-\infty}}^{\tau_\infty}e^{-\tau\lambda}d\tau\Big)\\
&\quad-\frac{|\lambda|}{2(1+\delta_0)\mu}\beta_0(\tau_{-\infty})-\frac{|\lambda|}{\mu}C(\mu^{-1}+e^{-\mu\delta'})\\
&\ge \beta(\tau_{-\infty})(1-\mathrm{Re}\,e^{-\lambda(\tau_\infty-\tau_{-\infty})})-\frac{1}{2}\beta_0(\tau_{-\infty})-C(\mu^{-1}+e^{-\mu\delta'})\\
&\ge\frac{\beta(\tau_{-\infty})}{2}-C(\mu^{-1}+e^{-\mu\delta'}+e^{-\mu(\tau_\infty-\tau_{-\infty})})\qquad(\lambda\in\mathbb{C}_{\delta_0})
\end{aligned}$$

since $|\lambda|/\mu\le 1+|\mathrm{Im}\,\lambda|/\mu\le 1+\delta_0$ for any $\lambda\in\mathbb{C}_{\delta_0}$. This implies that

$$\begin{aligned}
\mathrm{Re}\,(e^{\lambda\tau_{-\infty}}I(\lambda)) &\ge \mathrm{Re}\Big(e^{\lambda\tau_{-\infty}}\lambda\int_{\tau_{-\infty}}^{\tau_\infty}e^{-\tau\lambda}\beta_\lambda(\tau)d\tau\Big)-|e^{-\mu(\tau_\infty-\tau_{-\infty})}\beta_\lambda(\tau_\infty)|\\
&\ge\frac{\beta(\tau_{-\infty})}{2}-C(\mu^{-1}+e^{-\mu\delta'}+e^{-\mu(\tau_\infty-\tau_{-\infty})}).
\end{aligned}$$

Hence taking $\mu_0>0$ sufficiently large if it is necessary, for any $\delta_0>0$, we can find constants $C>0$ and $\mu_0>0$ such that $\mathrm{Re}\,(e^{\lambda\tau_{-\infty}}I(\lambda))\ge C$ $(\lambda\in\mathbb{C}_{\delta_0},\mu=\mathrm{Re}\,\lambda\ge\mu_0)$.

Case 3. In this case, $\tau_{-\infty}<\tau_\infty$ and $\beta_0(\tau_{-\infty})=0$. Note that $\tau_{-\infty}=\inf_{\sigma\in V}S(\sigma)\ge 0$. We need the following lemma: □

Lemma 4.3. *There exists a constant $C>0$ such that for $\tau_{-\infty}=0$, it follows that*

$$\int_{E_\tau}\eta(\sigma)d\sigma\ge C\tau^2\qquad(0\le\tau\le\tau_\infty),$$

and for $\tau_{-\infty}\ne 0$, it follows that

$$\int_{E_\tau}\eta(\sigma)d\sigma\ge C(\tau-\tau_{-\infty})\qquad(\tau_{-\infty}\le\tau\le\tau_\infty).$$

Proof. Since $\eta(0)>0$, we can choose $r_2>0$ satisfying $\sigma\in U$ and $\eta(\sigma)\ge\eta(0)/2$ for $|\sigma|\le r_2$. First, consider the case $\tau_{-\infty}=0$. In this case, it follows that $0\le\mathrm{dist}(p,\partial\Omega)\le\tau_{-\infty}=0$ since $0\le S(0)=|q-y_0|=\mathrm{dist}(q,\partial\Omega)\le|q-s(\sigma)|=S(\sigma)$ $(\sigma\in U)$. Hence (4.5) yields $|\sigma|\le S(\sigma)=\sqrt{|\sigma|^2+(g(\sigma))^2}\le C_1|\sigma|$ $(\sigma\in\tilde U)$. From these estimates, $\sigma\in E_\tau$ holds for $|\sigma|\le\tau/C_1$ and $\tau\le r_2C_1$. This implies that

$$\int_{E_\tau}\eta(\sigma)d\sigma\ge\int_{|\sigma|\le\tau/C_1}\eta(\sigma)d\sigma\ge\frac{\eta(0)}{2}\int_{|\sigma|\le\tau/C_1}d\sigma=\frac{\pi\eta(0)}{2}(\tau/C_1)^2\quad(0\le\tau\le r_2C_1).$$

Thus, for $\tau_{-\infty}=0$, Lemma 4.3 is obtained if $r_2C_1\ge\tau_\infty$. If $r_2C_1<\tau_\infty$, we note that $(\tau/C_1)^2\ge(r_2/\tau_\infty)^2\tau^2$ $(0\le\tau\le r_2C_1)$ and $r_2^2\ge(r_2/\tau_\infty)^2\tau^2$ $(r_2C_1\le\tau\le\tau_\infty)$. The above estimates and the fact that $\int_{E_\tau}\eta(\sigma)d\sigma$ is non-decreasing function in τ imply

$$\int_{E_\tau}\eta(\sigma)d\sigma\ge\frac{\pi\eta(0)}{2}\min\{(\tau/C_1)^2,r_2^2\}\ge\frac{\pi\eta(0)}{2}(r_2/\tau_\infty)^2\tau^2\quad(0\le\tau\le\tau_\infty).$$

Hence, we obtain Lemma 4.3 for $\tau_{-\infty}=0$.

Next, we consider the case of $\tau_{-\infty}>0$. In this case, from (4.5), there exists a constant $C_2>0$ such that for $\sigma\in\tilde U$

$$(S(\sigma))^2=(|p-y_0|-g(\sigma))^2+|\sigma|^2=\tau_{-\infty}^2-2\tau_{-\infty}g(\sigma)+|\sigma|^2+(g(\sigma))^2$$

$$\leq \tau_{-\infty}^2 + 2C_2\tau_{-\infty}|\sigma|^2 \leq (\tau_{-\infty} + C_2|\sigma|^2)^2,$$

which yields

$$S(\sigma) - \tau_{-\infty} \leq C_2|\sigma|^2 \qquad (\sigma \in \tilde{U}).$$

Hence, if $|\sigma| \leq \sqrt{(\tau - \tau_{-\infty})/C_2}$ with $\tau - \tau_{-\infty} \leq r_2^2 C_2$, then we have $|\sigma| \leq r_2$ and $S(\sigma) - \tau_{-\infty} \leq C_2|\sigma|^2 \leq \tau - \tau_{-\infty}$, i.e. $\sigma \in E_\tau$. Since $\eta(\sigma) \geq \eta(0)/2$ for $0 \leq \tau - \tau_{-\infty} \leq r_2^2 C_2$ and $|\sigma| \leq \sqrt{(\tau - \tau_{-\infty})/C_2}$, it follows that

$$\int_{E_\tau} \eta(\sigma)d\sigma \geq \int_{|\sigma| \leq \sqrt{(\tau-\tau_{-\infty})/C_2}} \eta(\sigma)d\sigma \geq \frac{\eta(0)}{2}\int_{|\sigma| \leq \sqrt{(\tau-\tau_{-\infty})/C_2}} d\sigma = \frac{\pi\eta(0)}{2C_2}(\tau - \tau_{-\infty}).$$

If $r_2^2 C_2 < \tau_\infty - \tau_{-\infty}$, as is in the case of $\tau_{-\infty} = 0$, we also obtain

$$\int_{E_\tau} \eta(\sigma)d\sigma \geq \frac{\pi\eta(0)}{2C_2}(\tau - \tau_{-\infty}) \geq \frac{\pi\eta(0)r_2^2}{2(\tau_\infty - \tau_{-\infty})}(\tau - \tau_{-\infty}) \quad (\tau_{-\infty} \leq \tau \leq \tau_\infty).$$

This completes the proof of Lemma 4.3. □

Since $E_\tau = \{\sigma \in U \,|\, S(\sigma) \leq \tau\}$, from (4.2) and Lemma 4.3, it follows that

$$\beta_0(\tau) = \int_{E_\tau} \frac{\eta(\sigma)}{S(\sigma)} h_1(\sigma)d\sigma \geq \frac{C_1}{\tau}\int_{E_\tau} \eta(\sigma)d\sigma \geq C(\tau - \tau_{-\infty}) \quad (\tau_{-\infty} \leq \tau \leq \tau_\infty) \tag{4.6}$$

for both cases $\tau_{-\infty} = 0$ and $\tau_{-\infty} > 0$. Using this estimate, we give an estimate of $\mathrm{Re}\,(e^{\lambda\tau_{-\infty}} I(\lambda))$. From (4.4) it follows that

$$\begin{aligned}\mathrm{Re}\Big(e^{\lambda\tau_{-\infty}}\lambda\int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\lambda}\beta_\lambda(\tau)d\tau\Big) &= \mathrm{Re}\Big(\lambda\int_{\tau_{-\infty}}^{\tau_\infty} e^{-(\tau-\tau_{-\infty})\lambda}\beta_\lambda(\tau)d\tau\Big) \\ &\geq \mathrm{Re}\Big(\lambda\int_{\tau_{-\infty}}^{\tau_\infty} e^{-(\tau-\tau_{-\infty})\lambda}\beta_0(\tau)d\tau\Big) - C\int_{\tau_{-\infty}}^{\tau_\infty} e^{-(\tau-\tau_{-\infty})\mu}\beta_0(\tau)d\tau \\ &= \mu\int_0^{\tau_\infty-\tau_{-\infty}} e^{-\tau\mu} h_2(\tau;\lambda)\beta_0(\tau + \tau_{-\infty})d\tau,\end{aligned} \tag{4.7}$$

where $h_2(\tau;\lambda) = \cos(\mathrm{Im}\,\lambda\tau) + \mu^{-1}(\mathrm{Im}\,\lambda)\sin(\mathrm{Im}\,\lambda\tau) - C\mu^{-1}$. Choose $0 < c_0 < 1$. From now on, we consider $\lambda \in \mathbb{C}$ satisfying $|\mathrm{Im}\,\lambda| \leq \delta_0\mu(\log\mu)^{-1}$ (i.e. $\lambda \in \Lambda_{\delta_0}$). Note that $\delta_0 > 0$ should be chosen small enough as is determined later.

Take $0 < \theta_0 < \pi/2$ with $c_0 < \cos\theta_0$. Then, choosing $\mu_0 > 0$ large enough, we obtain $h_2(\tau;\lambda) \geq c_0$ ($\lambda \in \Lambda_{\delta_0}, \mu \geq \mu_0, 0 \leq \tau \leq \gamma$), where $\gamma = \min\{\theta_0/|\mathrm{Im}\,\lambda|, \tau_\infty - \tau_{-\infty}\}$. Then from (4.6) and (4.7), it follows that

$$\begin{aligned}\mathrm{Re}\Big(e^{\lambda\tau_{-\infty}}\lambda\int_{\tau_{-\infty}}^{\tau_\infty} e^{-\tau\lambda}\beta_\lambda(\tau)d\tau\Big) &\geq \mu\int_0^{\gamma} e^{-\tau\mu}c_0C\tau d\tau - \mu(2+\delta_0)C'(\tau_\infty - \tau_{-\infty})e^{-\gamma\mu} \\ &\geq Cc_0\mu\{\mu^{-2}(1 - e^{-\gamma\mu}) - \mu^{-1}\gamma e^{-\gamma\mu}\} - C'\mu(2+\delta_0)(\tau_\infty - \tau_{-\infty})e^{-\gamma\mu}.\end{aligned}$$

From the above estimates and (4.3), there exist constants $C > 0$ and $C' > 0$ such that for any $\delta_0 > 0$, we can find $\mu_0 > 0$ satisfying

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}} I(\lambda)) \geq C\mu^{-1} - C'(1+\delta_0)\mu e^{-\gamma\mu} \qquad (\mu \geq \mu_0, \lambda \in \Lambda_{\delta_0}).$$

If $|\mathrm{Im}\,\lambda| \leq \theta_0(\tau_\infty - \tau_{-\infty})^{-1}$, it holds that $\gamma = \tau_\infty - \tau_{-\infty}$, which yields

$$e^{\gamma\mu} = e^{\mu(\tau_\infty - \tau_{-\infty})} \geq \frac{\mu^3(\tau_\infty - \tau_{-\infty})^3}{6} \geq \frac{2C'(1+\delta_0)}{C}\mu^2$$
$$(\mu \geq \max\{\frac{12C'(1+\delta_0)}{C(\tau_\infty - \tau_{-\infty})^3}, \mu_0\}).$$

Thus, one gets

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}} I(\lambda)) \geq 2^{-1}C\mu^{-1} \quad (\mu \geq \max\{\frac{12C'(1+\delta_0)}{C(\tau_\infty - \tau_{-\infty})^3}, \mu_0\}).$$

If $|\mathrm{Im}\,\lambda| > \theta_0(\tau_\infty - \tau_{-\infty})^{-1}$, it follows that $\gamma = \theta_0/|\mathrm{Im}\,\lambda|$. Since for any $\lambda \in \mathbb{C}$ with $|\mathrm{Im}\,\lambda| \leq \delta_0\mu(\log\mu)^{-1}$, we have $\gamma\mu = \theta_0\mu/|\mathrm{Im}\,\lambda| \geq \theta_0\delta_0^{-1}\log\mu = \log(\mu^{\theta_0\delta_0^{-1}})$. Hence for $\delta_0 < 2^{-1}\theta_0$, it follows that

$$e^{\gamma\mu} \geq \mu^{\theta_0\delta_0^{-1}} \geq 2C^{-1}C'(1+\delta_0)\mu^2 \quad (\mu \geq \max\{(2C^{-1}C'(1+\delta_0))^{(\theta_0\delta_0^{-1}-2)^{-1}}, \mu_0\}).$$

Thus if we choose $\delta_0 < \theta_0/2$, we obtain

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}} I(\lambda)) \geq 2^{-1}C\mu^{-1} \quad (\mu \geq \max\{(2C^{-1}C'(1+\delta_0))^{(\theta_0\delta_0^{-1}-2)^{-1}}, \mu_0\}).$$

Hence, there exist $\delta_0 > 0$, $\mu_0 > 0$ and $C > 0$ such that

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}} I(\lambda)) \geq 2^{-1}C\mu^{-1} \qquad (\lambda \in \Lambda_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq \mu_0)$$

is shown, which completes the proof of Proposition 4.1. □

Next is non-degenerate case, which is for Remark 1.2. First, we check the following fact.

Proposition 4.4. *If $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is a non-degenerate or degenerate critical point of finite order for $\partial\Omega \ni y \to |y-q| \in \mathbb{R}$, then* $\mathrm{dist}(q, \partial\Omega) > 0$, *i.e.* $\tau_{-\infty} > 0$ *holds.*

Proof. Note that from (1.6), there exist $l_0 > 0$ and $C' > C > 0$ such that

$$\mathrm{dist}\,(q, \partial\Omega) + C|y - y_0|^{2+l_0} \leq |y - q| \leq \mathrm{dist}\,(q, \partial\Omega) + C'|y - y_0|^{2+l_0} \quad (y \in \partial\Omega \cap B(y_0, 2r_0))$$

for sufficiently small $r_0 > 0$. The above estimate implies that

$$\tau_{-\infty} + C|\sigma|^{2+l_0} \leq S(\sigma) \leq \tau_{-\infty} + C'|\sigma|^{2+l_0} \quad (\sigma \in U). \tag{4.8}$$

In this case, one gets $\tau_{-\infty} > 0$. Indeed, if $\tau_{-\infty} \leq 0$, i.e. $\tau_{-\infty} = 0$, then from the above estimate and (4.5), it follows that

$$|\sigma|^2 + (g(\sigma))^2 = (S(\sigma))^2 \leq (C'|\sigma|^{2+l_0})^2 \quad (\sigma \in U),$$

which yields $1 \leq (C'|\sigma|^{1+l_0})^2$ for $0 \neq \sigma \in U$. Since $l_0 \geq 0$, this leads a contradiction. This completes the proof of Proposition 4.4. □

For the non-degenerate case, as is in the below, the order of the lower bound estimates is the same as for Proposition 4.1. The sets belonging the parameter λ for which the estimate valid are different. In the non-degenerate case, it is given by $\mathbb{C}_{\delta_0}$, while we can only take a smaller set Λ_{δ_1} for general cases as Proposition 4.1.

Proposition 4.5. *Assume that the same assumption as for* Proposition 4.1 *holds, and $q \in \Omega \setminus \overline{D}$ and $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is non-degenerate. Then there exist constants $\delta_0 > 0$, $C_1 > 0$ and $\mu_0 > 0$ such that*

$$\mathrm{Re}\,\{e^{\lambda \mathrm{dist}(p,\partial\Omega)}I(\lambda)\} \geq C_1\mu^{-1} \quad (\lambda \in \mathbb{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq \mu_0).$$

Proof. For $r_1 > 0$ in (1.6), we choose $r_0 > 0$ in the standard local coordinate $U \ni \sigma \mapsto s(\sigma) \in \partial\Omega\cap B(y_0, 2r_0)$ as $2r_0 < r_1$. From (1.6) and $q \in \Omega \setminus \overline{D}$, for Proposition 4.4, it follows that $S(0) = \tau_{-\infty} = \mathrm{dist}(q, \partial\Omega) > 0$, $S(\sigma) > \tau_{-\infty}$ $(\sigma \in U \setminus \{0\})$. Since $S(\sigma)$ is a class of C^2, there exist constants $0 < r_3, \mu_1, \mu_2 > 0$ and an orthogonal matrix $P \in O(2)$ such that

$$S(\sigma) = \tau_{-\infty} + (\mu_1\tilde{\sigma}_1^2 + \mu_2\tilde{\sigma}_2^2)(1 + k(\tilde{\sigma})) \qquad (\sqrt{\mu_1\tilde{\sigma}_1^2 + \mu_2\tilde{\sigma}_2^2} \leq r_3, \tilde{\sigma} = P\sigma),$$

where $k(\tilde{\sigma})$ is a function satisfying $\lim_{\tilde{\sigma}\to 0} k(\tilde{\sigma}) = 0$. Note also that for this $r_3 > 0$, there exists a constant $c_0 > 0$ such that $S(\sigma) \geq \tau_{-\infty} + c_0$ $(\sigma \in U, \sqrt{\mu_1\tilde{\sigma}_1^2 + \mu_2\tilde{\sigma}_2^2} \geq r_3)$. Noting these facts and changing variables, we can see that there exist constants $C > 0$ and $\tau_0 > \tau_{-\infty}$ with $\tau_{-\infty} < \tau_0 \leq \tau_\infty$ such that

$$\beta_\lambda(\tau) = \frac{1}{\sqrt{\mu_1\mu_2}}\int_{\tilde{E}_\tau} \frac{\tilde{\eta}_\lambda(\sigma)}{\tilde{S}(\sigma)}d\sigma \quad (\tau_{-\infty} \leq \tau \leq \tau_0(\leq \tau_\infty), \lambda \in \mathbb{C}_{\delta_0}, \mu \geq 1),$$
$$|\beta_\lambda(\tau)| \leq C \quad (\tau_0 \leq \tau, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq 1),$$

where $\tilde{S}(\sigma) = \tau_{-\infty} + |\sigma|^2(1 + k(K\sigma))$ with $K\sigma = (\sigma_1/\sqrt{\mu_1}, \sigma_2/\sqrt{\mu_2})$, $\tilde{E}_\tau = \{\sigma \in \mathbb{R}^2 \,|\, |\sigma| \leq r_2, \tilde{S}(\sigma) \leq \tau\}$ and $\tilde{\eta}_\lambda$ is defined by $\tilde{\eta}_\lambda(\sigma) = \eta(P^{-1}K\sigma)h(P^{-1}K\sigma, \lambda)$.

Since $\tau_{-\infty} > 0$, it holds that $1/\tilde{S}(\sigma)$ is also continuous on $\tilde{E_{\tau_0}}$. Hence for $1/3 > \varepsilon > 0$, there exists a constant $0 < \delta_\varepsilon \leq r_2$ such that

$$\frac{1}{\sqrt{\mu_1\mu_2}}\Big|\frac{\tilde{\eta}_0(\sigma)}{\tilde{S}(\sigma)} - \frac{\tilde{\eta}_0(0)}{\tau_{-\infty}}\Big| < \varepsilon, \quad |k(K\sigma)| < \varepsilon \qquad (|\sigma| < \delta_\varepsilon),$$

where $\tilde{\eta}_0(\sigma) = \eta(P^{-1}K\sigma)h_1(P^{-1}K\sigma)$.

We put $\tau_\varepsilon = \min\{\tau_0, \tau_{-\infty} + (1 - 2^{-1}\varepsilon)\delta_\varepsilon^2\}$. Note that $\tilde{E}_{\tau_\varepsilon} \subset \{\sigma \in \mathbb{R}^2 \,|\, |\sigma| < \delta_\varepsilon\}$ since

$$\tau_{-\infty} + (1 - \varepsilon)|\sigma|^2 \leq \tau_{-\infty} + |\sigma|^2(1 + k(K\sigma)) = \tilde{S}(\sigma) \leq \tau_\varepsilon \leq \tau_{-\infty} + (1 - 2^{-1}\varepsilon)\delta_\varepsilon^2$$

implies $|\sigma|^2 \leq (1 - 2^{-1}\varepsilon)(1 - \varepsilon)^{-1}\delta_\varepsilon^2 < \delta_\varepsilon^2$. From (4.2), there exists a constant $C > 0$ such that

$$|\tilde{\eta}_\lambda(\sigma) - \tilde{\eta}_0(\sigma)| \leq C\mu^{-1} \quad (P^{-1}K\sigma \in U, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq 1).$$

Hence we obtain

$$\Big|\beta_\lambda(\tau) - \frac{\tilde{\eta}_0(0)}{\tau_{-\infty}\sqrt{\mu_1\mu_2}}\int_{\tilde{E}_\tau} d\sigma\Big| \leq \varepsilon\int_{\tilde{E}_\tau} d\sigma + \frac{1}{\sqrt{\mu_1\mu_2}}\int_{\tilde{E}_\tau}\frac{|\tilde{\eta}_\lambda(0) - \tilde{\eta}_0(0)|}{\tilde{S}(\sigma)}d\sigma$$
$$\leq C(\varepsilon + \mu^{-1})\int_{\tilde{E}_\tau} d\sigma \qquad (\tau_{-\infty} \leq \tau \leq \tau_\varepsilon, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq 1).$$

We put $E^0_\tau = \{\sigma \in \mathbb{R}^2 \,|\, |\sigma|^2 \leq \tau\}$. Since $\sigma \in \tilde{E}_\tau$ implies $|\sigma| < \delta_\varepsilon$ for any $\tau \leq \tau_\varepsilon$, it holds that $(1 - \varepsilon)|\sigma|^2 \leq |\sigma|^2(1 + k(K\sigma)) \leq \tau - \tau_{-\infty}$, which yields $\tilde{E}_\tau \subset E^0_{(\tau-\tau_{-\infty})/(1-\varepsilon)}$. We can also show $E^0_{(\tau-\tau_{-\infty})/(1+\varepsilon)} \subset \tilde{E}_\tau$ for $\tau \leq \tau_\varepsilon$. Indeed, for $\sigma \in E^0_{(\tau-\tau_{-\infty})/(1+\varepsilon)}$, it follows that $|\sigma|^2 \leq (1 + \varepsilon)^{-1}(\tau - \tau_{-\infty}) < \tau_\varepsilon - \tau_{-\infty} < \delta_\varepsilon^2$, which yields that

$$\tilde{S}(\sigma) = \tau_{-\infty} + |\sigma|^2(1 + k(K\sigma)) \leq \tau_{-\infty} + |\sigma|^2(1 + \varepsilon) \leq \tau_{-\infty} + (\tau - \tau_{-\infty}) = \tau.$$

Hence we obtain

$$E^0_{(\tau-\tau_{-\infty})/(1+\varepsilon)} \subset \tilde{E}_\tau \subset E^0_{(\tau-\tau_{-\infty})/(1-\varepsilon)} \qquad (\tau_{-\infty} \leq \tau \leq \tau_\varepsilon).$$

We put $\tilde{\tau} = \tau - \tau_{-\infty}$. The above properties imply that

$$\Big|\int_{\tilde{E}_\tau} d\sigma - \int_{E^0_{\tilde{\tau}}} d\sigma\Big| \leq \int_{E^0_{\tilde{\tau}/(1-\varepsilon)}\setminus E^0_{\tilde{\tau}/(1+\varepsilon)}} d\sigma = \pi\tilde{\tau}\Big\{\frac{1}{1-\varepsilon} - \frac{1}{1+\varepsilon}\Big\} = \frac{2\varepsilon\pi\tilde{\tau}}{1-\varepsilon^2} < 3\pi\varepsilon\tilde{\tau}$$

since $\int_{E^0_\tau} d\sigma = \pi\tau$. Summarizing the above estimates, we see that there exists a constant $C > 0$ independent of $0 < \varepsilon < 1/3$ such that

$$\Big|\beta_\lambda(\tau) - \frac{\pi\tilde{\eta}_0(0)}{\tau_{-\infty}\sqrt{\mu_1\mu_2}}(\tau - \tau_{-\infty})\Big| \leq C(\varepsilon + \mu^{-1})(\tau - \tau_{-\infty})$$
$$(\tau_{-\infty} \leq \tau \leq \tau_\varepsilon, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq 1).$$

The above estimate and (4.3) imply that

$$\Big|I(\lambda) - \frac{\lambda\pi\tilde{\eta}_0(0)}{\tau_{-\infty}\sqrt{\mu_1\mu_2}}\int_{\tau_{-\infty}}^{\tau_\varepsilon} e^{-\tau\lambda}(\tau - \tau_{-\infty})d\tau\Big|$$
$$\leq C|\lambda|e^{-\mu\tau_\varepsilon} + C(\varepsilon + \mu^{-1})|\lambda|\int_{\tau_{-\infty}}^{\tau_\varepsilon} e^{-\tau\mu}(\tau - \tau_{-\infty})d\tau,$$

which yields

$$\Big|I(\lambda) - \frac{\pi\tilde{\eta}_0(0)}{\lambda\tau_{-\infty}\sqrt{\mu_1\mu_2}}e^{-\lambda\tau_{-\infty}}\Big| \leq C|\lambda|e^{-\mu\tau_\varepsilon} + C(\varepsilon + \mu^{-1})\frac{1}{|\lambda|}e^{-\mu\tau_{-\infty}} \tag{4.9}$$
$$(0 < \varepsilon < 1/3, \lambda \in \mathbb{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq 1).$$

Hence choosing $\varepsilon > 0$ sufficiently small and $\mu_0 \geq 1$ sufficiently large, we get

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}}I(\lambda)) \geq \mathrm{Re}\Big(\frac{\pi\tilde{\eta}(0)}{2\lambda\tau_{-\infty}\sqrt{\mu_1\mu_2}}\Big) - C\mu e^{-\mu(\tau_\varepsilon - \tau_{-\infty})} \qquad (\lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0).$$

Noting $\mathrm{Re}\,(1/\lambda) = \mu/|\lambda|^2 \geq \mu^{-1}(1 + \delta_0)^{-2}$ for $\lambda \in \mathbb{C}_{\delta_0}$, we obtain

$$\mathrm{Re}\,(e^{\lambda\tau_{-\infty}}I(\lambda)) \geq \frac{\pi\tilde{\eta}(0)}{4(1+\delta_0)^2\tau_{-\infty}\sqrt{\mu_1\mu_2}}\mu^{-1} \qquad (\lambda \in \mathbb{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq \mu_0)$$

if we choose $\mu_0 \geq 1$ sufficiently large again. This completes the proof of Proposition 4.5. □

Remark 4.6. When $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is a non-degenerate critical point of the function $\partial\Omega \ni y \mapsto |y - p| \in \mathbb{R}$, from (4.9), we can also obtain

$$|I(\lambda)| \leq C\mu^{-1}e^{-\mu\tau_{-\infty}} \qquad (\lambda \in \mathbb{C}_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq 1)$$

for some constant $C > 0$. In Section 6, this is used to show Theorem 1.3.

Last, we consider the case of the degenerate case of finite order.

Proposition 4.7. *If $y_0 \in \mathcal{M}_{\partial\Omega}(q)$ is a degenerate critical point of finite order for $\partial\Omega \ni y \mapsto |y - q| \in \mathbb{R}$, then for any $r_0 > 0$ sufficiently small, there exists a constant $l_0 > 0$ such that the following estimates hold:*
1) *there exist a constant $C > 0$ and a sufficiently small constant $\delta_0 > 0$ such that*

$$C^{-1}\mu^{-\frac{2}{l_0+2}}e^{-\mu\tau_{-\infty}} \leq \mathrm{Re}\,I(\lambda) \qquad (\lambda \in \Lambda_{\delta_0}, \mu = \mathrm{Re}\,\lambda \geq \mu_0),$$

2) *for any* $\delta_0 > 0$*, there exists a constant* $C > 0$ *such that*

$$|I(\lambda)| \le C\mu^{-\frac{2}{l_0+2}} e^{-\mu\tau_{-\infty}} \qquad (\lambda \in \mathbb{C}_{\delta_0}, \mu = \operatorname{Re}\lambda \ge \mu_0).$$

Proof. From Proposition 4.4, $\tau_{-\infty} > 0$. Note also that $|\sigma| \le \left(C'^{-1}(\tau - \tau_{-\infty})\right)^{1/(l_0+2)}$ for any $\sigma \in E_\tau$ with $\tau_{-\infty} \le \tau \le \tau_\infty$ since (4.8) implies $\tau_{-\infty} + C|\sigma|^{2+l_0} \le S(\sigma) \le \tau$. From these facts and

$$|\beta_\lambda(\tau)| \le \int_{E_\tau} \frac{\eta(\sigma)}{S(\sigma)}|h(\sigma,\lambda)|d\sigma \le \int_{E_\tau} \frac{\eta(\sigma)}{\tau_{-\infty}}|h(\sigma,\lambda)|d\sigma \le \frac{\sup_{\sigma\in U}|\eta(\sigma)h(\sigma,\lambda)|}{\tau_{-\infty}}\int_{E_\tau} d\sigma,$$

there exists a constant $M > 0$ such that

$$|\beta_\lambda(\tau)| \le M(\tau - \tau_{-\infty})^{2/(l_0+2)} \quad (\lambda \in \mathbb{C}_{\delta_0}, \mu \ge \mu_0, \tau_{-\infty} \le \tau \le \tau_\infty).$$

Combining the above estimate with (4.3), we obtain

$$\begin{aligned}|e^{\lambda\tau_{-\infty}} I(\lambda)| &\le |\lambda| \int_{\tau_{-\infty}}^{\tau_\infty} e^{-(\tau-\tau_{-\infty})\mu}|\beta_\lambda(\tau)|d\tau + e^{\mu(\tau_{-\infty}-\tau_\infty)}|\beta_\lambda(\tau_\infty)| \\ &\le M\frac{|\lambda|}{\mu}\mu^{-2/(l_0+2)} \int_0^\infty e^{-\tau}\tau^{2/(l_0+2)}d\tau + Me^{\mu(\tau_{-\infty}-\tau_\infty)}(\tau_\infty - \tau_{-\infty})^{2/(l_0+2)},\end{aligned}$$

which yields

$$|e^{\lambda\tau_{-\infty}} I(\lambda)| \le \tilde{C}\mu^{-2/(l_0+2)} \quad (\lambda \in \mathbb{C}_{\delta_0}, \mu \ge \mu_0)$$

for some constant $\tilde{C} > 0$.

Next, we show the estimate of $\operatorname{Re} I(\lambda)$. Take $r_2 > 0$ satisfying $\sigma \in U$ and $\eta(\sigma) \ge \eta(0)/2$ for $|\sigma| \le r_2$. Note that if $|\sigma| \le \left((\tau - \tau_{-\infty})/C'\right)^{1/(l_0+2)}$ with $\tau - \tau_{-\infty} \le r_2^{l_0+2}C'$, then from (4.8), we have $|\sigma| \le r_2$ and $S(\sigma) - \tau_{-\infty} \le C'|\sigma|^{l_0+2} \le \tau - \tau_{-\infty}$, which yields $\eta(\sigma) \ge \eta(0)/2$ and $\sigma \in E_\tau$. Hence as is in the proof of Lemma 4.3, we obtain

$$\int_{E_\tau} \eta(\sigma)d\sigma \ge C(\tau - \tau_{-\infty})^{2/(l_0+2)} \qquad (\tau_{-\infty} \le \tau \le \tau_\infty).$$

Using the above estimates and tracing the argument showing Proposition 4.1, we obtain

$$\operatorname{Re}\left(e^{\lambda\tau_{-\infty}} I(\lambda)\right) \ge C\mu^{-2/(l_0+2)} \qquad (\lambda \in \Lambda_{\delta_0}, \mu = \operatorname{Re}\lambda \ge \mu_0)$$

for some constants $\delta_0 > 0$, $C > 0$ and $\mu_0 > 0$ since

$$\int_0^\gamma e^{-\tau\mu}\tau^{2/(l_0+2)}d\tau \ge \mu^{-1-2/(l_0+2)} \int_0^{\min\{\theta_0\delta_0^{-1}\mu_0, (\tau_\infty-\tau_{-\infty})\mu_0\}} e^{-\tau}\tau^{2/(l_0+2)}d\tau$$
$$(\mu \ge \mu_0, \mu \in \mathbb{C}_{\delta_0}).$$

This completes the proof of Proposition 4.7. □

5. Proof of Theorem 1.1

We begin with showing the terms $V_\Omega(\lambda)\varphi_{12}(p;\lambda)$ and $V_D(\lambda)\varphi_{21}(p;\lambda)$ are negligible comparing to $V_\Omega(\lambda)\varphi_1(p;\lambda)$ and $V_D(\lambda)\varphi_2(p;\lambda)$ when both g_1 and g_2 are positive.

We put $d_0 = \inf_{(y,\xi)\in\partial\Omega\times\partial D}|y - \xi| > 0$, and recall $\operatorname{dist}(q,\partial\Omega) = \inf_{y\in\partial\Omega}|q - y|$ and $\operatorname{dist}(q,\partial D) = \inf_{\xi\in\partial D}|q - \xi|$ for $q \in \overline{\Omega} \setminus D$.

Proposition 5.1. *For any $\delta_0 > 0$, there exists constants $C > 0$ and $\mu_0 \geq 1$ such that the following estimates hold uniformly in $q \in \overline{\Omega} \setminus D$:*

$$|V_\Omega(\lambda)\varphi_{12}(q;\lambda)| \leq C|\lambda|e^{-(\mathrm{Re}\,\lambda)\mathrm{dist}(q,\partial\Omega)}e^{-d_0\mathrm{Re}\,\lambda}\|g_2\|_{C(\partial D)} \quad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0),$$
$$|V_D(\lambda)\varphi_{21}(q;\lambda)| \leq C|\lambda|e^{-(\mathrm{Re}\,\lambda)\mathrm{dist}(q,\partial D)}e^{-d_0\mathrm{Re}\,\lambda}\|g_1\|_{C(\partial\Omega)} \quad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0).$$

Proof. From (2.4) and the definition of $V_\Omega(\lambda)$, it follows that

$$|V_\Omega(\lambda)\varphi_{12}(q;\lambda)| \leq C\int_{\partial\Omega}\frac{e^{-\mu|q-y|}}{|q-y|}\|(I-Y_{11}(\lambda))^{-1}\|_{B(C(\partial\Omega))}\|Y_{12}(\lambda)\varphi_2(\cdot;\lambda)\|_{C(\partial\Omega)}dS_y,$$

where $\mu = \mathrm{Re}\,\lambda$. Since the kernel $Y_{12}(\xi, y;\lambda)$ of $Y_{12}(\lambda)$ is also given by

$$Y_{12}(y,\zeta;\lambda) = \frac{\partial}{\partial\nu_y}E_\lambda(y,\zeta) + \rho_1(y)E_\lambda(y,\zeta),$$

similarly to (3.26), (3.21) implies

$$\begin{aligned}|Y_{12}(\lambda)\varphi_2(y;\lambda)| &\leq C\int_{\partial D}e^{-\mu|\xi-y|}\Big(|\lambda| + \frac{1}{|\xi-y|}\Big)|\varphi_2(\xi;\lambda)|dS_\xi \\ &\leq C|\lambda|e^{-\mu d_0}\|\varphi_2(\cdot;\lambda)\|_{C(\partial D)} \quad (y \in \partial\Omega, \lambda \in \mathbb{C}, \mu = \mathrm{Re}\,\lambda > 0).\end{aligned}$$

Note that $(I-Y_{11}(\lambda))^{-1} \in B(C(\partial\Omega))$ and (2.2) yields

$$\|(I-Y_{11}(\lambda))^{-1}\|_{B(C(\partial\Omega))} \leq C \qquad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0).$$

These estimates imply

$$|V_\Omega(\lambda)\varphi_{12}(q;\lambda)| \leq C|\lambda|e^{-\mu d_0}\|\varphi_2(\cdot;\lambda)\|_{C(\partial D)}e^{-\mu\mathrm{dist}(q,\partial D)}\int_{\partial\Omega}\frac{1}{|q-y|}dS_y. \tag{5.1}$$

Note also that

$$\|\varphi_2(\cdot;\lambda)\|_{C(\partial D)} \leq C\|g_2\|_{C(\partial D)} \qquad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0) \tag{5.2}$$

since $\varphi_2(x;\lambda) = -(I-Y_{22}(\lambda))^{-1}(I-Z_2(\lambda))^{-1}g_2(x)$ and

$$\|(I-Y_{22}(\lambda))^{-1}\|_{B(C(\partial D))} + \|(I-Z_2(\lambda))^{-1}\|_{B(C(\partial D))} \leq C \qquad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0)$$

holds as is in Section 2. From (5.1), (5.2) and Lemma 3.2, we obtain the estimate of $V_\Omega(\lambda)\varphi_{12}(q;\lambda)$. We can show the estimate for $V_D(\lambda)\varphi_{21}(q;\lambda)$ similarly. □

To show Theorem 1.1, we need to obtain estimates of $|w(q;\lambda)|$ from the above and below. For the estimate from the above, note that we also have

$$\|\varphi_1(\cdot;\lambda)\|_{C(\partial\Omega)} \leq C\|g_1\|_{C(\partial\Omega)} \qquad (\lambda \in \mathbb{C}, \mathrm{Re}\,\lambda \geq \mu_0) \tag{5.3}$$

which can be shown similarly to (5.2). Thus Lemma 3.2 and definition of $V_\Omega(\lambda)\varphi_1(q;\lambda)$ imply that

$$|V_\Omega(\lambda)\varphi_1(q;\lambda)| = \frac{1}{2\pi}\int_{\partial\Omega}\frac{e^{-\mathrm{Re}\,\lambda|q-y|}}{|q-y|}|\varphi_1(y;\lambda)|dS_y \leq C\|g_1\|_{C(\partial\Omega)}e^{-(\mathrm{Re}\,\lambda)\mathrm{dist}(q,\partial\Omega)}. \tag{5.4}$$

Since we can get $|V_D(\lambda)\varphi_2(q;\lambda)| \leq C\|g_2\|_{C(\partial D)}e^{-(\mathrm{Re}\,\lambda)\mathrm{dist}(q,\partial D)}$ similarly, from these estimates and Proposition 5.1, we obtain

$$(5.5)\qquad |w(q;\lambda)| \le C(\|g_1\|_{C(\partial\Omega)} + \|g_2\|_{C(\partial D)})e^{-d(q)\mathrm{Re}\,\lambda} \quad (q \in \overline{\Omega}\setminus D, \lambda\in\mathbb{C}, \mathrm{Re}\lambda \ge \mu_0),$$

where $d(q) = \min\{\mathrm{dist}(q,\partial\Omega), \mathrm{dist}(q,\partial D)\}$.

To finish the proof of Theorem 1.1, it suffices to show the following estimates of $V_\Omega(\lambda)\varphi_1(q;\lambda)$ and $V_D(\lambda)\varphi_2(q;\lambda)$ from the below:

Proposition 5.2. *Let Ω, D and ρ_j $(j = 1,2)$ be as in* Theorem 1.1 *and take $q \in \overline{\Omega}\setminus D$. Assume that $g_1 \in C(\partial\Omega)$ and $g_2 \in C(\partial D)$ satisfy (1.4). Then there exist constants $\delta_0 > 0$, $\mu_0 > 0$ and $C > 0$ such that*

$$\mathrm{Re}\,\{e^{\lambda\mathrm{dist}(q,\partial\Omega)}V_\Omega(\lambda)\varphi_1(q;\lambda)\} \ge C\mu^{-1} \qquad (\lambda\in\Lambda_{\delta_0},\ \mu = \mathrm{Re}\,\lambda \ge \mu_0),$$
$$\mathrm{Re}\,\{e^{\lambda\mathrm{dist}(q,\partial D)}V_D(\lambda)\varphi_2(q;\lambda)\} \ge C\mu^{-1} \qquad (\lambda\in\Lambda_{\delta_0},\ \mu = \mathrm{Re}\,\lambda \ge \mu_0),$$

where $\Lambda_{\delta_0} = \{\,\lambda\in\mathbb{C}\,|\,|\mathrm{Im}\,\lambda| \le \delta_0(\mathrm{Re}\,\lambda)(\log\mathrm{Re}\,\lambda)^{-1}, \mathrm{Re}\,\lambda \ge e\,\}$. Further, if $q \in \Omega\setminus\overline{D}$, and all points $y \in \mathcal{M}_{\partial\Omega}(q)$ and $x \in \mathcal{M}_{\partial D}(q)$ are non-degenerate critical points of the functions $y \mapsto |y-q|$ and $\xi\mapsto|\xi - q|$, the set Λ_{δ_0} can be replaced to $\mathbb{C}_{\delta_0}$ for any fixed $\delta_0 > 0$.

Here, we proceed to finish the proof of Theorem 1.1. From Propositions 2.1 and 5.1, it follows that

$$\begin{aligned}
\mathrm{Re}\,\{e^{d(q)\lambda}w(q;\lambda)\} &\ge \mathrm{Re}\,\{e^{d(q)\lambda}(V_\Omega(\lambda)\varphi_1(q;\lambda) + e^{d(q)\lambda}V_D(\lambda)\varphi_2(q;\lambda))\}\\
&\qquad - |e^{d(q)\lambda}V_\Omega(\lambda)\varphi_{12}(q;\lambda)| - |e^{d(q)\lambda}V_\Omega(\lambda)\varphi_{21}(q;\lambda)|\\
&\ge \mathrm{Re}\,\{e^{(d(q)-\mathrm{dist}(q,\partial\Omega))\lambda}e^{\lambda\mathrm{dist}(q,\partial\Omega)}V_\Omega(\lambda)\varphi_1(q;\lambda)\}\\
&\qquad + \mathrm{Re}\,\{e^{(d(q)-\mathrm{dist}(q,\partial D))\lambda}e^{\lambda\mathrm{dist}(q,\partial D)}V_D(\lambda)\varphi_2(q;\lambda)\}\\
&\qquad - C'e^{\mu d(q)}\{\mu e^{-\mu d_0}e^{-\mu\mathrm{dist}(q,\partial\Omega)} + \mu e^{-\mu d_0}e^{-\mu\mathrm{dist}(q,\partial D)}\}.
\end{aligned}$$

Thus, if $\mathrm{dist}(q,\partial D) < \mathrm{dist}(q,\partial\Omega)$, i.e. $d(q) = \mathrm{dist}(q,\partial D)$ and $d(q) < \mathrm{dist}(q,\partial\Omega)$, estimate (5.4) and Proposition 5.2 for $V_D(\lambda)\varphi_2(q,\lambda)$ imply that

$$(5.6)\qquad \begin{aligned}
\mathrm{Re}\,\{e^{d(q)\lambda}w(q;\lambda)\} &\ge C\mu^{-1} - C''e^{\mu(d(q)-\mathrm{dist}(q,\partial\Omega))} - 2C'\mu e^{-\mu d_0}\\
&\ge \tilde{C}\mu^{-1} \qquad (\lambda\in\Lambda_{\delta_0}, \mu = \mathrm{Re}\,\lambda \ge \mu_0),
\end{aligned}$$

when we choose $\mu_0 > 0$ large enough if it is necessary. If $\mathrm{dist}(q,\partial D) > \mathrm{dist}(q,\partial\Omega)$, i.e. $d(q) = \mathrm{dist}(q,\partial\Omega)$ and $d(q) < \mathrm{dist}(q,\partial D)$, or $\mathrm{dist}(q,\partial\Omega) = \mathrm{dist}(q,\partial D)$, i.e. $d(q) = \mathrm{dist}(q,\partial D) = \mathrm{dist}(q,\partial\Omega)$, the same argument as above also gives estimate (5.6). Hence in any case, we get estimate (5.6). Combining (5.6) with (5.5), we obtain

$$\tilde{C}\mu^{-1}e^{-d(q)\mu} \le |w(q;\lambda)| \le Ce^{-d(q)\mu} \qquad (\lambda\in\Lambda_{\delta_0}, \mu = \mathrm{Re}\,\lambda \ge \mu_0),$$

which implies Theorem 1.1.

In what follows, we show the estimate of $V_\Omega(\lambda)\varphi_1(q;\lambda)$ in Proposition 5.2. Note that $V_D(\lambda)\varphi_2(q;\lambda)$ can be treated similarly to $V_\Omega(\lambda)\varphi_1(q;\lambda)$. For $\tau \ge 0$, we put

$$\mathcal{M}_{\partial\Omega,\tau}(q) = \{y\in\partial\Omega\,|\,|q-y| \le \mathrm{dist}(q,\partial\Omega) + \tau\}.$$

Lemma 5.3. *For any open set W satisfying $\mathcal{M}_{\partial\Omega}(q)\subset W$, there exists a constant $\delta_1 > 0$ such that $\mathcal{M}_{\partial\Omega,\tau}(q)\subset W$ $(0\le\tau\le\delta_1)$.*

Proof. Assume that Lemma 5.3 does not hold. Then for any $l\in\mathbb{N}$, we can find $0\le\tau_l\le 1/l$ satisfying $\mathcal{M}_{\partial\Omega,\tau_l}(q)\not\subset W$. Choose $y_l\in\mathcal{M}_{\partial\Omega,\tau_l}(q)$ with $y_l\notin W$. Since $\partial\Omega$ is compact, we can assume that $y_l \to y$ as $l \to \infty$ for some $y \in \partial\Omega$. This implies $y \notin W$ since W is a open set.

On the other hand, noting that $y_l \in \mathcal{M}_{\partial\Omega,\tau_l}(q)$, we obtain

$$\operatorname{dist}(q, \partial\Omega) \le |q - y_l| \le \operatorname{dist}(q, \partial\Omega) + \tau_l \to \operatorname{dist}(q, \partial\Omega) \qquad (l \to \infty).$$

Hence $\operatorname{dist}(q, \partial\Omega) = |q-y|$ holds. This means that $y \in \mathcal{M}_{\partial\Omega}(q) \subset W$, which is a contradiction. This completes the proof of Lemma 5.3. □

From continuity of $\partial\Omega \ni y \mapsto |q - y| \in \mathbb{R}$, it follows that $\mathcal{M}_{\partial\Omega}(q)$ is compact. Hence (ii) of Lemma 3.1 implies that there exist finitely many points $y_1, y_2, \ldots, y_N \in \mathcal{M}_{\partial\Omega}(q)$ and the standard local coordinates $U_j \ni \sigma \mapsto s^{(j)}(\sigma) \in \partial\Omega \cap B(y_j, 2r_0)$ around y_j $(j = 1, 2, \ldots, N)$ such that $\mathcal{M}_{\partial\Omega}(q) \subset \cup_{j=1}^{N} B(y_j, r_0)$ and $\partial\Omega \cap B(y_j, 2r_0) = s^{(j)}(U_j)$ $(j = 1, 2, \ldots, N)$. From assumption (1.4), we can take $\{U_j\}_{j=1,2,\ldots,N}$ satisfying $g_1(y) \ge C_0/2$ for all $y \in \partial\Omega \cap \cup_{j=1}^{N} B(y_j, 2r_0)$ since $r_0 > 0$ in Lemma 3.1 can be chosen as small as necessary. Lemma 5.3 implies that $\mathcal{M}_{\partial\Omega,\tau}(q) \subset \cup_{j=1}^{N} B(y_j, r_0)$ $(0 \le \tau \le \delta_1)$ holds for some $\delta_1 > 0$. We put $\tilde{U}_j = \partial\Omega \cap B(y_j, r_0)$ $(j = 1, 2, \ldots, N)$ and $\tilde{U}_{N+1} = \{x \in \partial\Omega \,|\, |q - y| > \operatorname{dist}(q, \partial\Omega) + \delta_1\}$. Since $\{\tilde{U}_j\}_{j=1,2,\ldots,N+1}$ is a open covering of $\partial\Omega$, we can choose functions $\psi_j \in C_0^2(\tilde{U}_j)$ $(j = 1, 2, \ldots, N + 1)$ satisfying $0 \le \psi_j \le 1$ and $\sum_{j=1}^{N+1} \psi_j = 1$ on $\partial\Omega$. We put

$$I_j(\lambda) = \int_{\tilde{U}_j} e^{-\lambda|q-y|} \frac{\psi_j(y)}{|q - y|} \varphi_1(y; \lambda) dS_y \qquad (j = 1, 2, \ldots, N + 1).$$

Definition of $\tilde{U}_{N+1}$, (5.3) and (3.1) imply that

$$\begin{aligned} |I_{N+1}(\lambda)| &\le e^{-\mu\delta_1} e^{-\mu \operatorname{dist}(q,\partial\Omega)} C \|g_1\|_{C(\partial\Omega)} \int_{\partial\Omega} \frac{1}{|q - y|} dS_y \\ &\le C e^{-\mu\delta_1} e^{-\mu \operatorname{dist}(q,\partial\Omega)} \|g_1\|_{C(\partial\Omega)} \qquad (\lambda \in \mathbb{C}, \operatorname{Re}\lambda \ge \mu_0). \end{aligned}$$

From this fact and $V_\Omega(\lambda)\varphi_1(q; \lambda) = \sum_{j=1}^{N+1} I_j(\lambda)$, it follows that

$$\operatorname{Re}\{e^{\lambda \operatorname{dist}(q,\partial\Omega)} V_\Omega(\lambda)\varphi_1(q; \lambda)\} \ge \sum_{j=1}^{N} \operatorname{Re}\{e^{\lambda \operatorname{dist}(p,\partial\Omega)} I_j(\lambda)\} - C e^{-\mu\delta_1} \qquad (\lambda \in \mathbb{C}, \operatorname{Re}\lambda \ge \mu_0). \tag{5.7}$$

For $I_j(\lambda)$ $(j = 1, 2, \ldots, N)$, each $I_j(\lambda)$ is reduced to the following integral:

$$I_j(\lambda) = \int_{U_j} e^{-\lambda S^{(j)}(\sigma)} \eta_0^{(j)}(\sigma) \beta^{(j)}(\sigma; \lambda) d\sigma \qquad (j = 1, 2, \ldots, N),$$

where $S^{(j)}(\sigma)$, $\eta_0^{(j)}(\sigma)$ and $\beta^{(j)}(\sigma, \lambda)$ are defined by

$$S^{(j)}(\sigma) = |p - s^{(j)}(\sigma)|, \qquad \eta_0^{(j)}(\sigma) = \frac{\psi_j(s^{(j)}(\sigma))}{|p - s^{(j)}(\sigma)|}$$
$$\beta^{(j)}(\sigma, \lambda) = J^{(j)}(\sigma)\varphi_1(s^{(j)}(\sigma); \lambda) = J^{(j)}(\sigma) g_1(s^{(j)}(\sigma)) + \lambda^{-1} J^{(j)}(\sigma) \tilde{g}_1(s^{(j)}(\sigma); \lambda).$$

In the above, $J^{(j)}(\sigma)$ denotes the volume elements of $\partial\Omega$ and $\tilde{g}_1(\cdot; \lambda) \in C(\partial\Omega)$ is the function given by

$$\tilde{g}_1(y; \lambda) = \lambda\{Y_{11}(\lambda)(I - Y_{11}(\lambda))^{-1}(I - Z_1(\lambda))^{-1} g_1(y) + Z_1(\lambda)(I - Z_1(\lambda))^{-1} g_1(y)\}.$$

Note that $\tilde{g}_1(\cdot;\lambda) \in C(\partial\Omega)$ are bounded in $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda \geq \mu_0$ for some $\mu_0 > 0$ large enough.

From assumption (1.4), it follows that there exist constants $C_1 > 0$ and $C_1' > 0$ such that

$$J^{(j)}(\sigma)g_1(s^{(j)}(\sigma)) \geq C_1, \quad |J^{(j)}(\sigma)\tilde{g}_1(s^{(j)}(\sigma);\lambda)| \leq C_1' \quad (\sigma \in U_j, \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0).$$

Thus these Laplace integrals $I_j(\lambda)$ $(j = 1, 2, \ldots, N)$ are of the form (4.1), and satisfy (4.2). Hence, Proposition 4.1 yields that there exists a constant $C_2 > 0$ such that $\mathrm{Re}\,\{e^{\lambda \mathrm{dist}(p,\partial\Omega)} I_j(\lambda)\} \geq C_2\mu^{-1}$ holds if we take $\delta_0 > 0$ and $\mu_0 > 0$ sufficiently small. Combining this estimate with (5.7), we obtain

$$\mathrm{Re}\,\{e^{\lambda \mathrm{dist}(q,\partial\Omega)} V_\Omega(\lambda)\varphi_1(q;\lambda)\} \geq C_2 N\mu^{-1} - Ce^{-\mu\delta_1} \geq 2^{-1}C_2\mu^{-1}$$

for any $\mu \geq \max\{\mu_0, 4C(\delta_1^2 C_2)^{-1}\}$. Thus, we obtain the estimate of $V_\Omega(\lambda)\varphi_1(q;\lambda)$ in Proposition 5.2. Note that $V_D(\lambda)\varphi_2(q;\lambda)$ can be treated similarly, which completes the proof of Theorem 1.1.

6. Proof of Theorem 1.3

For the solution $w(x;\lambda)$ of (1.5) (i.e. $g_2(x) = 0$ in (1.1)), it follows that $\varphi_2(x;\lambda) = 0$ and $\varphi_{12}(x;\lambda) = 0$ for $g_2 = 0$. Hence Proposition 2.1 implies that

$$w(q;\lambda) = V_\Omega(\lambda)\varphi_1(q;\lambda) + V_D(\lambda)\varphi_{21}(q;\lambda) \qquad (q \in \Omega \setminus D). \tag{6.1}$$

First, we show that (6.1) still holds even for $q \in \partial D$. This gives a kernel representation of $w(p;\lambda)$ for $p \in \partial D$, which is a basis of the proof of Theorem 1.3.

From the definition of $V_\Omega(\lambda)\varphi_1(q;\lambda)$, it is continuous in $q \in \mathbb{R}^3$. To obtain

$$\lim_{h\to+0} V_D(\lambda)\varphi_{21}(p + h\nu_p;\lambda) = V_D(\lambda)\varphi_{21}(p;\lambda)$$

for any $p \in \partial D$, it suffices to show the following lemma:

Lemma 6.1. *There exist constants $C_\alpha > 0$ and $\delta > 0$ such that for any $0 \leq \alpha < 1$, it holds that*

$$|V_D(\lambda)\varphi_{21}(p + h\nu_p;\lambda) - V_D(\lambda)\varphi_{21}(p;\lambda)| \leq C_\alpha(1 + |\lambda|)^3 e^{\mu h} h^\alpha$$
$$(p \in \partial D, 0 \leq h < \delta, \lambda \in \mathbb{C}_{\delta_0}).$$

Proof. We put $q = p + h\nu_p$ with $0 < h < 1$ small enough. From (i) of Lemma 3.1, it follows that

$$\begin{aligned}|q - \xi|^2 &= h^2 + |p - \xi|^2 - 2h\nu_p \cdot (\xi - p)\\ &\geq |p - \xi|^2(1 - 2Ch) \geq 2^{-2}|\xi - p|^2 \qquad (\xi, p \in \partial D, 0 < h \leq 3/(8C))\end{aligned}$$

for some constant $C > 0$. Thus we get

$$|\xi - q| \geq 2^{-1}|\xi - p| \qquad (\xi, p \in \partial D, 0 \leq h \leq \delta)$$

if we choose $\delta = \min\{1, 3/(8C)\}$. From this estimate, it follows that

$$\Big|\frac{1}{|\xi - q|} - \frac{1}{|\xi - p|}\Big| \leq \Big(\frac{|p - q|}{|\xi - q||\xi - p|}\Big)^\alpha \Big(\frac{1}{|\xi - q|} + \frac{1}{|\xi - p|}\Big)^{1-\alpha} \leq C\frac{|p - q|^\alpha}{|\xi - p|^{1+\alpha}}.$$

Noting that $|e^X - e^Y| \leq |X - Y|e^{\max\{\mathrm{Re}\,X, \mathrm{Re}\,Y\}}$, we also get $|e^{-\lambda|\xi-q|} - e^{-\lambda|\xi-p|}| \leq |\lambda| e^{\mu h} e^{-\mu|\xi-p|}|p -$

$q| \le (\sup_{\xi,\zeta\in\partial D}|\xi-\zeta|)^{\alpha}|\lambda|e^{\mu h}e^{-\mu|\xi-p|}|p-q|^{\alpha}/|\xi-p|^{\alpha}$ for $p,\xi\in\partial D$. These estimates yield

$$|E_\lambda(q,\xi)-E_\lambda(p,\xi)|\le C(1+|\lambda|)e^{\mu h}e^{-\mu|\xi-p|}\frac{h^{\alpha}}{|\xi-p|^{1+\alpha}}\quad(\xi,p\in\partial D, 0\le h\le\delta).$$

From (2.6), (3.25), estimate (3.28) and the above imply that

$$\begin{aligned}&|V_D(\lambda)\varphi_{21}(p+h\nu_p;\lambda)-V_D(\lambda)\varphi_{21}(p;\lambda)|\\&\quad\le\int_{\partial\Omega}|\varphi_1(y;\lambda)|\int_{\partial D}|M_\lambda(\xi,y)||E_\lambda(q,\xi)-E_\lambda(p,\xi)|dS_\xi dS_y\\&\quad\le C(1+|\lambda|)^3\int_{\partial\Omega}|\varphi_1(y;\lambda)|\int_{\partial D}e^{-\mu|\xi-y|}e^{\mu h}e^{-\mu|\xi-p|}\frac{h^{\alpha}}{|\xi-p|^{1+\alpha}}dS_\xi\\&\quad\le C(1+|\lambda|)^3h^{\alpha}e^{\mu h}e^{-\mu\mathrm{dist}(p,\partial D)}\|\varphi_1(\cdot;\lambda)\|_{C(\partial\Omega)}\int_{\partial D}\frac{dS_\xi}{|\xi-p|^{1+\alpha}}.\end{aligned}$$

This estimate, Remark 3.3 and (5.3) show Proposition 6.1. □

Thus, (6.1) still holds for $p\in\partial D$. From this fact with Propositions 2.1 and (2.6), for any $p\in\partial D$, the solution $w(p;\lambda)$ of (1.5) are represented as

$$w(p;\lambda)=\frac{1}{2\pi}\int_{\partial\Omega}e^{-\lambda|p-y|}\varphi_1(y;\lambda)\Big\{\frac{1}{|y-p|}+A(y,p;\lambda)\Big\}dS_y,\tag{6.2}$$

where $A(y,p;\lambda)$ is given by

$$A(y,p;\lambda)=e^{\lambda|p-y|}\int_{\partial D}M_\lambda(\xi,y)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi,\tag{6.3}$$

since (3.25) implies that the integral kernel of ${}^tY_{21}(\lambda)^t((I-Y_{22}(\lambda))^{-1})$ is $M_\lambda(\xi,y)$ ($y\in\partial\Omega$, $\xi\in\partial D$). From (3.28) and Remark 3.3, we obtain

$$|A(y,p;\lambda)|\le C\mu^2e^{\mu|p-y|}\int_{\partial D}e^{-\mu|\xi-y|}\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi\le C\mu^2.\tag{6.4}$$

This rough estimate tells that only the points belonging to $\mathcal{M}_{\partial\Omega}(p)$ surely contribute to the main term of (6.3) as $|\lambda|\to\infty$. Hence, to pick up the main part of $A(y,p;\lambda)$, we need to study for structures of $\mathcal{M}_{\partial\Omega}(p)$. Here, we recall the definitions of $\mathcal{M}_{\partial\Omega}(p)$, $\mathcal{M}^{\pm}_{\partial\Omega}(p)$ and $\mathcal{M}^{g}_{\partial\Omega}(p)$, which are introduced in Introduction. Note that they are also written by

$$\mathcal{M}^{\pm}_{\partial\Omega}(p)=\{y\in\mathcal{M}_{\partial\Omega}(p)\,|\,p\in\mathcal{G}^{\pm}(y)\},\qquad\mathcal{M}^{g}_{\partial\Omega}(p)=\{y\in\mathcal{M}_{\partial\Omega}(p)\,|\,p\in\mathcal{G}(y)\},$$

where for $y\in\partial\Omega$, $\mathcal{G}(y)$ and $\mathcal{G}^{\pm}(y)$ are defined by

$$\mathcal{G}(y)=\{\,\xi\in\partial D\,|\,\nu_\xi\cdot(y-\xi)=0\,\},\qquad\mathcal{G}^{\pm}(y)=\{\,\xi\in\partial D\,|\,\pm\nu_\xi\cdot(y-\xi)>0\,\}.$$

Since ∂D is strictly convex, for any point $y\in\mathcal{M}_{\partial\Omega}(p)$, only the following three cases occur: the line segment py has a single common point p in ∂D, the line segment py has a single different point $p^*=p^*(y,p)$ from p in ∂D, and the line segment py tangent to ∂D. The first case and the second one correspond to $\mathcal{M}^{+}_{\partial\Omega}(p)$ and $\mathcal{M}^{-}_{\partial\Omega}(p)$, respectively. From assumption 2) of Theorem 1.3, there is no point for the third case, i.e. $\mathcal{M}^{g}_{\partial\Omega}(p)=\emptyset$.

For $\delta>0$ and $y\in\partial\Omega$, we put

$$\mathcal{G}_\delta(y)=\{\,\xi\in\partial D\,|\,\mathrm{dist}\,(\xi,\mathcal{G}(y))\ge\delta\,\},\qquad\mathcal{G}^{\pm}_\delta(y)=\mathcal{G}_\delta(y)\cap\mathcal{G}^{\pm}(y).$$

As below, $\mathcal{M}^{\pm}_{\partial\Omega}(p)$ are disjoint to each other if $\mathcal{M}^{g}_{\partial\Omega}(p) = \emptyset$.

Proposition 6.2. *Assume that ∂D is of class C^2 and strictly convex. For a fixed $p \in \partial D$, assume that $\mathcal{M}^{g}_{\partial\Omega}(p) = \emptyset$. Then $\mathcal{M}^{\pm}_{\partial\Omega}(p) \subset \partial\Omega$ are closed sets. Further, there exist a constant $\delta_2 > 0$ and open sets $\mathcal{U}^{\pm}(p) \subset \partial\Omega$ such that $\mathcal{M}^{\pm}_{\partial\Omega}(p) \subset \mathcal{U}^{\pm}(p)$, $\overline{\mathcal{U}^{+}(p)} \cap \overline{\mathcal{U}^{-}(p)} = \emptyset$ and $\mathcal{U}^{\pm}(p)$ have the following properties:*

(6.5) $$p \in \mathcal{G}^{\pm}_{\delta_2}(y) \ \ \text{for any} \ \ y \in \overline{\mathcal{U}^{\pm}(p)}.$$

Proof. We set $\delta_\pm = \inf\limits_{y \in \mathcal{M}^{\pm}_{\partial\Omega}(p)} \pm\nu_p \cdot (y-p)/|y-p|$ and show $\delta_\pm > 0$. If this is not true, there exists a sequence $\{y^{\pm}_n\} \subset \mathcal{M}^{\pm}_{\partial\Omega}(p)$ such that $\pm\nu_p \cdot (y^{\pm}_n - p) \to 0$ $(n \to \infty)$. Since $\partial\Omega$ is compact, we may assume that $\{y^{\pm}_n\}$ itself converge to some point $y^{\pm}_0 \in \partial\Omega$ as $n \to \infty$. Hence it yields that $\pm\nu_p \cdot (y^{\pm}_0 - p) = 0$ and $y^{\pm}_0 \in \mathcal{M}_{\partial\Omega}(p)$ since $\mathcal{M}_{\partial\Omega}(p)$ is closed. This means that $y^{\pm}_0 \in \mathcal{M}^{g}_{\partial\Omega}(p)$, which is a contradiction.

We put $\mathcal{U}^{\pm}(p) = \{y \in \partial\Omega \,|\, \pm\nu_p \cdot (y-p)/|y-p| > \delta_\pm/2\}$. Then $\mathcal{U}^{\pm}(p)$ are open set in $\partial\Omega$, $\overline{\mathcal{U}^{+}(p)} \cap \overline{\mathcal{U}^{-}(p)} = \emptyset$ and $\mathcal{M}^{\pm}_{\partial\Omega}(p) \subset \mathcal{U}^{\pm}(p)$. From these facts, it follows that $\mathcal{M}^{\pm}_{\partial\Omega}(p) = \mathcal{M}_{\partial\Omega}(p) \cap \overline{\mathcal{U}^{\pm}(p)}$, which implies that $\mathcal{M}^{\pm}_{\partial\Omega}(p)$ are closed.

Next we show (6.5) by using contradiction argument. Assume that (6.5) does not hold, then for any $n \in \mathbb{N}$, there exists $\tilde{y}^{\pm}_n \in \overline{\mathcal{U}^{\pm}(p)}$ with $p \notin \mathcal{G}^{\pm}_{1/n}(\tilde{y}^{\pm}_n)$, i.e. $\mathrm{dist}(p, \mathcal{G}(\tilde{y}^{\pm}_n)) < 1/n$ or $\pm\nu_p \cdot (\tilde{y}^{\pm}_n - p) \le 0$ hold for any $n = 1, 2, \ldots$. From $\tilde{y}^{\pm}_n \in \overline{\mathcal{U}^{\pm}(p)}$, it follows that $\pm\nu_p \cdot (\tilde{y}^{\pm}_n - p) \ge 2^{-1}\delta_\pm |\tilde{y}^{\pm}_n - p| > 0$. Hence $\mathrm{dist}(p, \mathcal{G}(\tilde{y}^{\pm}_n)) < 1/n$ holds for any $n \in \mathbb{N}$, which implies that $|\tilde{z}^{\pm}_n - p| < \mathrm{dist}(p, \mathcal{G}(\tilde{y}^{\pm}_n)) + 1/n$ for some $\tilde{z}^{\pm}_n \in \mathcal{G}(\tilde{y}^{\pm}_n)$. Thus we obtain $\tilde{z}^{\pm}_n \to p$ $(n \to \infty)$. Since $\partial\Omega$ is compact, we can choose a subsequence $\{\tilde{y}^{\pm}_{n_j}\}$ such that $\tilde{y}^{\pm}_{n_j} \to \tilde{y}^{\pm}_0$ as $j \to \infty$ for some $\tilde{y}^{\pm}_0 \in \partial\Omega$. Thus, $\tilde{y}^{\pm}_0 \in \overline{\mathcal{U}^{\pm}(p)}$, i.e. $\pm\nu_p \cdot (\tilde{y}^{\pm}_0 - p) > 0$ since $\overline{\mathcal{U}^{\pm}(p)}$ is closed. On the other hand, we can get $\nu_p \cdot (\tilde{y}^{\pm}_0 - p) = 0$ since $\tilde{z}^{\pm}_n \in \mathcal{G}(\tilde{y}^{\pm}_n)$ means $\nu_{\tilde{z}^{\pm}_n} \cdot (\tilde{y}^{\pm}_n - \tilde{z}^{\pm}_n) = 0$ for any $n \in \mathbb{N}$, which is a contradiction. Thus we obtain Proposition 6.2. □

We also use the following property:

Lemma 6.3. *Assume that ∂D is of class C^2 and strictly convex. Then, the function defined by $(\xi, y) \in \partial D \times \partial\Omega \mapsto$ dist $(\xi, \mathcal{G}(y))$ is Lipschitz continuous on $\partial D \times \partial\Omega$.*

Proof. We put $F(\xi, y) = \mathrm{dist}\,(\xi, \mathcal{G}(y))$ $(\xi \in \partial D, y \in \partial\Omega)$. It suffices to show that F is Lipschitz continuous in y since the definition of F easily implies $|F(\xi', y) - F(\xi, y)| \le |\xi' - \xi|$ $(\xi, \xi' \in \partial D, y \in \partial\Omega)$. We choose a bounded open set $U \subset \mathbb{R}^3$ satisfying $\partial\Omega \subset U$ and $\overline{D} \cap U = \emptyset$. For the purpose, we need the following claim:

Claim: there exist $\delta > 0$ and $C > 0$ such that for any $y, y' \in U$ with $|y' - y| \le \delta$, and any $\xi \in \mathcal{G}(y)$, $\{\, \eta \in \mathcal{G}(y') \,|\, |\eta - \xi| \le C|y' - y| \,\} \ne \emptyset$.

If the claim is true, we can get Lipschitz continuous property of F as follows: Take any $\xi \in \partial D$ and any $y, y' \in \partial\Omega (\subset U)$ with $|y' - y| \le \delta$, where $\delta > 0$ is the constant in the claim. Since $\mathcal{G}(y) \subset \partial D$ is a bounded closed set, there exists a $\zeta \in \mathcal{G}(y)$ such that $F(\xi, y) = |\xi - \zeta|$. From the claim in the above, we can find $\eta \in \mathcal{G}(y')$ satisfying $|\eta - \zeta| \le C|y' - y|$. Hence, it follows that

$$F(\xi, y') - F(\xi, y) \le |\xi - \eta| - |\xi - \zeta| \le |\zeta - \eta| \le C|y' - y|.$$

This implies $|F(\xi, y') - F(\xi, y)| \le C|y' - y|$ $(y, y' \in \partial\Omega, \xi \in \partial D)$ by changing the role of y

and y'.

To show the claim, we take $y \in U$ and $\xi \in \mathcal{G}(y)$, and choose a standard system of local coordinates around x given by

$$U \ni \sigma = {}^t(\sigma_1, \sigma_2) \mapsto s(\sigma) = \xi + \sigma_1 e_1 + \sigma_2 e_2 + g(\sigma)(-\nu_\xi) \in B(\xi, 2r_0) \cap \partial D,$$

where $r_0 > 0$ depends only on ∂D (cf. (ii) of Lemma 3.1). Here, we can take $e_1 = (y - \xi)/|y - \xi|$ and $e_2 = e_1 \times \nu_\xi$. For the frame $\{e_1, e_2, -\nu_\xi\}$, we put $y' = y + \tau_1 e_1 + \tau_2 e_2 + t(-\nu_\xi)$ ($\tau = {}^t(\tau_1, \tau_2) \in \mathbb{R}^2$, $t \in \mathbb{R}$), and define

$$\begin{aligned} G(\sigma, \tau, t) &= \nu_{s(\sigma)} \cdot (y' - s(\sigma)) \\ &= (|y - \xi| + \tau_1 - \sigma_1) e_1 \cdot \nu_{s(\sigma)} + (\tau_2 - \sigma_2) e_2 \cdot \nu_{s(\sigma)} - (t - g(\sigma)) \nu_\xi \cdot \nu_{s(\sigma)}. \end{aligned}$$

Since $e_1 \cdot \nu_\xi = e_2 \cdot \nu_\xi = 0$ and $g(0) = 0$, we have $G(0, \tau, 0) = 0$ and $\partial_t G(0, \tau, 0) = -\nu_\xi \cdot \nu_\xi = -1$, the usual implicit function theorem implies that there exist a constant $\delta_0 > 0$ and a function $\varphi(\sigma, \tau)$ defined for $|\sigma| \le \delta_0$ and $|\tau| \le \delta_0$ such that $\varphi(0, \tau) = 0$ and $G(\sigma, \tau, \varphi(\sigma, \tau)) = 0$ for $|\sigma| \le \delta_0$ and $|\tau| \le \delta_0$. Since

$$\partial_{\sigma_1} G(0, \tau_1, 0, 0) = (|y - \xi| + \tau_1) e_1 \cdot \frac{\partial \nu_{s(\sigma)}}{\partial \sigma_1}\Big|_{\sigma=0} = -(|y - \xi| + \tau_1) \nu_\xi \cdot \frac{\partial^2 s}{\partial \sigma_1^2}(0),$$

estimate (3.5), being strict convexity of ∂D, yields

$$\partial_{\sigma_1} G(0, \tau_1, 0, 0) \ge R_2 (\inf_{y \in \partial\Omega, \zeta \in \partial D} |y - \zeta| + \tau_1).$$

Hence it follows that there exists a constant $C_0 > 0$ such that

$$\partial_{\sigma_1} \varphi(\sigma, \tau) = -\frac{\partial_{\sigma_1} G(\sigma, \tau, \varphi(\sigma, \tau))}{\partial_t G(\sigma, \tau, \varphi(\sigma, \tau))} \ge C_0 \quad (|\sigma| \le \delta_0, |\tau| \le \delta_0)$$

if we choose $\delta_0 > 0$ sufficiently small if it is necessary. From this estimate and $\varphi(0, 0, \tau) = 0$, it follows that $\varphi(\sigma_1, 0, \tau) \ge C_0 \sigma_1$ $(0 \le \sigma_1 \le \delta_0)$ and $\varphi(\sigma_1, 0, \tau) \le C_0 \sigma_1$ $(0 \ge \sigma_1 \ge -\delta_0)$. Hence for any $|t| \le C_0 \delta_0$ and $|\tau| \le \delta_0$, there exists $|\sigma_1| \le \delta_0$ such that $\varphi(\sigma_1, 0, \tau) = t$.

We choose $\delta = \min\{\delta_0, C_0 \delta_0\} > 0$. Then, for $y' = y + \tau_1 e_1 + \tau_2 e_2 + t(-\nu_\xi) \in \mathbb{R}^3$ with $|y' - y| \le \delta$, there exists $|\sigma_1| \le \delta_0$ such that $\varphi(\sigma_1, 0, \tau) = t$. We put $\eta = s(\sigma_1, 0) = \xi + \sigma_1 e_1 + g(\sigma_1, 0)(-\nu_\xi) \in \partial D$. From the property of φ, it follows that $\eta \in \mathcal{G}(y')$ since $\nu_\eta \cdot (y' - \eta) = G(\sigma_1, 0, \tau, \varphi(\sigma_1, 0, \tau)) = 0$. Further, (3.3) implies that

$$\begin{aligned} |\eta - \xi| &\le |\sigma_1| + |g(\sigma_1, 0)| \le |\sigma_1| + R_1 |\sigma_1|^2 \le \frac{1 + R_1 \delta_0}{C_0} |\varphi(\sigma_1, 0, \tau)| \\ &= \frac{1 + R_1 \delta_0}{C_0} |t| \le \frac{1 + R_1 \delta_0}{C_0} |y' - y|. \end{aligned}$$

Note that the above $\delta_0 > 0$, $C_0 > 0$ and $\delta > 0$ can be chosen as constants independent of $y \in \partial\Omega$ and $\xi \in \mathcal{G}(y)$ since as is in Lemma 3.1, the function g in σ belongs to $\mathcal{B}^2(\mathbb{R}^2)$ with the norm $\|g\|_{\mathcal{B}^2(\mathbb{R}^2)}$ having an upper bound independent of $\xi \in \partial D$. This shows the claim, which completes the proof of Lemma 6.3. □

In what follows, we fix $p \in \partial D$. For the sets $\mathcal{V}^{\pm}(p)$ introduced in Proposition 6.2, we put $\mathcal{V}^{-\infty}(p) = \partial D \setminus \{\mathcal{V}^{+}(p) \cup \mathcal{V}^{-}(p)\}$ and decompose $w(p; \lambda)$ into the following three parts:

$$w(p;\lambda) = w^+(p;\lambda) + w^-(p;\lambda) + w^{-\infty}(p;\lambda), \tag{6.6}$$

where

$$w^\gamma(p;\lambda) = \frac{1}{2\pi}\int_{\mathcal{U}^\gamma(p)} e^{-\lambda|p-y|}\varphi_1(y;\lambda)\Big\{\frac{1}{|y-p|} + A(y,p;\lambda)\Big\}dS_y \qquad (\gamma = +,-,-\infty).$$

First we show the term $w^{-\infty}(p;\lambda)$ in (6.6) is negligible. Note that there exists a constant $c_1 > 0$ such that

$$|y-p| \ge \mathrm{dist}(p,\partial\Omega) + c_1 \qquad (y \in \partial\Omega \setminus (\mathcal{U}^+(p) \cup \mathcal{U}^-(p)))$$

holds since $\mathcal{M}_{\partial\Omega}(p) \subset \mathcal{U}^+(p) \cup \mathcal{U}^-(p)$, $\mathcal{U}^\pm(p)$ are open sets in $\partial\Omega$, and $\mathcal{M}_{\partial\Omega}(p)$ is a closed set as is in Proposition 6.2. Combining the above estimate with (6.4) and (5.3), we obtain

$$|w^{-\infty}(p;\lambda)| \le C\mu^2 \int_{\mathcal{U}^{-\infty}(p)} e^{-\mu|p-y|}|\varphi_1(y;\lambda)|dS_y \le C\mu^2 e^{-c_1\mu}e^{-\mu\mathrm{dist}(p,\partial\Omega)}. \tag{6.7}$$

To obtain estimates for $w^\pm(p;\lambda)$, we need the following facts:

Proposition 6.4. *Assume that ∂D is of class C^2 and strictly convex. For a fixed $p \in \partial D$, assume that $\mathcal{M}^g_{\partial\Omega}(p) = \emptyset$. Then the term $A(y,p;\lambda)$ satisfies*

$$A(y,p;\lambda) = \frac{1}{|y-p|} + O(\mu^{-1}) \quad \text{as } |\lambda| \to \infty \quad (\text{uniformly in } y \in \mathcal{U}^+(p), \lambda \in \mathbb{C}_{\delta_0}),$$

and

$$A(y,p;\lambda) = -\frac{1}{|y-p|} + O(\mu^{-1}) \quad \text{as } |\lambda| \to \infty \quad (\text{uniformly in } y \in \mathcal{U}^-(p), \lambda \in \mathbb{C}_{\delta_0}).$$

Proposition 6.4 is the key estimates for Theorem 1.3. The proof will be given in the next section. Here we proceed to show Theorem 1.3 by using Proposition 6.4.

Proposition 6.4 leads an upper bound estimates of $|w(p;\lambda)|$ easily. Since Proposition 6.4 yields

$$|w^\pm(p;\lambda)| \le C\int_{\mathcal{U}^\pm(p)} e^{-\mu|p-y|}|\varphi_1(y;\lambda)A(y,p;\lambda)|dS_y \le Ce^{-\mu\mathrm{dist}(p,\partial\Omega)},$$

decomposition (6.6) of $w(p;\lambda)$ and (6.7) imply that

$$|w(p;\lambda)| \le Ce^{-\mu\mathrm{dist}(p,\partial\Omega)} \qquad (\lambda \in \mathbb{C}_{\delta_0}, \mu \ge \mu_0). \tag{6.8}$$

Thus we obtain an upper bound for $|w(p;\lambda)|$.

Next, consider estimates of $|w^\pm(p;\lambda)|$ from the below. For the constant $C_0 > 0$ in assumption 3) of Theorem 1.3, we put $W = \{y \in \partial\Omega \,|\, g_1(y) > C_0/2\}$ so that $W \subset \partial\Omega$ is open and $\mathcal{M}^+_{\partial\Omega}(p) \subset W$. From assumption 2) of Theorem 1.3, and Proposition 6.2, it follows that $\mathcal{M}_{\partial\Omega}(p) \subset W \cup \mathcal{U}^-(p)$. Since $W \cup \mathcal{U}^-(p)$ is open, Lemma 5.3 implies that $\mathcal{M}_{\partial\Omega,\tau}(p) \subset W \cup \mathcal{U}^-(p)$ $(0 \le \tau \le \delta_1)$ for some constant $\delta_1 > 0$. Noting $\mathcal{U}^+(p) \cap \mathcal{U}^-(p) = \emptyset$, we obtain

$$g_1(y) \ge C_0/2 \quad (y \in \mathcal{M}_{\partial\Omega,\tau}(p) \cap \mathcal{U}^+(p), 0 \le \tau \le \delta_1).$$

Hence from the same argument as in the proof of Proposition 5.2, we get constants $C > 0$ and $\mu_0 > 0$ satisfying

$$\text{Re}\,[e^{\lambda\text{dist}(p,\partial\Omega)}w^+(p;\lambda)] \geq C\mu^{-1} \qquad (\mu \geq \mu_0, \lambda \in \Lambda_{\delta_0}). \tag{6.9}$$

Next, let us consider $w^-(p;\lambda)$. Since Proposition 6.4 implies $\frac{1}{|y-p|} + A(y,p;\lambda) = O(\mu^{-1})$ as $|\lambda| \to \infty$ uniformly in $y \in \mathcal{U}^-(p)$ and $\lambda \in \mathbb{C}_{\delta_0}$, from (6.2), it follows that

$$|w^-(p;\lambda)| \leq C\mu^{-1}\int_{\mathcal{U}^-(p)} e^{-\mu|p-y|}dS_y. \qquad (\mu \geq \mu_0). \tag{6.10}$$

Note that using assumption 4) of Theorem 1.3 and applying Remark 4.6 or Proposition 4.7 in Section 4, we obtain

$$\int_{\mathcal{U}^-(p)} e^{-\mu|p-y|}dS_y \leq C\mu^{-\frac{2}{l_0+2}}e^{-\mu\text{dist}(p,\partial\Omega)} \qquad (\mu \geq \mu_0, \lambda \in \mathbb{C}_{\delta_0})$$

with some constants $C > 0$ and $l_0 \geq 0$. This estimate and (6.10) yield

$$|w^-(p;\lambda)| \leq C\mu^{-1-\frac{2}{l_0+2}}e^{-\mu\text{dist}(p,\partial\Omega)} \qquad (\mu \geq \mu_0, \lambda \in \mathbb{C}_{\delta_0}).$$

From the above estimate, (6.7), (6.9) and (6.6), it follows that

$$\begin{aligned}\text{Re}\,[e^{\lambda\text{dist}(p,\partial\Omega)}w(p;\lambda)] &\geq C\{\mu^{-1} - C'\mu^{-1-\frac{2}{l_0+2}} - C'\mu e^{-c_1\mu}\}\\ &\geq 2^{-1}C\mu^{-1} \quad (\mu \geq \mu_0, \lambda \in \Lambda_{\delta_0})\end{aligned}$$

if we choose a constant $\mu_0 > 0$ sufficiently large. Combining the above estimate with (6.8), we obtain Theorem 1.3. □

Concluding this section, we give a proof of Remark 1.4. If there exists a degenerate critical point $y_0 \in \mathcal{M}^+_{\partial\Omega}(p)$, from Proposition 4.7, one gets

$$\text{Re}\,w^+(p;\mu) \geq C\mu^{-\frac{2}{l_0+2}}e^{-\mu\text{dist}(p,\partial\Omega)} \qquad (\mu \in \Lambda_{\delta_0}, \mu \geq \mu_0)$$

for some constants $C > 0$ and $l_0 > 0$. This estimate is better than (6.9). Note also that from (6.10) and

$$\int_{\mathcal{U}^-(p)} e^{-\mu|p-y|}dS_y \leq C\mu^{-1}e^{-\mu\text{dist}(p,\partial\Omega)} \qquad (\mu \geq \mu_0),$$

it follows that

$$|w^-(p;\lambda)| \leq C\mu^{-1}e^{-\mu\text{dist}(p,\partial\Omega)} \qquad (\mu \geq \mu_0).$$

Combining these estimates with (6.7), we obtain

$$\text{Re}\,w(p;\mu) \geq Ce^{-\mu\text{dist}(p,\partial\Omega)}\mu^{-\frac{2}{l_0+2}} \qquad (\mu \geq \mu_0)$$

if we choose a constant $\mu_0 > 0$ sufficiently large. This shows that Remark 1.4 holds.

7. Asymptotic behavior of $A(y,p;\lambda)$

In this section, we show Proposition 6.4 giving the main terms of $A(y,p;\lambda)$. We need the following properties of the broken path in the boundary integrals:

Lemma 7.1. *Assume that ∂D is of class C^2 and strictly convex. Choose $(\xi, y) \in \partial D \times \partial\Omega$. If $\xi \in \mathcal{G}^+(y) \cup \mathcal{G}(y)$, then the function $|\xi - \eta| + |\eta - y|$ in $\eta \in \partial D$ attains the minimum only*

at $\eta = p$. If $\xi \in \mathcal{G}^-(y)$, then the points on ∂D that attain the minimum are given by only two points $\eta = \xi$ and $\xi^* = \xi^*(y,\xi)(\neq \xi)$, which are the cross points between the line segment py and ∂D. Moreover the following statements holds:
1) *Given $\delta > 0$ there exists a positive constant C_δ such that*

$$|\xi - \eta| + |\eta - y| \geq |\xi - y| + C_\delta|\xi - \eta| \qquad (\xi \in \mathcal{G}^+_\delta(y), \eta \in \partial D).$$

2) *Given $\delta > 0$, there exists constant $0 < \delta'_0 \leq \delta$ such that $|\xi - \xi^*| \geq 2\delta'_0$ for $\xi \in \mathcal{G}^-_\delta(y)$. Further, for any $0 < \delta' \leq \delta'_0$, there exists a constant $C_{\delta'} > 0$ such that*

$$|\xi - \eta| + |\eta - y| \geq |\xi - y| + C_\delta|\eta - \xi| \qquad (\xi \in \mathcal{G}^-_\delta(y), \eta \in \partial D, |\eta - \xi^*(y,\xi)| \geq \delta').$$

3) *Given $\delta > 0$, there exists constants $C_\delta > 0$ and $C'_\delta > 0$ such that for any $0 < \delta' \leq C'_\delta$, it holds that*

$$|\xi - \eta| + |\eta - y| \geq |\xi - y| + C_\delta|\eta - \xi^*(y,\xi)|^2 \quad (\xi \in \mathcal{G}^-_\delta(y), \eta \in \partial D, |\eta - \xi^*(y,\xi)| \leq \delta').$$

Proof. Lemma 7.1 is shown by the same argument as for Lemma 5.2 of [2]. To adjust Lemma 7.1, we need to replace p to y, x to p and z to η in Lemma 5.2 of [2]. Here we give a brief explanation for the proof.

First, note that similarly to (3.7), we get

$$|\xi - \eta| + |\eta - y| \geq |\xi - y| + |\xi - \eta|\Big(1 - \frac{\eta - \xi}{|\eta - \xi|} \cdot \frac{y - \xi}{|y - \xi|}\Big) \tag{7.1}$$

since it follows that

$$|y - \xi| = (y - \eta) \cdot \frac{y - \xi}{|y - \xi|} + (\eta - \xi) \cdot \frac{y - \xi}{|y - \xi|} \leq |y - \eta| + \frac{(\eta - \xi) \cdot (y - \xi)}{|y - \xi|}.$$

For $\xi \in \mathcal{G}^+_\delta(y)$, $\nu_\xi \cdot (y - \xi) > 0$ holds. Since ∂D is strictly convex, $(\eta - \xi) \cdot \nu_\xi < 0$. Hence it follows that

$$\begin{aligned}\Big(\frac{(y - \xi) \cdot \nu_\xi}{|y - \xi|}\Big)^2 &\leq \Big|\Big(\frac{y - \xi}{|y - \xi|} - \frac{\eta - \xi}{|\eta - \xi|}\Big) \cdot \nu_\xi\Big|^2 \leq \Big|\frac{y - \xi}{|y - \xi|} - \frac{\eta - \xi}{|\eta - \xi|}\Big|^2 \\ &\leq 2 - 2\frac{\eta - \xi}{|\eta - \xi|} \cdot \frac{y - \xi}{|y - \xi|}.\end{aligned}$$

The above estimate and (7.1) yield

$$|\xi - \eta| + |\eta - y| \geq |\xi - y| + \frac{A_\delta^2}{2}|\xi - \eta| \quad (\xi \in \mathcal{G}^+_\delta(y), \eta \in \partial D),$$

where $A_\delta = \inf_{\xi \in \mathcal{G}^+_\delta(y)}((y - \xi) \cdot \nu_\xi)/|y - \xi|$. From Lemma 6.3, for any $\delta > 0$, the set $\{(\xi, y) \in \partial D \times \partial\Omega \,|\, \xi \in \mathcal{G}^+_\delta(y)\} \subset \partial D \times \partial\Omega$ is a closed set. Hence we have $A_\delta > 0$, which yields 1) of Lemma 7.1.

The former part of 2) can be obtained by showing $B_\delta = \inf_{\xi \in \mathcal{G}^-_\delta(y)} |\xi - \xi^*| > 0$. The latter part is given by (7.1) and

$$D_{\delta'} = \sup_{\xi \in \mathcal{G}^-_\delta(y)} \; \sup_{\eta \in (\partial D \setminus \{\xi\}) \setminus \overline{B(\xi^*, \delta')}} \frac{\eta - \xi}{|\eta - \xi|} \cdot \frac{\xi^* - \xi}{|\xi^* - \xi|} < 1,$$

since $(\xi^* - \xi)/|\xi^* - \xi| = (y - \xi^*)/|y - \xi^*|$. From Lemma 6.3, for any $\delta > 0$, the set $\{(\xi, y) \in \partial D \times \partial\Omega \,|\, \xi \in \mathcal{G}^-_\delta(y)\} \subset \partial D \times \partial\Omega$ is a closed set. Note also that the mapping $(\xi, y) \mapsto \xi^*(y, \xi)$ is continuous. Using these facts and tracing the argument for showing Lemma 5.2 of [2], we can obtain $B_\delta > 0$ and $D_{\delta'} < 1$, which yields 2) of Lemma 7.1.

For 3), we also follow the argument of [2]. We take $y \in \partial\Omega$, $\eta \in \partial D$ and $\xi \in \mathcal{G}^-(y) \subset \partial D$ and put $L(x) = |x-\eta| + |x-\xi^*|$ for $x \in \mathbb{R}^3$, and $z = \eta - \xi^*$. Since $|y-\xi^*| + |\xi^* - \xi| = |y-\xi|$, it follows that

$$|y-\eta| + |\eta-\xi| - |y-\xi| = \frac{|z|^2 + 2(\xi^* - y)\cdot z}{L(y)} + \frac{|z|^2 + 2(\xi^* - \xi)\cdot z}{L(\xi)}.$$

Note that for any $x \in \mathbb{R}^3$, $x \neq \xi^*$,

$$\frac{(\xi^* - x)\cdot z}{L(x)} = \frac{(\xi^* - x)\cdot z}{2|x-\xi^*|} - \frac{\{(x-\xi^*)\cdot z\}^2}{(L(x))^2|x-\xi^*|} - \frac{|z|^2(\xi^* - x)\cdot z}{2(L(x))^2|x-\xi^*|},$$

and $(y-\xi^*)/|y-\xi^*| + (\xi - \xi^*)/|\xi - \xi^*| = 0$ holds. Since $y \in \mathcal{G}^-(p)$, we obtain

$$\begin{aligned}|y-\eta| + |\eta-\xi| - |y-\xi| = \Big(\frac{1}{L(y)} + \frac{1}{L(\xi)}\Big)|z|^2 - 2\Big(\frac{|y-\xi^*|}{(L(y))^2} + \frac{|\xi-\xi^*|}{(L(\xi))^2}\Big)(\theta\cdot z)^2 \\ - 2\Big(\frac{1}{(L(y))^2} - \frac{1}{(L(\xi))^2}\Big)|z|^2\theta\cdot z,\end{aligned}$$

where $\theta = (\xi - \xi^*)/|\xi - \xi^*|$.

By the same argument as for $A_\delta > 0$ and $B_\delta > 0$, it also follows that

$$\inf_{(\xi,\eta)\in\mathcal{G}_\delta^-(y)\times\partial D} |y-\eta| + |y-\xi^*| > 0, \quad \inf_{(\xi,\eta)\in\mathcal{G}_\delta^-(y)\times\partial D} |\eta-\xi| + |\xi^*-\xi| > 0,$$

$$\inf_{\xi\in\mathcal{G}_\delta^-(y)} |y-\xi^*| > 0, \quad \inf_{\xi\in\mathcal{G}_\delta^-(y)} |\xi^*-\xi| > 0, \quad \inf_{\xi\in\mathcal{G}_\delta^-(y)} \frac{y-\xi^*}{|y-\xi^*|}\cdot\nu_{\xi^*} > 0.$$

Hence, there exist constants $C > 0$ and $C' > 0$ such that

$$|y-\eta| + |\eta-\xi| - |y-\xi| \geq C(|z|^2 - (\theta\cdot z)^2) - C'|z|^3 \quad (\xi \in \mathcal{G}_\delta^-(y), \eta \in \partial D),$$

where we used $1/L(y) = 1/(2|y-\xi^*|) + O(|z|)$ and $1/L(\xi) = 1/(2|\xi-\xi^*|) + O(|z|)$ uniformly in $\xi \in \mathcal{G}^-(y)$. Since $(y-\xi^*)\cdot\nu_{\xi^*} = -(|y-\xi^*|/|\xi-\xi^*|)(\xi-\xi^*)\cdot\nu_{\xi^*} > 0$ and $(\eta-\xi^*)\cdot\nu_{\xi^*} < 0$, the same argument for 1), we obtain

$$\begin{aligned}|y-\eta| + |\eta-\xi| - |y-\xi| \geq \frac{C}{2}\Big(\inf_{\xi\in\mathcal{G}_\delta^-(y)} \frac{y-\xi^*}{|y-\xi^*|}\cdot\nu_{\xi^*}\Big)^2|z|^2 - C'|z|^3 \\ (\xi \in \mathcal{G}_\delta^-(y), \eta \in \partial D),\end{aligned}$$

which implies 3) of Lemma 7.1. This completes the proof of Lemma 7.1. □

From Proposition 6.2 and Lemma 7.1, it follows that for $p \in \partial D$ given in Proposition 6.2, there exist constants $C_1 > 0$, $\tilde{\delta}_2 > 0$ and $\delta_2' > 0$ with $\tilde{\delta}_2 > \delta_2'$ such that

(7.2) $\quad |p-\eta| + |\eta-y| \geq |p-y| + C_1|\eta-p| \quad (y \in \mathcal{U}^+(p), \eta \in \partial D),$

(7.3) $\quad |p-p^*(y,p)| \geq 2\tilde{\delta}_2 (> 2\delta_2') \quad (y \in \mathcal{U}^-(p)),$

(7.4) $\quad |p-\eta| + |\eta-y| \geq |p-y| + C_1|\eta-p| \quad (y \in \mathcal{U}^-(p), \eta \in \partial D, |\eta-p^*| \geq \delta_2'),$

(7.5) $\quad |p-\eta| + |\eta-y| \geq |p-y| + C_1|\eta-p^*|^2 \quad (y \in \mathcal{U}^-(p), \eta \in \partial D, |\eta-p^*| \leq \delta_2').$

Note that in (7.2)-(7.5), $\delta_2' > 0$ can be chosen as small as necessary.

Now, we treat the case $y \in \mathcal{U}^+(p)$. In this case, the main part $A_0^+(y,p;\lambda)$ of $A(y,p;\lambda)$ is given by

$$A_0^+(y,p;\lambda) = e^{\lambda|y-p|}\int_{\partial D} K_\lambda(\xi,y)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi$$

as is in the following lemma:

Lemma 7.2. *There exist constants* $\mu_0 > 0$ *and* $C > 0$ *such that*

$$\int_{\partial D}|M_\lambda(\xi,y)-K_\lambda(\xi,y)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi \le C\mu^{-1}e^{-\mu|y-p|} \quad (y\in\mathcal{U}^+(p), \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0).$$

Proof. Since $y\in\mathcal{U}^+(p)$, estimate (7.2) holds. From (7.2), (3.25), (3.26), (3.21), (3.31) and (3.27), it follows that

$$\begin{aligned}
\int_{\partial D}|M_\lambda(\xi,y)-K_\lambda(\xi,y)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi &\le C\Big\{\int_{\partial D}|\tilde K_\lambda(\xi,y)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi \\
&\qquad + \int_{\partial D}dS_\zeta\int_{\partial D}|Y_\lambda^\infty(\xi,\zeta)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi|Y_{21}(\zeta,y;\lambda)|\Big\} \\
&\le C\Big\{\int_{\partial D}\frac{e^{-\mu(|\xi-y|+|\xi-p|)}}{|\xi-p|}dS_\xi \\
&\qquad + \mu\int_{\partial D}e^{-\mu(|\zeta-p|+|\zeta-y|)}\Big(1+\frac{1}{\mu|\zeta-p|}+(\mu|\zeta-p|)^{1/2}\Big)dS_\zeta\Big\} \\
&\le Ce^{-\mu|y-p|}\int_{\partial D}e^{-\mu C_1|\zeta-p|/2}\Big(\mu+\frac{1}{|\zeta-p|}\Big)dS_\zeta.
\end{aligned}$$

Note that in the last inequality, $e^{-\mu C_1|\xi-p|/2}(\mu|\xi-p|)^{1/2}\le 1+2C_1^{-1}$ is used. Hence, Remark 3.3 implies

$$\int_{\partial D}e^{-\mu C_1|\xi-p|/2}\Big(\mu+\frac{1}{|\xi-p|}\Big)dS_\xi \le C\Big(\mu\cdot\mu^{-2}+\mu^{-1}\Big) = 2C\mu^{-1},$$

which completes the proof of Lemma 7.2. □

From (6.3) and Lemma 7.2, it suffices to show

$$A_0^+(y,p;\lambda) = \frac{1}{|y-p|} + O(\mu^{-1}) \quad \text{as } |\lambda|\to\infty \qquad \text{(uniformly in } y\in\mathcal{U}^+(p), \lambda\in\mathbb{C}_{\delta_0}) \tag{7.6}$$

to obtain the estimate of $A(y,p;\lambda)$ for the case $y\in\mathcal{U}^+(p)$ in Proposition 6.4. For $p\in\partial D$, we choose sufficiently small standard local coordinate $U\ni\sigma\mapsto s(\sigma)\in\partial D\cap B(p,2r_0)$ around $p\in\partial D$. Take $\chi\in C^2(\partial D)$ with $0\le\chi\le1$, $\chi(z)=1$ for $z\in\partial D\cap B(p,r_0)$ and $\chi(z)=0$ for $z\in\partial D\setminus B(p,4r_0/3)$. From (3.21), (7.2) and Remark 3.3, it follows that

$$\begin{aligned}
\Big|e^{\mu|y-p|}\int_{\partial D}(1-\chi(\xi))K_\lambda(\xi,y)\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi\Big| &\le C\mu\int_{\partial D}(1-\chi(\xi))\frac{e^{-\mu C_1|\xi-p|}}{|p-\xi|}dS_\xi \\
&\le Cr_0^{-1}\mu\int_{\partial D}e^{-\mu C_1|\xi-p|}dS_\xi \le Cr_0^{-1}\mu^{-1} \quad (y\in\mathcal{U}^+(p), \lambda\in\mathbb{C}_{\delta_0}, \mu\ge\mu_0).
\end{aligned}\tag{7.7}$$

Thus, by using the above standard coordinate, the main term of (7.6) is given by

$$e^{\lambda|y-p|}\int_{\partial D}\chi(\xi)K_\lambda(\xi,y)\frac{e^{-\lambda|p-\xi|}}{|p-\xi|}dS_\xi=\lambda\int_U e^{\lambda\varphi(\sigma;y)}\frac{H(\sigma;y)}{|s(\sigma)-p|}d\sigma, \tag{7.8}$$

where $\varphi(\sigma;y)=|y-p|-|p-s(\sigma)|-|s(\sigma)-y|$, and $H(\sigma;y)$ is defined by

$$H(\sigma;y)=\frac{1}{2\pi}\chi(s(\sigma))G(\sigma)\frac{\nu_{s(\sigma)}\cdot(y-s(\sigma))}{|s(\sigma)-y|^2}$$

with $G(\sigma)=\det\Big(\dfrac{\partial s}{\partial\sigma_j}(\sigma)\cdot\dfrac{\partial s}{\partial\sigma_k}(\sigma)\Big)$. Note that $H(\cdot;y)\in C_0^1(U)$ since ∂D is C^2 boundary. Thus, the main term of (7.6) is deduced by the following Lemma:

Lemma 7.3. *Assume that ∂D is of class C^2 and strictly convex. Then for any $h\in C_0^1(U)$, the integral $J(\lambda;y)$ defined by*

$$J(\lambda;y)=\int_U e^{\lambda\varphi(\sigma;y)}\frac{h(\sigma)}{|s(\sigma)-p|}d\sigma \tag{7.9}$$

has the following property:

$$J(\lambda;y)=2\pi\Big|\frac{|y-p|}{\nu_p\cdot(p-y)}\Big|h(0)\lambda^{-1}+O(\mu^{-2})\qquad(as\ \ |\lambda|\to\infty)$$

uniformly in $y\in\mathcal{U}^\pm(p)$, where $\varphi(\sigma;y)=|y-p|-|p-s(\sigma)|-|s(\sigma)-y|$.

Note that Lemma 7.3 is also valid for the case $y\in\mathcal{U}^-(p)$, which is used to treat the case $y\in\mathcal{U}^-(p)$. The proof of Lemma 7.3 is given in the last part of this section.

We apply Lemma 7.3 to (7.8). Noting $G(0)=1$, $\nu_p\cdot(y-p)>0$ for $y\in\mathcal{U}^+(p)$, we obtain

$$\begin{aligned}e^{\lambda|y-p|}\int_{\partial D}\chi(\xi)K_\lambda(\xi,y)\frac{e^{-\lambda|p-\xi|}}{|p-\xi|}dS_\xi&=\lambda\Big(2\pi\frac{|y-p|}{\nu_p\cdot(y-p)}\frac{1}{2\pi}\frac{\nu_p\cdot(y-p)}{|y-p|^2}\lambda^{-1}+O(\mu^{-2})\Big)\\&=\frac{1}{|y-p|}+O(\mu^{-1})\end{aligned} \tag{7.10}$$

as $|\lambda|\to\infty$ uniformly in $y\in\mathcal{U}^+(p)$, which yields (7.6). Thus, the case of $y\in\mathcal{U}^+(p)$ in Proposition 6.4 is shown.

Next lemma is for the case $y\in\mathcal{U}^-(p)$ where the main part $A_0^-(y,p;\lambda)$ of $A(y,p;\lambda)$ is given by

$$A_0^-(y,p;\lambda)=e^{\lambda|y-p|}\int_{\partial D}\Big(K_\lambda(\xi,y)+\int_{\partial D}K_\lambda(\xi,\eta)K_\lambda(\eta,y)dS_\eta\Big)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi.$$

Lemma 7.4. *There exist constants $\mu_0>0$ and $C>0$ such that*

$$\begin{aligned}&\int_{\partial D}\Big|M_\lambda(\xi,y)-K_\lambda(\xi,y)-\int_{\partial D}K_\lambda(\xi,\eta)K_\lambda(\eta,y)dS_\eta\Big|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi\\&\qquad\le C\mu^{-1}e^{-\mu|y-p|}\quad(y\in\mathcal{U}^-(p),\lambda\in\mathbb{C}_{\delta_0},\mu\ge\mu_0).\end{aligned} \tag{7.11}$$

Proof. From (3.25), the left side of (7.11) is estimated by

(7.12)

$$\begin{aligned}&\int_{\partial D}|\tilde K_\lambda(\xi,y)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi+\int_{\partial D}dS_\eta\int_{\partial D}|Y_\lambda^\infty(\xi,\eta)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi|\tilde K_\lambda(\eta,y)|\\&\qquad+\int_{\partial D}dS_\eta\Big|\frac{K_\lambda(\eta,y)}{\lambda}\Big|\int_{\partial D}|\lambda||Y_\lambda^\infty(\xi,\eta)-K_\lambda(\xi,\eta)|\frac{e^{-\mu|\xi-p|}}{|\xi-p|}dS_\xi.\end{aligned}$$

From (7.3), if $y \in \mathcal{U}^-(p)$ and $\xi \in \partial D$ satisfies $|\eta - p^*(y,\xi)| \le \delta_2'$, then $|\eta - p| \ge |p - p^*| - |\eta - p^*| \ge 2\delta_2' - \delta_2' = \delta_2'$. Hence, (3.31) and (3.32) yield

$$\begin{aligned}
&\int_{\partial D} |Y_\lambda^\infty(\eta,\xi)| \frac{e^{-\mu|\xi-p|}}{|\xi-p|} dS_\xi + \int_{\partial D} |\lambda||Y_\lambda^\infty(\xi,\eta) - K_\lambda(\xi,\eta)| \frac{e^{-\mu|\xi-p|}}{|\xi-p|} dS_\xi \\
&\qquad \le Ce^{-\mu|\eta-p|}\Big\{1 + \frac{1}{\mu|\eta-p|} + \frac{1}{|\eta-p|} + \frac{1}{|\eta-p|}\Big(1 + \frac{1}{|\eta-p|^3}\Big)\Big\} \\
&\qquad \le Ce^{-\mu|\eta-p|} \qquad (\eta \in \partial D, |\eta - p^*| \le \delta_2', \lambda \in \mathbb{C}_{\delta_0}, \mu = \operatorname{Re}\lambda \ge \mu_0).
\end{aligned}$$

From the above estimate, (3.31), (3.21), (7.4), (7.5) and Remark 3.3, it follows that (7.12) is estimated by

$$\begin{aligned}
&C\mu \int_{|\xi-p^*|\ge\delta_2'} e^{-C_1\mu|\xi-p|}\Big(1 + \frac{1}{\mu|\xi-p|} + (\mu|\xi-p|)^{\frac12}\Big) dS_\xi + C\int_{|\xi-p^*|\le\delta_2'} e^{-C_1\mu|\xi-p^*|^2} dS_\xi \\
&\qquad \le C\mu^{-1} \quad (\mu \ge \mu_0),
\end{aligned}$$

which completes the proof of Lemma 7.4. □

From (6.3) and Lemma 7.4, to obtain the estimate of $A(y,p;\lambda)$ for the case $y \in \mathcal{U}^-(p)$ in Proposition 6.4, it suffices to show

$$A_0^-(y,p;\lambda) = \frac{-1}{|y-p|} + O(\mu^{-1}) \quad \text{as } |\lambda| \to \infty \tag{7.13}$$
$$\text{(uniformly in } y \in \mathcal{U}^-(p), \lambda \in \mathbb{C}_{\delta_0}).$$

To treat $A_0^-(y,p;\lambda)$, we need to get an asymptotic behavior of the integral given by replacing $y \in \mathcal{U}^\pm(p)$ with a point $\xi \in \partial D$ satisfying $|\nu_p \cdot (\xi - p)| \ge 3r_0$ in (7.9), where $r_0 > 0$ is the constant appeared in the standard local coordinate $U \ni \sigma \mapsto s(\sigma) \in \partial D \cap B(p, 2r_0)$ around $p \in \partial D$ used to define (7.9) in Lemma 7.3.

Lemma 7.5. *Assume that ∂D is of class C^2 and strictly convex. Then for any $h \in C_0^1(U)$, the integral $J(\lambda;\xi)$ given by* (7.9) *has the following property:*

$$J(\lambda;\xi) = 2\pi\Big|\frac{|\xi-p|}{\nu_p\cdot(p-\xi)}\Big| h(0)\lambda^{-1} + O(\mu^{-2}) \qquad (\text{as } |\lambda| \to \infty)$$

uniformly in $\xi \in \partial D$ with $|(\xi - p)\cdot \nu_p| \ge 3r_0$.

The proof of Lemma 7.5 will be also given in the last part of this section. Using Lemma 7.5 and Lemma 7.3, we first show the case of $y \in \mathcal{U}^-(p)$ in Proposition 6.4.

For the case $y \in \mathcal{U}^-(p)$, we need the following properties of the broken lines:

Lemma 7.6. *Assume that ∂D is of class C^2 and strictly convex. Given $\delta > 0$ there exists a positive constant c_0 such that*
(i) for all $\xi, \zeta, p \in \partial D$ with $|\xi - p| \ge 2\delta$, $|\zeta - \xi| \ge \delta$, and $|\zeta - p| \ge \delta$ we have

$$|\xi - \zeta| + |\zeta - p| \ge |\xi - p| + c_0;$$

(ii) for all $\xi, \zeta, p \in \partial D$ with $|\xi - p| \ge 2\delta$, $|\zeta - p| \le \delta$ we have

$$|\xi - \zeta| + |\zeta - p| \ge |\xi - p| + c_0|\zeta - p|.$$

Note that these estimates are given by simple properties of the Euclid distance. Hence we show Lemma 7.6 although Lemma 7.6 is given as Proposition 2.2 of [1].

Proof of Lemma 7.6. For (i), c_0 is given the infimum of $|\xi-\zeta|+|\zeta-p|-|\xi-p|>0$ taken on the set $\{(\xi,\zeta,p)\in\partial D\times\partial D\times\partial D \| \xi-p|\geq 2\delta, |\zeta-\xi|\geq\delta, |\zeta-p|\geq\delta\}$. Note that since ∂D is strictly convex, $|\xi-\zeta|+|\zeta-p|-|\xi-p|$ is positive for these three points ξ, ζ and p.

For (ii), if the estimate of (ii) does not hold, for any $n\in\mathbb{N}$, there exist points ξ_n, ζ_n, $p_n\in\partial D$ such that $|\xi_n-p_n|\geq 2\delta$, $|\zeta_n-p_n|\leq\delta$ and $|\xi_n-\zeta_n|+|\zeta_n-p_n|<|\xi_n-p_n|+n^{-1}|\zeta_n-p_n|$. Note that $\zeta_n\neq p_n$ $(n\in\mathbb{N})$ since $n^{-1}|\zeta_n-p_n|>|\xi_n-\zeta_n|+|\zeta_n-p_n|-|\xi_n-p_n|\geq 0$. Hence we can put $\theta_n=(\zeta_n-p_n)/|\zeta_n-p_n|$. Since ∂D is compact, we can assume that $\xi_n\to\xi_0$, $\zeta_n\to\zeta_0$, $p_n\to p_0$ and $\theta_n\to\theta_0$ as $n\to\infty$ respectively. Then $|\xi_0-p_0|\geq 2\delta$, $|\zeta_0-p_0|\leq\delta$ and $|\xi_0-\zeta_0|+|\zeta_0-p_0|=|\xi_0-p_0|$ hold. Since ∂D is strictly convex, the points p_0, ζ_0 and ξ_0 do not on a line. This implies $p_0=\zeta_0$. Here we note that

$$
\begin{aligned}
&|\xi_n-p_n|^2+|\zeta_n-p_n|^2-2|\zeta_n-p_n|\theta_n\cdot(\xi_n-p_n)\\
&\qquad=|\xi_n-\zeta_n|^2\leq(|\xi_n-p_n|+(\frac{1}{n}-1)|\zeta_n-p_n|)^2,
\end{aligned}
$$

which yields

$$
-2\theta_n\cdot(\xi_n-p_n)\leq 2\Big(\frac{1}{n}-1\Big)|\xi_n-p_n|+\Big(\frac{1}{n^2}-\frac{2}{n}\Big)|\zeta_n-p_n|.
$$

Taking $n\to\infty$, we obtain $\theta_0\cdot(\xi_0-p_0)\geq|\xi_0-p_0|$, which implies $\theta_0=(\xi_0-p_0)/|\xi_0-p_0|$. Further, from (i) of Lemma 3.1, it follows that $|\nu_{p_n}\cdot\theta_n|\leq C|\zeta_n-p_n|\to 0$ $(n\to\infty)$, which yields $\nu_{p_0}\cdot\theta_0=0$. These facts mean that $0\neq\xi_0-p_0\in T_{p_0}(\partial D)$, and $\xi_0\in\partial D$, however this does not occur since ∂D is strictly convex. Thus we obtain Lemma 7.6.

From Lemma 7.5 and Lemma 7.6, we show the following property:

Lemma 7.7. *For any $\delta>0$, it holds that*

$$
\begin{aligned}
e^{\lambda|\eta-p|}\int_{\partial D}K_\lambda(\eta,\xi)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi=\frac{-1}{|p-\eta|}+O(\mu^{-1})\quad &\textit{as } |\lambda|\to\infty\\
&(\eta,p\in\partial D, |(\eta-p)\cdot\nu_p|\geq\delta, \lambda\in\mathbb{C}_{\delta_0}).
\end{aligned}
$$

Proof. As is in (7.7), we choose a standard local coordinate around $p\in\partial D$. Since ∂D is compact, we can take $r_0>0$ independent of $p\in\partial D$ and arbitrary small if it is necessary. Thus, for any fixed $\delta>0$, we choose $r_0>0$ as $\delta>3r_0$ and consider the case that $\eta,p\in\partial D$ with $|(\eta-p)\cdot\nu_p|\geq\delta$. If this is the case, since $|\eta-p|\geq\delta$ holds, Lemma 7.6 implies

$$
\begin{aligned}
&|\eta-\xi|+|\xi-p|\geq|\eta-p|+c_0\quad(\eta,\xi,p\in\partial D, |\eta-\xi|\geq\delta/2 \text{ and } |p-\xi|\geq\delta/2),\\
&|\eta-\xi|+|\xi-p|\geq|\eta-p|+c_0|\eta-\xi|\quad(\eta,\xi,p\in\partial D, |\eta-\xi|\leq\delta/2),\\
&|\eta-\xi|+|\xi-p|\geq|\eta-p|+c_0|p-\xi|\quad(\eta,\xi,p\in\partial D, |p-\xi|\leq\delta/2)
\end{aligned}
$$

for some constant $c_0>0$ depending only on ∂D and $\delta>0$. We take $\chi\in C^2(\partial D)$ with $0\leq\chi\leq 1, \chi(\xi)=1$ for $|\xi-p|<r_0/2$ and $\chi(\xi)=0$ for $|\xi-p|>r_0$. These properties of the broken line, (3.22) and Remark 3.3 yield that

$$
(7.14)\qquad \Big|e^{\lambda|\eta-p|}\int_{\partial D}(1-\chi(\xi))K_\lambda(\eta,\xi)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi\Big|
$$

$$\leq C|\lambda| \int_{\partial D} \{e^{-\mu c_0|\xi-p|} + e^{-\mu c_0|\xi-\eta|} + e^{-\mu c_0}\} dS_\xi$$
$$\leq C\mu^{-1} \quad (\lambda \in C_{\delta_0}, \mu \geq \mu_0).$$

Next, note that as is in (7.8), using the standard local coordinates, we have

$$e^{\lambda|\eta-p|} \int_{\partial D} \chi(\xi) K_\lambda(\eta,\xi) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi = \lambda \int_U e^{\lambda\varphi(\sigma;\eta)} \frac{H(\sigma;\eta)}{|s(\sigma)-p|} d\sigma,$$

where $\varphi(\sigma; y) = |\eta - p| - |p - s(\sigma)| - |s(\sigma) - \eta|$ and $H(\sigma; \eta)$ are given by replacing y with η in (7.8). Note that $H(\cdot; \eta) \in C_0^1(U)$ since $|s(\sigma) - p| \leq 2r_0$ for $\sigma \in U$ and $|\eta - p| \geq \delta > 3r_0$. Noting $G(0) = 1$, $\nu_p \cdot (\eta - p) < 0$ for $\eta \in \partial D$ with $|(\eta - p) \cdot \nu_p| \geq \delta$, from Lemma 7.5 we obtain

$$\begin{aligned} e^{\lambda|\eta-p|} \int_{\partial D} \chi(\xi) K_\lambda(\eta,\xi) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi &= \lambda\Big(2\pi \frac{|\eta-p|}{\nu_p\cdot(p-\eta)} \frac{1}{2\pi} \frac{\nu_p\cdot(\eta-p)}{|\eta-p|^2} \lambda^{-1} + O(\mu^{-2})\Big) \\ &= -\frac{2\pi}{|\eta-p|} + O(\mu^{-1}) \qquad (\text{as } |\lambda| \to \infty). \end{aligned}$$

Combining the above result with (7.14), we obtain Lemma 7.7. □

Using Lemma 7.7, we show (7.13). Choose a standard local coordinate $U \ni \sigma \mapsto s(\sigma) \in \partial D \cap B(p, 2r_0)$ around p. Note that for $\tilde{\delta}_2 > 0$ in (7.3)-(7.5), we can choose $r_0 > 0$ and $\delta_2' > 0$ in (7.3)-(7.5) satisfying $\delta_2' < r_0 < \tilde{\delta}_2$. Note that there exists a constant $0 < \delta' \leq \delta_2'$ such that

$$(7.15) \qquad |(\eta - p) \cdot \nu_p| \geq \delta' \qquad \text{for all } \eta \in \partial D \text{ with } |\eta - p| \geq \delta_2'.$$

Indeed, from strict convexity of ∂D, it follows that $|(\eta - p) \cdot \nu_p| > 0$ for any $\eta \in \partial D$, $\eta \neq p$. Since $\partial D \ni \eta \mapsto |(\eta - p) \cdot \nu_p| \in \mathbb{R}$ is continuous and $\{\eta \in \partial D \,|\, |\eta - p| \geq \delta_2'\}$ is compact, we obtain (7.15).

From (7.15) and Lemma 7.7, it follows that

$$(7.16) \qquad \Big|\int_{\partial D} K_\lambda(\eta,\xi) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi + \frac{e^{-\lambda|\eta-p|}}{|\eta-p|}\Big| \leq C\mu^{-1} e^{-\mu|\eta-p|}$$
$$(\eta \in \partial D, |\eta - p| \geq \delta_2', \lambda \in \mathbb{C}_{\delta_0}, \mu \geq \mu_0).$$

Take $\psi \in C^2(\partial D)$ with $0 \leq \psi \leq 1$, $\psi(\eta) = 1$ for $\eta \in \partial D \cap B(p, \delta_2')$, $\psi(\eta) = 0$ for $\eta \in \partial D \backslash B(p, 4\delta_2'/3)$. Using ψ and (7.16), we can show that the main part of $A_0^-(y, p; \lambda)$ is given by

$$(7.17) \qquad e^{\lambda|y-p|} \int_{\partial D} \psi(\xi) K_\lambda(\xi, y) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi.$$

To check it, we estimate the following integral:

$$\begin{aligned} (7.18) \qquad &\Big|A_0^-(y,p;\lambda) - e^{\lambda|y-p|} \int_{\partial D} \psi(\xi) K_\lambda(\xi,y) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi\Big| \\ &= \Big|e^{\lambda|y-p|} \int_{\partial D} (1-\psi(\eta)) K_\lambda(\eta,y) \frac{e^{-\lambda|\eta-p|}}{|\eta-p|} dS_\eta \\ &\qquad + e^{\lambda|y-p|} \int_{\partial D} dS_\eta \int_{\partial D} K_\lambda(\eta,\xi) K_\lambda(\eta,y) \frac{e^{-\lambda|\xi-p|}}{|\xi-p|} dS_\xi\Big| \end{aligned}$$

$$\le e^{\mu|y-p|}\int_{\partial D}(1-\psi(\eta))|K_\lambda(\eta,y)|\Big|\frac{e^{-\lambda|\eta-p|}}{|\eta-p|}+\int_{\partial D}K_\lambda(\eta,\xi)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi\Big|dS_\eta$$
$$+e^{\mu|y-p|}\int_{\partial D}\psi(\eta)\Big|\int_{\partial D}K_\lambda(\eta,\xi)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi\Big||K_\lambda(\eta,y)|dS_\eta.$$

From (7.16), (3.21), (7.4), (7.5) and Remark 3.3, the first term of (7.18) is estimated by

$$(7.19)\qquad Ce^{\mu|y-p|}\Big\{\int_{S^-_{\delta_2'}(p)\cap S^-_{\delta_2'}(p^*)}e^{-\mu(|\eta-y|+|y-p|)}dS_\eta+\int_{S_{\delta_2'}(p^*)}e^{-\mu(|\eta-y|+|y-p|)}dS_\eta\Big\}$$
$$\le C\Big\{\int_{S^-_{\delta_2'}(p)}e^{-C_1\mu|\eta-p|}dS_\eta+\int_{S_{\delta_2'}(p^*)}e^{-C_1\mu|\eta-p^*|^2}dS_\eta\Big\}\le C\mu^{-1}\quad(y\in\mathcal{U}^-(p)).$$

From (3.22), (3.31), (7.4) and Remark 3.3, the second term of (7.18) is estimated by

$$(7.20)\qquad Ce^{\mu|y-p|}\mu\int_{S_{2\delta_2'}(p)}e^{-\mu(|\eta-y|+|y-p|)}\Big(1+\frac{1}{\mu|\eta-p|}+(\mu|\eta-p|)^{1/2}\Big)dS_\eta$$
$$\le C\int_{S_{2\delta_2'}(p)}e^{-C_1|\eta-p|/2}\Big(\mu+\frac{1}{|\eta-p|}\Big)dS_\eta\le C\mu^{-1}\quad(y\in\mathcal{U}^-(p)).$$

From (7.18), (7.19) and (7.20), it follows that

$$(7.21)\qquad \Big|A_0^-(y,p;\lambda)-e^{\lambda|y-p|}\int_{\partial D}\psi(\xi)K_\lambda(\xi,y)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi\Big|\le C\mu^{-1}\quad(y\in\mathcal{U}^-(p)).$$

To obtain asymptotics of (7.17), we use Lemma 7.3 for the case $y\in\mathcal{U}^-(p)$. Since $\operatorname{supp}\psi\subset B(p,2r_0)\subset B(p,2\delta_0')$ and $\nu_p\cdot(y-p)<0$ for $y\in\mathcal{U}^-(p)$, the same argument as for (7.10) implies that

$$e^{\lambda|y-p|}\int_{\partial D}\psi(\xi)K_\lambda(\xi,y)\frac{e^{-\lambda|\xi-p|}}{|\xi-p|}dS_\xi=\frac{-1}{|y-p|}+O(\mu^{-1})$$

as $|\lambda|\to\infty$ uniformly in $y\in\mathcal{U}^-(p)$. Combining the above fact with (7.21), we obtain the case of $y\in\mathcal{U}^-(p)$ in Proposition 6.4, which completes the proof of Proposition 6.4. □

The last of this section, we give a proof of Lemma 7.3 and 7.5. These lemmas are unified to give an asymptotic formula for the integral $J(\lambda;q)$ given by

$$J(\lambda;q)=\int_U e^{\lambda\varphi(\sigma;q)}\frac{h(\sigma)}{|s(\sigma)-p|}d\sigma\qquad(q\in\mathbb{R}^3),$$

where $s(\sigma)$ is a standard local coordinate

$$U\ni\sigma\mapsto s(\sigma)=p+\sigma_1e_1+\sigma_2e_2-g(\sigma)\nu_p\in\partial D\cap B(p,2r_0)$$

around a fixed $p\in\partial D$ with an open set $U\subset\mathbb{R}^2$ and $r_0>0$. In the integral $J(\lambda;q)$, $\varphi(\sigma;q)$ is given by $\varphi(\sigma;q)=|q-p|-|q-s(\sigma)|-|s(\sigma)-p|$ for some $q\in\mathbb{R}^3$. Note that Lemma 7.3 is the case of $q=y\in\mathcal{U}^\pm(p)$, and Lemma 7.5 is for $q=\xi\in\partial D$ with $|(\xi-p)\cdot\nu_p|\ge 3r_0$. Thus it suffices to show the following lemma:

Lemma 7.8. *Assume that ∂D is of class C^2 and strictly convex. Then for any $h\in C_0^1(U)$, the integral $J(\lambda;q)$ stated above has the following property:*

$$J(\lambda;q) = 2\pi\Big|\frac{|q-p|}{\nu_p\cdot(p-q)}\Big|h(0)\lambda^{-1} + O(\mu^{-2}) \qquad (as\ \ |\lambda|\to\infty)$$

uniformly in $q\in\mathcal{U}^{\pm}(p)$ *or* $q\in\partial D$ *with* $|(q-p)\cdot\nu_p|\geq 3r_0$.

Proof. For the polar coordinate $\sigma_1 = r\cos\theta$ and $\sigma_2 = r\sin\theta$, we put $\tilde\varphi(r,\theta;q)=\varphi(\sigma;q)$, $\tilde h(r,\theta)=h(\sigma)$, $\tilde s(r,\theta)=s(r\cos\theta,r\sin\theta)$ and $\tilde g(r,\theta)=g(r\cos\theta,r\sin\theta)$. Then it follows that

$$J(\lambda;q)=\int_0^\infty\int_0^{2\pi}e^{\lambda\tilde\varphi(r,\theta;q)}\frac{\tilde h(r,\theta)}{|\tilde s(r,\theta)-p|}rdrd\theta.$$

Since $\tilde\varphi(r,\theta;q) = |q-p| - |q-s(r\cos\theta,r\sin\theta)| - r\sqrt{1+r^{-2}(\tilde g(r,\theta))^2}$ and $\tilde g(r,\theta)$ is a C^2 function with $\tilde g(r,\theta)=O(r^2)$ near $r=0$, we obtain $\partial_r\tilde\varphi(0,\theta) = -(1+\alpha_1\cos\theta+\alpha_2\sin\theta)$, where $\alpha_j=(p-q)\cdot e_j/|q-p|$ $(j=1,2)$.

First, we consider the case of $q\in\mathcal{U}^{\pm}(p)$. We put $\alpha_3=(p-q)\cdot\nu_p/|q-p|$ and $\beta=\sqrt{\alpha_1^2+\alpha_2^2}$. From the proof of Proposition 6.2, there exists a constant $0<\delta\leq 1$ such that $|(q-p)\cdot\nu_p|/|q-p|\geq\delta$ for all $q\in\mathcal{U}^{\pm}(p)$. Hence we obtain $|\alpha_1\cos\theta+\alpha_2\sin\theta|\leq\sqrt{\alpha_1^2+\alpha_2^2}=\sqrt{1-\alpha_3^2}\leq\sqrt{1-\delta^2}<1$. Thus, $\partial_r\tilde\varphi(0,\theta;q)<-(1-\sqrt{1-\alpha_3^2})\leq-\delta^2/2<0$ holds for any $\theta\in[0,2\pi]$, which implies that $\partial_r\tilde\varphi(r,\theta;q)\leq-\delta^2/4$ for $(r,\theta)\in[0,r_1]\times[0,2\pi]$ with some constant $0<r_1\leq 2r_0$ independent of $q\in\mathcal{U}^{\pm}(p)$. Take $\chi\in C^\infty(\mathbb{R})$ with $0\leq\chi\leq 1$ and $\chi(r)=1$ $(r\leq r_1/3)$ and $\chi(r)=0$ $(2r_1/3\leq r)$. Noting that (7.2) and (7.4) yield $-\tilde\varphi(r,\theta;q)\geq C_1|\tilde s(r,\theta)-p|\geq C_1 r$, and $r_1/3\leq r\leq|\tilde s(r,\theta)-p|\leq 2r_0$ for $\chi(r)\neq 1$, we obtain

$$|J(\lambda;q)-J_0(\lambda;q)|\leq C\int_{r_1/3}^{2r_0}\int_0^{2\pi}\frac{e^{-C_1\mu r}rdrd\theta}{r_1/3}\leq Ce^{-3^{-1}C_1r_1\mu}, \tag{7.22}$$

where

$$J_0(\lambda;q)=\int_0^\infty\int_0^{2\pi}e^{\lambda\tilde\varphi(r,\theta;q)}\chi(r)\frac{\tilde h(r,\sigma)}{|\tilde s(r,\theta)-p|}rdrd\theta.$$

For $J_0(\lambda;q)$, noting $r/|\tilde s(r,\theta)-p| = 1/\sqrt{1+r^{-2}(\tilde g(r,\theta))^2}$ is C^1 function near $r=0$ and $\chi(r)\tilde h(\cdot,\theta)\in C_0^1([0,r_1))$, we use integration by parts and obtain

$$\begin{aligned}J_0(\lambda;q)&=\int_0^{2\pi}\int_0^\infty\frac{1}{\lambda\partial_r\tilde\varphi(r,\theta;q)}\partial_r(e^{\lambda\tilde\varphi(r,\theta;q)})\frac{\chi(r)\tilde h(r,\theta)}{\sqrt{1+r^{-2}(\tilde g(r,\theta))^2}}drd\theta\\&=-\int_0^{2\pi}\frac{1}{\lambda\partial_r\tilde\varphi(0,\theta;q)}\frac{\tilde h(r,\theta)}{\sqrt{1+r^{-2}(\tilde g(r,\theta))^2}}\Big|_{r=0}d\theta\\&\qquad-\frac{1}{\lambda}\int_0^{2\pi}\int_0^\infty e^{\lambda\tilde\varphi(r,\theta;q)}\frac{\partial}{\partial r}\Big(\frac{1}{\partial_r\tilde\varphi(r,\theta;q)}\frac{\tilde h(r,\theta)}{\sqrt{1+r^{-2}(\tilde g(r,\theta))^2}}\Big)drd\theta.\end{aligned}$$

Noting that $\tilde\varphi(r,\theta;q)\leq-C_1r$, $\tilde g(r,\theta)=O(r^2)$ and $\tilde h(0,\theta)=h(0)$ in the above equality, and combining them with (7.22) we obtain

$$\Big|J(\lambda;q)-\frac{1}{\lambda}h(0)\int_0^{2\pi}\frac{-1}{\partial_r\tilde\varphi(0,\theta;q)}d\theta\Big|\leq C\mu^{-2} \tag{7.23}$$
$$(q\in\mathcal{U}^{\pm}(p),\lambda\in\mathbb{C}_{\delta_0},\mu=\operatorname{Re}\lambda\geq\mu_0)$$

for some constant $\mu_0>0$.

Here we also consider the case $q \in \partial D$ with $|(q-p)\cdot\nu_p| \geq 3r_0$. In this case, (7.23) also holds. Indeed, since $|q-p| \geq 3r_0 > 0$, it follows that

$$\inf_{|(q-p)\cdot\nu_p|\geq 3r_0, q\in\partial D} \frac{|(q-p)\cdot\nu_p|}{|q-p|} = \delta > 0.$$

Hence, as is in the case of $y \in \mathcal{U}^{\pm}(p)$, we obtain $\partial_r\tilde{\varphi}(r,\theta;q) \leq -\delta^2/4$ for $(r,\theta) \in [0,r_1]\times[0,2\pi]$ with some constant $0 < r_1 \leq 2r_0$ independent of $q \in \partial D$ with $|(q-p)\cdot\nu_p| \geq 3r_0$. Note that

$$|q-s(\sigma)| \geq |q-p| - |p-s(\sigma)| \geq |(q-p)\cdot\nu_p| - |p-s(\sigma)| \geq r_0 \quad (\sigma \in U)$$

since $\sigma \in U$ implies $|p-s(\sigma)| \leq 2r_0$. From these facts, and (i) and (ii) of Lemma 7.6 as $\delta = r_0$, we obtain $-\tilde{\varphi}(r,\theta;q) \geq c_0 r$. Hence, we can show (7.23) similarly to the case of $q \in \mathcal{U}^{\pm}(p)$.

We choose $\theta_0 \in [0,2\pi)$ satisfying $\cos\theta_0 = \beta^{-1}\alpha_1$ and $\sin\theta_0 = \beta^{-1}\alpha_2$. Then $\partial_r\tilde{\varphi}(0,\theta;q) = -(1+\beta\cos(\theta-\theta_0))$ holds. Thus, from (7.23) to obtain Lemma 7.8, it suffices to show

$$\int_0^{2\pi} \frac{-1}{\partial_r\tilde{\varphi}(0,\theta;q)}d\theta = \int_0^{2\pi} \frac{1}{1+\beta\cos(\theta-\theta_0)}d\theta = \frac{2\pi}{\sqrt{1-\beta^2}} = \frac{2\pi}{|\alpha_3|}. \tag{7.24}$$

If $\beta = 0$, it is obvious since $|\alpha_3| = 1$. If $0 < \beta$, using $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$, we can reduce the integral in (7.24) to

$$\int_0^{2\pi} \frac{1}{1+\beta\cos(\theta-\theta_0)}d\theta = \int_0^{2\pi} \frac{1}{1+\beta\cos\theta}d\theta = \frac{1}{i}\int_{|\zeta|=1} \frac{2}{\beta\zeta^2+2\zeta+\beta}d\zeta.$$

We write the roots $\gamma_{\pm} = -\beta^{-1} \pm \sqrt{\beta^{-2}-1}$ of $\zeta^2 + 2\beta^{-1}\zeta + 1 = 0$. From $0 < \beta < 1$ it follows that $\gamma_- < -1 < \gamma_+ < 1$. Hence residue theorem implies that

$$\frac{1}{i}\int_{|\zeta|=1} \frac{2}{\beta\zeta^2+2\zeta+\beta}d\zeta = \frac{2}{i\beta}2\pi i\frac{1}{\gamma_+-\gamma_-} = \frac{2\pi}{\sqrt{1-\beta^2}},$$

which yields (7.24). This completes the proof of Lemma 7.8. □

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