THE ARC METRIC ON TEICHMÜLLER SPACES OF SURFACES OF INFINITE TYPE WITH BOUNDARY

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Abstract

Let X_0 be a complete hyperbolic surface of infinite type with geodesic boundary which admits a countable pair of pants decomposition. As an application of the Basmajian identity for complete bordered hyperbolic surfaces of infinite type with limit sets of 1-dimensional measure zero, we define an asymmetric metric (which is called arc metric) on the quasiconformal Teichmüller space $\mathcal{T}(X_0)$ provided that X_0 satisfies a geometric condition. Furthermore, we construct several examples of hyperbolic surfaces of infinite type satisfying the geometric condition and discuss the relation between the Shiga's condition and the geometric condition.

1. Introduction

The Thurston metric was originally defined by Thurston [33] as an asymmetric metric to solve the extremal problem of finding the best Lipschitz map in the homotopy class of homeomorphisms between two hyperbolic surfaces of finite type without boundary. For surfaces of finite type with boundary, a modification of the Thurston metric, the so-called arc metric (see more precisely below) was studied [1, 18, 19, 26].

It's a natural problem whether the Thurston metric is well-defined in Teichmüller spaces of surfaces of infinite type (see [29] for example). In this paper, we define the arc metric, a modification of the Thurston metric, on the quasiconformal Teichmüller space $\mathcal{T}(X_0)$ of a complete hyperbolic surface X_0 of infinite type with geodesic boundary provided that X_0 satisfies the geometric condition $(\star)$ (see the definition below).

Let X_0 be a complete hyperbolic surface of infinite type with geodesic boundary which admits a countable pair of pants decomposition. The completeness for hyperbolic surfaces or hyperbolic structures that we consider throughout this paper means that each boundary component of this surface is a closed geodesic and each puncture of this surface has a neighbourhood which is isometric to a cusp, that is, a surface isometric to the quotient of the region $\{z = x + iy : y > a\}$ of the upper half-plane $\mathbb{H}^2$, for some $a > 0$, by the isometric group generated by $z \to z + 1$.

Denote the boundary of X_0 by ∂X_0 and denote the set of boundary components of X_0 by $\mathcal{B}(X_0) = \{\beta_1, \beta_2, ..., \beta_k, ...\}$. Note that the number of boundary components of X_0 and the number of cusps of X_0 can be countably infinite.

(X, f) is said to be a *marked hyperbolic surface* if X is a complete hyperbolic surface of infinite type and $f : X_0 \to X$ is a quasiconformal mapping which leaves each puncture

and each boundary component setwise fixed. Two marked hyperbolic surfaces (X_1, f_1) and (X_2, f_2) are said to be *equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to an isometry from X_1 to X_2. Denote the equivalence class of (X, f) by $[X, f]$. We denote by $\mathcal{T}(X_0)$ the *reduced quasiconformal Teichmüller space* of X_0 (see [14, 17, 32]), which is the set of equivalence classes of marked hyperbolic surfaces. It deserves to mention that all Teichmüller spaces that we consider here are reduced, which means that homotopies do not necessarily fix ∂X_0 pointwise.

For the sake of simplicity, we shall call $\mathcal{T}(X_0)$ the Teichmüller space of X_0 for short and denote a marked hyperbolic surface (X, f) or its equivalence class $[X, f]$ by X, without explicit reference to the marking.

The Teichmüller space $\mathcal{T}(X_0)$ has a complete distance d_T called *the Teichmüller distance* which is defined by

$$d_T([X_1, f_1], [X_2, f_2]) = \frac{1}{2} \log \inf_{g \simeq f_2 \circ f_1^{-1}} K[g],$$

where the infimum is taken over all quasiconformal mappings $g : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ and $K[g]$ is the maximal dilatation of g.

Recall that a *pair of pants* is a surface whose interior is homeomorphic to a sphere with three disjoint closed disks removed whose boundary is a (possibly empty) disjoint union of circles. A *generalized hyperbolic pair of pants* is a pair of pants equipped with a convex hyperbolic metric in which every topological hole corresponds to either a closed boundary geodesic or a cusp. In particular, a *hyperbolic pair of pants* is a generalized hyperbolic pair of pants with three closed geodesic boundary components.

A *pair of pants decomposition* of a hyperbolic surface X is a system of pairwise disjoint simple closed geodesics $\mathcal{P} = \{C_i\}_{i \in I}$ (for convenience, we ignore the degenerated ones which are homotopic to punctures) such that $X \setminus (\cup_{i \in I} C_i)$ is a disjoint union of the interior of generalized hyperbolic pairs of pants. Moreover, if $\mathcal{P}$ is countably infinite, we say that X admits a *countable pair of pants decomposition*. Note that the hyperbolic surfaces of infinite type in this paper are assumed to admit a countable pair of pants decomposition.

1.1. Related definitions and notations. Let S be a surface with negative Euler characteristic. A simple closed curve on S is said to be *interior* if it is contained in the interior of S. It is said to be *peripheral* if it is homotopic to a puncture. It is said to be *essential* if it is neither peripheral nor isotopic to a point. Let $\mathcal{S}(S)$ denote the set of homotopy classes of essential simple closed curves on S.

If S has non-empty boundary ∂S, we denote the set of boundary components of S by $\mathcal{B}(S)$. An *arc* on S is the image of a compact interval, which is immersed in S, with its interior (possibly with self-intersections) contained in the interior of S and its endpoints lying on ∂S. In particular, a *simple arc* is an arc without self-intersections. An arc is said to be *essential* if it is not isotopic (relative to ∂S) to a subset of ∂S. Note that we do not require the homotopies to fix ∂S pointwise. Denote by $\mathcal{A}(S)$ the set of homotopy classes of essential arcs on S and by $\mathcal{A}'(S)$ the subset of $\mathcal{A}(S)$ consisting of homotopy classes of essential simple arcs on S.

For any $\alpha \in \mathcal{S}(S) \cup \mathcal{A}(S)$ and any hyperbolic structure X on S, we denote by $\ell_\alpha(X)$ the *hyperbolic length* of α on X, that is, the length of the (unique) geodesic representative of α on the hyperbolic surface X.

If S is a surface of finite type without boundary, the *Thurston metric d_{Th}* (see [33]) is an asymmetric metric on the Teichmüller space $\mathcal{T}(S)$ defined by

$$d_{Th}(X, Y) = \log \sup_{\alpha \in S(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

for all $X, Y \in \mathcal{T}(S)$.

If S is a surface of finite type with boundary, the *arc metric d_A* (see [18, 19]) on $\mathcal{T}(S)$, as a modification of the Thurston metric, is defined by

$$d_A(X, Y) = \log \sup_{\alpha \in \mathcal{A}'(S) \cup S(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

for all $X, Y \in \mathcal{T}(S)$. It is essential to consider the union of closed curves and arcs in the definition of d_A for surfaces of finite type with boundary, since there exist two distinct hyperbolic structures X, Y (see [26, 27]) on S such that $\ell_Y(\alpha) < \ell_X(\alpha)$ for all $\alpha \in S(S)$. This implies that

$$\log \sup_{\alpha \in S(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} \leq 0.$$

Moreover, it was shown in [18] that

$$\log \sup_{\alpha \in \mathcal{A}'(S) \cup S(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \log \sup_{\alpha \in \mathcal{A}'(S) \cup B(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}.$$

Therefore, the arc metric d_A can be also defined by the following formula

$$d_A(X, Y) = \log \sup_{\alpha \in \mathcal{A}'(S) \cup B(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)}.$$

Recall that a *Fuchsian group* is a torsion-free discrete group of orientation-preserving isometries on $\mathbb{H}^2$. Let R be a hyperbolic Riemann surface. Denote by Γ_R the *Fuchsian group of R*, which is the Fuchsian group such that R is the quotient of $\mathbb{H}^2$ by Γ_R.

Denote by $\Lambda(\Gamma)$ the *limit set* of a Fuchsian group Γ acting on the upper half-plane $\mathbb{H}^2$, which is a set of points on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ where the orbit by Γ accumulates. Moreover, the complement of $\Lambda(\Gamma)$ in $\widehat{\mathbb{R}}$ is said to be the *set of discontinuity*, which is denoted by $\Omega(\Gamma)$. Γ is said to be of the *first kind* if $\Omega(\Gamma)$ is empty, otherwise it is said to be of the *second kind*. Note that the Fuchsian group of a bordered Riemann surface is of the second kind. The Fuchsian groups we consider in this paper are of the second kind and infinitely generated unless otherwise indicated.

Let $C(\Lambda(\Gamma))$ be the convex hull in $\mathbb{H}^2$ of the limit set $\Lambda(\Gamma)$ and let $\partial C(\Lambda(\Gamma))$ be the boundary of $C(\Lambda(\Gamma))$ in $\mathbb{H}^2$. The *convex core C_R* of a hyperbolic Riemann surface R is the quotient of $C(\Lambda(\Gamma_R))$ by Γ_R, which is the smallest closed convex subregion of R such that its inclusion map induces a homotopy equivalence. For a hyperbolic surface X, we denote by Γ_X the Fuchsian group of the Riemann surface with convex core X.

Definition 1.1. For a Fuchsian group Γ, we say that a disjoint union of regions $A = \cup_{n \in \mathbb{N}} A_n$ in $\mathbb{H}^2$ is *removable for Γ* (see [22]) if it satisfies the following conditions:

(1) Each A_n is a simply connected open set in $\mathbb{H}^2$ which is either a hyperbolic disk, a

horodisk tangent to $\widehat{\mathbb{R}}$ or an r-neighbourhood of a complete geodesic in $\mathbb{H}^2$ for some $r > 0$ (note that the radius r depends on the choice of the complete geodesic and is not necessarily uniformly bounded).

(2) The set A is invariant under the action of Γ.

DEFINITION 1.2. We say that X_0 satisfies *the geometric condition* ($\star$) (see [22]) if there is a positive constant L and a removable set A for Γ_{X_0} such that all points of $C(\Lambda(\Gamma_{X_0})) \setminus A$ lie within a distance L of $\partial C(\Lambda(\Gamma_{X_0}))$.

1.2. Main theorems.

Theorem 4.8. *Let X_0 be a complete hyperbolic surface of infinite type with boundary which satisfies the geometric condition* ($\star$). *Then the following two functions d and $\overline{d}$ on $\mathcal{T}(X_0) \times \mathcal{T}(X_0)$ are asymmetric metrics, where*

$$d(X, Y) = \log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

$$\overline{d}(X, Y) = \log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\alpha(X)}{\ell_\alpha(Y)},$$

for all $X, Y \in \mathcal{T}(X_0)$.

The asymmetric metric d is an analogue, for surfaces of infinite type with boundary, of the arc metrics defined for surfaces of finite type with boundary. We also call d the *arc metric* on $\mathcal{T}(X_0)$.

Theorem 4.11. *Let X_0 be a complete hyperbolic surface of infinite type with boundary, then the following equality still holds for all $X, Y \in \mathcal{T}(X_0)$.*

$$\sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \sup_{\gamma \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

In particular, if X_0 satisfies the geometric condition ($\star$), *then the following equality defines the same asymmetric metric.*

$$\log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \log \sup_{\gamma \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

1.3. Outline of the paper. In Section 2 we give the Basmajian identity and the generalized McShane identity for complete bordered hyperbolic surfaces of infinite type with limit sets of 1-dimensional measure zero. In Section 3, we consider the geometric condition ($\star$) and discuss its properties. In Section 4, we define an asymmetric metric on $\mathcal{T}(X_0)$ and give the proofs of Theorem 4.8 and Theorem 4.11. In the last section, we construct several examples of hyperbolic surfaces of infinite type satisfying the geometric condition ($\star$) and discuss the relation between the Shiga's condition and the geometric condition ($\star$).

2. Basmajian identity and generalized McShane identity for complete bordered hyperbolic surfaces of infinite type

In this section we present the Basmajian identity and the generalized McShane identity for a complete bordered hyperbolic surface X of infinite type with the limit set $\Lambda(\Gamma_X)$ of

1-dimensional measure zero. The Basmajian identity is a direct result of the orthogonal spectrum theorem given by Basmajian [3] if the limit set of the Fuchsian group Γ_X has 1-dimensional measure zero. We sketch the proof of the generalized McShane identity and refer to [8] for details.

2.1. Basmajian identity for complete bordered hyperbolic surfaces of infinite type. For the convenience of the exposition of the orthogonal spectrum theorem given by Basmajian, we introduce the related notations and terminology (see [3]).

Let M^n be an orientable hyperbolic manifold of dimension $n \geq 2$. A *hypersurface S* in M^n is a codimension one complete submanifold endowed with the induced metric. S is said to be *totally geodesic* if every geodesic on S is a geodesic in M^n.

Let S_1 be a totally geodesic hypersurface which is either disjoint from S or equal to S. Two paths from S to S_1 are said to be *freely homotopic relative to S and S_1* if there is a homotopy in M^n between them which keeps the initial point in S and the terminal point in S_1. The equivalence class of a path α is called the *relative free homotopy class of α* and it is said to be *trivial* if $S = S_1$ and α is homotopic to a single point in S.

Hypersurfaces S and S_1 are called *asymptotic* if there exists a path from S to S_1 such that its relative free homotopy class is nontrivial and contains paths of arbitrary short length. In this case, the length of the homotopy class is defined to be zero. If S and S_1 are not asymptotic, then each nontrivial relative free homotopy class of a path α from S to S_1 contains a shortest path which is the unique common orthogonal in the class $[\alpha]$. The length of this homotpy class $[\alpha]$ is defined to be the length of the common orthogonal in $[\alpha]$.

Let C be a (possibly infinite) set of mutually disjoint embedded totally geodesic hypersurfaces in M^n. For each non-negative integer k, the *k-th orthogonal spectrum of M^n related to S and C* is denoted by $\mathcal{O}_k(M^n; S, C)$, which is the ordered nondecreasing sequence of lengths of nontrivial relative free homotopy classes of paths which start in S and go in the direction of the normal to S, cross C along the way k times, and end in a hypersurface contained in C perpendicularly. Note that the direction of the normal to S here is chosen appropriately on one side, such that the lifts starting from $\tilde{S}$ of those paths lie to the same side of $\tilde{S}$ for a fixed connected component $\tilde{S}$ of a lift of S.

Denote by m_h the hyperbolic measure on S inherited from the volume element on M^n, and by $V_n(r)$ the hyperbolic volume of the n-dimensional ball of radius r.

Theorem 2.1. (Basmajian [3], The Orthogonal Spectrum Theorem) *Let C be a disjoint set of embedded totally geodesic hypersurfaces in the hyperbolic manifold M^n and let S be an embedded oriented hypersurface which is totally geodesic. Suppose further that S is either disjoint from C or one of the hypersurfaces in C, and that no nontrivial relative free homotopy class from S to C has length zero. Then the k-th orthogonal spectrum,*

$$\mathcal{O}_k(M^n; S, C) = \{d_i\},$$

satisfies:

$$(1) \qquad Vol_{n-1}(S) = m_h(F_k) + \sum_{i=1}^{\infty} V_{n-1}(r(d_i)),$$

where F_k is the subset of S consisting of all points whose corresponding oriented normal ray to S intersects C at most k times, and $r(x) = \log \coth(\frac{x}{2})$.

Applying Theorem 2.1 and the method introduced by Basmajian for the proof of Corollary 1.2 in [3], we have the following proposition.

Proposition 2.2. *Let X be a complete bordered hyperbolic surface of infinite type with the limit set $\Lambda(\Gamma_X)$ of 1-dimensional measure zero. Then for any $\beta_j \in \mathcal{B}(X)$, we have*

$$(2) \qquad \ell_{\beta_j}(X) = \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(X)}{2}),$$

where $\{d_i^j(X)\}_{i=1}^{\infty}$ denotes the 0-th orthogonal spectrum $\mathcal{O}_0(X; \beta_j, \mathcal{B}(X))$ of X related to β_j and $\mathcal{B}(X)$.

Proof. As in Theorem 2.1, we let $M^n = X$, $C = \mathcal{B}(X) = \{\beta_1, \beta_2, ..., \beta_k, ...\}$, $S = \beta_j$. Consider the orthogonal spectrum $\mathcal{O}_0(X; \beta_j, \mathcal{B}(X)) = \{d_i^j(X)\}_{i=1}^{\infty}$ and it follows from Theorem 2.1 that

$$\ell_{\beta_j}(X) = m_h(F_0^j) + \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(X)}{2}),$$

where F_0^j is the subset of β_j consisting of the points from which the oriented geodesics starting perpendicularly never hit ∂X.

Denote by $\mathcal{G}_j$ the set of all the complete geodesics which start perpendicularly from β_j and never hit ∂X. It is not hard to see that for any geodesic $g \in \mathcal{G}_j$, the endpoint at infinity of a lift of g must lie on the limit set of the Fuchsian group Γ_X. Fix a connected component $\widetilde{\beta_j}$ of a lift of β_j and denote by V_j the set of the endpoints at infinity of the lifts starting from $\widetilde{\beta_j}$ of all the geodesics in $\mathcal{G}_j$. It is clear that $V_j \subset \Lambda(\Gamma_X)$.

Observe that the endpoints of $\widetilde{\beta_j}$ divides the circle at infinity S_{∞}^1 into two disjoint open components. We endow S_{∞}^1 with 1-dimensional Lebesgue measure and let $\mathcal{R}^+$ be the open component for which the normal to $\widetilde{\beta_j}$ points. Consider the map $p_j : \mathcal{R}^+ \to \beta_j$ given by orthogonal projection to $\widetilde{\beta_j}$ followed by the covering map into the quotient surface X. Then F_0^j is exactly $p_j(V_j)$. By the assumption that the limit set $\Lambda(\Gamma_X)$ has 1-dimensional measure zero and by the fact that p_j preserves sets of measure zero (see Proposition 3.3 in [3]), we derive that $m_h(F_0^j) = m_h(p_j(V_j)) = 0$. Hence,

$$\ell_{\beta_j}(X) = \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(X)}{2}).$$

$\square$

2.2. Generalized McShane identity for complete bordered hyperbolic surfaces of infinite type. The generalized McShane identity for bordered hyperbolic surfaces of finite type is given by Mirzakhani [23]. To generalize it to the case of a complete hyperbolic surface X of infinite type with boundary, we apply the method given by Bridgeman and Tan [8]. The way is to consider the boundary flows on the surface X.

Indeed, let $T_1(X)$ be the unit tangent bundle of X and $\pi : T_1(X) \to X$ be the projective map. Fix a boundary component β_1 of X and denote by W the subset of $T_1(X)$ consisting of the vectors with basepoints on β_1 which are perpendicular to β_1 and point to the interior of X. It is obvious that π is a bijection from W to β_1. We identify W with β_1 under π and define

the measure μ on W to be the pull back of the 1-dimensional Lebesgue measure on β_1 under π. In particular, $\mu(W) = \ell_{\beta_1}(X)$. Then we consider the geodesic g_v starting at $p = \pi(v) \in \beta_1$ obtained by exponentiating v, where g_v is assumed to stop when it hits itself or the boundary ∂X.

Let $Z \subset W$ be the set of vectors in which g_v starts has infinite length. It is not hard to see that for every $v \in W \backslash Z$, g_v is a geodesic arc contained in a unique generalized hyperbolic pair of pants embedded in X bounded by β_1 and a pair of simple closed curves γ_1 and γ_2 (where either γ_1, γ_2 are both interior simple closed geodesics, or exactly one of them is an interior simple closed geodesic while the other is a geodesic boundary component or a cusp distinct from β_1). Denote by $\mathcal{P}$ the set of all such pairs of pants embedded in X. For each $P \in \mathcal{P}$, let $X_P = \{v \in W \backslash Z : g_v \subset P\}$, then $W = Z \cup (\cup_{P \in \mathcal{P}} X_P)$. Hence, $\ell_{\beta_1}(X) = \sum\limits_{P \in \mathcal{P}} \mu(X_P) + \mu(Z)$.

If γ_1 and γ_2 are both interior simple closed geodesics, it can be computed by elementary hyperbolic geometry that $\mu(X_P) = \mathcal{D}(\ell_{\beta_1}(X), \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))$. Otherwise, assume that γ_1 is a geodesic boundary component (may be a cusp) and γ_2 is an interior simple closed geodesic. It can be computed that $\mu(X_P) = \mathcal{R}(\ell_{\beta_1}(X), \ell_{\gamma_1}(X), \ell_{\gamma_2}(X))$. Here the functions $\mathcal{D}$ and $\mathcal{R}$ are respectively defined by

$$\mathcal{D}(x_1, x_2, x_3) = 2 \log \left(\frac{e^{\frac{x_1}{2}} + e^{\frac{x_2+x_3}{2}}}{e^{\frac{-x_1}{2}} + e^{\frac{x_2+x_3}{2}}} \right),$$

$$\mathcal{R}(x_1, x_2, x_3) = x_1 - \log \left(\frac{\cosh \frac{x_2}{2} + \cosh \frac{x_1+x_3}{2}}{\cosh \frac{x_2}{2} + \cosh \frac{x_1-x_3}{2}} \right).$$

The difficulty is how to ensure that $\mu(Z) = 0$. However, if the limit set of the Fuchsian group Γ_X has 1-dimensional measure zero, it is true that $\mu(Z) = 0$. This proof is similar to the proof for $m_h(F_0^j) = 0$ in Proposition 2.2.

Therefore, the generalized McShance identity still holds for X if Γ_X has 1-dimensional measure zero. Then we have the following proposition.

Proposition 2.3. *Let X be a complete bordered hyperbolic surface of infinite type with the limit set $\Lambda(\Gamma_X)$ of 1-dimensional measure zero. Let β_1 be a boundary component of X with $\ell_{\beta_1}(X) > 0$. Then we have*

$$(3) \qquad \sum_{\{\gamma_1, \gamma_2\} \in \mathcal{F}_1} \mathcal{D}(L_1, \ell_{\gamma_1}, \ell_{\gamma_2}) + \sum_{i=2}^{\infty} \sum_{\gamma \in \mathcal{F}_{1,i}} \mathcal{R}(L_1, L_i, \ell_{\gamma}) = L_1.$$

Here $L_i = \ell_{\beta_i}(X), \ell_{\gamma_i} = \ell_{\gamma_i}(X)$ and $\mathcal{B}(X) = \{\beta_1, \beta_2, ..., \beta_k, ...\}$. In particular, we include the cusps as geodesic boundary components of length zero in $\mathcal{B}(X)$. $\mathcal{F}_1$ denotes the set of all the unordered pairs of isotopy classes of interior simple closed curves which bound a pair of pants with β_1. $\mathcal{F}_{1,i}$ denotes the set of all the isotopy classes of interior simple closed curves which bound a pair of pants with β_1 and β_i.

Recall that a class $\mathcal{O}$ of Fuchsian groups is *quasiconformally invariant* [21] if it satisfies that for any Fuchsian group $\Gamma \in \mathcal{O}$, if there is a quasiconformal homeomorphism f of $\mathbb{H}^2$ such that $\Gamma' = f\Gamma f^{-1}$ is Fuchsian, then Γ' belongs to $\mathcal{O}$.

REMARK 2.4. It was remarked in [21] by Matsuzaki that the class of Fuchsian groups whose limit set has vanishing 1-dimensional measure is not quasiconformally invariant (see

Example 2 in [30] and Theorem 3 in [7]). Thus it's possible that the Basmajian identity and the generalized McShane identity fail to hold for the Teichmüller space of a hyperbolic surface of infinite type with the limit set of 1-dimensional measure zero. To overcome this difficulty, we consider the geometric condition (⋆) in the next section.

3. A geometric condition

In this section, we aim to show that the Basmajian identity and the generalized McShane identity hold for $\mathcal{T}(X_0)$ provided that X_0 satisfies the geometric condition (⋆). The key is to show the 1-dimensional measure of the limit set $\Lambda(\Gamma_X)$ of each $X \in \mathcal{T}(X_0)$ is zero.

First we discuss some properties of the geometric condition (⋆). To state and verify the related results, we fix some terminology and notations first.

We say that a map $f : (X_1, d_1) \to (X_2, d_2)$ between two metric spaces is *bi-Lipschitz* if there exists a real number $L \geq 1$ satisfying

$$\frac{1}{L} d_1(x, y) \leq d_2(x, y) \leq L d_1(x, y)$$

for any $x, y \in X_1$. The real number L is called a *bi-Lipschitz constant of f*. Two metric spaces are said to be *bi-Lipschitz equivalent* if there exists a bi-Lipschitz homeomorphism between them.

Let X_0 be a complete hyperbolic surface of infinite type with geodesic boundary. We denote by $\mathcal{T}_{bL}(X_0)$ the *bi-Lipschitz Teichmüller space* of X_0, which is the set of equivalence classes of pairs (X, f), where X is a complete hyperbolic surface of infinite type and $f : X_0 \to X$ is a bi-Lipschitz homeomorphism with respect to the hyperbolic metrics which leaves each puncture and each boundary component setwise fixed. Here two pairs (X_1, f_1) and (X_2, f_2) are said to be *equivalent* if $f_2 \circ f_1^{-1}$ is homotopic to an isometry from X_1 to X_2. Denote the equivalence class of (X, f) by $[X, f]$. It deserves to mention that the homotopies do not necessarily fix ∂X_0 pointwise.

In $\mathcal{T}_{bL}(X_0)$, we consider the *bi-Lipschitz metric* d_{bL} (see [17]) which is defined by

$$d_{bL}([X_1, f_1], [X_2, f_2]) = \frac{1}{2} \log \inf_{g \simeq f_2 \circ f_1^{-1}} L(g),$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $g : X_1 \to X_2$ homotopic to $f_2 \circ f_1^{-1}$ and $L(g)$ is the bi-Lipschitz constant of g.

Theorem 3.1. (Matsuzaki [22]) *Let Γ be a Fuchsian group acting on the upper half-plane $\mathbb{H}^2$. If there is a positive constant L and a removable set A for Γ such that all points of $C(\Lambda(\Gamma)) \setminus A$ lie within a distance L of $\partial C(\Lambda(\Gamma))$, then there is a constant $\alpha \in (0, 1)$ depending only on L such that the Hausdorff dimension of the limit set of Γ satisfies $\dim \Lambda(\Gamma) \leq \alpha < 1$.*

REMARK 3.2. The condition in Theorem 3.1 is exactly the geometric condition (⋆) in Definition 1.1. In the estimate of the Hausdorff dimension $\dim \Lambda(\Gamma)$ in Theorem 3.1 (see [22, Theorem 1]), the author aimed to show that only the depth of the convex core $C(\Lambda(\Gamma))/\Gamma$ without the removable set is important. For a removable set A for Γ, the components as horodisks and neighbourhoods of complete geodesics in $\mathbb{H}^2$ are used to deal with the thin parts of $C(\Lambda(\Gamma))/\Gamma$, while the components as hyperbolic disks are used to deal with the thick parts of $C(\Lambda(\Gamma))/\Gamma$. We give the corresponding examples in Section 5, see Example 5.10,

Example 5.11 and Example 5.12 respectively.

Definition 3.3. For a Fuchsian group Γ, we say that a disjoint union of regions $A = \cup_{n \in \mathbb{N}} A_n$ in $\mathbb{H}^2$ is *weakly removable for* Γ (see [22]) if it satisfies the following conditions:

(1) Each A_n is an open set in $\mathbb{H}^2$ whose euclidean closure intersects $\widehat{\mathbb{R}}$ with the set of 1-dimensional measure zero.

(2) The set A is invariant under the action of Γ.

Definition 3.4. We say that X_0 satisfies *the weak geometric condition* $(\diamond)$ if there is a positive constant L and a weakly removable set A for Γ_{X_0} such that all points of $C(\Lambda(\Gamma_{X_0})) \setminus A$ lie within a distance L of $\partial C(\Lambda(\Gamma_{X_0}))$.

Remark 3.5. It was proved in [22, Theorem 5] that if X_0 satisfies the weak geometric condition $(\diamond)$, then the 1-dimensional measure of $\Lambda(\Gamma_{X_0})$ is zero. In other words, if the conclusion in Theorem 3.1 is weaken to be that the 1-dimensional measure of $\Lambda(\Gamma)$ is zero, it suffices to consider a weakly removable set for Γ instead of a removable set for Γ.

Theorem 3.6. (Matsuzaki [22]) *Let N_Γ be a hyperbolic surface of infinite topological type and let $\{c_n\}_{n=1,2,\ldots}$ be the components of the boundary of the convex core $\partial C_\Gamma \subset N_\Gamma$. If the hyperbolic lengths $\ell(c_n)$ satisfy*

$$\sum_n \ell(c_n)^{\frac{1}{2}} < \infty,$$

then the Hausdorff dimension of the limit set of Γ is equal to 1.

Theorem 3.7. (Liu, Papadopoulos [17]) *For every complete hyperbolic surface X_0 of infinite type, we have the set-theoretic equality*

$$\mathcal{T}(X_0) = \mathcal{T}_{bL}(X_0),$$

and there exists a constant C such that for every X and Y in $\mathcal{T}(X_0)$, we have

(4) $$d_T(X,Y) \leq d_{bL}(X,Y) \leq Cd_T(X,Y).$$

It is an alternative statement of Theorem 4.3 in [17]. The idea was originally introduced by Thurston (see [32] p. 268).

Lemma 3.8. *Let X_0 be a complete hyperbolic surface of infinite type with boundary. Let $X \in \mathcal{T}(X_0)$ and let $A = \cup_{n \in \mathbb{N}} A_n$ be a removable set for Γ_X. Then for any subsurface Σ of X which contains an essential self-intersecting closed curve that is not γ^n for any simple closed curve γ and $n \in \mathbb{Z}$, the projection $\pi(A)$ on X of A under Γ_X fails to cover Σ.*

Proof. Let Σ be such a subsurface of X and let α be such an essential self-intersecting closed curve on Σ. Assume that $\pi(A)$ covers Σ, then $\pi(A)$ covers α. This implies that A contains a connected component of a lift of α in $\mathbb{H}^2$, called $\tilde{\alpha}$. Note that α is self-intersecting and cannot be written as γ^n for any simple closed curve γ and $n \in \mathbb{Z}$, then $\tilde{\alpha}$ intersects the boundary at infinity $\widehat{\mathbb{R}}$ with at least four points. Moreover, since A_i and A_j are disjoint for all $i \neq j$, then A has a component A_n which contains $\tilde{\alpha}$ and intersects $\widehat{\mathbb{R}}$ with at least four points. By Definition 1.1, each component of a removable set intersects $\widehat{\mathbb{R}}$ with at most two points. This contradiction proves that A fails to cover Σ. $\square$

Lemma 3.9. *Let X_0 be a complete hyperbolic surface of infinite type with boundary. For any $X = [X, f_1]$, $Y = [Y, f_2]$ in $\mathcal{T}(X_0)$, let $f = f_2 \circ f_1^{-1}$ and let $K = K[f]$ be the maximal dilatation of f, then*

$$\frac{1}{K} \leq \frac{\ell_{f(\alpha)}(Y)}{\ell_\alpha(X)} \leq K,$$

for all $\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)$.

Proof. We recall a result of Wolpert (see [34]), which says that given any K'-quasiconformal map h between two hyperbolic surfaces X' and Y' without boundary, we have

$$\frac{1}{K'} \leq \frac{\ell_{h(\alpha)}(Y')}{\ell_\alpha(X')} \leq K',$$

for all isotopy classes of essential closed curves α on X'. Note that this result also holds for isotopy classes of essential closed curves and essential arcs on hyperbolic surfaces with boundary, by applying an argument of doubling (see e.g. Theorem 2.1 in [19]). Therefore, for any $X = [X, f_1]$, $Y = [Y, f_2]$ in $\mathcal{T}(X_0)$, since $f = f_2 \circ f_1^{-1} : X \to Y$ is a K-quasiconformal map, we have

$$\frac{1}{K} \leq \frac{\ell_{f(\alpha)}(Y)}{\ell_\alpha(X)} \leq K,$$

for all $\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)$. $\qquad\square$

Now we give some properties of the geometric condition $(\star)$ as follows.

Proposition 3.10. *Let X_0 be a complete hyperbolic surface of infinite type with boundary which satisfies the geometric condition $(\star)$. Then for any $X \in \mathcal{T}(X_0)$, the following statements hold:*

(1) The number of all boundary components of X is countably infinite.
(2) X satisfies the weak geometric condition $(\diamond)$.
(3) The limit set of the Fuchsian group Γ_X has 1-dimensional measure zero.
(4) The sum of the lengths of all boundary components of X is infinite.

Proof. *The proof of (1)*. Recall that the hyperbolic surfaces of infinite type in this paper admit a countable pair of pants decomposition. This implies that if X has infinitely many boundary components, then the number of its boundary components is countably infinite. Note that $X = [X, f]$ for a quasiconformal map $f : X_0 \to X$ which leaves each puncture and each boundary component setwise fixed.

It suffices to prove that the number of all boundary components of X_0 is infinite. Note that X_0 is complete and thus each boundary component of X_0 is a simple closed geodesic, which implies that X_0 has no boundary component of infinite length.

We argue by contradiction. Assume that X_0 has finitely many geodesic boundary components and denote them by $\beta_1, \beta_2, ..., \beta_n$. Then

$$\sum_{i=1}^n \ell_{\beta_i}(X_0)^{\frac{1}{2}} < \infty.$$

By Theorem 3.6, the Hausdorff dimension of the limit set $\Lambda(\Gamma_{X_0})$ is 1. However, note that X_0 satisfies the geometric condition ($\star$) and by Theorem 3.1, the Hausdorff dimension of the limit set $\Lambda(\Gamma_{X_0})$ is less than 1. This leads to contradiction.

The proof of (2). By Theorem 3.7, there exists a constant C such that for every $[X, f] \in \mathcal{T}(X_0)$, we have $d_{bL}([X_0, id], [X, f]) \leq Cd_T([X_0, id], [X, f])$. By the definition of d_{bL}, there exists a bi-Lipschitz homeomorphism $g : X_0 \to X$ homotopic to f with the bi-Lipschitz constant $L(g) \leq e^{2Cd_T(X_0, X)} + \epsilon_0$ where ϵ_0 is a sufficiently small positive number. Let $M = e^{2Cd_T(X_0, X)} + \epsilon_0$. We obtain that

$$(5) \qquad \frac{1}{M}\rho_{X_0}(x, y) \leq \rho_X(g(x), g(y)) \leq M\rho_{X_0}(x, y),$$

for any two points x, y on X_0, where ρ_{X_0} (resp. ρ_X) denotes the hyperbolic distance on X_0 (resp. X) induced by the hyperbolic structure of X_0 (resp. X).

Since X_0 satisfies the geometric condition ($\star$), then there exists a positive constant L and a removable set $A = \cup_{n \in \mathbb{N}} A_n \subset \mathbb{H}^2$ for Γ_{X_0} such that

$$\rho_{X_0}(x, \partial X_0) \leq L,$$

for any point x on X_0 except the image of the removable set $A \subset \mathbb{H}^2$ under the universal covering map π_0 of X_0, where $\rho_{X_0}(x, \partial X_0) = \inf_{y \in \partial X_0} \rho_{X_0}(x, y)$.

It follows directly from (5) that

$$(6) \qquad \rho_X(p, \partial X) \leq ML,$$

for any point p on X except the set $g(\pi_0(A)) \subset X$.

Let $\tilde{g}$ be a lift of the map g to the universal covering space of X. Set $A' = \tilde{g}(A)$. First we claim that $A' = \cup_{n \in \mathbb{N}} \tilde{g}(A_n)$ is a weakly removable set for Γ_X. Indeed, A' is a disjoint union of open sets in $\mathbb{H}^2$, since $\tilde{g}$ is a homeomorphism and A is a disjoint union of open sets in $\mathbb{H}^2$. Note that A is Γ_{X_0}-invariant, and $\tilde{g}$ is equivariant with respect to Γ_{X_0} and Γ_X, then A' is invariant under the action of Γ_X. By Definition 1.1, for each $n \in \mathbb{N}$, A_n is possibly a hyperbolic disk, a horodisk tangent to $\widehat{\mathbb{R}}$, or an r-neighbourhood of a complete geodesic in $\mathbb{H}^2$ for some $r > 0$.

Note that $\tilde{g} : \mathbb{H}^2 \to \mathbb{H}^2$ is a bi-Lipschitz homeomorphism with respect to the hyperbolic metrics, then it induces a homeomorphism from $\widehat{\mathbb{R}}$ to $\widehat{\mathbb{R}}$. If A_n is a hyperbolic disk, then $\tilde{g}(A_n)$ is a topological disk in $\mathbb{H}^2$ whose euclidean closure does not intersect $\widehat{\mathbb{R}}$. If A_n is a horodisk tangent to $\widehat{\mathbb{R}}$ at $\xi \in \widehat{\mathbb{R}}$, then the euclidean clousre of $\tilde{g}(A_n)$ intersects $\widehat{\mathbb{R}}$ exactly at $\tilde{g}(\xi) \in \widehat{\mathbb{R}}$. If A_n is a neighbourhood of a complete geodesic with two distinct endpoints $\xi_1, \xi_2 \in \widehat{\mathbb{R}}$, then the euclidean clousre of $\tilde{g}(A_n)$ intersects $\widehat{\mathbb{R}}$ exactly at two distinct points $\tilde{g}(\xi_1), \tilde{g}(\xi_2) \in \widehat{\mathbb{R}}$. Therefore, the euclidean closure of each $\tilde{g}(A_n)$ intersects $\widehat{\mathbb{R}}$ with the set of 1-dimensional measure zero. By Definition 3.3, A' is a weakly removable set for Γ_X.

By (6) and the fact that $g(\pi_0(A)) = \pi(A')$, there exists a constant $L' = ML > 0$ and a removable set $A' = \tilde{g}(A)$ for Γ_X such that

$$\rho_X(p, \partial X) \leq L',$$

for all $p \in X \setminus \pi(A')$. This implies that X satisfies the weak geometric condition ($\diamond$).

The proof of (3). By Statement (2) and Remark 3.5, the 1-dimensional measure of the limit set $\Lambda(\Gamma_X)$ is zero.

The proof of (4). First we prove that it suffices to show this statement for the special case $X = X_0$. Indeed, by Lemma 3.9, for any $X = [X, f] \in \mathcal{T}(X_0)$, let K be the maximal dilatation of f, we have

$$(7) \qquad \frac{1}{K} \leq \frac{\ell_{f(\alpha)}(X)}{\ell_\alpha(X_0)} \leq K,$$

for all $\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)$.

By Statement (1), X_0 has infinitely many boundary components and denote the set of boundary components of X_0 by $\mathcal{B}(X_0) = \{\beta_1, \beta_2, ..., \beta_k, ...\}$. By (7), we have $\sum_{i=1}^{\infty} \ell_{\beta_i}(X) < \infty$ if and only if $\sum_{i=1}^{\infty} \ell_{\beta_i}(X_0) < \infty$. Hence, we only need to consider $X = X_0$.

Denote $b_i = \ell_{\beta_i}(X_0)$. We argue by contradiction. Suppose

$$\sum_{i=1}^{\infty} b_i < \infty.$$

Then $b_i \to 0$, as $i \to \infty$. By the collar lemma (see [9]), there exists a collar neighbourhood $\mathcal{N}(\beta_i) = \{p \in X_0 : \rho_{X_0}(p, \beta_i) \leq r(b_i)\}$ of β_i such that $\mathcal{N}(\beta_i)$ does not intersect any other simple closed geodesics disjoint from β_i, where ρ_{X_0} denotes the hyperbolic distance on X_0 and $r(b_i) = \text{arcsinh}\{1/\sinh(\frac{1}{2}b_i)\}$.

Note that $r(b_i) \to \infty$, as $i \to \infty$. For any $L > 0$ and any removable set A for Γ_{X_0}, there exists an integer $n_0 > 0$ (depending on L) such that $r(b_i) > L$ for all $i \geq n_0$. Denote by $B(\partial X_0; L)$ the set consisting of the points on X_0 lying within the distance L of ∂X_0. Let $\Omega = X_0 \setminus B(\partial X_0; L)$.

Note that Ω has a connected component which contains at least two distinct isotopy classes of simple closed curves, then it contains an essential self-intersecting curve α which is not γ^n for any simple closed curve γ and $n \in \mathbb{Z}$. By Lemma 3.8, $\pi_0(A)$ fails to cover Ω, where π_0 is the universal covering map of X_0. This contradicts the assumption that X_0 satisfies the geometric condition $(\star)$.   □

REMARK 3.11. The assumption that X_0 is complete is necessary for Statement (1) of Proposition 3.10. Otherwise, there exists a hyperbolic surface of infinite type called *tight flute surface* by Basmajian (see [4, 5]) satisfying the geometric condition $(\star)$ but has only one geodesic boundary component which is a simple open infinite geodesic (see Example 5.7).

Combining Proposition 2.2, Proposition 2.3 and Statement (3) of Proposition 3.10, we have the following corollary.

Corollary 3.12. *Let X_0 be a complete hyperbolic surface of infinite type with boundary which satisfies the geometric condition $(\star)$. Then both the Basmajian identity and the generalized McShane identity hold for $\mathcal{T}(X_0)$.*

QUESTION 3.13. Is the geometric condition $(\star)$ quasiconformally invariant? That is, if X_0 satisfies the geometric condition $(\star)$, then for any $X \in \mathcal{T}(X_0)$, does X also satisfy the geometric condition $(\star)$? We can also ask the same question for the weak geometric condition $(\diamond)$.

The construction of examples of hyperbolic surfaces of infinite type which satisfy the geometric condition $(\star)$ will be given in Section 5.

4. An asymmetric metric on $\mathcal{T}(X_0)$

DEFINITION 4.1. An *asymmetric metric* on a set M is a function $\delta : M \times M \rightarrow [0, +\infty)$ satisfying the following conditions.

(a) The separation axiom: for any $x, y \in M$, $\delta(x, y) = 0$ if and only if $x = y$.
(b) The triangle inequality: $\delta(x, y) \leq \delta(x, z) + \delta(z, y)$, for all $x, y, z \in M$.
(c) The asymmetric condition: there exists $x, y \in M$, such that $\delta(x, y) \neq \delta(y, x)$.

The pair (M, δ) defined as above is said to be an *asymmetric metric space* (see [25, 33]). In particular, a function $f : M \times M \rightarrow [0, +\infty]$ is said to be *positive definite* if it satisfies the separation axiom (a).

For a Nielsen convex hyperbolic surface X (equivalently, X can be constructed by gluing some generalized hyperbolic pairs of pants along their boundary components), the *Fenchel-Nielsen coordinates* of X associated with a pair of pants decomposition $\mathcal{P} = \{C_i\}_{i=1}^{\infty}$ (see [2]) is defined to be $\{\ell_{C_i}(X), t_{C_i}(X)\}_{i=1}^{\infty}$ consisting of the hyperbolic lengths with respect to X of all the simple closed curves in $\mathcal{P}$ and the twisting parameters used to glue the pairs of pants, where the positive direction of twisting means turning left. It is understood that if α_i is peripheral, then there is no associated twisting parameter, and instead of a pair $(\ell_{C_i}(X), t_{C_i}(X))$, we take a single parameter $\ell_{C_i}(X)$.

Now we recall some elementary knowledge about measured laminations (see [1, 31]) for the completeness of exposition.

A *geodesic lamination* λ on a hyperbolic surface X is a closed subset of X that is the disjoint union of simple complete geodesics (note that the geodesic with one end or both ends transversely hitting the boundary ∂X is also considered to be complete) called the *leaves* of λ. By the definition, a leaf L of λ on $X \in \mathcal{T}(X_0)$ may be a geodesic boundary component of X, a geodesic ending at a cusp or a boundary component of X (L may transversely hit a boundary component or spiral around it), or even a geodesic with one or both of its ends never stay in any compact subset of X if X is a surface of infinite type. Note that if L is a geodesic that hits ∂X at a point $p \in \partial X$, we require that L is perpendicular to ∂X at p.

Let λ be a geodesic lamination on X. A *transverse measure* for λ is an assignment of a finite positive Borel measure μ on each embedded arc k on X (transverse to λ and with endpoints contained in the complement of λ), such that μ satisfies the following conditions:

(1) The support of μ is $\lambda \cap k$.

(2) μ is invariant under homotopies relative to the leaves of λ, that is, $\mu(k) = \mu(k')$ for any two transverse arcs k and k' that are homotopic through embedded arcs which move their endpoints within fixed complementary components of μ.

A *measured geodesic lamination* is a pair (λ, μ), where λ is a geodesic lamination and μ is a transverse measure. For simplicity, we call a "measured lamination" instead of a "measured geodesic lamination" and sometimes denote (λ, μ) by μ. Denote by $\mathcal{ML}(X)$ the space of all measured laminations on X and denote by $\mathcal{PML}(X)$ the space of projective classes of measured laminations on X.

Let X_0 be a complete hyperbolic surface of infinite type with boundary. Let $\{\mu_n\}_{n=0}^{\infty}$ be a

sequence of measured laminations in $\mathcal{ML}(X_0)$. We say that μ_n *converges* to μ_0 in $\mathcal{ML}(X_0)$ if $i(\mu_n, \alpha) \to i(\mu_0, \alpha)$ for all $\alpha \in \mathcal{A}'(X_0) \cup \mathcal{S}(X_0)$, where $i(\mu_n, \alpha) = \inf_{\alpha' \sim \alpha} \int_{\alpha'} d\mu_n$.

Let $\mathcal{S}^{int}(X_0) \subset \mathcal{S}(X_0)$ be the set of homotopy classes of essential interior simple closed curves on X_0. Denote by $\mathcal{ML}^{\mathcal{A}'}(X_0)$ the closure of $\mathbb{R}^+ \times \mathcal{A}'(X_0)$ in $\mathcal{ML}(X_0)$. That is, for any $\mu \in \mathcal{ML}^{\mathcal{A}'}(X_0)$, there exists a sequence $\{\gamma_n\}_{n=1}^{\infty}$ in $\mathcal{A}'(X_0)$ with a corresponding sequence of positive weights $\{t_n\}_{n=1}^{\infty}$, such that $\{t_n\gamma_n\}_{n=1}^{\infty}$ converges to μ in $\mathcal{ML}(X_0)$. The *hyperbolic length of* μ is defined to be $\mathcal{L}_\mu(X_0) = \lim_{n\to\infty} t_n\ell_{\gamma_n}(X_0)$ (see [24,32] for more details). In particular, $\mathcal{L}_\mu(X_0) = \ell_\mu(X_0)$ for all $\mu \in \mathcal{A}'(X_0) \cup \mathcal{S}(X_0)$. It is known that $\mathcal{L}_\mu(X_0)$ is independent of the choice of the sequence which converges to it. Therefore,

$$\log \sup_{\gamma \in \mathcal{A}'(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \log \sup_{\mu \in \mathcal{ML}^{\mathcal{A}'}(X_0)} \frac{\mathcal{L}_\mu(Y)}{\mathcal{L}_\mu(X)}.$$

In this section, we consider the following two functions on $\mathcal{T}(X_0) \times \mathcal{T}(X_0)$:

$$d(X, Y) = \log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

$$\overline{d}(X, Y) = \log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{S}(X_0)} \frac{\ell_\alpha(X)}{\ell_\alpha(Y)},$$

for all $X, Y \in \mathcal{T}(X_0)$.

Lemma 4.2. (Thurston [33], Proposition 3.5) *For any two complete hyperbolic structures* X, Y *on a surface* S *of finite type without boundary, we have*

$$\sup_{\alpha \in \mathcal{S}(S)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \sup_{\alpha \in \pi_1(S) - \{0\}} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

where $\pi_1(S) - \{0\}$ *is the set of homotopy classes of essential closed curves.*

Indeed, the proof presented by Thurston [33] is independent of the topological types of hyperbolic surfaces, thus this equality holds for hyperbolic surfaces of all topological types.

Lemma 4.3. (Proposition 2.8 in [18]) *For any two complete hyperbolic structures* X, Y *on a surface* S *of finite type with boundary, we have*

$$\sup_{\gamma \in \mathcal{A}'(S) \cup \mathcal{S}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\gamma \in \mathcal{S}(S^d)} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)},$$

where S^d *denotes the double of* S *which carries a canonical involution such that the set of fixed points is* ∂S. X^d, Y^d *are respectively the doubled structure of* X, Y *on* S^d.

Proposition 4.4. *For any two complete hyperbolic structures* X, Y *on a surface* S *of finite type with boundary, we have*

$$(8) \qquad \sup_{\gamma \in \mathcal{A}(S) \cup \mathcal{S}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\gamma \in \mathcal{A}'(S) \cup \mathcal{S}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

Proof. Observe that $\mathcal{A}'(S) \subset \mathcal{A}(S)$. It suffices to verify that for any non-simple essential arc $\gamma_0 \in \mathcal{A}(S) - \mathcal{A}'(S)$, the following inequality holds.

$$(9) \qquad \frac{\ell_{\gamma_0}(Y)}{\ell_{\gamma_0}(X)} \leq \sup_{\gamma \in \mathcal{A}'(S) \cup \mathcal{S}(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

By Lemma 4.3, we have

$$(10) \qquad \sup_{\gamma \in \mathcal{A}'(S) \cup S(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\gamma \in S(S^d)} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)}.$$

By Lemma 4.2, it follows that

$$(11) \qquad \sup_{\gamma \in S(S^d)} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)} = \sup_{\gamma \in \pi_1(S^d) - \{0\}} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)}.$$

Denote by γ_0^d the double of γ_0 with respect to ∂S. Combining (10) and (11), we derive that

$$\frac{\ell_{\gamma_0}(Y)}{\ell_{\gamma_0}(X)} = \frac{\ell_{\gamma_0^d}(Y^d)}{\ell_{\gamma_0^d}(X^d)} \leq \sup_{\gamma \in \pi_1(S^d) - \{0\}} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)} = \sup_{\gamma \in \mathcal{A}'(S) \cup S(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

This implies (9). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

REMARK 4.5. Proposition 4.4 shows that the essential simple arcs taken in the definition of the arc metric d_A for surfaces of finite type with boundary can be replaced by essential arcs. However, the method for the proof of Proposition 2.8 in [18] is not valid if S is a surface of infinite type with boundary. The reason is that the set $\mathcal{PML}^S(S^d)$ which is the closure of $S(S^d)$ in $\mathcal{PML}(S^d)$ is not compact and it is possible that the value

$$\sup_{\gamma \in S(S^d)} \frac{\ell_\gamma(Y^d)}{\ell_\gamma(X^d)}$$

cannot be realized by any measured lamination in $\mathcal{ML}^S(S^d)$. As a result, the method for the proof of Proposition 4.4 also fails for the case of surfaces of infinite type with boundary.

QUESTION 4.6. For any two complete hyperbolic structures X, Y on a surface S of infinite type with boundary, does the following equality still hold?

$$\sup_{\gamma \in \mathcal{A}(S) \cup S(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\gamma \in \mathcal{A}'(S) \cup S(S)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

Lemma 4.7. *Let X_0 be a complete hyperbolic surface with boundary which has at least one interior simple closed curve. Then for any $\alpha \in S^{int}(X_0)$, the length of α can be approximated by a sequence of lengths of weighted simple geodesic arcs $\gamma_n \in \mathcal{A}'(X_0)$.*

Proof. It is equivalent to show that $S^{int}(X_0) \subset \mathcal{ML}^{\mathcal{A}'}(X_0)$. Note that X_0 has at least one geodesic boundary component, then for any $\alpha \in S^{int}(X_0)$, we can find a geodesic arc $\gamma \in \mathcal{A}'(X_0)$ that essentially intersects α in one or two points. See Figure 1 (resp. Figure 2) for an example of γ corresponding to a separable (resp. non-separable) interior simple closed curve α.

Let γ_n be the weighted geodesic arc obtained by taking a power n of a positive Dehn-twist along α with the weight $1/(i(\gamma, \alpha) n)$ on γ. It is obvious that $\{\frac{1}{i(\gamma,\alpha)n} \gamma_n\}_{n=1}^\infty$ converges to α in $\mathcal{ML}(X_0)$, hence $\ell_\alpha(X_0) = \lim_{n \to \infty} \frac{1}{i(\gamma,\alpha)n} \ell_{\gamma_n}(X_0)$ and we obtain that $S^{int}(X_0) \subset \mathcal{ML}^{\mathcal{A}'}(X_0)$. □

Theorem 4.8. *Let X_0 be a complete hyperbolic surface of infinite type with boundary which satisfies the geometric condition $(\star)$. Then the two functions d and $\overline{d}$ on $\mathcal{T}(X_0) \times \mathcal{T}(X_0)$*
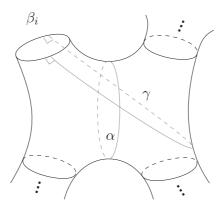
Fig. 1. An example of γ corresponding to α (separable), where the complement of α in X_0 is disconnected.
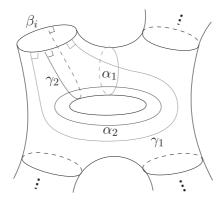

Fig. 2. An example of γ_i corresponding to α_i (non-separable).

are asymmetric metrics.

Proof. By Lemma 3.9, d and $\bar{d}$ are valued in $[0, +\infty)$. Observe that $\bar{d}(X, Y) = d(Y, X)$, for all $X, Y \in \mathcal{T}(X_0)$, it suffices to consider d. Note that the triangle inequality naturally holds for d. Now we prove the separation axiom for d by showing that if $X \neq Y \in \mathcal{T}(X_0)$, then $d(X, Y) > 0$.

Assume that $d(X, Y) \leq 0$, then $\ell_\alpha(Y) \leq \ell_\alpha(X)$ for all $\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)$. In particular, we have that

$$(12) \qquad\qquad\qquad \ell_{\beta_j}(Y) \leq \ell_{\beta_j}(X),$$

for all $\beta_j \in \mathcal{B}(X_0)$.

Moreover, for each $j \in \mathbb{N}$, we have that

$$(13) \qquad\qquad\qquad d_i^j(Y) \leq d_i^j(X),$$

for all $i \in \mathbb{N}$. Here $\{d_i^j(X)\}_{i=1}^\infty$ denotes the 0-th orthogonal spectrum of X related to $\beta_j(X)$ and $\mathcal{B}(X)$, and $\{d_i^j(Y)\}_{i=1}^\infty$ denotes the 0-th orthogonal spectrum of Y related to $\beta_j(Y)$ and $\mathcal{B}(Y)$, where $\beta_j(X)$ (resp. $\beta_j(Y)$) is the corresponding geodesic boundary component on X (resp. Y).

By Corollary 3.12 and the Basmajian identity (2), it follows that for each $j \in \mathbb{N}$ and each $X' \in \mathcal{T}(X_0)$,

$$(14) \qquad \ell_{\beta_j}(X') = \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(X')}{2}).$$

By (13), (14) and the monotonically decreasing of the function $\log \coth(\frac{x}{2})$,

$$(15) \qquad \ell_{\beta_j}(Y) = \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(Y)}{2}) \geq \ell_{\beta_j}(X) = \sum_{i=1}^{\infty} 2 \log \coth(\frac{d_i^j(X)}{2}).$$

Since the geodesic arc which minimizes the lengths of all the arcs in a given homotopy class is unique and hits the boundary perpendicularly, then there is a bijection between $\mathcal{A}(X_0)$ and the set of geodesic arcs (possibly with self-intersections) in X_0 which are orthogonal to ∂X_0 at their endpoints. Therefore,

$$(16) \qquad \begin{aligned} \cup_j \{d_i^j(X)\} &= \{\ell_\alpha(X) : \alpha \in \mathcal{A}(X_0)\}, \\ \cup_j \{d_i^j(Y)\} &= \{\ell_\alpha(Y) : \alpha \in \mathcal{A}(X_0)\}. \end{aligned}$$

Combining (12) (15) and (16), we have

$$(17) \qquad \ell_\alpha(Y) = \ell_\alpha(X),$$

for all $\alpha \in \mathcal{B}(X_0) \cup \mathcal{A}(X_0)$.

Note that X_0 admits a countable pair of pants decomposition $\mathcal{P} = \{C_i\}_{i=1}^{\infty}$, then X_0 can be parameterized by the Fenchel-Nielsen coordinates with respect to $\mathcal{P}$ (see [2]). Moreover, for each interior simple closed geodesic C_i in $\mathcal{P}$, the twisting parameter can be uniquely determined by the length of the shortest simple closed geodesic γ_i which intersects C_i and the length of the geodesic $T_{C_i}(\gamma_i)$ obtained by taking a positive Dehn-twist along C_i on γ_i.

By Lemma 4.7, the length of an interior simple closed geodesic can be approximated by a sequence of lengths of weighted geodesic arcs $\gamma_n \in \mathcal{A}(X_0)$. From this and (17), we have $\ell_\alpha(Y) = \ell_\alpha(X)$ for all $\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)$. Then X and Y have the same Fenchel-Nielsen coordinates and hence $X = Y$, which implies the assumption is false.

The asymmetric condition of d can be deduced from the example constructed by Thurston (see [33]). Let X be a complete hyperbolic surface of infinite type which satisfies the geometric condition ($\star$) and the following conditions:

(1) X contains an embedded hyperbolic X-piece S (that is, a hyperbolic surface whose interior is homeomorphic to a sphere with four disjoint closed disks removed) with four geodesic boundary components $\beta_1, \beta_2, \beta_3, \beta_4$ of the same length l satisfying $\sinh \frac{l}{2} = 1$.

(2) Let γ_1 (resp. γ_2) be the shortest geodesic arc connecting β_1 and β_2 (resp. β_2 and β_3). Denote by α_1 (resp. α_2) the third boundary component of the hyperbolic pair of pants determined by β_1, β_2 and γ_1 (resp. β_2, β_3 and γ_2). We choose X such that $\ell_{\alpha_1}(X)$ is sufficiently small and the twisting parameter of α_1 is zero, as indicated in Figure 3.

Note that X satisfies the geometric condition ($\star$). And such a surface X always exists. Now we deform X by contracting the length of α_1 appropriately on X, while keeping the lengths of β_i ($i = 1, 2, 3, 4$) and the hyperbolic structure of the complement of S in X unchanged. Denote by Y the obtained surface, as presented in Figure 4. It is not hard to see that $d(X, Y) \neq d(Y, X)$ if we can choose Y with the contracted length $\ell_{\alpha_1}(Y)$ satisfying
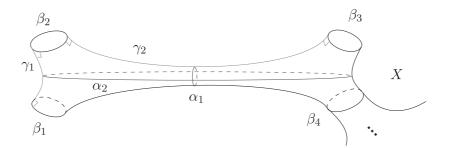
Fig. 3. A hyperbolic surface X of infinite type which satisfies the geometric condition ($\star$) and the conditions (1) and (2).
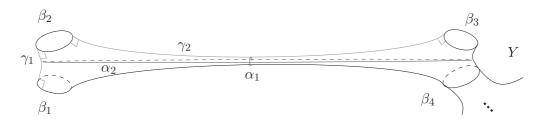

Fig. 4. A deformed surface Y of X.

$$(18) \qquad \frac{\ell_{\gamma_2}(Y)}{\ell_{\gamma_2}(X)} \leq \frac{\ell_{\alpha_2}(Y)}{\ell_{\alpha_2}(X)}, \qquad \frac{\ell_{\gamma_1}(X)}{\ell_{\gamma_1}(Y)} \leq \frac{\ell_{\alpha_1}(X)}{\ell_{\alpha_1}(Y)}.$$

Indeed, if Y satisfies the property (18), we have

$$d(X, Y) = \log \sup_{\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \log \frac{\ell_{\alpha_2}(Y)}{\ell_{\alpha_2}(X)},$$

$$d(Y, X) = \log \sup_{\alpha \in \mathcal{S}(X_0) \cup \mathcal{A}(X_0)} \frac{\ell_\alpha(X)}{\ell_\alpha(Y)} = \log \frac{\ell_{\alpha_1}(X)}{\ell_{\alpha_1}(Y)} \approx \frac{1}{2}(\ell_{\alpha_2}(Y) - \ell_{\alpha_2}(X)).$$

Choosing $\ell_{\alpha_2}(Y)$ appropriately large (equivalently, contracting the length of α_1 appropriately on X), we have $d(X, Y) \neq d(Y, X)$.

Now we show that such a surface Y with the property (18) always exists. For simplicity, still denote the length of α_i (resp. β_i, γ_i) by α_i (resp. β_i, γ_i) for $i = 1, 2$. By the formulae for right-angled pentagons and right-angled hexagons respectively (see [9]) and the assumption that $\sinh \frac{\beta_i}{2} = \sinh \frac{l}{2} = 1$ for $i = 1, 2, 3, 4$, we have

$$(19) \qquad \begin{aligned} \cosh \frac{\alpha_2}{4} &= \sinh \frac{\beta_2}{2} \sinh \frac{\gamma_2}{2} = \sinh \frac{\gamma_2}{2}, \\ \cosh \gamma_1 &= \frac{\cosh \frac{\alpha_1}{2} + \cosh \frac{\beta_1}{2} \cosh \frac{\beta_2}{2}}{\sinh \frac{\beta_1}{2} \sinh \frac{\beta_2}{2}} = \cosh \frac{\alpha_1}{2} + 2. \end{aligned}$$

By (19) and the growth trends of the two functions $y = \cosh x$ and $y = \sinh x$ (resp. $y = \cosh x$ and $y = \cosh x + 2$), as presented in Figure 5 (resp. Figure 6), we can always find such a deformed surface Y with the property (18).
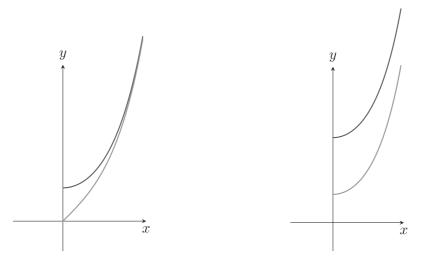
$\square$

Fig. 5. Two functions: $y = \cosh x$ (above) and $y = \sinh x$ (below), where $x \geq 0$.

Fig. 6. Two functions: $y = \cosh x + 2$ (above) and $y = \cosh x$ (below), where $x \geq 0$.

REMARK 4.9. By Proposition 4.4, the asymmetric metric d can be viewed as an analogue, for surfaces of infinite type with boundary, of the arc metric defined for surfaces of finite type with boundary. That's why we also call d the arc metric.

The problem that if the function d in Theorem 4.8 is positive definite can be viewed as a particular version of the marked length spectrum rigidity problem (see e.g. [11, 13]). In general, let (M, g) be a Riemannian manifold and let Σ be a set of homotopy classes of the curves on M one wants to consider. The Σ-*marked length spectrum* of (M, g) is the length vector $(\ell_\gamma(g))_{\gamma \in \Sigma}$ indexed over Σ, where $\ell_\gamma(g)$ is the infimum of the lengths under the metric g of all the curves in the homotopy class $[\gamma] \in \Sigma$. The *marked length spectrum rigidity problem* asks whether an inequality between the marked length spectra of two Riemannian manifolds implies an isometry homotopic to the identity between them.

In our case, the rigidity problem is the marked $\mathcal{S}(X_0) \cup \mathcal{A}(X_0)$-spectrum rigidity problem in the special case of complete hyperbolic surfaces of infinite type with geodesic boundary.

It is necessary to take arcs into consideration in the definition of d, since for any complete hyperbolic surface X_0 of infinite type with geodesic boundary components whose lengths are uniformly bounded above, we can find two distinct elements X, Y in $\mathcal{T}(X_0)$ such that $\ell_\alpha(Y) \leq \ell_\alpha(X)$ for all $\alpha \in \mathcal{S}(X_0)$. To see this, let $X = X_0$. Denote by $\bar{X}$ the Riemann surface such that its convex core is exactly the hyperbolic surface X. Let $\bar{Y}$ be the Nielsen extension of $\bar{X}$. Note that there exists a quasiconformal homeomorphism from $\bar{X}$ to $\bar{Y}$. Then we obtain another hyperbolic surface $Y \in \mathcal{T}(X_0)$ which is the convex core of $\bar{Y}$. By generalized Schwarz lemma, we have $\ell_\alpha(Y) \leq \ell_\alpha(X)$ for all $\alpha \in \mathcal{S}(X_0)$. This implies that

$$\log \sup_{\alpha \in \mathcal{S}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} \leq 0.$$

REMARK 4.10. Let X_0 be a complete hyperbolic surface of infinite type with geodesic boundary. Recall that the set of boundary components of X_0 is $\mathcal{B}(X_0) = \{\beta_1, \beta_2, ..., \beta_k, ...\}$. Let $L = (L_\alpha)_{\alpha \in \mathcal{B}(X_0)} \in \mathbb{R}_{>0}^{|\mathcal{B}(X_0)|}$, where $|\mathcal{B}(X_0)|$ denotes the number of the elements in $\mathcal{B}(X_0)$ which is finite or countably infinite. Denote by $\mathcal{T}(X_0, L)$ the subspace of $\mathcal{T}(X_0)$ which consists of

the equivalence classes of marked hyperbolic surfaces with geodesic boundary components of fixed lengths, that is, the geodesic length $\ell_{\beta_i}(X)$ of β_i under each element X of $\mathcal{T}(X_0, L)$ is L_{β_i} for each $i \in \mathbb{N}$. For convenience, we denote L_{β_i} by L_i.

If X_0 satisfies the geometric condition ($\star$), as discussed in Theorem 4.8, by applying the generalized McShane identity (3) and the Basmajian identity (2), the following two functions are asymmetric metrics on $\mathcal{T}(X_0, L)$.

$$d_1(X, Y) = \log \sup_{\alpha \in S(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

$$d_2(X, Y) = \log \sup_{\alpha \in \mathcal{A}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)},$$

for all $X, Y \in \mathcal{T}(X_0, L)$.

The following theorem shows that one can obtain the same asymmetric metric by taking the supremum over $\mathcal{A}(X_0) \cup \mathcal{B}(X_0)$ instead of $\mathcal{A}(X_0) \cup S(X_0)$ in the formula which defines the arc metric d on $\mathcal{T}(X_0)$ in Theorem 4.8.

Theorem 4.11. *Let X_0 be a complete hyperbolic surface of infinite type with boundary, then the following equality still holds for all $X, Y \in \mathcal{T}(X_0)$.*

$$(20) \qquad \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \sup_{\gamma \in \mathcal{A}(X_0) \cup S(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

In particular, if X_0 satisfies the geometric condition ($\star$), then the following equality defines the same asymmetric metric on $\mathcal{T}(X_0)$.

$$(21) \qquad \log \sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \log \sup_{\gamma \in \mathcal{A}(X_0) \cup S(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

Proof. Obviously,

$$\sup_{\alpha \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} \le \sup_{\gamma \in \mathcal{A}(X_0) \cup S(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

It suffices to verify that

$$(22) \qquad \sup_{\alpha \in \mathcal{A}(X_0) \cup S(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} \le \sup_{\gamma \in \mathcal{A}(X_0) \cup \mathcal{B}(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

By Lemma 4.7, we have

$$S^{int}(X_0) \subset \mathcal{ML}^{\mathcal{A}}(X_0).$$

Observe that $S(X_0) = S^{int}(X_0) \cup \mathcal{B}(X_0)$ and

$$\sup_{\gamma \in \mathcal{A}(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)} = \sup_{\mu \in \mathcal{ML}^{\mathcal{A}}(X_0)} \frac{\mathcal{L}_\mu(Y)}{\mathcal{L}_\mu(X)}.$$

Therefore, the inequality (22) holds. If X_0 satisfies the geometric condition ($\star$), it follows from Theorem 4.8 that the equality (21) defines the same asymmetric metric. This completes the proof of this theorem. $\qquad\square$

REMARK 4.12. Using the same method for the proof of the equality (20) in Theorem 4.11, we give an affirmative answer to the following question (see Problem 5.5 in [20]): does the equality (23) hold on Teichmüller spaces of surfaces of infinite type with boundary?

$$(23) \qquad \log \sup_{\alpha \in \mathcal{A}'(X_0) \cup B(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} = \log \sup_{\gamma \in \mathcal{A}'(X_0) \cup S(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)}.$$

QUESTION 4.13. For a complete hyperbolic surface X_0 of infinite type with geodesic boundary, is the following function δ an asymmetric metric on $\mathcal{T}(X_0)$?

$$\delta(X, Y) = \log \sup_{\gamma \in \mathcal{A}(X_0)} \frac{\ell_\gamma(Y)}{\ell_\gamma(X)},$$

for any $X, Y \in \mathcal{T}(X_0)$.

It is clear that δ is positive definite for hyperbolic surfaces of finite type with boundary, by an application of the Bridgeman identity [8].

QUESTION 4.14. Let X_0 be a complete hyperbolic surface X_0 of infinite type with geodesic boundary, are there two elements X, Y in $\mathcal{T}(X_0)$ satisfying the following condition?

$$\log \sup_{\alpha \in S(X_0)} \frac{\ell_\alpha(Y)}{\ell_\alpha(X)} < 0.$$

For a complete hyperbolic surface of finite type with geodesic boundary, it is true. We can construct some examples by Nielsen extension (see [6, 10]) or strip deformation (see [12, 26]). Does it still work for the case of surfaces of infinite type with boundary?

5. Several examples of hyperbolic surfaces of infinite type which satisfy the geometric condition (⋆)

In this section, we construct several examples of hyperbolic surfaces of infinite type which satisfy the geometric condition (⋆). We find that these hyperbolic surfaces may be incomplete. And we prove that there is no direct relation between the geometric condition (⋆) and the Shiga's condition.

5.1. The construction of examples. In order to construct the desired examples, we first give the following two lemmas.

Lemma 5.1. *Let $\mathbb{P}_n$ be a geodesically convex hyperbolic n-polygon in $\mathbb{H}^2$ with consecutive edges $\alpha_1, \alpha_2, ..., \alpha_n$, where the endpoints of the edge α_i are denoted by Q_i and Q_{i-1}, here $Q_0 = Q_n$. Then $\rho(x, \alpha_1) \leq \sup_{2 \leq i \leq n-1} \{\rho(Q_i, \alpha_1)\}$ for any point $x \in \mathbb{P}_n$, where ρ denotes the hyperbolic distance on $\mathbb{H}^2$ and $\rho(x, \alpha_1) = \inf_{y \in \alpha_1} \rho(x, y)$.*

Proof. By the continuity of the hyperbolic distance on $\mathbb{P}_n$, we only need to consider the hyperbolic distance from each point of the piecewise geodesic boundary $\partial \mathbb{P}_n$ to α_1. Note that $\mathbb{P}_n$ is geodesically convex, the function $f : \partial \mathbb{P}_n \backslash \alpha_1 \to \mathbb{R}_{\geq 0}$ which assigns $\rho(x, \alpha_1)$ to x restricted to each smooth edge except α_1 attains its maximum only if x is one of the two endpoints. Therefore,

$$\sup_{x \in \partial \mathbb{P}_n \setminus \alpha_1} \rho(x, \alpha_1) = \sup_{2 \leq i \leq n-1} \{\rho(Q_i, \alpha_1)\},$$

which implies the desired result. □

Lemma 5.2. *Let $H_n (n \geq 1)$ be a right-angled hexagon in $\mathbb{H}^2$ with pairwise non-adjacent edges $\alpha_n, \beta_n, \gamma_n$ whose lengths are respectively l_0, l_n, l_n, where $l_0 > 0$, $\{l_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive numbers and $l_n \to \infty$ as $n \to \infty$. Then there exists a constant $M > 0$, such that*

$$\sup_n \{\sup_{x \in H_n} \rho(x, \beta_n)\} = \sup_n \{\sup_{x \in H_n} \rho(x, \gamma_n)\} \leq M,$$

where ρ denotes the hyperbolic distance on $\mathbb{H}^2$.

Proof. Denote the vertices of H_n which are not on the edge γ_n by A_n, C_n, D_n, B_n respectively in the counter-clockwise order as presented in Figure 7. Denote $\rho(A_n, \gamma_n) = a_n$, $\rho(B_n, \gamma_n) = b_n$, $\rho(C_n, \gamma_n) = c_n$, $\rho(D_n, \gamma_n) = d_n$.
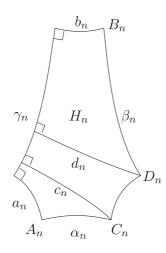

Fig. 7. The right-angled hexagon H_n in Lemma 5.2.

Note that $\sup_{x \in H_n} \rho(x, \beta_n) = \sup_{x \in H_n} \rho(x, \gamma_n)$ for all $n \geq 1$. By Lemma 5.1, it suffices to show that $\sup_n \{a_n, b_n, c_n, d_n\} \leq M$ for a constant $M > 0$.

By the formula for a right-angled hexagon and the formula for a trirectangle (that is, a quadrilateral with three right angles) [9], we have

$$(24) \qquad \cosh a_n = \frac{\cosh l_n + \cosh l_n \cosh l_0}{\sinh l_n \sinh l_0} \leq \frac{\coth l_1}{\sinh l_0} + \coth l_1 \coth l_0.$$

$$(25) \qquad \cosh b_n = \frac{\cosh l_0 + \cosh l_n \cosh l_n}{\sinh l_n \sinh l_n} \leq \frac{\cosh l_0}{(\sinh l_1)^2} + (\coth l_1)^2.$$

$$(26) \qquad \sinh c_n = \sinh a_n \cosh l_0 \leq \cosh a_n \cosh l_0.$$

$$(27) \qquad \sinh d_n = \sinh b_n \cosh l_n = \sqrt{(\cosh b_n)^2 - 1} \cosh l_n.$$

Substitute (25) into (27), we have

(28)
$$\sinh d_n = \sqrt{\frac{\cosh^2 l_0 \coth^2 l_n}{\sinh^2 l_n} + 2\coth^4 l_n \cosh l_0 + \cosh^2 l_n(\coth^4 l_n - 1)}$$

$$\leq \sqrt{\frac{\cosh^2 l_0 \coth^2 l_1}{\sinh^2 l_1} + 2\coth^4 l_1 \cosh l_0 + \cosh^2 l_n(\coth^4 l_n - 1)}$$

Note that $\coth x \to 1$, $\operatorname{sech} x \to 0$ as $x \to \infty$ and $(\coth x)' = -\operatorname{csch}^2 x$, $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$, we have

(29)
$$\lim_{x\to\infty} \cosh^2 x(\coth^4 x - 1)$$
$$= \lim_{x\to\infty} \frac{\coth^4 x - 1}{\operatorname{sech}^2 x}$$
$$= \lim_{x\to\infty} \frac{-4\coth^3 x \operatorname{csch}^2 x}{-2\operatorname{sech}^2 x \tanh x}$$
$$= \lim_{x\to\infty} 2\coth^6 x$$
$$= 2$$

Combining (24), (25), (26), (28) and (29), we have the desired result. □

EXAMPLE 5.3. Now we construct a complete hyperbolic surface X_0 of infinite type which satisfies the geometric condition ($\star$).

Let $\{l_n\}_{n=1}^{\infty}$ be a strictly increasing divergent sequence of positive numbers. Let P_n be a hyperbolic pair of pants with boundary lengths $(2l_0, 2l_n, 2l_n)$. Then we glue P_n with its copy P_n' along the geodesic boundary component of common length $2l_n$. Denote by X_n the obtained X-piece for $n \geq 1$. Let X_0 be the surface obtained by gluing the sequence $\{X_n\}_{n=1}^{\infty}$ in succession along the geodesic boundary component α_n of common length $2l_0$, as indicated in Figure 8. Note that the amount of the twisting along the gluing curves can be taken arbitrarily in the above process.
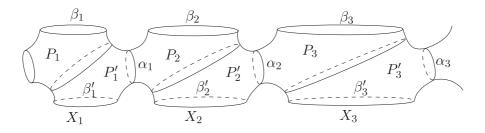

Fig. 8. The hyperbolic surface X_0 of infinite type in Example 5.3.

Since any closed ball of radius 1 on the surface X_0 is contained in a finite number of pairs of pants of the given decomposition as show in Figure 8, then it is compact. By the Hopf-Rinow Theorem, X_0 is complete.

We claim that X_0 satisfies the geometric condition ($\star$). Indeed, P_n can be constructed by pasting two copies of the right-angled hexagon H_n with pairwise non-adjacent edges of lengths l_0, l_n, l_n along the remaining edges. Denote by β_n (resp. β_n') the boundary component

of P_n (resp. P'_n) which is contained in ∂X_0 and has length $2l_n$. For each $x \in X_0$, there exists an integer $N \geq 1$, such that x lies in P_N or P'_N. Without lost of generality, we assume that $x \in P_N$. By Lemma 5.2, there exists a constant $M > 0$ independent of N such that

$$\rho(x, \partial X_0) \leq \rho(x, \beta_N) \leq M,$$

which implies the claim.

In order to construct more examples which satisfy the geometric condition $(\star)$, we introduce the following notations and propositions given by Basmajian (see [4, 5]).

A *flute surface* is a hyperbolic surface of infinite type obtained by gluing a sequence of generalized hyperbolic pairs of pants $\{P_i\}_{i=0}^{\infty}$ in succession along the common length boundary components, that is, any two adjacent pairs of pants P_i, P_{i+1} have exactly one common geodesic boundary component which is denoted by α_{i+1} for $i \geq 0$. Note that P_0 has at least one geodesic boundary component α_1 and P_i has at least two geodesic boundary components α_i, α_{i+1} for $i \geq 1$. We say that a flute surface is *tight* if all the pants holes that have not been glued along are in fact cusps. In this case, denote by α_0 the image of a horocycle under the universal covering of this surface, which is a simple closed curve of length one and homotopic to a cusp of P_0 (see Figure 9). We say a subsurface S is a *spike* if it is isometric to the region $\{z = x + iy : 0 \leq x \leq 1, y > a\}$ of $\mathbb{H}^2$, for some $a > 0$.

Let ℓ_i be the length of α_i for $i \geq 0$. Denote by d_i the hyperbolic distance from α_i to α_{i+1} and denote by s_i the amount of the twisting along α_{i+1} for $i \geq 0$. Here the amount of a positive Dehn-twist along α_{i+1} is defined to be the hyperbolic length of α_{i+1}.
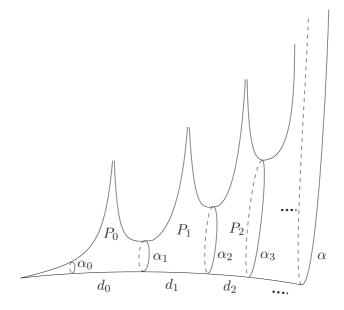

Fig. 9. A tight flute surface Y_0 (where α_0 is the image of a horocycle under the universal covering and has length one).

Let $\{L_i\}_{i \geq 0}$ be a sequence of geodesics in $\mathbb{H}^2$. We say that $\{L_i\}$ is a *nested sequence of geodesics* if L_{i-1} and L_{i+1} lie in different components of $\mathbb{H}^2 - L_i$ for each $i \geq 1$, and if the L_i are disjoint in $\overline{\mathbb{H}}^2$. $\{L_i\}$ *converges to the geodesic L* if the endpoints of L_i converge to the endpoints of L on $\partial \mathbb{H}^2$. If the endpoints of L_i converge to a single point of $\partial \mathbb{H}^2$, then we

say that the sequence $\{L_i\}$ *converges to a point* on the boundary of the hyperbolic plane. It is known that the limit of a nested sequence of geodesics in $\mathbb{H}^2$ is unique and is one of the above two possibilities.

Introduce a positive direction for each L_i by designating L_{i+1} to lie to the right of L_i. Let σ_i be the unique common perpendicular between the geodesics L_i and L_{i+1}. The distance from σ_i to σ_{i+1} is measured by traversing L_{i+1}. Let $s_i = \rho(\sigma_i, \sigma_{i+1})$ if L_{i+1} is traversed in the positive direction, and let $s_i = -\rho(\sigma_i, \sigma_{i+1})$ if L_{i+1} is traversed in the negative direction. Set $d_i = \rho(L_i, L_{i+1})$. Here ρ is the hyperbolic distance on $\mathbb{H}^2$.

Proposition 5.4. (Basmajian [4]) *Let the sequence $\{L_i\}_{i \geq 0}$ be a nested sequence of geodesics in $\mathbb{H}^2$. Then L_i converges to a geodesic if and only if*

$$\lim_{i \to \infty} \rho(L_1, L_i) < \infty,$$

where $\rho(L_1, L_i)$ is the hyperbolic distance between L_1 and L_i.

Proposition 5.5. (Basmajian [4], The Pair of Pants Theorem) *Suppose γ and β are nonelliptic elements. Let d be the hyperbolic distance between the axes of γ and β (if γ is parabolic, the axis of γ is the horocycle based at the fixed point of γ whose projection to $\mathbb{H}^2/\langle\gamma\rangle$ has length one). Then (γ, β) form standard generators for a tight pair of pants (that is, the third boundary component of this pair of pants is a cusp) if and only if $c(\gamma) + c(\beta) = d$. Here $c(\gamma) = \log 2$ if γ is parabolic, and $c(\gamma) = \log \coth \frac{T(\gamma)}{4}$ if γ is hyperbolic, where $T(\gamma)$ is the translation length of γ.*

We first give an example of incomplete hyperbolic surfaces of infinite type which satisfy the geometric condition $(\star)$ by the following proposition.

Proposition 5.6. *Let Y_0 be a tight flute surface with $\sum d_i < \infty$ and $\sum |s_i| < \infty$, where the sum is taken over all $i \geq 0$. If there exists a constant $M > 0$ such that*

$$(30) \qquad \sinh\left(\sum_{i=n-1}^{\infty} d_i\right) \cosh \frac{l_n}{2} \leq M,$$

for all $n \geq 1$, then Y_0 is incomplete and satisfies the geometric condition $(\star)$.

Proof. By the assumptions that $\sum d_i < \infty$ and $\sum |s_i| < \infty$, it follows from Proposition 5.4 that the nested sequence $\{\alpha_i\}$ converges to a geodesic. We denote it by α. We claim that the length of α must be infinity. Otherwise, assume that the length of α is a finite positive number l. Then $l_i \leq l + 1$ for all $i \geq N$, where N is a sufficiently big integer. By the formula for a pentagon with four right angles and an angle of zero (see [9]), we have

$$\cosh d_i = \frac{1 + \cosh \frac{l_i}{2} \cosh \frac{l_{i+1}}{2}}{\sinh \frac{l_i}{2} \sinh \frac{l_{i+1}}{2}} \geq \frac{1}{\sinh^2 \frac{l+1}{2}} + \coth^2 \frac{l+1}{2},$$

for all $i \geq N$, which contradicts the assumption that $\sum d_i < \infty$. Therefore, $\ell_i \to \infty$, as $i \to \infty$ and Y_0 is an incomplete hyperbolic surface with a simple open infinite geodesic boundary.

Now we prove that Y_0 satisfies the geometric condition $(\star)$ if it satisfies the condition (30).

First we consider the special case that $s_i = 0$ for all $i \geq 0$. In this case, Y_0 can be constructed by pasting two copies of the geodesically convex ideal region R with infinitely

many geodesic edges along all the edges of common lengths except α', which is half of the geodesic α.
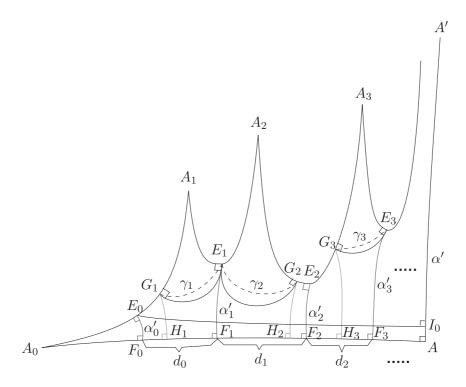

Fig. 10. The geodesically convex ideal region R (where each spike S_n is an open subset of R, the dashed lines are geodesics γ_n between G_n and E'_n and the angle at E'_n between the finite edge $\overline{G_n E'_n}$ of the spike S_n and the common perpendicular α'_n is zero. In this figure, $E'_1 = E'_2 = E_1, E'_3 = E_3$).

Now we consider the geodesically convex ideal region R with ideal vertices $\{A_i\}_{i=0}^{\infty}$ corresponding to the cusps of Y_0, as shown in Figure 10. Denote by A, A' the two endpoints of α' (where A' is an ideal vertex). Let α'_0 be half of the simple closed curve α_0 and let α'_i be the common perpendicular between the infinite geodesic edge $\overline{A_i A_{i+1}}$ and the infinite geodesic edge $\overline{A_0 A}$ for $i \geq 1$.

It suffices to find a constant $M' > 0$ and a disjoint union S of spikes in R such that any point in $R \setminus S$ is within the distance M' of α'.

To see this, we denote by E_i, F_i the two endpoints of α'_i (where F_i lies in the edge $\overline{A_0 A}$). Since the length of α_i is l_i, then the hyperbolic length of α'_i is $\frac{1}{2}\ell_i$. For the ideal vertex A_0, we take a spike S_0 which has a finite edge α'_0 of length $\frac{1}{2}l_0 = \frac{1}{2}$. For each ideal vertex A_i ($i \geq 1$), we take a spike S_i such that it goes through the point E'_i, where $E'_i = E_i$ if $l_i \geq l_{i-1}$, and $E'_i = E_{i-1}$ if $l_i < l_{i-1}$. Denote by G_i the third vertex of S_i except the two vertices A_i and E'_i (note that each spike S_i is an open subset of R and the finite edge $\overline{E'_i G_i}$ is not a geodesic). Draw a geodesic segment which starts at G_i and intersects the edge $\overline{A_0 A}$ perpendicularly at the point H_i for $i \geq 1$.

We claim that each spike S_i is disjoint from any other spikes. Indeed, we represent the geodesically convex ideal region R in the upper half-plane model of $\mathbb{H}^2$ (see Figure 11). It suffices to consider the position of S_{i+1} in the special case that $l_i = l_{i+1}$. In this case, the

vertex G_{i+1} of S_{i+1} coincides with the point E_i (see Figure 12). By the construction of S_i, we have that S_i and S_j are disjoint for all $i \neq j$.
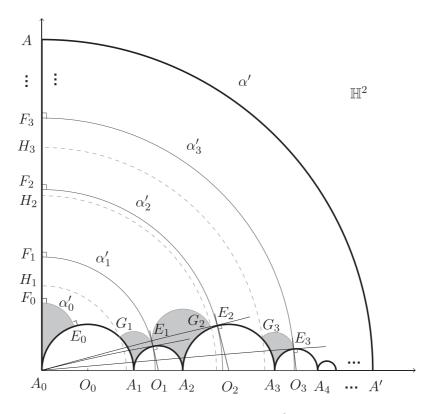

Fig. 11. The geodesically convex ideal region R in $\mathbb{H}^2$, where the boundary of R is drawn in bold lines, and O_i is the Euclidean center of the semi-circle corresponding to the infinite geodesic edge $\overline{A_i A_{i+1}}$ of R (in this figure, $E_1' = E_2' = E_1, E_3' = E_3$).
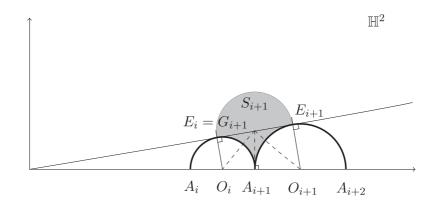

Fig. 12. The position of the spike S_{i+1} when $l_i = l_{i+1}$ (in this figure, $E_{i+1}' = E_{i+1}$).

Let c_i be the hyperbolic distance between H_i and F_i for $i \geq 1$. Let d be the hyperbolic distance between α_0' and α'. Then $\sum c_i < \sum d_i = d < \infty$.

Let $S = \cup_{i \geq 0} S_i$. We need to show that there exists a constant $M' > 0$ such that any point in $R \setminus S$ is within the distance M' of α'.

Note that Lemma 5.1 can be generalized to the case of a geodesically convex simply connected region in $\mathbb{H}^2$ with infinitely many geodesic edges, and each geodesic arc γ_n connecting E_n and G_n is contained in S_n, it suffices to consider $\rho(E_n, \alpha')$ for $n \geq 0$ and $\rho(G_n, \alpha')$ for $n \geq 1$, where ρ is the hyperbolic distance on R.

Now we compute $\rho(E_n, \alpha')$. Draw a geodesic from E_n to α' such that it intersects α' at the point I_n perpendicularly. Then $\rho(E_n, \alpha') = \rho(E_n, I_n)$. The geodesic segment $\overline{E_n I_n}$ is an edge of the trirectangle with consecutive vertices A, I_n, E_n, F_n (see the trirectangle with consecutive vertices A, I_0, E_0, F_0 in Figure 10 as an example). By the formula for trirectangles (see [9]), for each $n \geq 1$, we have

$$(31) \qquad \sinh \rho(E_n, \alpha') = \sinh \left(\sum_{i=n}^{\infty} d_i \right) \cosh \frac{\ell_n}{2}.$$

To estimate $\rho(G_n, \alpha')$, we need to estimate the length b_n of the geodesic segment $\overline{G_n H_n}$. Note that $b_n \leq \frac{l_n}{2}$ if $E'_n = E_n$ and $b_n \leq \frac{l_{n-1}}{2}$ if $E'_n = E_{n-1}$. Then $b_n \leq \max\{\frac{l_n}{2}, \frac{l_{n-1}}{2}\}$ for all $n \geq 1$.

Similarly, we compute $\rho(G_n, \alpha')$ in a trirectangle. For each $n \geq 2$,

$$
\begin{aligned}
(32) \qquad \sinh \rho(G_n, \alpha') &= \sinh \left(\sum_{i=n}^{\infty} d_i + c_n \right) \cosh b_n \\
&= \left(\sinh \left(\sum_{i=n}^{\infty} d_i \right) \cosh c_n + \cosh \left(\sum_{i=n}^{\infty} d_i \right) \sinh c_n \right) \cosh b_n \\
&\leq 2 \cosh d \sinh \left(\sum_{i=n-1}^{\infty} d_i \right) \max \left\{ \cosh \frac{l_n}{2}, \cosh \frac{l_{n-1}}{2} \right\} \\
&\leq 2 \cosh d \max \left\{ \sinh \left(\sum_{i=n-1}^{\infty} d_i \right) \cosh \frac{l_n}{2}, \sinh \left(\sum_{i=n-2}^{\infty} d_i \right) \cosh \frac{l_{n-1}}{2} \right\}.
\end{aligned}
$$

By (31), (32) and the given condition (30), for all $n \geq 1$, we get

$$\sinh \rho(E_n, \alpha') \leq M,$$

and for all $n \geq 2$, we have

$$\sinh \rho(G_n, \alpha') \leq 2M \cosh d.$$

Besides,

$$\sinh \rho(E_0, \alpha') \leq \sinh d \cosh \frac{1}{2} \leq \cosh d \cosh \frac{1}{2},$$

$$\sinh \rho(G_1, \alpha') = \sinh \left(c_1 + \sum_{i=1}^{\infty} d_i \right) \cosh \frac{l_1}{2} \leq \cosh d \cosh \frac{l_1}{2}.$$

Note that $x < \sinh x$ for all $x > 0$. Let $M' = 2 \cosh d (M + \cosh \frac{1}{2} + \cosh \frac{l_1}{2})$. Then any point in $R \setminus S$ is within the distance M' of α'.

Now we consider the general case that $\sum |s_i| < \infty$. Denote by $\bar{\rho}(E_n, \alpha')$ (resp. $\bar{\rho}(G_n, \alpha')$) the hyperbolic distance between E_n (resp. G_n) and α'. Then

$$\bar{\rho}(E_n, \alpha') \leq \rho(E_n, \alpha') + \sum_{i=n}^{\infty} |s_i|,$$

$$\bar{\rho}(G_n, \alpha') \leq \rho(G_n, \alpha') + \sum_{i=n}^{\infty} |s_i|.$$

Note that $\sum |s_i| < \infty$, the statement is also true for the general case. This completes the proof of this proposition. $\square$

EXAMPLE 5.7. Now we construct an incomplete hyperbolic surface Y_0 of infinite type which satisfies the geometric condition $(\star)$.

Consider a tight flute surface with the sequence $\{P_i\}_{i \geq 0}$ of glued generalized hyperbolic pairs of pants. Since each pair of pants P_i is tight for $i \geq 0$, it follows from Proposition 5.5 that $c(\alpha_i) + c(\alpha_{i+1}) = d_i$ for $i \geq 0$. For each $n \geq 2$, we obtain that

$$\sinh\Big(\sum_{i=n-1}^{\infty} d_i \Big) \cosh \frac{l_n}{2} = \sinh\Big\{ \sum_{i=n-1}^{\infty} (c(\alpha_i) + c(\alpha_{i+1})) \Big\} \cosh \frac{l_n}{2}$$

$$= \sinh\Big\{ \log \coth \frac{l_{n-1}}{4} + 2 \sum_{i=n}^{\infty} \log \coth \frac{l_i}{4} \Big\} \cosh \frac{l_n}{2}$$

$$\leq \sinh\Big(2 \sum_{i=n-1}^{\infty} \log \coth \frac{l_i}{4} \Big) \cosh \frac{l_n}{2}.$$

Note that the sequence $\{d_0, d_1, d_2, ...\}$ is completely determined by the sequences $\{l_0, l_1, l_2, ...\}$. Let Y_0 be a tight flute surface with $\sum |s_i| < \infty$ and the sequence $\{l_0, l_1, l_2, ...\}$ satisfying $\log \coth \frac{l_i}{4} = \frac{1}{2^i}$ for each $i \geq 0$. Then

$$d = \sum_{i=0}^{\infty} d_i = \log 2 + 2 \sum_{i=1}^{\infty} \log \coth \frac{l_i}{4} = \log 2 + 2 \sum_{i=1}^{\infty} \frac{1}{2^i} = \log 2 + 2 < \infty.$$

For $n = 1$, $\sinh\Big(\sum_{i=n-1}^{\infty} d_i \Big) \cosh \frac{l_n}{2} = \sinh d \cosh \frac{l_1}{2} < \infty$. For $n \geq 2$, we get

$$\sinh\Big(2 \sum_{i=n-1}^{\infty} \log \coth \frac{l_i}{4} \Big) \cosh \frac{l_n}{2} = \sinh\Big(2 \sum_{i=n-1}^{\infty} \frac{1}{2^i} \Big) \cosh \frac{l_n}{2}$$

$$= \sinh \frac{8}{2^n} \cosh \frac{l_n}{2}$$

$$= \sinh\Big(8 \log \coth \frac{l_n}{4} \Big) \cosh \frac{l_n}{2}.$$

Observe that $l_i = 4 \operatorname{arcoth} e^{\frac{1}{2^i}} \to \infty$ as $i \to \infty$, $\coth x \to 1$, $\operatorname{sech} x \to 0$ as $x \to \infty$ and $(\coth x)' = -\operatorname{csch}^2 x$, $(\operatorname{sech} x)' = -\operatorname{sech} x \tanh x$, we have

$$\lim_{x\to\infty} \sinh(8\log\coth\frac{x}{4})\cosh\frac{x}{2} = \lim_{x\to\infty}\frac{\sinh(8\log\coth\frac{x}{4})}{\operatorname{sech}\frac{x}{2}}$$

$$= \lim_{x\to\infty}\frac{4\cosh(8\log\coth\frac{x}{4})\operatorname{csch}^2\frac{x}{4}}{\coth\frac{x}{4}\operatorname{sech}\frac{x}{2}\tanh\frac{x}{2}}$$

$$= \lim_{x\to\infty} 8(\coth\frac{x}{2})^2$$

$$= 8.$$

Hence, the surface Y_0 constructed above satisfies the condition (30). By Proposition 5.6, it is incomplete and satisfies the geometric condition ($\star$).

To construct some other examples of complete hyperbolic surfaces of infinite type which satisfy the geometric condition ($\star$), we prove the following proposition.

Proposition 5.8. *Let X_0 be a flute surface of which all the pants holes that have not been glued are boundary components and the series $\sum d_i$ is divergent. If there exists a positive constant L such that*

$$\sup_{n\in\mathbb{N}}\{a_n, b_n, c_n, d_n\} \le L,$$

then X_0 is complete and satisfies the geometric condition ($\star$). Here a_n, b_n, c_n, d_n satisfy that

$$\cosh a_n = \frac{\cosh\alpha'_{n+1} + \cosh\alpha'_n\cosh\beta'_n}{\sinh\alpha'_n\sinh\beta'_n},$$

$$\cosh b_n = \frac{\cosh\alpha'_n + \cosh\alpha'_{n+1}\cosh\beta'_n}{\sinh\alpha'_{n+1}\sinh\beta'_n},$$

$$\sinh c_n = \sinh a_n\cosh\alpha'_n,$$

$$\sinh d_n = \sinh b_n\cosh\alpha'_{n+1},$$

where $\alpha'_n = \frac{1}{2}\ell_{\alpha_n}(X_0)$, $\beta'_n = \frac{1}{2}\ell_{\beta_n}(X_0)$, $\alpha_0\cup\beta_0 = \partial P_0\cap\partial X_0$, $\beta_n = \partial P_n\cap\partial X_0$ for $n\ge 1$, as shown in Figure 13.

Proof. Since $\sum d_i$ diverges, it follows from Proposition 5.4 that $\{\widetilde{\alpha}_n\}_{n=0}^\infty$ converges to a point of $\partial\mathbb{H}^2$, where $\widetilde{\alpha}_n$ is a lift of α_n in $\mathbb{H}^2$. Hence, each geodesic boundary component of X_0 is a simple closed geodesic and X_0 is complete.
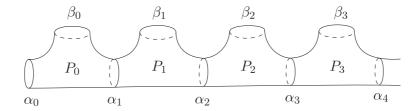

Fig. 13. A flute surface with geodesic boundary components α_0, β_n for $n\ge 0$ and the series $\sum d_i$ divergent (this figure is a special case that the lengths of all α_n are equal).

Note that P_n can be constructed by pasting two copies of the right-angled geodesic hexagon H_n with pairwise non-adjacent edges $\frac{1}{2}\beta_n$, $\frac{1}{2}\alpha_n$, $\frac{1}{2}\alpha_{n+1}$ along the remaining three

edges. Denote the vertices of H_n not on the edge $\frac{1}{2}\beta_n$ by A_n, C_n, D_n, B_n respectively in the anticlockwise order, as indicated in Figure 14.
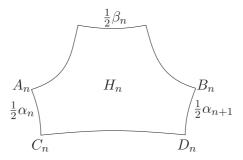

Fig. 14. A right-angled hexagon H_n.

By Lemma 5.1, for any $y \in H_n$,

$$\rho(y, \frac{1}{2}\beta_n) = \max \{\rho(A_n, \frac{1}{2}\beta_n), \rho(B_n, \frac{1}{2}\beta_n), \rho(C_n, \frac{1}{2}\beta_n), \rho(D_n, \frac{1}{2}\beta_n\}.$$

For simplicity, denote $\frac{1}{2}\ell_{\alpha_n}(X_0) = \alpha'_n$, $\frac{1}{2}\ell_{\beta_n}(X_0) = \beta'_n$, $\rho(A_n, \frac{1}{2}\beta_n) = a_n$, $\rho(B_n, \frac{1}{2}\beta_n) = b_n$, $\rho(C_n, \frac{1}{2}\beta_n) = c_n$, $\rho(D_n, \frac{1}{2}\beta_n) = d_n$.

Then we have

$$\cosh a_n = \frac{\cosh \alpha'_{n+1} + \cosh \alpha'_n \cosh \beta'_n}{\sinh \alpha'_n \sinh \beta'_n},$$
$$\cosh b_n = \frac{\cosh \alpha'_n + \cosh \alpha'_{n+1} \cosh \beta'_n}{\sinh \alpha'_{n+1} \sinh \beta'_n},$$
$$\sinh c_n = \sinh a_n \cosh \alpha'_n,$$
$$\sinh d_n = \sinh b_n \cosh \alpha'_{n+1}.$$

For any point $x \in X_0$, there exists an integer $N \geq 0$ such that $x \in P_N$. In particular, $x \in H_N$. By assumption, we obtain that

$$\rho(x, \beta_N) \leq \max\{a_N, b_N, c_N, d_N\} \leq L.$$

Therefore, $\rho(x, \partial X_0) \leq \rho(x, \beta_N) \leq L$, which implies that X_0 satisfies the geometric condition ($\star$). □

EXAMPLE 5.9. Let X'_0 be a flute surface. Let α'_n, β'_n denote the lengths of the corresponding simple geodesic segments in Proposition 5.8. Suppose that α'_n, β'_n satisfy the following conditions:
  (1) $\alpha'_n = l_0$ for all $n \geq 0$;
  (2) $\{\beta'_n\}_{n=0}^\infty$ is a strictly increasing sequence of positive numbers such that
$$\beta'_n \to \infty \text{ as } n \to \infty.$$
We claim that X'_0 is complete and satisfies the geometric condition ($\star$). Indeed, using the same notations a_n, b_n, c_n, d_n as in Proposition 5.8, by direct computation, we obtain that

$$\cosh a_n = \cosh b_n = \frac{\cosh l_0 + \cosh \beta_n' \cosh l_0}{\sinh \beta_n' \sinh l_0}$$

$$= \operatorname{csch} \beta_n' \coth l_0 + \coth \beta_n' \coth l_0$$

$$\le 2 \coth \beta_n' \coth l_0,$$

$$\sinh c_n = \sinh d_n = \sinh a_n \cosh l_0 \le \cosh a_n \cosh l_0.$$

Note that the sequence $\{\coth \beta_n'\}$ strictly decreases, we have that

$$\sup_{n \in \mathbb{N}} \{a_n, b_n, c_n, d_n\} \le L,$$

for a constant $L > 0$. Moreover, it is easy to see that $\sum d_i = \infty$. By Proposition 5.8, X_0' is complete and satisfies the geometric condition $(\star)$, as indicated in Figure 15.
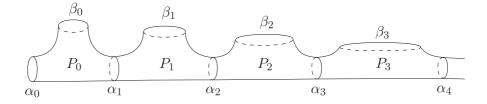

Fig. 15. The flute surface X_0' in Example 5.9.

EXAMPLE 5.10. We construct a hyperbolic surface X_1 of infinite type, which satisfies the geometric condition $(\star)$ with a removable set A for Γ_{X_1} consisting of horodisks.

Let Z_0 be a flute surface with $\alpha_n = \beta_n = 1$ for $n \ge 0$, where α_n and β_n denote the same simple closed geodesics as in Proposition 5.8. Then we construct X_1 by inserting a generalized hyperbolic pair of pants with one cusp and two boundary components α_n^l, α_n^r of lengths 1 along both sides of α_n for $n \ge 1$, as shown in Figure 16. Let A be the removable set for Γ_{X_1}, whose projection $\pi(A)$ on X_1 under Γ_{X_1} is a disjoint union of open cusps with the boundary γ_n' of length 1. Note that the geodesics $\alpha_n^l, \alpha_n^r, \beta_n$ and γ_n' have the same length 1 for all $n \ge 1$. It is not hard to see that $X_1 \setminus \pi(A)$ is contained in a bounded distance of ∂X_1. This implies that X_1 satisfies the geometric condition $(\star)$.
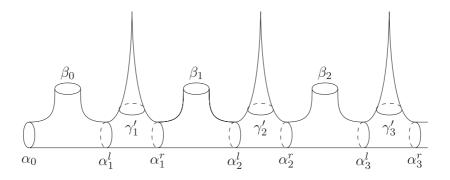

Fig. 16. The flute surface X_1 in Example 5.10 for the case that the removable set for Γ_{X_1} is a disjoint union of horodisks.

Example 5.11. We construct a hyperbolic surface X_2 of infinite type, which satisfies the geometric condition ($\star$) with a removable set A for Γ_{X_2} consisting of neighbourhoods of complete geodesics in $\mathbb{H}^2$ whose radii tend to infinity.

Let Z_0 be a flute surface with $\alpha_n = \beta_n = 1$ for $n \geq 0$, where α_n and β_n denote the same simple closed geodesics as in Proposition 5.8. Then we construct X_2 by inserting a hyperbolic pair of pants with two boundary components α_n^l, α_n^r of lengths 1 and the other boundary component γ_n of length $\frac{1}{2n}$ along both sides of α_n for $n \geq 1$, as indicated in Figure 17. Let A be a removable set for Γ_{X_2}, whose projection $\pi(A)$ on X_2 under Γ_{X_2} is a disjoint union of relatively open annuli with two boundary components γ_n and γ_n', where γ_n' is an equidistant curve of the geodesic γ_n for a distance $r_n = \operatorname{arcsinh}\{1/\sinh(\frac{1}{2}\ell_{\gamma_n}(X_2))\} = \operatorname{arcsinh}\{1/\sinh(\frac{1}{4n})\}$ (this is ensured by the collar lemma, see [9]). Note that γ_n' is not a geodesic and the relation between $\ell_{\gamma_n'}(X_2)$ and $\ell_{\gamma_n}(X_2)$ is given by the following formula (see [9, Example 1.3.2]):

$$\ell_{\gamma_n'}(X_2) = \ell_{\gamma_n}(X_2)\cosh d_{X_2}(\gamma_n, \gamma_n'),$$

where $d_{X_2}(\gamma_n, \gamma_n')$ is the distance between γ_n and γ_n' on X_2. By computation,

$$\ell_{\gamma_n'}(X_2) = \frac{1}{2n}\cosh r_n = \sqrt{\frac{1}{4n^2} + \frac{4}{(4n\sinh(\frac{1}{4n}))^2}} \to 2,$$

as $n \to \infty$. Hence, there exists $n_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ such that

$$2 - \epsilon_0 < \ell_{X_2}(\gamma_n') \leq 2 + \epsilon_0,$$

for all $n \geq n_0$. Combined with the fact that the geodesics α_n^l, α_n^r and β_n have the same length 1 for all $n \geq 1$, it follows that $X_2 \setminus \pi(A)$ is contained in a bounded distance of ∂X_2. This implies that X_2 satisfies the geometric condition ($\star$). We obtain the desired surface X_2.
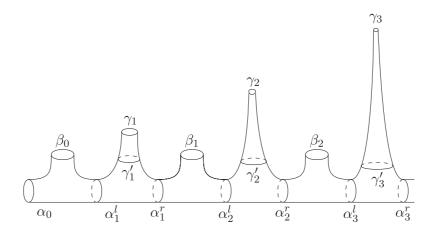

Fig. 17. The flute surface X_2 in Example 5.11 for the case that the removable set for Γ_{X_2} is a disjoint union of neighbourhoods of complete geodesics in $\mathbb{H}^2$.

Example 5.12. We construct a hyperbolic surface X_3 of infinite type, which satisfies the geometric condition ($\star$) with a removable set A for Γ_{X_3} consisting of hyperbolic disks whose radii tend to infinity.

For each integer $n \geq 3$, let T_n be a trirectangle with one angle $\theta_n = \pi/n$ and three right angles. Denote the four consecutive edges of T_n by α_n, b_n, a_n and β_n. Let θ_n be the angle bounded by α_n and β_n (see Figure 18). For convenience, we also denote the lengths of α_n, b_n, a_n, β_n by α_n, b_n, a_n, β_n, respectively. Note that T_n can be uniquely (up to isometries) determined by θ_n and α_n. We choose α_n such that $\sin \theta_n \cosh \alpha_n = 2$ for all $n \geq 3$. By the formula $\cosh a_n = \cosh \alpha_n \sin \theta_n$ for a trirectangle T_n (see [9]), $\cosh a_n = 2$ for all $n \geq 3$.

We claim that in each T_n we have

$$\text{(33)} \qquad \beta_n < \alpha_n.$$

Indeed, by the formulae for a trirectangle T_n (see [9]), we get

$$\text{(34)} \qquad \begin{aligned} \cos \theta_n &= \sinh a_n \sinh b_n, \\ \frac{\cosh a_n}{\cosh b_n} &= \frac{\cosh \alpha_n}{\cosh \beta_n}. \end{aligned}$$

Hence,

$$\sinh b_n = \frac{\cos \theta_n}{\sinh a_n} = \frac{\cos \theta_n}{\sqrt{\cosh^2 a_n - 1}} = \frac{\cos \theta_n}{\sqrt{3}} < 1 < \sqrt{3} = \sinh a_n.$$

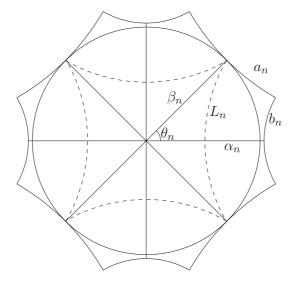Combined with (34), we have $\beta_n < \alpha_n$ for all $n \geq 3$.

Fig. 18. The trirectangle T_n in the right-angled $2n$-polygon P_n for $n = 4$.

Denote by L_n the length of the geodesic perpendicular to α_n through the intersection point of β_n and a_n, as shown in Figure 18. By (33) and a formula for a right-angled triangle (see [9]), we obtain that

$$\sinh L_n = \sin \theta_n \sinh \beta_n < \sin \theta_n \sinh \alpha_n < \sin \theta_n \cosh \alpha_n = \cosh a_n = 2,$$

for all $n \geq 3$. Therefore,

$$\text{(35)} \qquad L_n < \operatorname{arcsinh} 2,$$

for all $n \geq 3$.

Now we construct a right-angled $2n$-polygon P_n by gluing $2n$ copies of T_n along the edges of the common lengths α_n and β_n alternately (see Figure 18). Then P_n has n sides of lengths $2a_n$ and the other n sides of lengths $2b_n$. Denote the $2b_n$-length sides of P_n by e_1, e_2, ..., e_n in the anti-clockwise order. Let B_n be the maximal embedding open hyperbolic disk in P_n whose center is the center of P_n.

By the inequality (33) and the construction of P_n, the radius of B_n is β_n. Note that $\theta_n = \pi/n$, $\cosh a_n = \cosh \alpha_n \sin \theta_n \equiv 2$, it follows that $\alpha_n \to \infty$ as $n \to \infty$. Combined with the formula for a trirectangle T_n (see [9]) that $\cosh \beta_n \sinh a_n = \sinh \alpha_n$, the radius β_n of B_n tends to infinity as $n \to \infty$.

Take another copy P'_n of P_n and denote by e'_1, e'_2, ..., e'_n the $2b_n$-length sides corresponding to the sides e_1, e_2, ..., e_n of P_n. Let B'_n be the maximal embedded open hyperbolic disk in P'_n. Denote by S_n the surface obtained by gluing P_n and P'_n along e_i and e'_i for $i = 1, 2, ..., n$. Then S_n is a hyperbolic surface with n consecutive boundary components $\gamma_1^{(n)}$, $\gamma_2^{(n)}$, ..., $\gamma_n^{(n)}$ of the same length $4a_n = 4 \operatorname{arccosh} 2 > 0$ for all $n \geq 3$. The hyperbolic disks B_n and B'_n are disjoint from each other and tangent to $\gamma_i^{(n)}$ for $i = 1, 2, ..., n$. Moreover, they have the same radius β_n, which tends to infinity as $n \to \infty$.

It is not hard to see that $S_n \setminus (B_n \cup B'_n)$ is within the distance L_n of the boundary ∂S_n of S_n. By (35), we have

(36) $$d(p, \partial S_n) < \operatorname{arcsinh} 2,$$

for all $p \in S_n \setminus (B_n \cup B'_n)$ and all $n \geq 3$.

We construct X_3 by pasting the boundary component $\gamma_1^{(n)}$ of S_n and the boundary component $\gamma_n^{(n+1)}$ of S_{n+1} one by one for $n \geq 3$ (see Figure 19). Let A be a removable set for Γ_{X_3} whose projection $\pi(A)$ on X_3 under Γ_{X_3} is a disjoint union of hyperbolic disks B_n and B'_n over $n \geq 3$. By inequality (36) and the construction of X_3, it is not hard to see that $X_3 \setminus \pi(A)$ is within the distance $L = 2 \operatorname{arcsinh} 2$ of ∂X_3. This implies that X_3 satisfies the geometric condition $(\star)$ and we obtain the desired surface X_3.
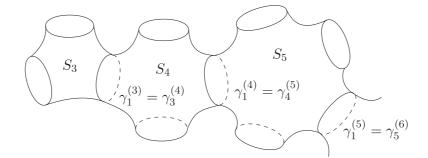

Fig. 19. The hyperbolic surface X_3 in Example 5.12 for the case that the removable set for Γ_{X_3} is a disjoint union of hyperbolic disks.

5.2. The relation between the Shiga's condition and the geometric condition $(\star)$.

To investigate the relation between the two conditions, we first recall some terminology as follows.

We say that $\mathcal{P} = \{C_i\}_{i=1}^{\infty}$ is an *upper-bounded pants decomposition* of X_0 if there exists a constant $M > 0$ such that $\ell_{C_i}(X_0) \leq M$ for each $i \in \mathbb{N}$. Similarly, we say that $\mathcal{P} = \{C_i\}_{i=1}^{\infty}$

is a *lower-bounded pants decomposition* of X_0 if there exists a constant $m > 0$ such that $\ell_{C_i}(X_0) \geq m$ for each $i \in \mathbb{N}$. Furthermore, $\mathcal{P} = \{C_i\}_{i=1}^{\infty}$ is said to be a *bounded pants decomposition* of X_0 if it is both an upper-bounded pants decomposition and a lower-bounded pants decomposition of X_0. Recall that a hyperbolic surface X of infinite type satisfies the *Shiga's condition* (see [28]) if it admits a bounded pants decomposition.

We claim that there is no direct relation between the Shiga's condition and the geometric condition $(\star)$.

Indeed, consider the surface X_0 in Example 5.3 and the flute surface X_0' in Example 5.9. The length of the boundary component β_n of X_0 (resp. X_0') tends to infinity, as $n \to \infty$. In Example 5.11, the surface X_2 has a subsequence of boundary components $\{\gamma_n\}$ whose lengths tend to zero. Therefore, these surfaces X_0, X_0' and X_2 do not satisfy the Shiga's condition while they satisfy the geometric condition $(\star)$.

On the other hand, we can find a complete hyperbolic surface Y_0 of infinite type which has infinitely many geodesic boundary components and satisfies the Shiga's condition but does not satisfy the geometric condition $(\star)$. The surface Y_0 is constructed as follows:
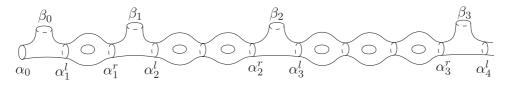

Fig. 20. An example Y_0 which satisfies Shiga's condition but does not satisfy the geometric condition $(\star)$.

Let Z_0 be a flute surface with $\alpha_n = \beta_n = 1$ for $n \geq 0$, where α_n and β_n denote the same simple closed geodesics as in Proposition 5.8. Then we construct Y_0 by inserting a hyperbolic surface of genus n with two geodesic boundary components α_n^l, α_n^r (which admits a pair of pants decomposition with all decomposing curves of length 1) along both sides of α_n for $n \geq 1$, as indicated in Figure 20. It follows easily that Y_0 satisfies the Shiga's condition. However, it follows from the construction of Y_0 and Lemma 3.8 that for any $L > 0$ and any removable set A for Γ_{Y_0}, the projection $\pi(A)$ fails to cover $Y_0 \setminus B(\partial Y_0; L)$, where $B(\partial Y_0; L)$ consists of the points on Y_0 lying within the distance L of ∂Y_0. This implies that Y_0 does not satisfy the geometric condition $(\star)$.

In particular, there exist complete hyperbolic surfaces of infinite type which satisfy both the Shiga's condition and the geometric condition $(\star)$. The surface Z_0 mentioned above is such an example.

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