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Osaka J. Math.
54 (2017), 807–822

GROUPS OF AUTOMORPHISMS OF BORDERED ORIENTABLE KLEIN SURFACES OF TOPOLOGICAL GENUS 2

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(Received June 14, 2016, revised October 6, 2016)

Abstract

In this paper, we obtain the groups of automorphisms of orientable bordered Klein surfaces of topological genus 2. For each of those groups G we determine the values of k such that G acts on a surface with k boundary components. Besides, for each given k we exhibit the groups acting on a surface with k boundary components.

1. Introduction and Preliminaries

A natural extension of the definition of compact Riemann surfaces, a concept that implies orientable and unbordered surfaces, is to allow dianalytic transition functions. This leads to surfaces that may be bordered and, or, non-orientable, endowed with a dianalytic structure. These surfaces are called Klein surfaces. They were already studied by F. Klein himself, and a modern treatment of them can be seen in [1] or [6].

In this work we consider surfaces with non-empty boundary. Klein surfaces are determined topologically by three data, the topological genus g, the number of boundary components k, and $\alpha = 2$ or 1, according to whether the surface is orientable or not. Then $p = \alpha g + k - 1$ is the algebraic genus of the surface. If $p \geq 2$ the automorphism group of the surface is finite, and its order is bounded above by $12(p - 1)$.

A major problem is to determine the groups acting on surfaces of a given topological type. A first step is to obtain bounds for the maximal order of an automorphism group acting for each genus. For bordered surfaces this problem was studied by M. Heins in [15] for $g = 0$. For $k \leq 3$, results were obtained by T. Kato in [16], where an additional list of references can be found. Bounds for punctured surfaces were obtained by T. Arakawa in [2] in the case of Riemann surfaces, and by E. Bujalance and G. Gromadzki in [8] for unbordered Klein surfaces.

Returning to the general problem of obtaining the groups of automorphisms of surfaces according to the topological type, it has been solved for Riemann surfaces of low topological genus. For bordered Klein surfaces, results have been obtained for low algebraic genus. In terms of the topological type, the list of groups was obtained for orientable surfaces of topological genus 0 in [4] and for surfaces with connected boundary, that is to say $k = 1$, in Chapter 5 of [6].

2010 Mathematics Subject Classification. Primary 57M60; Secondary 20F05, 20H10, 30F50.

The first and third authors are partially supported by MTM2014-55812 and the second one by UCM910444, MTM2014-55565.

We deal here with the next feasible step, orientable surfaces of topological genus $g = 2$. The Abelian groups acting on them were obtained in [7], and the complete list is given in the present paper.

Due to the functorial equivalence between bordered Klein surfaces and real algebraic curves, these results apply directly to real algebraic curves of genus 2 which disconnect its complexified, see the Appendix in [6].

In the study of Klein surfaces and their automorphism groups the non-Euclidean crystallographic groups (NEC groups in short) play an essential role. An NEC group Γ is a discrete subgroup of $\mathcal{G}$ (the full group of isometries of the hyperbolic plane $\mathcal{H}$) with compact quotient $\mathcal{H}/\Gamma$. For a Klein surface X with $p \geq 2$ there exists an NEC group Γ, such that $X = \mathcal{H}/\Gamma$, [21].

A finite group G of order N is an automorphism group of a Klein surface $X = \mathcal{H}/\Gamma$ if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ with index N and $G = \Lambda/\Gamma$. If G is a finite group there exists a bordered Klein surface X such that G is an automorphism group of X, [18].

First, we give some preliminaries about NEC groups and Klein surfaces.

An NEC group Γ is a discrete subgroup of isometries of the hyperbolic plane $\mathcal{H}$, including orientation-reversing elements, with compact quotient $X = \mathcal{H}/\Gamma$. Each NEC group Γ has associated a signature [17]:

$$\sigma(\Gamma) = (g, \pm, [m_1, \dots, m_r], \{(n_{i,1}, \dots, n_{i,s_i}), i = 1, \dots, k\}), \tag{1}$$

where g, k, r, m_i, $n_{i,j}$ are integers verifying g, k, $r \geq 0$, m_i, $n_{i,j} \geq 2$. If the sign is "$-$", then $g \geq 1$. The numbers m_i are the *proper periods*. The brackets $(n_{i,1}, \dots, n_{i,s_i})$ are the *period-cycles*. Numbers $n_{i,j}$ are the periods of the period-cycle $(n_{i,1}, \dots, n_{i,s_i})$, also called *link-periods*. We will denote by $[-]$, $(-)$ and $\{-\}$ the cases when $r = 0$, $s_i = 0$ and $k = 0$, respectively. When a proper period or link-period t is repeated r_t times we will write t^{r_t}. Analogously $(-)^s$ means s empty period-cycles.

The signature determines a presentation of Γ, [22]:
Generators:

$$\begin{array}{lll}
x_i & i = 1, \dots, r; & \\
e_i & i = 1, \dots, k; & \\
c_{i,j} & i = 1, \dots, k; & j = 0, \dots, s_i; \\
a_i, b_i & i = 1, \dots, g; & (\text{if } \sigma \text{ has sign " + "}); \\
d_i & i = 1, \dots, g; & (\text{if } \sigma \text{ has sign " − "}).
\end{array}$$

Relations:

$$\begin{array}{ll}
x_i^{m_i} = 1; & i = 1, \dots, r; \\
c_{i,j-1}^2 = c_{i,j}^2 = (c_{i,j-1}c_{i,j})^{n_{i,j}} = 1 & i = 1, \dots, k;\ j = 1, \dots, s_i; \\
e_i^{-1} c_{i,0} e_i c_{i,s_i} = 1; & i = 1, \dots, k; \\
\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g (a_i b_i a_i^{-1} b_i^{-1}) = 1; & (\text{if } \sigma \text{ has sign " + "}); \\
\prod_{i=1}^r x_i \prod_{i=1}^k e_i \prod_{i=1}^g d_i^2 = 1; & (\text{if } \sigma \text{ has sign " − "}).
\end{array}$$

The isometries x_i are elliptic, e_i, a_i, b_i are hyperbolic, $c_{i,j}$ are reflections and d_i are glide-reflections.

Every NEC group Γ with signature (1) has associated a fundamental region whose area

$\mu(\Gamma)$, called *area of the group*, is

$$\mu(\Gamma) = 2\pi\left(\alpha g + k - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1 - \frac{1}{n_{i,j}}\right)\right), \tag{2}$$

with $\alpha = 2$ or 1 according to the sign be "+" or "−". An NEC group with signature (1) actually exists if and only if the right hand side of (2) is greater than 0.

We denote by $|\Gamma|$ the expression $\mu(\Gamma)/2\pi$ and call it the *reduced area* of Γ.

If Γ is a subgroup of an NEC group Γ' of finite index N, then also Γ is an NEC group and the following Riemann-Hurwitz formula holds:

$$\mu(\Gamma) = N\mu(\Gamma'). \tag{3}$$

Let X be a Klein surface of topological genus g with $k > 0$ boundary components. Then by [21] there exists an NEC group Γ with signature

$$\sigma(\Gamma) = (g, \pm, [-], \{(-), \overset{k}{\ldots}, (-)\}),$$

such that $X = \mathcal{H}/\Gamma$. The sign is "+" or "−" according to X be orientable or not. An NEC group with this signature is called a *bordered surface group*.

For each automorphism group G of a surface $X = \mathcal{H}/\Gamma$ of algebraic genus $p \geq 2$ there exists an NEC group Λ such that $G = \Lambda/\Gamma$ where $\Gamma \subset \Lambda \subset N_{\mathcal{G}}$, [18], and $N_{\mathcal{G}}$ denotes the normalizer of Γ in the group $\mathcal{G}$, the full group of isometries of $\mathcal{H}$.

We recall that the signature of Λ must contain an empty period-cycle or a period-cycle with two consecutive link-periods equal to 2, see [9].

2. The groups of automorphisms

In this paper we obtain the automorphism group of compact orientable Klein surfaces of topological genus 2. The corresponding result for genus 0 was obtained by Bujalance in [4]. In that work he made use of the following result which relates bordered and unbordered surfaces, see [14, Theorem D].

Proposition 2.1. *Let X be a bordered Klein surface. Then it is possible to embed X in a Klein surface X^* without boundary of the same topological genus, so that every automorphism of X extends to an automorphism of X^*.*

By using this result, our starting point is the list of orientation-preserving automorphism groups of compact Riemann surfaces of genus 2 which was obtained by Broughton in [3]. The list of groups is as in Table 1 below, where C_n denotes the cyclic group or order n, D_n the dihedral group of order $2n$, Q the quaternion group and DC_3 the dicyclic group of order 12. For the group of order 16 we follow the Hall-Senior notation, and the group $(4, 6 \mid 2, 2)$ is defined in [12].

The second step is to consider automorphism groups including orientation-reversing involutions, called symmetries. Bujalance and Singerman obtained in [10] the full automorphism group of all Riemann surfaces of genus 2 admitting symmetries. Their list provides the new groups of Table 2.

In the list in [10] the group of order 48 is called G^*_{24} but its presentation in page 517 makes it clear that the group is $D_3 \times D_4$. In the same way, the group of order 96 is called G^*_{48}. Since

Table 1.

G	o(G)
C_2	2
C_3	3
C_4	4
$C_2 \times C_2$	4
C_5	5
C_6	6
D_3	6
C_8	8
Q	8
D_4	8
C_{10}	10
$C_2 \times C_6$	12
DC_3	12
D_6	12
$\Gamma_3 a_2$	16
$(4, 6 \mid 2, 2)$	24
$SL(2, 3)$	24
$GL(2, 3)$	48

Table 2.

G	o(G)
$C_2 \times C_2 \times C_2$	8
$C_2 \times D_4$	16
D_8	16
D_{10}	20
$C_2 \times D_6$	24
D_{12}	24
$D_3 \times D_4$	48
$GL(2, 3) \rtimes C_2$	96

the authors indicate how to obtain a presentation, this allows to identify the group. The details will be given when this group be studied.

The groups in Table 2 are the full automorphism group of a surface. Hence their subgroups also act on the same surface and so new groups need still to be added, and they are as follows

Table 3.

G	o(G)	Subgroup of
$C_2 \times C_4$	8	$C_2 \times D_4$
D_5	10	D_{10}
C_{12}	12	D_{12}
$C_3 \times D_4$	24	$D_3 \times D_4$
$C_4 \times D_3$	24	$D_3 \times D_4$
$\Gamma_6 a_1$	32	$GL(2, 3) \rtimes C_2$
P_{48}	48	$GL(2, 3) \rtimes C_2$

The group of order 32 is denoted according to Hall and Senior. The group of order 48 is a semidirect product $SL(2, 3) \rtimes C_2$ called P_{48} in [20]. Details on it will be given in the adequate place.

Finally, it is necessary to consider groups including orientation-reversing automorphisms but not symmetries. However, the unique such group is C_4, see [5], and so no new group appears.

In the following sections we consider each of these thirty three groups and we obtain for which values of k they act on an orientable Klein surface of topological genus 2 with k boundary components.

The notation throughout the paper will be established here. The main tools in order to obtain k appear in [6]. The first one is Theorem 2.4.4 there, which was obtained in [11].

We partially restate it as follows.

Theorem 2.2. *Let Γ be a bordered surface NEC group with signature $\sigma(\Gamma) = (g, \pm, [-], \{(-)^k\})$. Let N be an even integer. Let Γ' be another NEC group with signature $\sigma(\Gamma') = (g', \pm, [m_1, \ldots, m_r], \{(-)^{s'}\})$ containing Γ as a normal subgroup with factor of order N. Let $c_{i,0} \in \Gamma$ for $1 \le i \le p'$ and $c_{i,0} \notin \Gamma$ for $p' + 1 \le i \le s'$. Let l_i be the order of $\Gamma e_i \in \Gamma'/\Gamma$ for $i \le p'$. Then*

i) *Each m_j divides N, for $j = 1, \ldots, r$.*
ii) $k = N \displaystyle\sum_{i=1}^{p'} \frac{1}{l_i}.$

In the general case, the signature of Γ' has k' period-cycles not necessarily all empty. We split then the k' period-cycles of Γ' as follows. There are s' empty period-cycles and h non-empty period-cycles. In order to simplify expressions we fix now some notations. We call $\{\tau_1, \ldots, \tau_\lambda\}$ the increasing ordered set of orders of the elements of the group G. Now

$$s' = t_{\tau_1} + t_{\tau_2} + \cdots + t_{\tau_\lambda} + t_0,$$

where t_{τ_i} is the number of period-cycles for which the reflection belongs to Γ and the class of the hyperbolic generator has order N/τ_i in Γ'/Γ, and t_0 is the number of period-cycles such that the corresponding reflection does not belong to Γ. Let us observe that $t_0 = s' - p'$. Throughout the paper the value of t_0 will always turn to be 0. In the same way the signature of Γ' will have sign "+".

If there are non-empty period-cycles in the signature of Γ' the computation of k is a little more involved. The key point is that the non-empty period-cycles in Γ' can produce period-cycles of Γ. The number of the latter was obtained in Theorem 2.3.3 (see also the last sentence of Remark 2.3.7) in the book [6]. This result can be restated in the following way.

Theorem 2.3. *Let Γ be a bordered surface NEC group, N an even integer, and Γ' an NEC group containing Γ as a normal subgroup with quotient of order N. Let $(c_{i,0}, \ldots, c_{i,s_i})$ be one of the period-cycles of Γ', such that the reflection $c_{i,j} \in \Gamma$ while $c_{i,j-1}, c_{i,j+1} \notin \Gamma$ for $j \in J \subset \{1, 2, \ldots, s_i - 1\}$. Then the number of period-cycles of Γ produced by this period-cycle of Γ' is $\displaystyle\sum_{j \in J} \frac{N}{2n(j)}$, where we call $n(j)$ the order of $\Gamma(c_{i,j-1}c_{i,j+1})$ as an element of Γ'/Γ.*

3. The main result

The twelve Abelian groups appearing in Tables 1, 2 and 3, have been studied in [7]. Since the results about these groups will be used in what follows, we include in the forthcoming main Theorem.

The non-Abelian groups are the following. The dihedral groups D_3, D_4, D_5, D_6, D_8, D_{10} and D_{12}; and the remaining fourteen groups: Q, DC_3, $\Gamma_3 a_2$, $C_2 \times D_4$, $(4, 6 \mid 2, 2)$, $SL(2, 3)$,

$C_2 \times D_6$, $C_3 \times D_4$, $C_4 \times D_3$, $\Gamma_6 a_1$, $GL(2,3)$, $D_3 \times D_4$, P_{48} and $GL(2,3) \rtimes C_2$.

The main result of this paper is the following:

Theorem 3.1. *Let G be a group of automorphisms of an orientable Klein surface of topological genus* 2 *with* $k > 0$ *boundary components. Then the group G and the admissible values of k for each G are given in the next Table.*

Table 4.

g	k
C_2	All k
C_3	All k
C_4	All k
$C_2 \times C_2$	All k
C_5	$k \equiv 0, 1, 2, 3 \pmod 5$
C_6	All k
D_3	All k
C_8	$k \equiv 0, 1, 2 \pmod 4$
Q	k even
D_4	$k \equiv 0, 1, 2 \pmod 4$
$C_2 \times C_4$	k even
$C_2 \times C_2 \times C_2$	k even
C_{10}	$k \equiv 0, 1, 2, 3 \pmod 5$
D_5	$k \equiv 0, 1, 2, 3 \pmod 5$
C_{12}	$k \equiv 0, 4, 6, 10 \pmod{12}$
$C_2 \times C_6$	$k \equiv 0, 2, 3, 4, 6, 7, 8, 10 \pmod{12}$
DC_3	$k \equiv 0, 3, 4, 6, 7, 10 \pmod{12}$
D_6	All k
$\Gamma_3 a_2$	k even
$C_2 \times D_4$	k even
D_8	$k \equiv 0, 1, 2 \pmod 4$
D_{10}	$k \equiv 0, 1, 2, 3 \pmod 5$
$(4, 6 \mid 2, 2)$	$k \equiv 0, 3, 4, 6, 7, 10 \pmod{12}$
$SL(2,3)$	$k \equiv 0, 6, 8, 14, 16, 22 \pmod{24}$
$C_2 \times D_6$	k even
D_{12}	$k \equiv 0, 4, 6, 10 \pmod{12}$
$C_3 \times D_4$	$k \equiv 0, 4, 6, 10, 12, 16 \pmod{24}$
$C_4 \times D_3$	$k \equiv 0, 4, 6, 10, 12, 18 \pmod{24}$
$\Gamma_6 a_1$	k even
$GL(2,3)$	$k \equiv 0, 6, 8, 14, 16, 22 \pmod{24}$
$D_3 \times D_4$	$k \equiv 0, 4, 6, 12 \pmod{24}$
P_{48}	$k \equiv 0, 6, 16, 22 \pmod{24}$
$GL(2,3) \rtimes C_2$	$k \equiv 0, 6, 16, 22 \pmod{24}$

Proof of Theorem 3.1.

The proof splits into a case-by-case study for each group. Since it involves twenty-one groups we have chosen a significant sample of groups in order to show the techniques used to obtain the epimorphisms. In particular, if a group G in the sample contains a subgroup H, we will use the values of k stated in the Theorem for H, in order to prove the result for G. We shall point it out when necessary.

We are going to give a complete proof of the result for the following groups: one dihedral group, namely D_5, the group $(4, 6 \mid 2, 2)$, the direct product $C_3 \times D_4$, and the semidirect products P_{48} and $GL(2, 3) \rtimes C_2$. For the remaining groups, one can apply adequately similar techniques.

Group D_5

We take two generators X and Y of order 2, such that XY has order 5. We begin by taking signatures of Γ' with all period-cycles empty. Then $\sigma(\Gamma')$ is

$$(g', \pm, [2^{r_2}, 5^{r_5}], \{(-)^{t_2}, (-)^{t_5}, (-)^{t_{10}}\})$$

with $k = 2t_2 + 5t_5 + 10t_{10}$. By (3) we have

$$22 = 10\alpha g' + 5(r_2 + t_5) + 8(r_5 + t_2)$$

which has no solution. Hence there must be non-empty period-cycles in Γ'. Since the elements of order 2 in D_5 have product of order 1 or 5, by Theorem 2.3 they can give $\frac{10}{2\cdot 1} = 5$ or $\frac{10}{2\cdot 5} = 1$ period-cycles in Γ. We look first for 1 period-cycle. Then, Γ' has $h \geq 1$ non-empty period-cycles, and there are at least two consecutive $n_{i,j}$ equal to 2 in one period-cycle. Applying (3), we have

$$23 = 10\alpha g' + 5(r_2 + t_5) + 8(r_5 + t_2) + 10h + 5\sum(1 - \frac{1}{n_{i,j}})$$

and we obtain the following solutions: $\alpha = 2$, $g' = 0$, $h = 1$, and either $r_5 + t_2 = 1$ and the non-empty period-cycle is $(2, 2)$, or all $r_i = t_j = 0$ and the period-cycle is $(2, 2, 5, 5)$. Among all those possibilities we take the following two signatures which cover the two searched values of k:

a1	$(0, +, [5], \{(2, 2), (-)^{t_{10}}\})$	$k = 10t_{10} + 1$
a2	$(0, +, [-], \{(2, 2), (-)^{t_2=1}, (-)^{t_{10}}\})$	$k = 10t_{10} + 3$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}) \to (YX, XY, X, 1, YXY)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}) \to (XY, YX, X, 1, YXY)$$

where the canonical generators of Γ' that are omitted are mapped onto 1. We follow this convention throughout the paper.

Now we try to get 5 period-cycles. Acting as above we have $\alpha = 2$, $g' = 0$, $h = 1$, and either $r_5 + t_2 = 1$ and the period-cycle is $(2, 2, 5)$, or all $r_i = t_j = 0$ and the period-cycle is $(2, 2, 5, 5, 5)$. Then we apply Theorem 2.3 and choose the two following signatures.

b1	$(0, +, [5], \{(2, 2, 5), (-)^{t_{10}}\})$	$k = 10t_{10} + 5$
b2	$(0, +, [-], \{(2, 2, 5), (-)^{t_2=1}, (-)^{t_{10}}\})$	$k = 10t_{10} + 7$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (YX, XY, X, 1, X, YXY)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (XY, YX, X, 1, X, YXY)$$

Now we suppose that several reflections in the period-cycle of Γ' belong to Γ. In order to get the classes 6, 8 (mod 10) we take the signatures

c1	$(0, +, [-], \{(2, 2, 2, 2, 5, 5), (-)^{t_{10}}\})$	$k = 10t_{10} + 6$
c2	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2, 2), (-)^{t_{10}}\})$	$k = 10t_{10} + 8$

The respective epimorphisms are defined by

$$\theta_1 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (X, 1, XYX, 1, XYX, Y, X)$$
$$\theta_2 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, c_{1,8}) \to (X, 1, X, 1, Y, 1, XYX, 1, X)$$

Finally, we consider that Γ' has non-empty period-cycles but all period-cycles of Γ come from empty period-cycles in Γ'. Then by (3) we have $\alpha = 2$, $g' = 0$, $r_5 + t_2 = 1$, $h = 1$, and the non-empty period-cycle is (5). So we have the signatures

d1	$(0, +, [5], \{(5), (-)^{t_{10}}\})$	$k = 10t_{10}$
d2	$(0, +, [-], \{(5), (-)^{t_2=1}, (-)^{t_{10}}\})$	$k = 10t_{10} + 2$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}) \to (YX, XY, X, YXY)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}) \to (XY, YX, X, YXY)$$

The classes 4, 9 (mod 10) cannot be obtained because the group C_5 does not act on surfaces with $k \equiv 4 \pmod 5$.

Group $(4, 6 \mid 2, 2)$

This group has elements of orders 2, 3, 4 and 6, and a presentation given by generators X and Y and relations $X^4 = Y^6 = (XY)^2 = (X^{-1}Y)^2 = 1$.

Also we begin with empty period-cycles in the signature of Γ' which is

$$(g', \pm, [2^{r_2}, 3^{r_3}, 4^{r_4}, 6^{r_6}], \{(-)^{t_4}, (-)^{t_6}, (-)^{t_8}, (-)^{t_{12}}, (-)^{t_{24}}\})$$

with $k = 4t_4 + 6t_6 + 8t_8 + 12t_{12} + 24t_{24}$. Applying (3) we have

$$50 = 24\alpha g' + 12(r_2 + t_{12}) + 16(r_3 + t_8) + 18(r_4 + t_6) + 20(r_6 + t_4).$$

This equality has two solutions:

i)	$\alpha = 2,$	$g' = 0,$	$r_2 + t_{12} = 1,$	$r_4 + t_6 = 1,$	$r_6 + t_4 = 1$
ii)	$\alpha = 2,$	$g' = 0,$		$r_4 + t_6 = 1,$	$r_3 + t_8 = 2$

We are going to discard the solution ii). The group $(4, 6 \mid 2, 2)$ has only two elements of order 3, so that their product cannot have order 4, and so the desired epimorphism does not exist.

For solution i) we have the following signatures:

a1	$(0, +, [2, 4, 6], \{(-)^{t_{24}}\})$	$k = 24t_{24}$
a2	$(0, +, [2, 4], \{(-)^{t_4=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 4$
a3	$(0, +, [2, 6], \{(-)^{t_6=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 6$
a4	$(0, +, [2], \{(-)^{t_4=1}, (-)^{t_6=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 10$
a5	$(0, +, [4, 6], \{(-)^{t_{12}=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 12$
a6	$(0, +, [4], \{(-)^{t_4=1}, (-)^{t_{12}=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 16$
a7	$(0, +, [6], \{(-)^{t_6=1}, (-)^{t_{12}=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 18$
a8	$(0, +, [-], \{(-)^{t_4=1}, (-)^{t_6=1}, (-)^{t_{12}=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 22$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, x_2, x_3) \to (XY, X, Y)$$
$$\theta_2 : (x_1, x_2, e_1) \to (XY, X, Y)$$
$$\theta_3 : (x_1, x_2, e_1) \to (YX, Y, X)$$
$$\theta_4 : (x_1, e_1, e_2) \to (YX, Y, X)$$
$$\theta_5 : (x_1, x_2, e_1) \to (X, Y, XY)$$
$$\theta_6 : (x_1, e_1, e_2) \to (X, Y, XY)$$
$$\theta_7 : (x_1, e_1, e_2) \to (Y, X, YX)$$
$$\theta_8 : (e_1, e_2, e_3) \to (Y, X, YX)$$

This group contains the group DC_3. Since DC_3 does not act on surfaces with $k \equiv 1, 2, 5, 8, 9, 11 \pmod{12}$, we need only to check the classes 3, 7, 15 and 19 (mod 24).

In order to make clear the statements, we use that the group $(4, 6 \mid 2, 2)$ is a subgroup of $D_3 \times D_4$. For, let A and B be the generators of order 2 of D_3, and C and D the generators of order 2 of D_4. Then $X = ACD$, $Y = ABC$ satisfy the relations of the presentation of $(4, 6 \mid 2, 2)$. We now define the elements of order 2 $R = Y^3 = C$, $S = XY^3 = ACDC$, $T = XY = BCDC$. These elements generate $(4, 6 \mid 2, 2)$ because $SR = ACD = X$, $RST = ABC = Y$.

We can already take the following signatures:

b1	$(0, +, [-], \{(2, 2, 3, 4), (-)^{t_{24}}\})$	$k = 24t_{24} + 3$
b2	$(0, +, [-], \{(2, 2, 2, 2, 4), (-)^{t_{24}}\})$	$k = 24t_{24} + 7$
b3	$(0, +, [-], \{(2, 2, 2, 2, 3, 4), (-)^{t_{24}}\})$	$k = 24t_{24} + 15$
b4	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 4), (-)^{t_{24}}\})$	$k = 24t_{24} + 19$

The respective epimorphisms are defined by

$$\theta_1 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (R, 1, S, T, R)$$
$$\theta_2 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}) \to (R, 1, S, 1, T, R)$$
$$\theta_3 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (R, 1, R, 1, S, T, R)$$
$$\theta_4 : (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}) \to (R, 1, R, 1, S, 1, T, R)$$

Group $C_3 \times D_4$

We denote by A the generator of C_3, and by X and Y the generators of D_4 of order 2. If Γ' has only empty period-cycles, its signature is

$$(g', \pm, [2^{r_2}, 3^{r_3}, 4^{r_4}, 6^{r_6}, 12^{r_{12}}], \{(-)^{t_2}, (-)^{t_4}, (-)^{t_6}, (-)^{t_8}, (-)^{t_{12}}, (-)^{t_{24}}\})$$

with $k = 2t_2 + 4t_4 + 6t_6 + 8t_8 + 12t_{12} + 24t_{24}$. Then by (3) we have

$$50 = 24\alpha g' + 12(r_2 + t_{12}) + 16(r_3 + t_8) + 18(r_4 + t_6) + 20(r_6 + t_4) + 22(r_{12} + t_2)$$

The solutions of this equality satisfy $\alpha = 2$, $g' = 0$ and the remaining parameters are

$r_{12} + t_2 = 1$	$r_3 + t_8 = 1$	$r_2 + t_{12} = 1$
$r_6 + t_4 = 1$	$r_4 + t_6 = 1$	$r_2 + t_{12} = 1$
$r_4 + t_6 = 1$	$r_3 + t_8 = 2$	

In none of the cases the epimorphism exists because there are not two elements of orders 2 and 3 with product of order 12, nor of orders 2 and 4 with product of order 6, nor two elements of order 3 with product of order 4.

Hence there are non-empty period-cycles in Γ'. Two elements of order 2 in G have product with order 1, 2 or 4, and so they provide 12, 6 or 3 period-cycles in Γ. We begin looking for 3 period-cycles. The relation (3) does not provide any solution.

Now we try to get 6 period-cycles. Then we obtain the unique solution $\alpha = 2$, $g' = 0$, $h = 1$, $r_6 + t_4 = 1$ and the period-cycle is $(2, 2)$, and so we take the signatures

a1	$(0, +, [6], \{(2, 2), (-)^{t_{24}}\})$	$k = 24t_{24} + 6$
a2	$(0, +, [-], \{(2, 2), (-)^{t_4=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 10$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}) \to (AX, A^2X, Y, 1, XYX)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}) \to (AX, A^2X, Y, 1, XYX)$$

Finally we look for 12 period-cycles. We have $\alpha = 2$, $g' = 0$, $h = 1$, $r_6 + t_4 = 1$ and the period-cycle is $(2, 2, 2)$. So we take the signatures

b1	$(0, +, [6], \{(2, 2, 2), (-)^{t_{24}}\})$	$k = 24t_{24} + 12$
b2	$(0, +, [-], \{(2, 2, 2), (-)^{t_4=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 16$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (AX, A^2X, Y, 1, Y, XYX)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (AX, A^2X, Y, 1, Y, XYX)$$

If there are non-empty period-cycles in Γ' not providing period-cycles in Γ, applying once again (3) we obtain $\alpha = 2$, $g' = 0$, $h = 1$, $r_6 + t_4 = 1$ and the period-cycle is (2). We take the signatures

c1	$(0, +, [6], \{(2), (-)^{t_{24}}\})$	$k = 24t_{24}$
c2	$(0, +, [-], \{(2), (-)^{t_4=1}, (-)^{t_{24}}\})$	$k = 24t_{24} + 4$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}) \to (AX, A^2X, Y, XYX)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}) \to (AX, A^2X, Y, XYX)$$

We have obtained the classes 0, 4, 6, 10, 12 and 16 (mod 24). This group contains $C_2 \times C_6$, which does not act on surfaces with $k \equiv 1, 5, 9, 11 \pmod{12}$. Hence only the classes 2, 3, 7, 8, 14, 15, 18, 19, 20 and 22 (mod 24) must be checked. Keeping in mind that the group $C_3 \times D_4$ cannot be generated by involutions, it is easy to realize that the equality (3) has no solution for these values of k, allowing to define an epimorphism with kernel Γ.

Group P_{48}

The group P_{48} receives this name in [20]. It is a semidirect product $SL(2,3) \rtimes C_2$, different from $GL(2,3)$. Coxeter made a thorough study of this group in [13], describing it as the subgroup of S_{16} generated by the elements

$$A = (1,7,4)(2,5,8)(a,b,f)(c,h,g), \quad C = (1,h,8,a)(2,c,6,g,4,d,7,f,3,b,5,e).$$

The element $AC = (1,f)(2,e)(3,b)(4,h)(5,a)(6,g)(7,d)(8,c)$ has order 2 and evidently A and AC generate the group G.

This group has elements of orders 2, 3, 4, 6 and 12. Hence if Γ' has only empty period-cycles, its signature is

$$(g', \pm, [2^{r_2}, 3^{r_3}, 4^{r_4}, 6^{r_6}, 12^{r_{12}}], \{(-)^{t_4}, (-)^{t_8}, (-)^{t_{12}}, (-)^{t_{16}}, (-)^{t_{24}}, (-)^{t_{48}}\})$$

with $k = 4t_4 + 8t_8 + 12t_{12} + 16t_{16} + 24t_{24} + 48t_{48}$, and so applying (3) we obtain

$$98 = 48\alpha g' + 24(r_2 + t_{24}) + 32(r_3 + t_{16}) + 36(r_4 + t_{12}) + 40(r_6 + t_8) + 44(r_{12} + t_4)$$

what has no solutions. Thus Γ' must have non-empty period-cycles. Since the product of two elements of order 2 in P_{48} has order 1, 2, or 4, they can give 24, 12 or 6 period-cycles in Γ.

We look first for 6 period-cycles. Then the relation (3) gives $\alpha = 2$, $g' = 0$, $h = 1$, $r_3 + t_{16} = 1$ and the period-cycle is $(2,2)$. We show the signatures

a1	$(0, +, [3], \{(2,2), (-)^{t_{48}}\})$	$k = 48t_{48} + 6$
a2	$(0, +, [-], \{(2,2), (-)^{t_{16}=1}, (-)^{t_{48}}\})$	$k = 48t_{48} + 22$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}) \to (A^{-1}, A, AC, 1, CA)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}) \to (A, A^{-1}, AC, 1, CA)$$

Observe that $(AC)(CA) = (1,3,8,6)(2,5,7,4)(a,e,h,d)(b,f,g,c)$ has order 4 as desired.

If we look for 12 period-cycles, the relation (3) has no solution. So we go to 24 period-cycles. Then $\alpha = 2$, $g' = 0$, $h = 1$, $r_3 + t_{16} = 1$ and the period-cycle is $(2,2,4)$. We have the signatures

b1	$(0, +, [3], \{(2,2,4), (-)^{t_{48}}\})$	$k = 48t_{48} + 24$
b2	$(0, +, [-], \{(2,2,4), (-)^{t_{16}=1}, (-)^{t_{48}}\})$	$k = 48t_{48} + 40$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (A^{-1}, A, AC, 1, AC, CA)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (A, A^{-1}, AC, 1, AC, CA)$$

Finally, if the non-empty period-cycles in Γ' do not produce period-cycles in Γ, we obtain $\alpha = 2$, $g' = 0$, $h = 1$, $r_3 + t_{16} = 1$ and the period-cycle is (4), and so we have

c1	$(0, +, [3], \{(4), (-)^{t_{48}}\})$	$k = 48t_{48}$
c2	$(0, +, [-], \{(4), (-)^{t_{16}=1}, (-)^{t_{48}}\})$	$k = 48t_{48} + 16$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}) \to (A^{-1}, A, AC, CA)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}) \to (A, A^{-1}, AC, CA)$$

The group P_{48} contains $SL(2,3)$. So we need to look for classes 8, 14, 30, 32, 38 and 46 (mod 48), by allowing more than one reflection in the non-empty period-cycle of Γ' to belong to Γ. If we take two elements of order 2 in P_{48}, their product has order 1, 2 or 4, and so they produce 6, 12 or 24 period-cycles. Hence only the class 30 is attainable in this way. Applying the equality (3) we get $\alpha = 2$, $g' = 0$, $r_3 + t_{16} = 1$, $h = 1$, and the period-cycle is $(2, 2, 2, 2)$. So we take the signatures:

d1	$(0, +, [3], \{(2, 2, 2, 2), (-)^{t_{48}}\})$	$k = 48t_{48} + 30$
d2	$(0, +, [-], \{(2, 2, 2, 2), (-)^{t_{16}=1}(-)^{t_{48}}\})$	$k = 48t_{48} + 46$

The respective epimorphisms are defined by

$$\theta_1 : (x_1, e_1, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (A^2, A, AC, 1, AC, 1, CA)$$
$$\theta_2 : (e_1, e_2, c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (A, A^2, AC, 1, AC, 1, CA)$$

Group $GL(2,3) \rtimes C_2$

This group was obtained in [10] as an extension of $GL(2,3)$. The authors call it G^*_{48}, and indicate how to obtain for it a presentation, which turns to be given by generators A, B, C satisfying

$$A^2 = B^2 = C^2 = (AB)^8 = (BC)^3 = (AC)^2 = (AB)^{-3}(BC) = 1$$

From this presentation we know that this group is an M^*-group, and so it has real genus 9. The unique group of order 96 and real genus 9 is called T_{96} in [19], and was studied by Coxeter in [13]. The elements of the group have orders 2, 3, 4, 6, 8 and 12, and so if Γ' has only empty period-cycles, its signature is

$$(g', \pm, [2^{r_2}, 3^{r_3}, 4^{r_4}, 6^{r_6}, 8^{r_8}, 12^{r_{12}}], \{(-)^{t_8}, (-)^{t_{12}}, (-)^{t_{16}}, (-)^{t_{24}}, (-)^{t_{32}}, (-)^{t_{48}}, (-)^{t_{96}}\})$$

with $k = 8t_8 + 12t_{12} + 16t_{16} + 24t_{24} + 32t_{32} + 48t_{48} + 96t_{96}$. The relation (3) has no solution in this case. Hence there must be non-empty period-cycles in Γ'.

Since G contains P_{48}, if it acts on a surface with k boundary components, then the latter group must also act on that surface. Hence, we need only to look on $k \equiv 0, 6, 16, 22, 24, 30, 40, 46, 48, 54, 64, 70, 72, 78, 88, 94 \pmod{96}$.

Applying (3) we obtain the following list of signatures which provide the sixteen classes:

a1	$(0, +, [-], \{(2, 3, 8), (-)^{t_{96}}\})$	$k = 96t_{96}$
a2	$(0, +, [-], \{(2, 2, 2, 3), (-)^{t_{96}}\})$	$k = 96t_{96} + 6$
a3	$(0, +, [-], \{(2, 2, 2, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 16$
a4	$(0, +, [-], \{(2, 2, 2, 2, 2), (-)^{t_{96}}\})$	$k = 96t_{96} + 22$
a5	$(0, +, [-], \{(2, 2, 3, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 24$
a6	$(0, +, [-], \{(2, 2, 2, 2, 3), (-)^{t_{96}}\})$	$k = 96t_{96} + 30$
a7	$(0, +, [-], \{(2, 2, 2, 2, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 40$
a8	$(0, +, [-], \{(2, 2, 2, 2, 2, 2), (-)^{t_{96}}\})$	$k = 96t_{96} + 46$
a9	$(0, +, [-], \{(2, 2, 2, 3, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 48$
a10	$(0, +, [-], \{(2, 2, 2, 2, 2, 3), (-)^{t_{96}}\})$	$k = 96t_{96} + 54$
a11	$(0, +, [-], \{(2, 2, 2, 2, 2, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 64$
a12	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2), (-)^{t_{96}}\})$	$k = 96t_{96} + 70$
a13	$(0, +, [-], \{(2, 2, 2, 2, 3, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 72$
a14	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 3), (-)^{t_{96}}\})$	$k = 96t_{96} + 78$
a15	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 8), (-)^{t_{96}}\})$	$k = 96t_{96} + 88$
a16	$(0, +, [-], \{(2, 2, 2, 2, 2, 2, 2, 2), (-)^{t_{96}}\})$	$k = 96t_{96} + 94$

The respective epimorphisms are defined by

$$\begin{aligned}
\theta_1 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}) \to (A, C, B, A)\\
\theta_2 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (B, 1, A, C, B)\\
\theta_3 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (B, 1, C, A, B)\\
\theta_4 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}) \to (A, 1, B, 1, C, A)\\
\theta_5 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}) \to (A, 1, C, B, A)\\
\theta_6 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}) \to (B, 1, A, 1, C, B)\\
\theta_7 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}) \to (A, 1, C, 1, B, A)\\
\theta_8 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (A, 1, B, 1, C, 1, A)\\
\theta_9 &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}) \to (A, 1, A, C, B, A)\\
\theta_{10} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (B, 1, B, 1, A, C, B)\\
\theta_{11} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (B, 1, B, 1, C, A, B)\\
\theta_{12} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}) \to (C, 1, C, 1, B, 1, A, C)\\
\theta_{13} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}) \to (A, 1, A, 1, C, B, A)\\
\theta_{14} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}) \to (C, 1, C, 1, A, 1, B, C)\\
\theta_{15} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}) \to (A, 1, A, 1, C, 1, B, A)\\
\theta_{16} &: (c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, c_{1,8}) \to (A, 1, A, 1, C, 1, B, 1, A)
\end{aligned}$$

With this last group, we finish the proof of the theorem. As we indicated above, the remaining groups are dealt with the same techniques used in the samples shown. □

It is very convenient to write explicitly which groups act for a given value of k. It is obtained from Theorem 3.1 and for the sake of simplicity we express them separately in terms of congruence classes mod 24 and mod 5.

Corollary 3.2. *Let $k > 0$ be the number of boundary components of an orientable Klein surface of topological genus 2. Then, according to the congruence classes mod 24, x, of k the possible groups of automorphisms on the surface are as follows:*

x	G
$0, 6$	*All groups appearing in Theorem 3.1*
$1, 5, 9, 13, 17, 21$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, D_4, D_6, D_8$
$2, 20$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_2 \times C_6, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, C_2 \times D_6, \Gamma_6 a_1$
$3, 7, 15, 19$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_2 \times C_6, DC_3, D_6, (4, 6 \mid 2, 2)$
$4, 12$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_{12}, C_2 \times C_6, DC_3, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, (4, 6 \mid 2, 2), C_2 \times D_6,$ $D_{12}, C_3 \times D_4, C_4 \times D_3, \Gamma_6 a_1, D_3 \times D_4$
$8, 14$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_2 \times C_6, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, SL(2, 3), C_2 \times D_6, \Gamma_6 a_1,$ $GL(2, 3)$
10	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2, C_2, C_2,$ $C_{12}, C_2 \times C_6, DC_3, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, (4, 6 \mid 2, 2), C_2 \times D_6,$ $D_{12}, C_3 \times D_4, C_4 \times D_3, \Gamma_6 a_1$
$11, 23$	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, D_6$
16	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_{12}, C_2 \times C_6, DC_3, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, (4, 6 \mid 2, 2), SL(2, 3),$ $C_2 \times D_6, D_{12}, C_3 \times D_4, \Gamma_6 a_1, GL(2, 3), P_{48}, GL(2, 3) \rtimes C_2$
18	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_{12}, C_2 \times C_6, DC_3, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, (4, 6 \mid 2, 2), C_2 \times D_6,$ $D_{12}, C_4 \times D_3, \Gamma_6 a_1$
22	$C_2, C_3, C_4, C_2 \times C_2, C_6, D_3, C_8, Q, D_4, C_2 \times C_4, C_2 \times C_2 \times C_2,$ $C_{12}, C_2 \times C_6, DC_3, D_6, \Gamma_3 a_2, C_2 \times D_4, D_8, (4, 6 \mid 2, 2), SL(2, 3),$ $C_2 \times D_6, D_{12}, \Gamma_6 a_1, GL(2, 3), P_{48}, GL(2, 3) \rtimes C_2$

Besides, the groups C_5, C_{10}, D_5 and D_{10} are also possible if $k \not\equiv 4 \pmod 5$. □

Remark 3.3. Let us observe that the automorphism groups of Klein surfaces with one boundary component were obtained in Chapter 5 of [6]. The result in the above Theorem matches with theorems 5.2.3 and 5.2.5 of that book.

The authors wish to thank the referee for the helpful suggestions in order to improve the readability of the paper.

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