Hasegawa, M.
Osaka J. Math.
54 (2017), 707–721

PARABOLIC, RIDGE AND SUB-PARABOLIC CURVES ON IMPLICIT SURFACES WITH SINGULARITIES

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(Received February 4, 2015, revised September 16, 2016)

Abstract
We study parabolic, ridge and sub-parabolic curves on implicit surfaces defined by smooth functions $\mathcal{R}$-equivalent to A_1^--singularity. To investigate ridge and sub-parabolic curves, we present the local parameterizations of the implicit surfaces, and we show the asymptotic behavior of the principal curvatures and directions by using the parameterization. We also present height and distance squared functions on implicit surfaces in the appendix.

1. Introduction

Let $\mathcal{E}_n$ denote the local ring of smooth function germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ with the unique maximal ideal $\mathcal{M}_n = \{h \in \mathcal{E}_n \,|\, h(0) = 0\}$. The group $\mathcal{R}$ of diffeomorphisms $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ acts on $\mathcal{M}_n.\mathcal{E}(n, p)$ by $f \circ h^{-1}$, where $\mathcal{E}(n, p)$ denotes the p-tuple of elements in $\mathcal{E}_n$. Two smooth function germs $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are said to be *$\mathcal{R}$-equivalent* if there exists $h \in \mathcal{R}$ such that $g = f \circ h^{-1}$. Let X be a manifold and G a Lie group acting on X. We call *modality* of a point $x \in X$ under the action of G on X the least number m such that a sufficiently small neighborhood of x may be covered by a finite number of m-parameter families of orbits. The point x is said to be *simple* if its modality is 0. Arnold [1] has showed that any $\mathcal{R}$-simple function germ is $\mathcal{R}$-equivalent to one of the following types:

$$
\begin{aligned}
&A_k : \pm x_1^{k+1} \pm x_2^2 + \cdots \pm x_n^2, \quad k \geqslant 1,\\
&D_k : \pm x_1^{k-1} + x_1 x_2 \pm x_3^2 \cdots \pm x_n^2, \quad k \geqslant 4,\\
&E_6 : \pm x_1^4 + x_2^3 \pm x_3^2 + \cdots \pm x_n^2,\\
&E_7 : x_1^3 x_2 + x_2^3 \pm x_3^2 + \cdots \pm x_n^2,\\
&E_8 : x_1^5 + x_2^3 \pm x_3^2 + \cdots \pm x_n^2.
\end{aligned}
$$

An A_1-singularity is also known as a Morse or nondegenerate singularity. Here, a function germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ has a *Morse* or *nondegenerate singularity* at 0 if its first partial derivatives vanish at 0 and the determinant of the Hessian matrix $\mathcal{H}_f$ does not vanish at 0. The *index* of the Morse singularity 0 is the number of the negative eigenvalues of $\mathcal{H}_f(0)$.

Singular points of implicit surfaces defined by functions $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ having Morse singularities of index 0 or 3 are isolated. Furthermore, an implicit surface defined by a function f having a singularity of index 2 coincides with that of $-f$, which has a singularity

2010 Mathematics Subject Classification. Primary 53A05; Secondary 58K05.

*Supported by the FAPESP post-doctoral grant number 2013/02543-1 during a post doctoral period at ICMC-USP, São Carlos, Brazil.

of index 1. For this reason, when we study the geometry of implicit surfaces defined by zero sets of functions having Morse singularities, it is natural to consider only in case that the index of singularities is 1. In [5], lines of curvature on implicit surfaces with Morse singular points of index 1 were investigated. In this paper, we study parabolic, ridge and sub-parabolic curves on implicit surfaces defined by functions having Morse singularities of index 1.

The parabolic curve on a surface in $\mathbb{R}^3$ is the locus of points where the Gaussian curvature vanishes and divides the surface into the elliptic and hyperbolic regions. It is known that the parabolic curve is the locus of points where the surface has $A_{\geqslant 2}$-contact with it tangent plane. The parabolic curve on the surface is generically non-singular or there may be not parabolic curve when the surface is a convex like an ellipsoid. However, parabolic curves on singular surfaces have singularities. The parabolic curves on singular surfaces parameterized by smooth maps with corank 1 singularities were studied in [10].

There are two asymptotic directions in the hyperbolic region, and there is a unique asymptotic direction at parabolic points. A point at which the unique asymptotic direction is tangent to the parabolic curve is called a cusp of Gauss. If the parabolic curve is non-singular, then the image of this curve under the Gauss map is cusp at the cusp of Gauss.

The ridge curve of a surface in $\mathbb{R}^3$ was first studied in details by Porteous [11] as the locus of points where the surface has $A_{\geqslant 3}$-contact with its focal sphere. It is also the locus of points where one principal curvature has an extremum value along the corresponding line of curvature. The locus of points where one principal curvature has an extremum value along the other line of curvature is also important. This locus is called the sub-parabolic curve, which was first studied in details by Bruce and Wilkinson [2] in terms of folding maps. It is also the locus of points where the lines of curvature have geodesic inflections. Moreover, it is known that the ridge and sub-parabolic curves correspond respectively to singular point sets and parabolic curves of the focal set.

In section 2, we recall the ingredients of the differential geometry of implicit surfaces in $\mathbb{R}^3$ and the curvature formula of the implicit surfaces. In Section 3, we consider the parabolic curves and cusps of Gauss on implicit surfaces with Morse singularity of index 1. In Section 4, we present local parameterization of the surfaces and study ridge and sub-parabolic curves on the surfaces. In appendix A, we introduce the families of height and distance squared functions on implicit surfaces. Families of height and distance squared functions are introduced by Thom and fundamental tools to study of the geometry of submanifolds. Several authors studied the singularities of height and distance squared functions in order to investigate the geometry of submanifolds in Euclidean and other spaces (see, for example, [8, 9, 11]). Recently, in [4, 10] analyzing the singularities of these two functions are applied to the investigation of the geometry of singular surfaces in $\mathbb{R}^3$.

This work is part of a project of an investigation of geometry of surfaces defined by zero sets/fibers of functions having singularities supported by the FAPESP. The author would like to express his sincere gratitude to Toshizumi Fukui and Farid Tari for fruitful discussions and helpful comments.

2. Preliminaries

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let M be an implicit surface defined by the zero set of f, that is, $M = \{(x, y, z) | f(x, y, z) = 0\}$. Away from singular points, the unit normal vector to M is given by $N = \nabla f(x, y, z)/||\nabla f(x, y, z)||$. We denote the principal curvatures of M by κ_1 and κ_2. The eigenvalues λ_i $(i = 0, 1, 2)$ of dN are $\lambda_0 = 0$, $\lambda_1 = -\kappa_1$ and $\lambda_2 = -\kappa_2$. The principal directions $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are the unit eigenvectors corresponding respectively to κ_1 and κ_2. We may assume that $\kappa_1 \geqslant \kappa_2$. We remark that a surface $\{(x, y, z)| - f(x, y, z) = 0\}$ defines the same surface as M but the orientation and the magnitude correlation for the principal curvatures of $\{(x, y, z)| - f(x, y, z) = 0\}$ are differ from those of M. The Gaussian curvature K and mean curvature H of M are given by the following formulas:

$$K = -\frac{\begin{vmatrix} \mathcal{H}_f & (\nabla f)^{\mathrm{T}} \\ \nabla f & 0 \end{vmatrix}}{||\nabla f||^4}, \tag{2.1}$$

$$H = \frac{(\nabla f)\mathcal{H}_f(\nabla f)^{\mathrm{T}} - ||\nabla f||^2 \mathrm{tr}(\mathcal{H}_f)}{2||\nabla f||^3}, \tag{2.2}$$

where $\mathcal{H}_f$ is the Hessian matrix of f. Theses formulas appear, for example, in [6].

As mentioned in Introduction, we are interested in the case that the singular point of M is of Morse type of index 1. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a smooth function germ, and suppose it has a Morse singularity of index 1 at the origin. If $g : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ is singular at 0 then g can be written as

$$g = a_{11}x^2 + 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2 + +O(4).$$

The quadratic form g_2 of g can be expressed as

$$g_2 = \boldsymbol{x}^{\mathrm{T}} A \boldsymbol{x},$$

where

$$\boldsymbol{x} = \begin{pmatrix} x & y & z \end{pmatrix}^{\mathrm{T}}, \quad A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Real symmetric matrices are diagonalizable by orthogonal matrices, so there exists an orthogonal matrix P such that

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where λ_i $(i = 1, 2, 3)$ are the eigenvalues of A. If we set

$$\boldsymbol{x} = P\boldsymbol{X}, \quad \boldsymbol{X} = \begin{pmatrix} X & Y & Z \end{pmatrix}^{\mathrm{T}},$$

then we show that

$$g_2 = \boldsymbol{X}^{\mathrm{T}} P^{\mathrm{T}} A P \boldsymbol{X} = \boldsymbol{X}^{T} P^{-1} A P \boldsymbol{X} = \lambda_1 {X_1}^2 + \lambda_2 {X_2}^2 + \lambda_3 {X_3}^2.$$

Hence, f can be written as

$$f(x, y, z) = f_2(x, y, z) + \sum_{m=3}^{n} f_m(x, y, z) + O(n + 1) \tag{2.3}$$

where

$$f_2(x, y, z) = \frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - z^2 \quad (0 < a_1 \leqslant a_2), \quad f_m(x, y, z) = \sum_{i+j+k=m} a_{ijk} x^i y^j z^k.$$

3. Parabolic curves and cusps of Gauss

Given an implicit surface M, we set

$$P_f = -\begin{vmatrix} \mathcal{H}_f & (\nabla f)^{\mathrm{T}} \\ \nabla f & 0 \end{vmatrix}.$$

From (2.1), the parabolic set on M is the intersection between M and a set $\{(x, y, z) | P_f(x, y, z) = 0\}$.

Theorem 1. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a smooth function with an A_1^--singularity at at the origin. Then there are generically* 0, 2, 4 *or* 6 *parabolic curves of the implicit surface M passing through the singularity.*

Proof. We may assume that f is given by (2.3). Then we have

$$\begin{aligned} P_f = {} & \frac{16}{a_1^2 a_2^2} f_2 + \frac{16}{a_1^1 a_2^2} f_2((3a_1^2 a_{300} + a_2^2 a_{120} - a_{102})x \\ & + (a_1^2 a_{210} + 3a_2^2 a_{201} - a_{012})y + (a_1^2 a_{201} + a_2^2 a_{021} - 3a_{003})z) + O(4). \end{aligned}$$

Note that M and $\{(x, y, z) | P_f(x, y, z) = 0\}$ have same tangent cone. Here, the *tangent cone* of a set $X \subset \mathbb{R}^n$ at a point x consists of the limits of secants that pass through a sequence of points $x_i \in X \setminus \{x\}$ converging to x. Under the condition $(x, y, z) \in M$, P_f is given by

$$\begin{aligned} P_f = {} & \frac{16}{a_1^2 a_2^2}(-f_3 + O(4)) + \frac{16}{a_1^1 a_2^2}(-f_3 + O(4))((3a_1^2 a_{300} + a_2^2 a_{120} - a_{102})x \\ & + (a_1^2 a_{210} + 3a_2^2 a_{201} - a_{012})y + (a_1^2 a_{201} + a_2^2 a_{021} - 3a_{003})z) + O(4), \\ = {} & -\frac{16}{a_1^2 a_2^2} f_3 + O(4). \end{aligned} \tag{3.1}$$

By substituting $x = a_1 r \cos s$, $y = a_2 r \sin s$, $z = r$ into f and the above expression, we have

$$f = r^3 f_3(a_1 \cos s, a_2 \sin s, 1) + O(r^4), \quad P_f = -\frac{16}{a_1^2 a_2^2} r^3 f_3(a_1 \cos s, a_2 \sin s, 1) + O(r^4).$$

The vector $(a_1 \cos s_0, a_2 \sin s_0, 1)$ $(s_0 \in [0, 2\pi))$ is the tangent vector to the parabolic curve M passing through the origin. Hence, to see the number of such parabolic curves, we need to investigate the number of simple roots $s \in [0, 2\pi)$ of $f_3(a_1 \cos s, a_2 \sin s, 1)$. To do this, we prepare the following lemma:

Lemma 2. *The function*

$$f(s) = \sum_{k=0}^{n} \sum_{i=0}^{k} c_{i,k-i} \cos^i s \sin^{k-i} s$$

has generically $2k$ $(k = 0, 1, \cdots n)$ simple roots $s \in [0, 2\pi)$.

Proof of Lemma 2. Let $s \in [0, 2\pi)$. By substituting the relation

$$\cos s = \pm\frac{1}{\sqrt{1+\tan^2 s}}, \quad \sin s = \pm\frac{\tan s}{\sqrt{1+\tan^2 s}}$$

we set

$$f_1 = \sum_{k=0}^{n}\sum_{i=0}^{k}\frac{c_{i,k-i}\tan^{k-i} s}{(1+\tan^2 s)^{k/2}} \qquad (s \in [0,\pi/2)\cup(3\pi/2, 2\pi)),$$
$$f_2 = \sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^k\frac{c_{i,k-i}\tan^{k-i} s}{(1+\tan^2 s)^{k/2}} \qquad (s \in (\pi/2, 3\pi/2)),$$

and $g = f_1 f_2$. Since $\tan(s \pm \pi) = \tan s$, we have $f_i(s \pm \pi) = f_i(s)$ and $g(s \pm \pi) = g(\pi)$. Moreover, since f_1 and f_2 are defined respectively in $[0, \pi/2) \cup (3\pi/2, 2\pi)$ and $(\pi/2, 3\pi/2)$, the number of roots of g is twice the number of roots of f, counting their multiplicity.

Let us investigate the number of roots of g. In case $n = 2m$ $(m \geqslant 1)$, we have $g = g_1^2 - g_2^2$, where

$$g_1 = c_{0,0} + \sum_{k=1}^{m}\sum_{i=0}^{2k}\frac{c_{i,2k-i}\tan^{2k-i} s}{(1+\tan^2 s)^k}, \quad g_2 = \sum_{k=1}^{m}\sum_{i=0}^{2k-1}\frac{c_{i,2k-1-i}\tan^{2k-1-i} s}{(1+\tan^2 s)^{(2k-1)/2}}.$$

We see

$$\begin{aligned}
g_1^1 - g_2^2 &= \frac{1}{(1+\tan^2 s)^m}\left(c_{0,0}(1+\tan^2 s)^m + \sum_{k=1}^{m}\sum_{i=0}^{2k} c_{i,2k-i}\tan^{2k-i} s(1+\tan^2 s)^{m-k}\right)^2\\
&\quad - \frac{1}{(1+\tan^2 s)^{(2m-1)/2}}\left(\sum_{k=1}^{m}\sum_{i=0}^{2k-1} c_{i,2k-1-i}\tan^{2k-1-i} s(1+\tan^2 s)^{n-k}\right)^2\\
&= \frac{1}{(1+\tan^2 s)^{2m}}\left(\left(c_{0,0}+\sum_{k=1}^{m}c_{2k,0}\right)^2 + \cdots + \left(c_{0,0}+\sum_{k=1}^{m}c_{0,2k}\right)^2\tan^{4m} s\right)\\
&\quad - \frac{1}{(1+\tan^2 s)^{2m-1}}\left(\left(\sum_{k=1}^{m}c_{2k-1,0}\right)^2 + \cdots + \left(\sum_{k=1}^{m}c_{0,2k-1}\right)^2\tan^{4m-2} s\right)\\
&= \frac{1}{(1+\tan^2 s)^{2n}}\left(\left(c_{0,0}+\sum_{k=1}^{m}c_{2k,0}\right)^2 - \left(\sum_{k=1}^{m}c_{2k-1,0}\right)^2 + \cdots\right.\\
&\quad \left. + \left(\left(c_{0,0}+\sum_{k=1}^{m}c_{0,2k}\right)^2 - \left(\sum_{k=1}^{m}c_{0,2k-1}\right)^2\right)\tan^{4m} s\right),
\end{aligned}$$

and thus g has generically simple $4k$ $(k = 0, 1, \cdots, 2m)$ roots $s \in [0, 2\pi)$. In case that n is odd is similar. Hence, we complete the proof. □

Let us continue the proof of Theorem 1. Since

$$\begin{aligned}
f_3(a_1\cos s, a_2\sin s, 1) &= a_1^3 a_{300}\cos^3 s + a_1^2 a_2 a_{210}\cos^2 s\sin s + a_1 a_2^2 a_{120}\cos s\sin^2 s\\
&\quad + a_2^3 a_{030}\sin^3 s + a_1^2 a_{201}\cos^2 s + a_1 a_2 a_{111}\cos s\sin s + a_2^2 a_{021}\sin^2 s\\
&\quad + a_1 a_{102}\cos s + a_2 a_{012}\sin s + a_{003}
\end{aligned}$$

is the polynomial in $\cos s$ and $\sin s$ of degree 3, by Lemma 2 we see that $f_3(a_1 \cos s, a_2 \sin s, r)$ has generically 0, 2, 4 or 6 simple roots and we complete the proof. □

The normal curvature of M in the tangent direction $\boldsymbol{v}$ is given by

$$\kappa_n = \frac{\boldsymbol{v}\mathcal{H}_f\boldsymbol{v}^{\mathrm{T}}}{||\boldsymbol{v}||^2||\nabla f||}$$

(see, for example, [7]). Hence, the equation of asymptotic directions of M is given by

$$f_x dx + f_y dy + f_z dz = 0, \quad \begin{pmatrix} dx & dy & dz \end{pmatrix} \mathcal{H}_f \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = 0. \tag{3.2}$$

Proposition 3. *Let $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ be an asymptotic direction of M at a parabolic point p. Then the point p is a cusp of Gauss if and only if $\langle \boldsymbol{v}, \nabla P_f \rangle = 0$ at p.*

Proof. Recall that the cusp of Gauss is a parabolic point at which the parabolic curve is tangent to the unique asymptotic direction. Since $\boldsymbol{v}$ is tangent to M at p, $\boldsymbol{v}$ is tangent to the parabolic set at p if and only if $\boldsymbol{v}$ is tangent to $\{(x, y, z)|P_f(x, y, z) = 0\}$, that is, $\langle \boldsymbol{v}, \nabla P_f \rangle = 0$ at p, and the proof is completed. □

Theorem 4. *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a smooth function with an A_1^--singularity at the origin, and let $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$ be a vector in $\mathbb{R}^3$ satisfying* (3.2) *under the condition that $(x, y, z) \in \{(x, y, z)|P_f(x, y, z) = 0\}$. Then generically the set $M \cap \{(x, y, z)|C_f(x, y, z) = 0\}$ consists locally of curves passing through the singularity whose number is same as the number of parabolic curves passing through there, where $C_f = \langle \boldsymbol{v}, \nabla P_f \rangle$. Moreover, the branches are tangent to parabolic curves at the origin.*

Proof. When $f_z \neq 0$, we obtain

$$\begin{aligned}[dx : dy : dz] = [&-f_y^2 f_{xz} + f_x f_y f_{yz} + f_y f_z f_{xy} - f_x f_z f_{yy} : \\ &f_x f_y f_{xz} - f_x^2 f_{yz} + f_x f_z f_{xy} - f_y f_z f_{xx} : -2 f_x f_z f_{xy} + f_y^2 f_{xx} + f_x^2 f_{yy}],\end{aligned}$$

which satisfies (3.2) in case $f_z = 0$. Hence, $\langle \boldsymbol{v}, \nabla P_f \rangle = 0$ is equivalent to the conditions that

$$\begin{aligned} 0 = \frac{128}{a_1^8 a_2^8}(&a_2^4(a_2^2 a_{201} - 3a_{003})x^4 - a_1^2 a_2^2 a_{111} x^3 y + a_1^2 a_2^4(a_2^2 a_{120} - 3a_{102})x^3 z \\ &+ a_1^2 a_2^2(a_1^2 a_{201} + a_2^2 a_{021} - 6a_{003})x^2 y^2 - a_1^2 a_2^4(3a_1^2 a_{201} + a_2^2 a_{021})x^2 z^2 \\ &- a_1^2 a_2^4(2a_1^2 a_{210} - 3a_2^2 a_{030} + 3a_{012})x^2 yz - a_1^4 a_2^2 a_{111} xy^3 - 2a_1^4 a_2^4 a_{111} xyz^2 \\ &+ a_1^4 a_2^2(3a_1^2 a_{300} - 2a_2^2 a_{120} - 3a_{102})xy^2 z - a_1^4 a_2^4(3a_1^2 a_{300} + a_2^2 a_{120})xz^3 \\ &+ a_1^4(a_1^2 a_{201} - 3a_{003})y^4 + a_1^4 a_2^2(a_1^2 a_{201} - 3a_{012})y^3 z - a_1^4 a_2^2(a_1^2 a_{201} + 3a_2^2 a_{021})y^2 z^2 \\ &- a_1^4 a_2^4(a_1^2 a_{210} + 3a_2^2 a_{030})z^4) + O(5).\end{aligned}$$

By substituting $x = a_1 r \cos s$, $y = a_2 r \sin s$, $z = r$ into the above equation, we have

$$0 = -\frac{384}{a_1^4 a_2^4} f_3(a_1 r \cos s, a_2 r \sin s, 1) r^4 + O(r^5). \tag{3.3}$$

By a similar argument on parabolic curves, we show that near the origin $M \cap \{(x, y, z) | C_f(x, y, z) = 0\}$ consists of generically 0, 2, 4 or 6 smooth curves passing through the origin and the curves tangent to parabolic curves at the origin. □

4. Ridge and sub-parabolic curves

We consider a function $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ written as (2.3) and a map

$$\varphi : S^1 \times (\mathbb{R}, 0) \to (\mathbb{R}^3, 0), \quad \varphi(s, r) = (a_1 r \cos s, a_2 r \sin s, z(s, r)), \tag{4.1}$$

satisfying $f(a_1 r \cos s, a_2 r \sin s, z(s, r)) = 0$. The map φ is the local parameterization of the implicit surface $f = 0$ and $z(s, r)$ can be written as

$$z(s, r) = r + c_2(s)r^2 + c_3(s)r^3 + O(r^4), \tag{4.2}$$

where

$$\begin{aligned}
c_2(s) &= \frac{1}{2} f_3(a_1 \cos s, a_2 \sin s, 1),\\
c_3(s) &= \frac{1}{2} f_4(a_1 \cos s, a_2 \sin s, 1) - \frac{1}{8} f_3(a_1 \cos s, a_2 \sin s, 1)(a_1^3 a_{300} \cos^3 s\\
&\quad + a_1^2 a_2 a_{210} \cos^2 s \sin s + a_1 a_2^2 a_{120} \cos s \sin^2 s + a_2^3 a_{030} \sin^3 s - a_1^2 a_{201} \cos^2 s\\
&\quad - a_1 a_{111} \cos s \sin s - a_2^2 a_{021} \sin^2 s - 3a_1 a_{102} \cos s - 3a_2 a_{012} \sin s - 5a_{003}).
\end{aligned}$$

We remark that $c_2(s)$ and $c_3(s)$ are the third and sixth degree polynomial in $\cos s$ and $\sin s$, respectively.

We consider a curve $\alpha(s, r) = 0$ on $S^1 \times (\mathbb{R}, 0)$ which satisfies $\alpha(s_0, 0) = 0$ and $\alpha_s(s_0, 0) \neq 0$. Then the curve $\alpha = 0$ can be locally parameterized by $\gamma(r) = (s(r), r)$ with $\gamma(0) = (s_0, 0)$ and $\gamma'(0) = (-\alpha_r(s_0, 0)/\alpha_s(s_0, 0), 1)$. Since the Jacobian matrix of φ is given by

$$J_\varphi(s_0, 0) = \begin{pmatrix} 0 & a_1 \cos s_0 \\ 0 & a_2 \sin s_0 \\ 0 & 1 \end{pmatrix},$$

$\varphi(\gamma(r))$ is a space curve, on the surface $f(x, y, z) = 0$, tangent to the vector $(a_1 \cos s_0, a_2 \sin s_0, 1)$ at the origin.

Since

$$\varphi_s = (-a_1 r \sin s, a_2 r \cos s, c_2'(s) r^2 + O(r^3)), \tag{4.3}$$

$$\varphi_r = (a_1 \cos s, a_2 \sin s, 1 + 2c_2(s) r + O(r^2)), \tag{4.4}$$

the coefficients $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ of the first fundamental form of φ are expressed respectively as follows:

$$\begin{aligned}
\widetilde{E}(s, r) &= r^2(E_0(s) + O(r)),\\
\widetilde{F}(s, r) &= r(F_0(s) + O(r)),\\
\widetilde{G}(s, r) &= G_0(s) + G_1(s) r + O(r^2),
\end{aligned}$$

where

$$E_0 = a_2^2 \cos^2 s + a_1^2 \sin^2 s, \qquad F_0 = -(a_1^2 - a_2^2)\cos s \sin s,$$
$$G_0 = 1 + a_1^2 \cos^2 s + a_2^2 \sin^2 s, \qquad G_1 = 4c_2(s).$$

From (4.3) and (4.4), we have

$$\begin{aligned}\varphi_s \times \varphi_r = &(-a_2 r \cos s + a_2(c_2'(s)\sin s - 2c_2(s)\cos s)r^2 + O(r^3),\\ &- a_1 r \sin s - a_1(2c_2(s)\sin s + c_2'(s)\cos s)r^2 + O(r^3),\ a_1a_2 r),\end{aligned}$$

and we obtain

$$\begin{aligned}\|\varphi_s \times \varphi_r\|^2 = &(a_1^2a_2^2 + a_2^2\cos^2 s + a_1^2 \sin^2 s)r^2\\ &+ (4c_2(s)(a_2^2\cos^2 s + a_1^2\sin^2 s) + 2c_2'(s)(a_1^2 - a_2^2)\cos s \sin s)r^3 + O(r^4).\end{aligned}$$

Hence the unit normal vector $\widetilde{N}$ of φ can be written as

$$\begin{aligned}\widetilde{N} = &\frac{1}{A_s}(a_2\cos s,\ a_1 \sin s,\ -a_1a_2)\\ &+ \frac{1}{A_s^3}(a_1^2a_2(a_2^2c_2(s)\cos s - (1 + a_2^2)c_2'(s)\sin s),\\ &a_1a_2^2(2a_1^2c_2(s)\sin s + (1 + a_1^2)c_2'(s)\cos s),\\ &a_1a_2(2c_2(s)(a_2^2\cos^2 s + a_1^2\sin^2 s) + (a_1^2 - a_2^2)c_2'(s)\cos s \sin s))r + O(r^2),\end{aligned}$$

where $A_s = \sqrt{a_1^2a_2^2 + a_2^2\cos^2 s + a_1^2\sin^2 s}$. Since

$$\begin{aligned}\varphi_{ss} &= (-a_1 r\cos s, -a_2 r \sin s, c_2''(s)r^2 + O(r^3)),\\ \varphi_{rs} &= (-a_1 \sin s,\ a_2\cos s,\ 2c_2'(s)r + O(r^2)),\\ \varphi_{rr} &= (0,\ 0,\ 2c_2(s) + 6c_3(s)r + O(r^2)),\end{aligned}$$

the coefficients $\widetilde{l}$, $\widetilde{m}$ and $\widetilde{n}$ of the second fundamental form of φ are expressed respectively as follows:

$$\begin{aligned}\widetilde{l}(s, r) &= r(l_0(s) + O(r)),\\ \widetilde{m}(s, r) &= r(m_0(s) + O(r)),\\ \widetilde{n}(s, r) &= n_0(s) + n_1(s)r + O(r^2),\end{aligned}$$

where

$$\begin{aligned}l_0 &= -\frac{a_1a_2}{A_s}, \quad m_0 = -\frac{a_1a_2c_2'(s)}{A_s}, \quad n_0 = -\frac{2a_1a_2c_2(s)}{A_s},\\ n_1 &= \frac{1}{A_s^3}(2a_1a_2^3(2c_2(s)^2 - 3c_3(s))\cos^2 s\\ &\quad + 2a_1a_2(a_1^2 - a_2^2)c_2(s)c_2'(s)\cos s \sin s + 2a_1^3a_2(2c_2(s)^2 - 3c_3(s))\sin^2 s + 6a_1^3a_2^3).\end{aligned}$$

By using above expressions, we show that the Gaussian curvature $\widetilde{K}$ and mean curvature $\widetilde{H}$ of φ are expressed respectively as follows:

$$\widetilde{K} = \frac{1}{r}\left(\frac{2a_1a_2c_2(s)}{A_s} + O(r)\right),$$

$$\widetilde{H} = \frac{1}{r}\left(-\frac{a_1a_2(1+a_1^2\cos^2 s + a_2^2\sin^2 s)}{A_s^3} + O(r)\right).$$

Moreover, the principal curvature $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ can be written respectively as follows:

$$\widetilde{\kappa}_1 = \widetilde{H} + \sqrt{\widetilde{H}^2 - \widetilde{K}} = k_{10}(s) + k_{11}(s)r + O(r^2),$$
$$\widetilde{\kappa}_2 = \widetilde{H} - \sqrt{\widetilde{H}^2 - \widetilde{K}} = \frac{1}{r}(k_{20}(s) + O(r)),$$

where

$$k_{10} = \frac{n_0}{G_0}, \quad k_{11} = -\frac{1}{G_0^3 l_0}(G_0(G_1 l_0 - 2F_0 m_0)n_0 + F_0^2 n_0^2 + G_0^2(m_0^2 - l_0 n_1)),$$
$$k_{20} = \frac{G_0 l_0}{E_0 G_0 - F_0^2}.$$

We note that $\widetilde{\kappa}_1$ and $\widetilde{\kappa}_2$ are, respectively, the maximal and minimal principal curvatures of φ.

The vectors $(\widetilde{m} - \widetilde{\kappa}_i\widetilde{F})\partial_s + (-\widetilde{l} + \widetilde{\kappa}_i\widetilde{E})\partial_r$ $(i = 1,2)$ are along the principal direction corresponding to $\widetilde{\kappa}_i$. Set $\widetilde{\xi}_i = \widetilde{m} - \widetilde{\kappa}_i\widetilde{F}$ and $\widetilde{\eta}_i = -\widetilde{l} + \widetilde{\kappa}_i\widetilde{E}$, the principal directions

$$\widetilde{\boldsymbol{v}}_i = \frac{1}{\sqrt{\widetilde{E}\widetilde{\xi}_i^2 + 2\widetilde{F}\widetilde{\xi}_i\widetilde{\eta}_i + \widetilde{G}\widetilde{\eta}_i^2}}\left(\widetilde{\xi}_i\frac{\partial}{\partial s} + \widetilde{\eta}_i\frac{\partial}{\partial r}\right)$$

can be expressed as follows:

$$\widetilde{\boldsymbol{v}}_1 = (\xi_{10}(s) + O(r))\frac{\partial}{\partial s} + (\eta_{10}(s) + O(r))\frac{\partial}{\partial r}, \tag{4.5}$$

$$\widetilde{\boldsymbol{v}}_2 = \frac{1}{r}(\xi_{20}(s) + O(r))\frac{\partial}{\partial s} + r(\eta_{20}(s) + O(r))\frac{\partial}{\partial r}, \tag{4.6}$$

where

$$\xi_{10} = \frac{F_0 n_0 - G_0 m_0}{G_0^{3/2} l_0}, \quad \eta_{10} = \frac{1}{\sqrt{G_0}}, \quad \xi_{20} = \frac{1}{\sqrt{E_0}}, \quad \eta_{20} = \frac{F_0^2}{G_0^2\sqrt{E_0}}. \tag{4.7}$$

(From (4.5), (4.6) and (4.7), the vector $\widetilde{\boldsymbol{v}}_1$ can be extensible near $(s,0) \in S^1 \times (\mathbb{R},0)$. Moreover, the vector $r\widetilde{\boldsymbol{v}}_2$ can be extensible near $(s,0) \in S^1 \times (\mathbb{R},0)$ even if $\widetilde{\boldsymbol{v}}_2$ is not.)

Theorem 5. *Let $f : (\mathbb{R}^3,0) \to (\mathbb{R},0)$ be a smooth function having an A_1^--singularity at at the origin, and let the eigenvalues of $\mathcal{H}_f(0)$ with same sign be different from each other.*

1. *There are at most 20 ridge curves relative to the maximal principal curvature of M passing through the origin. On the other hand, there are 4 ridge curves relative to the minimal principal curvature of M passing though the origin.*
2. *There are at most 14 sub-parabolic curves relative to the maximal principal curvature of M passing through the origin. On the other hand, there is no sub-parabolic curve relative to the minimal principal curvature of M passing through the origin.*

Proof. Without loss of generality, we may assume that f is given by (2.3). Then M can be locally parameterized by (4.1), (4.2). Hence, by the expressions of the principal curvatures $\widetilde{\kappa}_i$ and principal directions $\widetilde{\boldsymbol{v}}_i$, the directional derivatives of $\widetilde{\kappa}_i$ in $\widetilde{\boldsymbol{v}}_j$ are given by follows:

$$
\begin{aligned}
(4.8) \qquad D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_1 &= k'_{10}\xi_{10} + k_{11}\eta_{10} + O(r) \\
&= \frac{3a_1^2a_2^2}{G_0^{7/2}A_s}\Big((4c_2(s)^2(1 + a_1^2\cos^2 s + a_2^2\sin^2 s + (a_1^2 - a_2^2)\cos^2 s\sin^2 s) \\
&\quad + 4(a_1^2 - a_2^2)c_2(s)c_2'(s)\cos s\sin s(1 + a_1^2\cos^2 s + a_2^2\sin^2 s) \\
&\quad - (2c_3(s) - c_2'(s)^2)(1 + a_1^2\cos^2 s + a_2^2\sin^2 s)^2)\Big) + O(r)
\end{aligned}
$$

$$
\begin{aligned}
(4.9) \qquad D_{\widetilde{\boldsymbol{v}}_2}\widetilde{\kappa}_2 &= \frac{1}{r^2}(k'_{20}\xi_{20} + O(r)) \\
&= \frac{1}{r^2}\left(\frac{a_1a_2(a_1^2 - a_2^2)}{E_0^{1/2}A_s^5}(2(1 + a_1^2)(1 + a_2^2) + G_0)\cos s\sin s + O(r)\right)
\end{aligned}
$$

$$
(4.10) \qquad D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_2 = \frac{1}{r^2}(-k_{20}\eta_{20} + O(r)) = \frac{1}{r^2}\left(\frac{a_1a_2G_0^{1/2}}{A_s^3} + O(r)\right),
$$

$$
\begin{aligned}
(4.11) \qquad D_{\widetilde{\boldsymbol{v}}_2}\widetilde{\kappa}_1 &= \frac{1}{r}(k'_{10}\xi_{20} + O(r)) \\
&= \frac{1}{r}\left(-\frac{a_1a_2}{2E_0^{1/2}G_0^2A_s^3}\Big(2c_2(s)(a_1^2 - a_2^2)(-2 + a_2^2 + a_1^2(1 + 4a_2^2)\right. \\
&\quad \left. -3(a_1^2 - a_2^2)\cos s)\cos s\sin s + c'(s)G_0A_s^2\Big)\right).
\end{aligned}
$$

From (4.8), the numerator of $D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_1(s, 0)$ is the polynomial of degree 10 in $\cos s$ and $\sin s$. It follows that the number of simple roots $s \in [0, 2\pi)$ of $D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_1(s, 0) = 0$ is at most twenty, that is, the ridge curves relative to $\widetilde{\boldsymbol{v}}_1$ intersects with $r = 0$ in $S^1 \times (\mathbb{R}, 0)$ at most twenty points. Moreover, from (4.9), the ridge curves relative to $\widetilde{\boldsymbol{v}}_2$ intersects with $r = 0$ at four points. If $s_0 \in [0, 2\pi)$ is a simple root of $D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_1(s, 0)$, that is, $D_{\widetilde{\boldsymbol{v}}_1}\widetilde{\kappa}_1(s, r) = 0$ intersects with $r = 0$ at $(s_0, 0)$, then the ridge curves relative to $\widetilde{\boldsymbol{v}}_1$ passes through the origin and is tangent to the vector $(a_1\cos s, a_2\sin s, 1)$ there. The other ridge is similar, and we complete the proof of (1). The proof of (2) is similar to that of (1), and will be omitted. □

Appendix A. Height and distance squared functions on implicit surfaces

In this section, we introduce the height and distance squared functions on implicit surfaces.

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $\nabla f(\boldsymbol{x}) \neq 0$. First, we consider the height functions on implicit surfaces. We define a family of functions on an implicit surface M by

$$
H_{\boldsymbol{v}} : \mathbb{R}^3 \times \mathbb{R} \times S^2 \to \mathbb{R}, \quad (\boldsymbol{x}, \lambda, \boldsymbol{v}) \mapsto \langle \boldsymbol{x}, \boldsymbol{v}\rangle + \lambda f(\boldsymbol{x})
$$

and $h_{\boldsymbol{v}}(\boldsymbol{x}, \lambda) = H_{\boldsymbol{v}}(\boldsymbol{x}, \lambda, \boldsymbol{v})$.

Theorem 6. (1) $\nabla h_{\boldsymbol{v}} = 0$ *at* $(\boldsymbol{x}_0, \lambda_0)$ *if and only if* $\boldsymbol{x}_0$ *is a point on* M (*i.e.,* $f(\boldsymbol{x}_0) = 0$), $\lambda_0 = \pm 1/\|\nabla f(\boldsymbol{x}_0)\|$ *and* $\boldsymbol{v} = \lambda\nabla f(\boldsymbol{x}_0) = \pm N(\boldsymbol{x}_0)$.

(2) $\nabla h_{\boldsymbol{v}} = 0$ *and* $\det(\mathcal{H}_{h_v}) = 0$ *at* $(\boldsymbol{x}_0, \lambda_0)$ *if and only if* $\boldsymbol{x}_0$ *is a parabolic point on* M, $\lambda_0 = \pm 1/\|\nabla f(\boldsymbol{x}_0)\|$ *and* $\boldsymbol{v} = \pm N(\boldsymbol{x}_0)$.

(3) $\nabla h_{\boldsymbol{v}} = 0$, $\mathrm{rank}(\mathcal{H}_{h_v}) = 3$ *and* $\mathrm{rank}(J_{\psi_h}) < 4$ *at* $(\boldsymbol{x}_0, \lambda_0)$ *if and only if* $\boldsymbol{x}_0$ *is a cusp of Gauss of* M, $\lambda_0 = \pm 1/\|\nabla f(\boldsymbol{x}_0)\|$ *and* $\boldsymbol{v} = \pm N(\boldsymbol{x}_0)$, *where* $\psi_h : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^5$ *defined by* $\psi_h(\boldsymbol{x}, \lambda) = (\nabla h_{\boldsymbol{v}}(\boldsymbol{x}, \lambda), \det(\mathcal{H}_{h_v})(\boldsymbol{x}, \lambda))$.

(4) $\nabla h_{\boldsymbol{v}} = 0$ *and* $\mathrm{rank}(\mathcal{H}_{h_v}) = 2$ *if and only if* $\boldsymbol{x}_0$ *is a flat umbilic of* M, $\lambda_0 = \pm 1/\|\nabla f(\boldsymbol{x}_0)\|$ *and* $\boldsymbol{v} = \pm N(\boldsymbol{x}_0)$.

Proof. We have $\nabla h_{\boldsymbol{v}} = (\boldsymbol{v} + \lambda \nabla f,\ f)$. The assertion (1) is immediately deduced from this. The Hessian matrix of $h_{\boldsymbol{v}}$ is given by

$$\mathcal{H}_{h_v} = \begin{pmatrix} \lambda \mathcal{H}_f & (\nabla f)^{\mathrm{T}} \\ \nabla f & 0 \end{pmatrix}.$$

It follows from (2.1) that

$$\det(\mathcal{H}_{h_v}) = -\lambda^2 \|\nabla f\|^4 K = -\lambda^2 P_f,$$

which establishes the assertion (2).

By a suitable rotation in $\mathbb{R}^3$ if necessary, we can take the z-axis as the normal direction to M at a point $\boldsymbol{x}_0$. So we may assume that $f_x(\boldsymbol{x}_0) = f_y(\boldsymbol{x}_0) = 0$ and $f_z(\boldsymbol{x}_0) \neq 0$. Suppose that $\nabla h_{\boldsymbol{v}}(\boldsymbol{x}_0) = 0$. Then we have $\lambda_0 = \pm 1$ and

$$\mathcal{H}_{h_v} = \begin{pmatrix} \lambda_0 f_{xx} & \lambda_0 f_{xy} & \lambda_0 f_{xz} & 0 \\ \lambda_0 f_{xy} & \lambda_0 f_{yy} & \lambda_0 f_{yz} & 0 \\ \lambda_0 f_{xz} & \lambda_0 f_{yz} & \lambda_0 f_{zz} & f_z \\ 0 & 0 & f_z & 0 \end{pmatrix} \quad \text{at } (\boldsymbol{x}_0, \lambda_0).$$

It turns out that $\mathcal{H}_{h_v}$ is of rank 3 (resp. 2) at $(\boldsymbol{x}_0, \lambda_0)$ if and only if $f_{xx}f_{yy} - f_{xy}^2 = 0$ and $(f_{xx}, f_{xy}, f_{yy}) \neq (0, 0, 0)$ (resp. $f_{xx} = f_{xy} = f_{yy} = 0$) at $\boldsymbol{x}_0$.

Let us consider the case $\mathrm{rank}(\mathcal{H}_{h_v}) = 2$. A point $\boldsymbol{x}_0$ is a flat umbilic if and only if $K = H = 0$ at $\boldsymbol{x}_0$. Straightforward calculations show that

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{f_z^2}, \quad H = -\frac{f_{xx} + f_{yy}}{2\sqrt{f_z^2}} \quad \text{at } \boldsymbol{x}_0, \tag{A.1}$$

and thus $\boldsymbol{x}_0$ is a flat umbilic if and only if $f_{xx} = f_{xy} = f_{yy} = 0$ at $\boldsymbol{x}_0$. We complete the proof of (4).

Let us prove (3). Suppose that $\mathrm{rank}(\mathcal{H}_{h_v}) = 3$ at $(\boldsymbol{x}_0, \lambda_0)$. Since $f_{xx}f_{yy} - f_{xy}^2 = 0$ and $(f_{xx}, f_{xy}, f_{yy}) \neq (0, 0, 0)$ at $\boldsymbol{x}_0$, there exist a non-zero real number s and a pair of real numbers $(\xi, \eta) \neq (0, 0)$ such that

$$\begin{pmatrix} f_{xx}(\boldsymbol{x}_0) & f_{xy}(\boldsymbol{x}_0) \\ f_{xy}(\boldsymbol{x}_0) & f_{yy}(\boldsymbol{x}_0) \end{pmatrix} = s \begin{pmatrix} \eta^2 & -\xi\eta \\ -\xi\eta & \eta^2 \end{pmatrix}. \tag{A.2}$$

Now we have

$$J_{\psi_h}=\begin{pmatrix}\lambda_0 f_{xx} & \lambda_0 f_{xy} & \lambda_0 f_{xz} & 0\\ \lambda_0 f_{xy} & \lambda_0 f_{yy} & \lambda_0 f_{yz} & 0\\ \lambda_0 f_{xz} & \lambda_0 f_{yz} & \lambda_0 f_{zz} & f_z\\ 0 & 0 & f_z & 0\\ \lambda_0^2(P_f)_x & \lambda_0^2(P_f)_y & \lambda_0^2(P_f)_z & 0\end{pmatrix}\quad \text{at } (\boldsymbol{x}_0,\lambda_0).$$

Since $f_z(\boldsymbol{x}_0)\neq 0$, $\operatorname{rank}(J_{\psi_h})<4$ at $(\boldsymbol{x}_0,\lambda_0)$ if and only if

$$\operatorname{rank}\begin{pmatrix}\lambda_0 s\eta^2 & -\lambda_0 s\xi\eta & 0 & 0\\ -\lambda_0 s\xi\eta & \lambda_0 s\xi^2 & 0 & 0\\ 0 & 0 & 0 & f_z\\ 0 & 0 & f_z & 0\\ \lambda_0^2(P_f)_x & \lambda_0^2(P_f)_y & 0 & 0\end{pmatrix}<4 \quad \text{at } \boldsymbol{x}_0,$$

which is equivalent to the condition that

$$\begin{vmatrix}\eta^2 & -\xi\eta\\ (P_f)_x & (P_f)_y\end{vmatrix}=\begin{vmatrix}-\xi\eta & \xi^2\\ (P_f)_x & (P_f)_y\end{vmatrix}=0 \quad \text{at } \boldsymbol{x}_0,$$

Hence, straightforward calculations show that $\operatorname{rank}(J_{\psi_h})<4$ at $(\boldsymbol{x}_0,\lambda_0)$ if and only if

$$\text{(A.3)}\qquad f_{xxx}(\boldsymbol{x}_0)\xi^3+3f_{xxy}(\boldsymbol{x}_0)\xi^2\eta+3f_{xyy}(\boldsymbol{x}_0)\xi\eta^2+f_{yyy}(\boldsymbol{x}_0)\eta^3=0.$$

From (3.2), the asymptotic direction (dx,dy,dz) at $f(\boldsymbol{x}_0)$ is given by

$$dz=0,\quad f_{xx}dx^2+2f_{xy}dxdy+f_{yy}dy^2=0.$$

If follows from (A.2) that $(\xi,\eta,0)$ is the asymptotic direction at $f(\boldsymbol{x}_0)$. The point $f(\boldsymbol{x}_0)$ is a cusp of Gauss if and only if the parabolic curve is tangent to the asymptotic direction at $f(\boldsymbol{x}_0)$, namely, $\langle(\xi,\eta,0),\nabla P_f\rangle=0$ at $\boldsymbol{x}_0$. Simple calculation shows that this condition is equivalent to (A.3), and (3) is proved. □

We conclude from this theorem that $h_{\boldsymbol{v}}$ can be considered as the height function on implicit surfaces.

Next, we consider the distance squared functions on implicit surfaces. We set a family of functions on M by

$$D_{\boldsymbol{p}}:\mathbb{R}^3\times\mathbb{R}\setminus\{0\}\times\mathbb{R}^3\to\mathbb{R},\quad (\boldsymbol{x},\lambda,\boldsymbol{p})\mapsto\|\boldsymbol{x}-\boldsymbol{p}\|^2+\lambda f(\boldsymbol{x}),$$

and we define $d_{\boldsymbol{p}}(\boldsymbol{x},\lambda)=D_{\boldsymbol{p}}(\boldsymbol{x},\lambda,\boldsymbol{p})$.

Theorem 7. (1) $\nabla d_{\boldsymbol{p}}=0$ *at* $(\boldsymbol{x}_0,\lambda_0)$ *if and only if* $f(\boldsymbol{x}_0)=0$ *and* $\boldsymbol{p}=\boldsymbol{x}_0+\lambda_0\nabla f(\boldsymbol{x}_0)/2$.

(2) $\nabla d_{\boldsymbol{p}}=0$ *and* $\det(\mathcal{H}_{d_{\boldsymbol{p}}})=0$ *at* $(\boldsymbol{x}_0,\lambda_0)$ *if and only if* $f(\boldsymbol{x}_0)=0$, $\boldsymbol{p}=\boldsymbol{x}_0+\lambda_0\nabla f(\boldsymbol{x}_0)/2$ *and* $\lambda_0=2/(\|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0))$.

(3) $\nabla d_{\boldsymbol{p}}=0$, $\operatorname{rank}(\mathcal{H}_{d_{\boldsymbol{p}}})=3$ *and* $\operatorname{rank}(J_{\psi_d})<4$ *at* $(\boldsymbol{x}_0,\lambda_0)$ *if and only if* $\boldsymbol{x}_0$ *is a ridge point of* M *with respect to* κ_i, $\boldsymbol{p}=\boldsymbol{x}_0+\lambda_0\nabla f(\boldsymbol{x}_0)/2$ *and* $\lambda_0=2/(\|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0))$, *where* ψ_d *is a map* $\psi_d:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}^5$ *defined by* $\psi_d(\boldsymbol{x},\lambda)=(\nabla d(\boldsymbol{x},\lambda),\det(\mathcal{H}_{d_{\boldsymbol{p}}})(\boldsymbol{x},\lambda))$.

(4) $\nabla d_{\boldsymbol{p}}=0$ *and* $\operatorname{rank}(\mathcal{H}_{d_{\boldsymbol{p}}})=2$ *if and only if* $\boldsymbol{x}_0$ *is an umbilic of* M, $\boldsymbol{p}=\boldsymbol{x}_0+\lambda_0\nabla f(\boldsymbol{x}_0)/2$ *and* $\lambda_0=2/(\|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0))$.

Proof. We have $\nabla d_{\boldsymbol{p}}(\boldsymbol{x},\lambda)=(2(\boldsymbol{x}-\boldsymbol{p})+\lambda\nabla f(\boldsymbol{x}),f(\boldsymbol{x}))$, which prove the assertion (1). Now we write $\det(\mathcal{H}_{d_{\boldsymbol{p}}})=H_0+H_1\lambda+H_2\lambda^2$. Straightforward calculations show that

(A.4) $$H_0 = -4\|\nabla f\|^2, \quad H_1 = 4\|\nabla f\|^3 H, \quad H_2 = -\|\nabla f\|^4 K.$$

From these relations we have

$$\begin{aligned}\det(\mathcal{H}_{d_p}) &= -4\|\nabla f\|^2 + 4\|\nabla f\|^3 H\lambda - \|\nabla f\|^4 K\lambda^2 \\ &= -\frac{1}{\|\nabla f\|^4\lambda^2}\left(\left(\frac{2}{\|\nabla f\|\lambda}\right)^2 - 2H\left(\frac{2}{\|\nabla f\|\lambda}\right) + K\right).\end{aligned}$$

Since the principal curvatures are the roots of $\kappa^2 - 2H\kappa + K = 0$, the condition that $\det(\mathcal{H}_{d_p}) = 0$ holds if and only if $2/(\|\nabla f\|\lambda) = \kappa_i$, and the assertion (2) follows this.

By the same reason as in the proof of Theorem 6, we may assume that $f_x = f_y = 0$ and $f_z \neq 0$ at $\boldsymbol{x}_0$. Suppose that $\nabla d_{\boldsymbol{p}}(\boldsymbol{x}_0, \lambda_0) = 0$. Then we have

$$\mathcal{H}_{d_p} = \begin{pmatrix} 2+\lambda_0 f_{xx} & \lambda_0 f_{xy} & \lambda_0 f_{xz} & 0 \\ \lambda_0 f_{xy} & 2+\lambda_0 f_{yy} & \lambda_0 f_{yz} & 0 \\ \lambda_0 f_{xz} & \lambda_0 f_{yz} & 2+\lambda_0 f_{zz} & f_z \\ 0 & 0 & f_z & 0 \end{pmatrix} \quad \text{at} \quad (\boldsymbol{x}_0, \lambda_0).$$

Since $f_z(\boldsymbol{x}_0) \neq 0$, $\operatorname{rank}(\mathcal{H}_{d_p}) = 3$ (resp. 2) if and only if $\operatorname{rank}(A) = 1$ (resp. 0), where

$$A = \begin{pmatrix} 2+\lambda_0 f_{xx}(\boldsymbol{x}_0) & \lambda_0 f_{xy}(\boldsymbol{x}_0) \\ \lambda_0 f_{xy}(\boldsymbol{x}_0) & 2+\lambda_0 f_{yy}(\boldsymbol{x}_0) \end{pmatrix}.$$

Let us consider the case $\operatorname{rank}(A) = 0$. From (A.1), we have

$$H^2 - K = \frac{(f_{xx} - f_{yy})^2 + 4f_{xy}^2}{4f_z^2} \quad \text{at } \boldsymbol{x}_0,$$

and thus $f(\boldsymbol{x}_0)$ is an umbilic if and only if $f_{xx} = f_{yy}$ and $f_{xy} = 0$ at $\boldsymbol{x}_0$, in which case the common principal curvature is given by $\kappa(\boldsymbol{x}_0) = -f_{xx}(\boldsymbol{x}_0)/\|f_z(\boldsymbol{x}_0)\|$. On the other hand, $\operatorname{rank}(A) = 0$ if and only if $f_{xx}(\boldsymbol{x}_0) = f_{yy}(\boldsymbol{x}_0)$, $f_{xy}(\boldsymbol{x}_0) = 0$ and

$$\lambda_0 = -\frac{2}{f_{xx}(\boldsymbol{x}_0)} = \frac{2}{\|\nabla f(\boldsymbol{x}_0)\|\kappa(\boldsymbol{x}_0)},$$

and the assertion (4) is proved.

Now we turn to (3). By a suitable rotation in $\mathbb{R}^3$ if necessary, we can take $(0, 0, 1)$ as the unit normal vector to M at $f(\boldsymbol{x}_0)$. So we may assume that $f_x = f_y = 0$ and $f_z > 0$ at $\boldsymbol{x}_0$. Suppose that $\nabla d_{\boldsymbol{p}} = 0$ and $\operatorname{rank}(\mathcal{H}_{d_p}) = 3$ at $(\boldsymbol{x}_0, \lambda_0)$. Then the point $f(\boldsymbol{x}_0)$ is not an umbilic. From the proof of (4), we have $f_{xx}(\boldsymbol{x}_0) \neq f_{yy}(\boldsymbol{x}_0)$. Loss of generality, we may assume that $f_{xx} > f_{yy}$ at $\boldsymbol{x}_0$. Since $\operatorname{rank}(A) = 1$, we show that

(A.5) $$\begin{aligned}&\begin{pmatrix} 2+\lambda_0 f_{xx}(\boldsymbol{x}_0) & \lambda_0 f_{xy}(\boldsymbol{x}_0) \\ \lambda_0 f_{xy}(\boldsymbol{x}_0) & 2+\lambda_0 f_{yy}(\boldsymbol{x}_0) \end{pmatrix} \\ &= \frac{2}{\|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0)} \begin{pmatrix} \|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0) + f_{xx}(\boldsymbol{x}_0) & f_{xy}(\boldsymbol{x}_0) \\ f_{xy}(\boldsymbol{x}_0) & \|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0) + f_{yy}(\boldsymbol{x}_0) \end{pmatrix} \\ &= \frac{2}{\|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0)} \begin{pmatrix} \eta^2 & -\xi\eta \\ -\xi\eta & \xi^2 \end{pmatrix}\end{aligned}$$

holds for some $(\xi, \eta) \neq (0, 0)$. Now the principal directions $\boldsymbol{v}_i = (dx, dy, dz)$ are given by

$$\langle \nabla f, \boldsymbol{v}_i \rangle = 0, \quad (DN + \kappa_i I_3)\boldsymbol{v}_i = 0,$$

where I_3 is the 3×3 identity matrix. Since $f_x = f_y = 0$ and $f_z > 0$ at $\boldsymbol{x}_0$, $\boldsymbol{v}_i$ is given by $dz = 0$ and

$$\frac{1}{\|\nabla f(\boldsymbol{x}_0)\|} \begin{pmatrix} \|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0) + f_{xx}(\boldsymbol{x}_0) & f_{xy}(\boldsymbol{x}_0) \\ f_{xy}(\boldsymbol{x}_0) & \|\nabla f(\boldsymbol{x}_0)\|\kappa_i(\boldsymbol{x}_0) + f_{yy}(\boldsymbol{x}_0) \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From (A.5) and the above equality we see $(\xi, \eta) = (dx, dy)$. We shall prove the case $\lambda_0 = 2/(\|f(\boldsymbol{x}_0)\|\kappa_1(\boldsymbol{x}_0))$ and omit the proof for the case $\lambda_0 = 2/(\|f(\boldsymbol{x}_0)\|\kappa_2(\boldsymbol{x}_0))$. By a suitable rotation around the z-axis, we can take $(1, 0, 0)$ as the principal direction $\boldsymbol{v}_1$, and thus we may assume that $f_{xy}(\boldsymbol{x}_0) = 0$. Then we have

$$\lambda_0 = -\frac{2}{f_{xx}(\boldsymbol{x}_0)} \quad \text{and} \quad \kappa_1(\boldsymbol{x}_0) = -\frac{f_{xx}(\boldsymbol{x}_0)}{f_z(\boldsymbol{x}_0)}.$$

Since J_{φ_d} at $(\lambda_0, \boldsymbol{x}_0)$ is given by

$$J_{\psi_d} = \begin{pmatrix} 2 + \lambda_0 f_{xx} & \lambda_0 f_{xy} & \lambda_0 f_{xz} & 0 \\ \lambda_0 f_{xy} & 2 + \lambda_0 f_{yy} & \lambda_0 f_{yz} & 0 \\ \lambda_0 f_{xz} & \lambda_0 f_{yz} & 2 + \lambda_0 f_{zz} & f_z \\ 0 & 0 & f_z & 0 \\ (\mathcal{H}_{d_p})_x & (\mathcal{H}_{d_p})_y & (\mathcal{H}_{d_p})_z & (\mathcal{H}_{d_p})_\lambda \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -2f_{xz}/f_{xx} & 0 \\ 0 & 2(f_{xx} - f_{yy})/f_{xx} & -2f_{yz}/f_{xx} & 0 \\ -2f_{xz}/f_{xx} & -2f_{yz}/f_{xx} & 2(f_{xx} - f_{zz})/f_{xx} & f_z \\ 0 & 0 & f_z & 0 \\ (\mathcal{H}_{d_p})_x & (\mathcal{H}_{d_p})_y & (\mathcal{H}_{d_p})_z & (\mathcal{H}_{d_p})_\lambda \end{pmatrix},$$

$\mathrm{rank}(J_{\psi_d}) < 4$ if and only if

$$\frac{2(f_{xx}(\boldsymbol{x}_0) - f_{yy}(\boldsymbol{x}_0))}{f_{xx}(\boldsymbol{x}_0)} \left((\mathcal{H}_{d_p})_x - (\mathcal{H}_{d_p})_\lambda \frac{1}{f_z(\boldsymbol{x}_0)} \left(-\frac{2f_{xz}(\boldsymbol{x}_0)}{f_{xx}(\boldsymbol{x}_0)} \right) \right)$$
$$= \frac{4f_z(\boldsymbol{x}_0)(f_{xx}(\boldsymbol{x}_0) - f_{yy}(\boldsymbol{x}_0))(3f_{xx}(\boldsymbol{x}_0)f_{xz}(\boldsymbol{x}_0) - f_{xxx}(\boldsymbol{x}_0)f_z(\boldsymbol{x}_0))}{f_{xx}(\boldsymbol{x}_0)^2} = 0.$$

Meanwhile, the necessary and sufficient condition for the point $f(\boldsymbol{x}_0)$ to be a ridge point with respect to κ_1 is

$$\text{(A.6)} \qquad \langle \boldsymbol{v}_1, \nabla \kappa_1(\boldsymbol{x}_0) \rangle = dx\kappa_{1x}(\boldsymbol{x}_0) + dy\kappa_{1y}(\boldsymbol{x}_0) + dz\kappa_{1z}(\boldsymbol{x}_0) = 0.$$

Since $f_z > 0$ and $f_{xx} > f_{yy}$ at $\boldsymbol{x}_0$, (A.1) implies that

$$(H - \sqrt{H^2 - K})(\boldsymbol{x}_0) = -\frac{f_{xx}(\boldsymbol{x}_0)}{f_z(\boldsymbol{x}_0)} = \kappa_1(\boldsymbol{x}_0).$$

Since $\boldsymbol{v}_1 = (dx, dy, dz) = (1, 0, 0)$, it follows that (A.6) is equivalent to

$$\frac{\partial}{\partial x}(H - \sqrt{H^2 - K})(\boldsymbol{x}_0) = \frac{3f_{xx}(\boldsymbol{x}_0)f_{xz}(\boldsymbol{x}_0) - f_{xxx}(\boldsymbol{x}_0)f_z(\boldsymbol{x}_0)}{f_z(\boldsymbol{x}_0)^2} = 0,$$

and we complete the proof. □

We conclude from this theorem that d_p can be considered as the distance squared function on implicit surfaces.

We denote the discriminant of $\det(\mathcal{H}_{d_p})(\boldsymbol{x},\lambda)$ with respect to λ by $\Delta(\boldsymbol{x})$. We have the following criterion for umbilics of M in terms of Δ (cf. [3, Theorem 3.3]).

Theorem 8. *A point $\boldsymbol{x}$ on M is an umbilic on M if and only if $\Delta(\boldsymbol{x}) = 0$.*

Proof. Let us write $\det(\mathcal{H}_{d_p}) = H_0 + H_1\lambda + H_2\lambda^2$. Then $\Delta = H_1^2 - 4H_0H_2$. A point on a surface is an umbilic if and only if $H^2 - K = 0$ at the point. By using (A.4), we have

$$H^2 - K = \left(-\frac{H_1}{4\|\nabla f\|^3}\right)^2 - \left(-\frac{H_2}{\|\nabla f\|^4}\right) = -\frac{4({H_2}^2 - 4H_0H_2)}{{H_0}^3},$$

and we complete the proof. □

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