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# DECOMPOSITION OF COMPLEX HYPERBOLIC ISOMETRIES BY TWO COMPLEX SYMMETRIES

XUE-JING REN, BAO-HUA XIE and YUE-PING JIANG

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**Abstract**

Let $\mathbf{PU}(2,1)$ denote the holomorphic isometry group of the 2-dimensional complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$, and the group $\mathbf{SU}(2,1)$ is a 3-fold covering of $\mathbf{PU}(2,1)$: $\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{\omega I : \omega^3 = 1\}$. We study how to decompose a given pair of isometries $(A, B) \in \mathbf{SU}(2,1)^2$ under the form $A = I_1 I_2$ and $B = I_3 I_2$, where the $I_k$'s are complex symmetries about complex lines. If $(A, B)$ can be written as above, we call it is $\mathbb{C}$-decomposable. The main results are decomposability criteria, which improve and supplement the result of [17].

## 1. Introduction

Let $\mathbf{H}^2_{\mathbb{C}}$ denote the 2-dimensional complex hyperbolic space, and $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ denote the full isometry group which consists of holomorphic, as well as anti-holomorphic isometries. The projective unitary group $\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{\omega I : \omega^3 = 1\}$ which is an index 2 subgroup of $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ denotes the holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$. There are two types of totally geodesic 2-dimensional submanifolds in $\mathbf{H}^2_{\mathbb{C}}$: complex lines and the $\mathbb{R}$-planes. These correspond to two kinds of isometric involutions of $\mathbf{H}^2_{\mathbb{C}}$. A complex line $C \subset \mathbf{H}^2_{\mathbb{C}}$ is fixed by a unique involutive holomorphic isometry. We call this isometry the complex symmetry about $C$, which is represented by an element $I_C \in \mathbf{SU}(2,1)$ that is given by

$$I_C(z) = -z + 2\frac{\langle z, \mathbf{c}\rangle}{\langle \mathbf{c}, \mathbf{c}\rangle}\mathbf{c}, \tag{1.1}$$

where $\mathbf{c}$ is a polar vector of $C$. Any $\mathbb{R}$-plane $P$ is fixed pointwise by a unique anti-holomorphic isometry of order 2: the Lagrangian reflection about $P$. There is another involution in $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$: the complex reflection about a point in $\mathbf{H}^2_{\mathbb{C}}$.

An element $T$ in $G$ is called reversible if $T$ is conjugate to $T^{-1}$. Furthermore, if $T$ is a product of two involutions, it is called strongly reversible. Reversible elements and strongly reversible elements have been extensive studied in several contexts (see [2], [3], [10], [11], [12], [15], [18]). In particular, when $G = \mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$ there are three kinds involutive elements as mentioned above. In [4], Falbel and Zocca proved that every element in $\mathbf{PU}(2,1)$ is strongly reversible in $\mathrm{Iso}(\mathbf{H}^2_{\mathbb{C}})$, since it can be expressed as a product of two Lagrangian reflections. Gongopadhyay and Parker [9] classified reversible and strongly reversible element in $\mathbf{PU}(2,1)$ and shown that $T \in \mathbf{SU}(2,1)$ is reversible if and only if it is strongly reversible.

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For simplicity of presentation, we call an element $A \in G$ $\mathbb{C}$-strongly reversible, if $A$ is a product of two complex symmetries about complex lines. The $\mathbb{C}$-strong reversibility of loxodromic elements has been considered in [17] (Theorem 1). In this paper, we give the $\mathbb{C}$-strong reversibility criteria for parabolic and elliptic elements (Theorem 2 and Theorem 3).

**Theorem 1** (Proposition 4 of [17])**.** *Let A be a loxodromic element of* $\mathbf{PU}(2,1)$*. A is* $\mathbb{C}$*-strongly reversible if and only if A admits a lift to* $\mathbf{SU}(2,1)$ *with real trace greater than* 3.

**Theorem 2.** *Let A be a parabolic element of* $\mathbf{SU}(2,1)$*. Then A is* $\mathbb{C}$*-strongly reversible if and only if A is a* 3*-step unipotent parabolic. In other words, A is* $\mathbb{C}$*-strongly reversible if and only if A is strongly reversible.*

**Theorem 3.** *Let A be an elliptic element of* $\mathbf{SU}(2,1)$*. A is* $\mathbb{C}$*-strongly reversible if and only if A is strongly reversible and A is not a complex symmetry.*

A pair of elements $(A, B) \in \mathbf{SU}(2,1)^2$ or $\mathbf{PU}(2,1)^2$ is said to be $\mathbb{C}$-decomposable (resp. $\mathbb{R}$-decomposable) if there exist three complex symmetries (resp. three Lagrangian reflections) $I_1, I_2$ and $I_3$ such that $A = I_1I_2$ and $B = I_3I_2$ holds. Note that when writing the two elements $A$ and $B$ as products of complex symmetries (or Lagrangian reflections), the order in which the involutions appear is not important. $\mathbb{C}$-decomposability (resp. $\mathbb{R}$-decomposability) is very closely related to triangle groups (groups generated by three involutions). In the setting of $\mathbf{H}^2_{\mathbb{C}}$, many of the examples known of discrete groups are related to triangle groups, see for instance [6] and [16]. It also turns out that since the group $\langle A, B\rangle$ has index two in $\Gamma = \langle I_1, I_2, I_3\rangle$, then $\langle A, B\rangle$ is discrete if and only if $\Gamma$ is. This can lead to considerable simplification in the study of the discreteness of 2-generator subgroups of $\mathbf{PU}(2,1)$. For example, Gilman has presented a new sufficient condition for a subgroup of $\mathbf{PSL}(2,\mathbb{C})$ to be discrete by using this idea in [5]. For these reasons, we wish to decompose a pair of elements $(A, B)$ of $\mathbf{SU}(2,1)^2$ or $\mathbf{PU}(2,1)^2$ such that $\langle A, B\rangle$ contained with index 2 in a triangle group.

Will [17] gave $\mathbb{C}$-decomposability criterion and $\mathbb{R}$-decomposability criterion for a pair of loxodromic isometries $(A, B)$ of $\mathbf{H}^2_{\mathbb{C}}$, which are expressed in terms of traces of elements of the group $\langle A, B\rangle$. Since an element of $\mathbf{PU}(2,1)$ admits 3 lifts to $\mathbf{SU}(2,1)$, the trace of an isometry is well defined up to this indetermination. We will say that an isometry has real trace if and only if it admits a lift to $\mathbf{SU}(2,1)$ which has real trace.

**Theorem 4** (Theorem 1 of [17])**.** *Let A and B be two loxodromic isometries of* $\mathbf{H}^2_{\mathbb{C}}$ *and* $G = \langle A, B\rangle$*. Assume that G does not preserve a totally geodesic subspace. Then*

(1). *The following two propositions are equivalent:*

(i) *The isometry* $[A, B]$ *has real trace.*

(ii) *The pair* $(A, B)$ *is* $\mathbb{R}$*-decomposable.*

(2). *The following two propositions are equivalent:*

(i) *The isometries* $A, B, AB$ *and* $A^{-1}B$ *all have real trace.*

(ii) *Either the pair* $(A, B)$ *is* $\mathbb{C}$*-decomposable, or the pair* $(A^2, B^2)$ *is* $\mathbb{C}$*-decomposable.*

In 2013, Paupert and Will [14] provided a criterion to determine whether any two given elements of $\mathbf{PU}(2,1)$ is $\mathbb{R}$-decomposable, which completed the $\mathbb{R}$-decomposability criterion of elements in $\mathbf{PU}(2,1)$.

**Theorem 5** (Theorem 4.1 of [14])**.** *Let $A, B \in \mathbf{PU}(2,1)$ be two isometries not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then: the pair $(A, B)$ is $\mathbb{R}$-decomposable if and only if the commutator $[A, B]$ has a fixed point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$ whose associated eigenvalue is real and positive.*

It should be pointed out that a number of issues related to $\mathbb{C}$-decomposability are still unclear. For example, a $\mathbb{C}$-decomposability criterion for a pair of parabolic or elliptic elements has never been considered. In this paper, we are concerned with how to decompose a pair elements $(A, B) \in \mathbf{SU}(2,1)^2$ under the form $A = I_1 I_2$ and $B = I_3 I_2$, where $I_k$'s are complex symmetries, and we investigate criteria to determine whether two given elements of $\mathbf{SU}(2,1)$ can be $\mathbb{C}$-decomposable. Moreover, we also obtain the necessary and sufficient condition of $\mathbb{C}$-decomposability when one is a loxodromic element and the other one is a parabolic element. Our main results are the followings:

**Theorem 6.** *Let $A, B \in \mathbf{SU}(2,1)$ be two elements of the same type not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then, the pair $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible, and* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$.

**Proposition 1.1.** *If $A, B \in \mathbf{SU}(2,1)$ have a common fixed point in $\mathbf{H}^2_{\mathbb{C}}$, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible.*

**Proposition 1.2.** *Let $A, B \in \mathbf{SU}(2,1)$ have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$.*

(i) *If $A$ and $B$ are both loxodromic elements, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible and* $\mathrm{fix}(A) = \mathrm{fix}(B)$.

(ii) *If $A$ or $B$ is a loxodromic element and the other one is a* 3*-step unipotent parabolic element, then $(A, B)$ is not $\mathbb{C}$-decomposable.*

(iii) *If $A$ and $B$ are both* 3*-step unipotent parabolic elements, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ don't commute or $A, B$ have the same invariant fan.*

**Theorem 7.** *Let $(A, B)$ be a pair of elements of* $\mathbf{SU}(2,1)$*, where $A$ is a loxodromic element and $B$ is a parabolic element. Then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible,* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$, *and $A, B$ have distinct fixed points.*

Our Theorem 6 contains the result of Will's Theorem 4 (2). Propositions 1.1 and 1.2 complement the conclusion of Theorem 4. Theorem 7 shows the $\mathbb{C}$-decomposability criterion for one element is loxodromic and the other one is parabolic, which hasn't been considered in [17].

This paper is organized as follows. We start with some geometric preliminaries is Section 2. The definition of invariant fan of a parabolic element in Proposition 1.2 is also in Section 2. The proofs of Theorems 2 and 3 will be given in Section 3. Finally the proofs of our main results are presented in Section 4.

## 2. Preliminaries

**2.1. Complex hyperbolic space and isometries.** We begin with some background material on complex hyperbolic geometry. Much of this is found in Goldman's book [7].

Let $\mathbb{C}^{2,1}$ be a complex vector space of dimension 3 with a Hermitian form of signature $(2,1)$. Consider the subspaces

$$V_- = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\},$$

$$V_0 = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\},$$
$$V_+ = \{\mathbf{z} \in \mathbf{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle > 0\}.$$

where $\mathbf{z}$ is the column vector $\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^T$. Let $\mathbf{P} : \mathbb{C}^{2,1} \setminus \{0\} \longrightarrow \mathbb{C}\mathbf{P}^2$ be the canonical projection onto complex projective space. The complex hyperbolic space is defined to be $\mathbf{H}^2_{\mathbb{C}} = \mathbf{P}(V_-)$, and $\partial \mathbf{H}^2_{\mathbb{C}} = \mathbf{P}(V_0)$ is its boundary.

For the projective model the metric on $\mathbf{H}^2_{\mathbb{C}}$, called the Bergman metric is given by the distance function $\rho(\cdot,\cdot)$ defined by the formula

$$\cosh^2\left(\frac{\rho(\mathbf{P}(\mathbf{z}), \mathbf{P}(\mathbf{w}))}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}. \tag{2.2}$$

There are two standard models of $\mathbf{H}^2_{\mathbb{C}}$. The first one is called the ball model of $\mathbf{H}^2_{\mathbb{C}}$, when the Hermitian form is given by $\langle \mathbf{z}, \mathbf{z} \rangle = -|z_1|^2 + |z_2|^2 + |z_3|^2$. The second one is called the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$, when the Hermitian form is given by $\langle \mathbf{z}, \mathbf{z} \rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$. From (2.2) it is easy to show that the projective unitary group $\mathbf{PU}(2,1)$ acts by isometries on $\mathbf{H}^2_{\mathbb{C}}$, which we identify with the holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$. The group $\mathbf{SU}(2,1)$ is a 3-fold covering of $\mathbf{PU}(2,1)$:

$$\mathbf{PU}(2,1) = \mathbf{SU}(2,1)/\{I, \omega I, \omega^2 I\},$$

where $\omega = (-1 + \sqrt{3}i)/2$ is a cube root of unity.

The familiar trichotomy from real hyperbolic geometry applies in the complex hyperbolic setting as well: $A \in \mathbf{PU}(2,1)$ is said to be:

- loxodromic if it fixes exactly two points of $\partial \mathbf{H}^2_{\mathbb{C}}$;
- parabolic if it fixes exactly one point of $\partial \mathbf{H}^2_{\mathbb{C}}$;
- elliptic if it fixes at least one point of $\mathbf{H}^2_{\mathbb{C}}$.

It is clear that a fixed point of an isometry $A$ lying in $\mathbf{H}^2_{\mathbb{C}}$ or its boundary corresponds to an eigenvector of the corresponding matrix lying in $V_-$ or $V_0$ respectively. So we have the following theorem.

**Theorem 8** ([13])**.** *Let $A$ be a matrix in $\mathbf{SU}(2,1)$. Then one of the following possibilities occurs:*

(i) *$A$ has two null eigenvectors with eigenvalues $\lambda$ and $\overline{\lambda}^{-1}$ where $|\lambda| \neq 1$, in which case $A$ is loxodromic;*

(ii) *$A$ has a repeated eigenvalue of unit modulus whose eigenspace is spanned by a null vector, in which case $A$ is parabolic;*

(iii) *$A$ has a negative eigenvector, in which case $A$ is elliptic.*

An eigenvalue $\lambda$ of $A \in \mathbf{SU}(2,1)$ is said to be of negative type, positive type or null if every eigenvector of $\lambda$ is in $V_-$, $V_+$ or $V_0$ respectively. The eigenvalue $\lambda$ is said to be of indefinite type if there are some eigenvectors of $\lambda$ in $V_-$ and some in $V_+$.

A parabolic element in $\mathbf{SU}(2,1)$ is called unipotent if it is a unipotent matrix. Unipotent parabolic elements are either 2-step or 3-step, according to whether the minimal polynomial of the matrix is $(x-1)^2$ or $(x-1)^3$. If a parabolic element is not unipotent, we call it screw-parabolic. It can be decomposed as $A = PE = EP$, where $P$ is a unipotent parabolic element and $E$ is an elliptic element.

An elliptic element in $\mathbf{SU}(2,1)$ is called regular if it has three distinct eigenvalues. A non-regular elliptic element is called special. Special elliptic elements have two kinds: An elliptic element is a complex reflection about complex line if it has 2 equal eigenvalues, and one of which has eigenvectors in $V_-$; An elliptic element is a complex reflection in a point if it has 2 equal eigenvalues, and the remaining one has eigenvectors in $V_-$. These reflections may not have order 2, and not even finite order.

Also, we can use the trace of $A \in \mathbf{SU}(2,1)$ to decide whether it is elliptic, parabolic or loxodromic.

**Lemma 9** ([7]). *Let $f$ be the polynomial $f(z) = |z|^4 - 8\mathfrak{R}(z^3) + 18|z|^2 - 27$, where $z \in \mathbb{C}$. Denote by $C_3$ is the set of cube roots of unity in $\mathbb{C}$. Let $A \in \mathbf{SU}(2,1)$. Then:*
(1) *$A$ is regular elliptic* $\Leftrightarrow f(\mathrm{tr}(A)) < 0$;
(2) *$A$ is loxodromic* $\Leftrightarrow f(\mathrm{tr}(A)) > 0$;
(3) *$A$ is screw parabolic or special elliptic* $\Leftrightarrow f(\mathrm{tr}(A)) = 0$ *and* $\mathrm{tr}(A) \notin 3C_3$;
(4) *$A$ is unipotent or the identity* $\Leftrightarrow \mathrm{tr}(A) \in 3C_3$.

**2.2. The ball model of $\mathbf{H}^2_{\mathbb{C}}$.** The ball model of $\mathbf{H}^2_{\mathbb{C}}$ arises from the choice of Hermitian form

$$H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.3}$$

The vector $\mathbf{z} = \begin{bmatrix} 1 & z_1 & z_2 \end{bmatrix}^T$ is the standard lift of $z \in \mathbf{H}^2_{\mathbb{C}}$ to $V_-$. Furthermore, we see that $z \in \mathbf{H}^2_{\mathbb{C}}$ provided

$$\langle \mathbf{z}, \mathbf{z} \rangle = -1 + |z_1|^2 + |z_2|^2 < 0.$$

It is obviously that any elliptic element of $\mathbf{H}^2_{\mathbb{C}}$ is conjugate to one given by the diagonal matrix

$$E_{(\alpha,\beta)} = \begin{bmatrix} e^{-i(\alpha+\beta)/3} & 0 & 0 \\ 0 & e^{i(2\alpha-\beta)/3} & 0 \\ 0 & 0 & e^{i(2\beta-\alpha)/3} \end{bmatrix}. \tag{2.4}$$

Projectively, the associated isometry is given by

$$(z_1, z_2) \mapsto (e^{i\alpha} z_1, e^{i\beta} z_2).$$

Sometimes it is more convenient to work with the lift to $\mathbf{U}(2,1)$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{bmatrix}. \tag{2.5}$$

**2.3. The Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$.** The Siegel domain model of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ corresponds to the Hermitian form given by the matrix :

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$ with horospherical coordinates is

$$\mathbf{H}^2_{\mathbb{C}} = \{(z, t, u) : z \in \mathbb{C}, t \in \mathbb{R}, u \in \mathbb{R}_+\}.$$

The boundary of the Siegel domain is

$$\partial\mathbf{H}^2_{\mathbb{C}} = \{(z, t, 0) : z \in \mathbb{C}, t \in \mathbb{R}\} \cup \{\infty\}.$$

Points in $\mathbf{H}^2_{\mathbb{C}}$ may be identified with negative vectors in $\mathbb{C}^{2,1}$ and points of $\partial\mathbf{H}^2_{\mathbb{C}}$ may be identified with null vectors in $\mathbb{C}^{2,1}$ by the map $\psi : \overline{\mathbf{H}^2_{\mathbb{C}}} \to \mathbb{C}^{2,1}$ given by

$$\psi : (z, t, u) \mapsto \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix}, \quad \psi : \infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.6}$$

The boundary $\partial\mathbf{H}^2_{\mathbb{C}} \backslash \{\infty\}$ is a copy of the Heisenberg group $\mathfrak{N}$ of dimension 3, with group law given in $(z, t)$ coordinates by:

$$(z_1, t_1) * (z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\Im(z_1\bar{z}_2)).$$

We conclude this subsection by considering the subgroup of $\mathbf{PU}(2, 1)$ stabilising the point at infinity. Such maps will be called Heisenberg similarities. The corresponding elements in $\mathbf{SU}(2, 1)$ are generated by the following 3 types: Heisenberg translations $T_{(z,t)}$ ($(z, t) \in \mathbb{C}\times\mathbb{R}$), Heisenberg rotations $R_\theta$ ($\theta \in \mathbb{R}/2\pi\mathbb{Z}$) and Heisenberg dilations $D_r$ ($r > 1$), where:

$$T_{(z,t)} = \begin{bmatrix} 1 & -\bar{z} & -(|z|^2 - it)/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \ R_\theta = \begin{bmatrix} e^{-i\theta/3} & 0 & 0 \\ 0 & e^{2i\theta/3} & 0 \\ 0 & 0 & e^{-i\theta/3} \end{bmatrix}, \ D_r = \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix}. \tag{2.7}$$

**2.4. The invariant fan of a 3-step unipotent parabolic.** The standard reference for the invariant fan of a 3-unipotent parabolic element is [8] (see also Section 2.3 of [14]).

For any $z \in \mathbb{C}$ there exists a unique complex line which contains $\infty$ and the point $(z, 0)$. This induces a projection $\widetilde{\Pi} : \overline{\mathbf{H}^2_{\mathbb{C}}} \setminus \{\infty\} \longmapsto \mathbb{C}$ whose fibers are the complex lines through $\infty$. In restriction to the boundary, this projection is just the vertical projection $\Pi : (z, t) \longmapsto (z, 0)$, which is given in Heisenberg coordinates.

A fan through $\infty$ is the preimage of any affine line in $\mathbb{C}$ under the projection $\widetilde{\Pi}$. A general fan is the image of a fan through $\infty$ by an element of $\mathbf{PU}(2, 1)$. As stated in [8], fans enjoy a double foliation, by $\mathbb{R}$-planes and complex lines. In [14], the authors make the foliation of fan explicit. See the following:

**Lemma 10** (Lemma 2.2 of [14])**.** *Let $L_{w,k}$ be the affine line in $\mathbb{C}$ parameterized by $L_{w,k} = \{w(s + ik), s \in \mathbb{R}\}$, for some unit modulus $w$ and $k \geq 0$. Then the boundary foliation of the fan above $L_{w,k}$ is given by the lines parameterized in Heisenberg coordinates by $L_{t_0}$ =*

$\{(w(s+ik), t_0+2sk), s \in \mathbb{R}\}$.

If $T$ is a 3-step unipotent parabolic element of $\mathbf{PU}(2,1)$, there exists a unique fan $F_T$ through the fixed point of $T$ such that it is stable under $T$ and every leaf of the foliation of $F_T$ by real planes is stable under $T$. We call this fan $F_T$ the invariant fan of $T$.

Remark 2.1. When $T = T_{(z,t)}$ with $z \neq 0$, the fan $F_T$ is the one above the affine line $L_{w,k}$, where $w = z/|z|$ and $k = t/(4|z|)$.

For future reference, let us state the following proposition.

**Proposition 2.1** (Lemma 2.3 of [14]). *Let $T_{(z_1,t_1)}$ and $T_{(z_2,t_2)}$ be two 3-step unipotent parabolic elements. Then these two translations commute if and only if $\bar{z}_1 z_2 \in \mathbb{R}$, which is equivalent to saying that their invariant fans are parallel.*

**2.5. $\mathbb{C}$-strong reversibility and $\mathbb{C}$-decomposability.** In [9], the authors described necessary and sufficient conditions of reversibility or strong reversibility of $A \in \mathbf{SU}(2,1)$, which is written in terms of trace and eigenvalue of $A$. Since reversibility is equivalent to strong reversibility for $A \in \mathbf{SU}(2,1)$ (see Theorem 4.2 of [9]), we have the following theorem:

**Theorem 11** (Corollary 4.10 of [9]). *Let $A$ be an element in $\mathbf{SU}(2,1)$.*

(1) *$A$ is a loxodromic element. $A$ is strongly reversible in $\mathbf{SU}(2,1)$ if and only if $\mathrm{tr}(A) \in \mathbb{R}$.*

(2) *$A = PE$ is a parabolic element. $A$ is strongly reversible in $\mathbf{SU}(2,1)$ if and only if the trace of $A$ is real, the null eigenvalue of $A$ is $1$ or $-1$ and the minimum polynomial of $P$ is $(x-1)^3$.*

(3) *$A$ is an elliptic element. $A$ is strongly reversible in $\mathbf{SU}(2,1)$ if and only if the trace of $A$ is real and the eigenvalue of negative type or indefinite type of $A$ is $1$ or $-1$.*

We define $A \in \mathbf{SU}(2,1)$ is $\mathbb{C}$-strongly reversible, if $A = I_1 I_2$. A pair of elements $(A, B) \in \mathbf{SU}(2,1)^2$, if $A = I_1 I_2$ and $B = I_3 I_2$, we call $(A, B)$ is $\mathbb{C}$-decomposable. The above $I_1, I_2, I_3$ are both elements of $\mathbf{SU}(2,1)$, which represent three complex symmetries about complex lines as (1.1). It is apparent that if $A$ is $\mathbb{C}$-strongly reversible, then $A$ is strongly reversible. Generally speaking, the converse implication is not true.

**Lemma 12.** *Let $A \in \mathbf{SU}(2,1)$ be $\mathbb{C}$-strongly reversible, then $A$ has real trace.*

Proof. If $A$ is $\mathbb{C}$-strongly reversible, then it may be written as $A = I_1 I_2$, where $I_1, I_2$ are two matrices in $\mathbf{SU}(2,1)$ corresponding two complex symmetries. Hence $A^{-1} = I_2 I_1 = (I_1)^{-1} A (I_1) = (I_2) A (I_2)^{-1}$. In particular, $A$ is conjugate to $A^{-1}$, so they have the same trace. Since in $\mathbf{SU}(2,1)$ we have $\mathrm{tr}(A^{-1}) = \overline{\mathrm{tr}(A)}$, we see $\mathrm{tr}(A) = \overline{\mathrm{tr}(A)}$, so $\mathrm{tr}(A)$ is real. □

The following proposition will be needed in the Section 4.

**Proposition 2.2** (Proposition 4 of [17]). *$A \in \mathbf{SU}(2,1)$ is a loxodromic element, if $I_1$ and $I_2$ are two complex symmetries such that $A = I_1 I_2$, both $I_1$ and $I_2$ permute the fixed points of $A$.*

If $(A, B) \in \mathbf{SU}(2,1)^2$ is $\mathbb{C}$-decomposable, that is $A = I_1 I_2$ and $B = I_3 I_2$, where $I_1, I_2, I_3 \in \mathbf{SU}(2,1)$ given by (1.1) which represent three complex symmetries. It follows that $AB = I_1(I_2 I_3 I_2)$ and $BA^{-1} = I_3 I_1$ are both $\mathbb{C}$-strongly reversible. According to Lemma 12, we obtain the following proposition:

**Proposition 2.3.** *If* $(A, B) \in \mathbf{SU}(2,1)^2$ *is* $\mathbb{C}$*-decomposable. Then* $A, B, AB$ *and* $BA^{-1}$ *all have real trace.*

From the above we know that elements of $\mathbf{SU}(2,1)$ with real trace are very important for us.

**Proposition 2.4** (Proposition 2.3 of [14])**.** *Let* $A \in \mathbf{SU}(2,1)$ *satisfy* $\mathrm{tr}(A) \in \mathbb{R}$. *Then A has an eigenvalue equal to* 1. *More precisely:*
• *If A is loxodromic then A has eigenvalues* $\{1, r, 1/r\}$ *for some* $r > 1$ *or* $r < -1$.
• *If A is elliptic then A has eigenvalues* $\{1, e^{i\theta}, e^{-i\theta}\}$ *for some* $\theta \in (0, \pi]$.
• *If A is parabolic then A has eigenvalues* $\{1, 1, 1\}$ *or* $\{1, -1, -1\}$.

The main purpose of this paper is to discuss the $\mathbb{C}$-strong reversibility and $\mathbb{C}$-decomposability of elements in $\mathbf{SU}(2,1)$. It is simple to show that the $\mathbb{C}$-strong reversibility for one element and the $\mathbb{C}$-decomposability for a pair elements of $\mathbf{SU}(2,1)$ are both invariant under conjugation, which make things a little easier.

## 3. $\mathbb{C}$-strong reversibility

In this section, we study the $\mathbb{C}$-strong reversibility of parabolic and elliptic elements. We have known the results about strong reversibility of elements of $\mathbf{SU}(2,1)$ from Theorem 11, then to investigate $\mathbb{C}$-strong reversibility one needs to rule out the case where at least one of $I_1$ and $I_2$ fixes a point.

**Lemma 13.** (1) *Suppose that* $A = I_1 I_2$ *where* $I_1$ *and* $I_2$ *are complex involutions in* $\mathbf{SU}(2,1)$ *with unique fixed points* $p_1$ *and* $p_2$ *respectively. Then*

$$\mathrm{tr}(A) = 2\cosh\left(\rho(p_1, p_2)\right) + 1.$$

*In particular, if A is not the identity map then* $\mathrm{tr}(A) > 3$, *so A is hyperbolic.*

(2) *Suppose that* $A = I_1 I_2$ *where* $I_1$ *and* $I_2$ *are complex involutions in* $\mathbf{SU}(2,1)$, $I_1$ *has a unique fixed points* $p_1$ *and* $I_2$ *fixes the complex line* $L_2$. *Then*

$$\mathrm{tr}(A) = -2\cosh\left(\rho(p_1, L_2)\right) + 1.$$

*In particular,* $\mathrm{tr}(A) \le -1$. *If* $p_1 \notin L_2$ *then* $\mathrm{tr}(A) \le -1$ *and A is hyperbolic. If* $p_1 \in L_2$ *then A is a complex symmetry fixing a complex line through* $p_1$ *orthogonal to* $L_2$.

The above result is easy to verify, so the proof is omitted.

**3.1. $\mathbb{C}$-strong reversibility of parabolic elements.** Owing to Proposition 2.4, in the Siegel domain model of $\mathbf{H}^2_{\mathbb{C}}$, any parabolic element of $\mathbf{SU}(2,1)$ which has real trace is conjugate in $\mathbf{SU}(2,1)$ to exactly one of the following:

• If it is 3-step unipotent parabolic: $\begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

• If it is 2-step unipotent parabolic: $\begin{bmatrix} 1 & 0 & i/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

• If it is screw parabolic: $\begin{bmatrix} -1 & 0 & -i/2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

From Theorem 11, any 2-step unipotent parabolic elements and screw parabolic elements are not $\mathbb{C}$-strongly reversible. Combined with Lemma 13, we can get the following result immediately.

**Theorem 14.** *Let $A$ be a parabolic element of* $\mathbf{SU}(2,1)$. *Then $A$ is $\mathbb{C}$-strongly reversible if and only if $A$ is a* 3*-step unipotent parabolic. In other words, $A$ is $\mathbb{C}$-strongly reversible if and only if $A$ is strongly reversible.*

As stated above, if $A$ is a 3-step unipotent parabolic, we can assume $A = T_{(1,0)} \in \mathbf{SU}(2,1)$, and the null eigenvalue of $A$ is 1. We can decompose $A$ as following:

$$A = T_{(1,0)} = \begin{bmatrix} -1 & -1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.8}$$

The first (resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} 1/2 & -1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$).

More generally,

$$T_{(z,0)} = \begin{bmatrix} -1 & -\bar{z} & |z|^2/2 \\ 0 & 1 & -z \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{3.9}$$

where $z \neq 0$. The first (resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} \bar{z}/2 & -1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$).

**Proposition 3.1.** *Let $A \in \mathbf{SU}(2,1)$ be a* 3*-step unipotent parabolic element fixing $p \in \partial \mathbf{H}^2_{\mathbb{C}}$, and $A = I_1 I_2$, where $I_1$ and $I_2$ are both complex symmetries. Then $I_1, I_2$ both fix the point $p$. Especially, the fixed lines of $I_1$ and $I_2$ lie in the invariant fan of $A$.*

Proof. Normalise $A = T_{(1,0)}$, suppose $I_2(\infty) = q \neq \infty$, where $q \in \partial \mathbf{H}^2_{\mathbb{C}}$. Since $A(\infty) = I_1 I_2(\infty) = \infty$, then $I_1(q) = \infty$. Because $I_1^2 = I_2^2 = \mathrm{Id}$, we get $A(q) = q$ which is a contradiction. Thus, $I_2$ fixes $\infty$. Similarly, $I_1$ also fixes $\infty$. Let $L_1$ and $L_2$ be two complex lines fixed pointwise by $I_1$ and $I_2$ respectively. Then $L_1$ and $L_2$ both through $\infty$, we can obtain

$$I_k = \begin{bmatrix} -1 & -2\bar{z}_k & 2|z_k|^2 \\ 0 & 1 & -2z_k \\ 0 & 0 & -1 \end{bmatrix},$$

where $z_k \in \mathbb{C}, k = 1, 2$. As $A = I_1 I_2$, we have $z_1, z_2 \in \mathbb{R}$. Thus, $L_1$ and $L_2$ both lie in the invariant fan of $A$. □

**3.2. $\mathbb{C}$-strong reversibility of elliptic elements.** In this subsection, we use the unit ball model of $\mathbf{H}^2_{\mathbb{C}}$ with the Hermitian form $H$ in (2.3). Let $A$ be an elliptic element with real trace. Combining Proposition 2.4 and Theorem 11, we know that if $A$ is $\mathbb{C}$-strongly reversible, $A$ may be conjugate in $\mathbf{SU}(2,1)$ to exactly one of the following:

- If it is regular elliptic: $E_{(\theta,-\theta)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{bmatrix}$, where $\theta \in (0, \pi)$;
- If it is a complex reflection about a complex line (or boundary elliptic): $E_l = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;
- If it is a complex reflection in a point: $E_{(\pi,-\pi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

From the above analysis and Lemma 13, we obtain the following theorem, which states the $\mathbb{C}$-strong reversibility of elliptic elements in $\mathbf{SU}(2, 1)$.

**Theorem 15.** *Let $A$ be an elliptic element of $\mathbf{SU}(2, 1)$. $A$ is $\mathbb{C}$-strongly reversible if and only if $A$ is a regular elliptic or a complex reflection in a point which is conjugate to one given by the matrix $E_{(\theta,-\theta)}$ ($\theta \in (0, \pi]$). In other words, $A$ is $\mathbb{C}$-strongly reversible if and only if $A$ is strongly reversible and $A$ is not a complex symmetry.*

We can decompose $E_{(\theta,-\theta)}$ as following:

$$E_{(\theta,-\theta)} = I_1 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & e^{i\theta} \\ 0 & e^{-i\theta} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \theta \in (0, \pi],$$

where $I_1$ represents the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & \sqrt{2}e^{i\theta}/2 & \sqrt{2}/2 \end{bmatrix}^T$, $I_2$ represents the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}^T$.

Remark 3.1. Any pair of elliptic elements which conjugates to $(E_{(\theta,-\theta)}, E_{(\alpha,-\alpha)})$ ($\theta, \alpha \in (0, \pi]$) is $\mathbb{C}$-decomposable.

We have known that if an elliptic element $A$ is $\mathbb{C}$-strongly reversible, either it is a regular elliptic element which conjugates to $E_{(\theta,-\theta)}$ ($\theta \in (0, \pi)$), or it is a complex reflection of order 2 about a point in $\mathbf{H}^2_{\mathbb{C}}$. Then the unique fixed point of $A$ is in $\mathbf{H}^2_{\mathbb{C}}$. The following proposition is well known.

**Proposition 3.2.** *Let $A \in \mathbf{SU}(2, 1)$ be an elliptic element fixing $p \in \mathbf{H}^2_{\mathbb{C}}$, and $A = I_1 I_2$, where $I_1$ and $I_2$ are both complex symmetries. Then $I_1, I_2$ both fix the point $p$.*

Let $E$ be any $\mathbb{C}$-strongly reversible regular elliptic element fixing the point 0 (or it is a complex reflection in the point 0). As $E$ is conjugate in $\mathbf{SU}(2, 1)$ to $E_{(\theta,-\theta)}$ ($\theta \in (0, \pi]$), we can represent such $E$ by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta \pm i\sqrt{1 - |b|^2}\sin\theta & i\bar{b}\sin\theta \\ 0 & ib\sin\theta & \cos\theta \mp i\sqrt{1 - |b|^2}\sin\theta \end{bmatrix}, \tag{3.10}$$

where $\theta \in (0, \pi]$, $b \in \mathbb{C}$ and $0 \le |b| \le 1$.

## 4. $\mathbb{C}$-decomposability

**4.1. Main results.** In this section, we give the $\mathbb{C}$-decomposability of two elements of the same type of $\mathbf{SU}(2,1)$ and get the necessary and sufficient condition of $\mathbb{C}$-decomposability when one is a loxodromic element and the other one is a parabolic element. Recall that a pair of elements $(A, B) \in \mathbf{SU}(2,1)^2$ is said to be $\mathbb{C}$-decomposable if there exist three complex symmetries $I_1, I_2, I_3$ such that $A = I_1 I_2$ and $B = I_3 I_2$. Now we are ready to prove our main result.

**Theorem 16.** *Let $A, B \in \mathbf{SU}(2,1)$ be two elements of the same type not fixing a common point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$. Then, the pair $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible, and* $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$.

Proof. (1). Let $(A, B)$ be a pair of loxodromic elements of $\mathbf{SU}(2,1)$ and $A, B$ have distinct fixed points. From Theorems 1 and 4, we get the result.

(2). Let $(A, B) \in \mathbf{SU}(2,1)^2$ be a pair of parabolic elements and $\mathrm{fix}(A) \cap \mathrm{fix}(B) = \emptyset$.

($\Rightarrow$) Assume $(A, B)$ is $\mathbb{C}$-decomposable, then $A$ and $B$ must be $\mathbb{C}$-strongly reversible and $A, B$ are both 3-step unipotent parabolic by Theorem 14. Thus, $\mathrm{tr}(AB) \in \mathbb{R}$ and $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$ by Proposition 2.3.

($\Leftarrow$) Now that $A$ and $B$ are both $\mathbb{C}$-strongly reversible, it follows that $A, B$ are both 3-step unipotent parabolic. For simplicity, we may take $A = T_{(z,t)}$, $B$ is a 3-step unipotent parabolic element fixing 0, where $B$ has the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ \zeta & 1 & 0 \\ \frac{-|\zeta|^2 + iv}{2} & -\overline{\zeta} & 1 \end{bmatrix}, \ (\zeta, v) \in \{\mathbb{C} \setminus 0\} \times \mathbb{R}.$$

Notice that $AB$ and $BA^{-1}$ both have real trace, we get:

$$\mathrm{tr}(AB) = 3 - 2\Re(\overline{z}\zeta) + \frac{|z|^2|\zeta|^2 - tv}{4} - \frac{i(|z|^2 v + |\zeta|^2 t)}{4} \in \mathbb{R}$$

and

$$\mathrm{tr}(BA^{-1}) = 3 + 2\Re(\overline{z}\zeta) + \frac{|z|^2|\zeta|^2 + tv}{4} - \frac{i(|z|^2 v - |\zeta|^2 t)}{4} \in \mathbb{R},$$

Due to $z \neq 0$ and $\zeta \neq 0$, it follows that $t = v = 0$. Thus we derived that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ \zeta & 1 & 0 \\ \frac{-|\zeta|^2}{2} & -\overline{\zeta} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ \zeta & 1 & 0 \\ |\zeta|^2/2 & \overline{\zeta} & -1 \end{bmatrix}. \tag{4.11}$$

The first(resp. second) matrix in the right hand side product corresponds to the complex symmetry about the complex line polar to $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ (resp. $\begin{bmatrix} 0 & 1 & \overline{\zeta}/2 \end{bmatrix}^T$ ).

Consequently, $(A, B)$ is $\mathbb{C}$-decomposable from (3.9) and (4.11).

(3). Let $A$ be a $\mathbb{C}$-strongly reversible elliptic element in $\mathbf{SU}(2,1)$ fixing the origin in the ball model. Then the origin corresponds to a 1-eigenvector of $A$, and so $A$ has the following form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_1 & -\overline{\mu}_1 \\ 0 & \mu_1 & \overline{\lambda}_1 \end{bmatrix},$$

where $\lambda_1 + \overline{\lambda}_1 = 2\cos(\theta_1)$ for some $\theta_1 \in (0, 2\pi)$. Without loss of generality, the fixed point of $B$ is $p = (\tanh(t), 0) \in \mathbb{B}^2$. A map in $\mathbf{SU}(2,1)$ sending the origin to $p$ is

$$\begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore we may suppose that $B$ has the form

$$\begin{aligned} B &= \begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & -\overline{\mu}_2 \\ 0 & \mu_2 & \overline{\lambda}_2 \end{bmatrix} \begin{bmatrix} \cosh(t) & -\sinh(t) & 0 \\ -\sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - (\lambda_2 - 1)\sinh^2(t) & (\lambda_2 - 1)\cosh(t)\sinh(t) & -\overline{\mu}_2 \sinh(t) \\ -(\lambda_2 - 1)\cosh(t)\sinh(t) & \lambda_2 + (\lambda_2 - 1)\sinh^2(t) & -\overline{\mu}_2 \cosh(t) \\ -\mu_2 \sinh(t) & \mu_2 \cosh(t) & \overline{\lambda}_2 \end{bmatrix}. \end{aligned}$$

Note that if $(A, B)$ is $\mathbb{C}$-decomposable as $A = I_1 I_2$ and $B = I_3 I_2$, then $I_2$ must fix the complex line passing through the fixed points of $A$ and $B$ from Proposition 3.2. In the case above,

$$I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $(A, B)$ is $\mathbb{C}$-decomposable if and only if $\mathrm{tr}(AI_2) = \mathrm{tr}(BI_2) = -1$.

$$\mathrm{tr}(AI_2) = -1 + \overline{\lambda}_1 - \lambda_1, \quad \mathrm{tr}(BI_2) = -1 + \overline{\lambda}_2 - \lambda_2,$$

so $(A, B)$ is $\mathbb{C}$-decomposable if and only if $\lambda_1$ and $\lambda_2$ are both real.

The necessity is obvious. Now if $\mathrm{tr}(AB)$ and $\mathrm{tr}(BA^{-1})$ are both real. A simple calculation shows that

$$\mathrm{tr}(AB) = 1 + \lambda_1\lambda_2 + \overline{\lambda}_1\overline{\lambda}_2 + (\lambda_1 - 1)(\lambda_2 - 1)\sinh^2(t) - (\overline{\mu}_1\mu_2 + \mu_1\overline{\mu}_2)\cosh(t),$$

$$\mathrm{tr}(BA^{-1}) = 1 + \overline{\lambda}_1\lambda_2 + \lambda_1\overline{\lambda}_2 + (\overline{\lambda}_1 - 1)(\lambda_2 - 1)\sinh^2(t) + (\overline{\mu}_1\mu_2 + \mu_1\overline{\mu}_2)\cosh(t).$$

Therefore

$$2i\Im(\mathrm{tr}(AB) + \mathrm{tr}(BA^{-1})) = (\lambda_1 + \overline{\lambda}_1 - 2)(\lambda_2 - \overline{\lambda}_2)\sinh^2(t),$$

$$2i\Im(\mathrm{tr}(AB) - \mathrm{tr}(BA^{-1})) = (\lambda_1 - \overline{\lambda}_1)(\lambda_2 + \overline{\lambda}_2 - 2)\sinh^2(t).$$

Since $\lambda_j + \overline{\lambda}_j = 2\cos(\theta_j) < 2$ $(j = 1, 2)$, we see that $\lambda_1$ and $\lambda_2$ must be real as required. Thus, $(A, B)$ is $\mathbb{C}$-decomposable. □

**4.2. Groups fixing a point.** In this subsection, we think about the case when $A$ and $B$ have a common fixed point in $\overline{\mathbf{H}^2_{\mathbb{C}}}$.

**Proposition 4.1.** *If $A, B \in \mathbf{SU}(2,1)$ have a common fixed point in $\mathbf{H}^2_{\mathbb{C}}$, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible.*

Proof. Let $(A, B) \in \mathbf{SU}(2,1)^2$ be a pair of elliptic elements and have a common point $p \in \mathbf{H}^2_{\mathbb{C}}$. The necessity is trivial.

Now suppose $A$ and $B$ are both $\mathbb{C}$-strongly reversible, thus $A$ and $B$ are both regular elliptic elements, or both complex symmetries in a point, or one of them is regular elliptic and the other one is complex reflection in a point by Theorem 15.

(i). If $A$ and $B$ are both complex symmetries in a point $p$, then $A = B$ and $(A, B)$ is $\mathbb{C}$-decomposable.

(ii). If $A$ and $B$ are both regular elliptic elements, because $A(p) = B(p) = p$, we may assume $A = E_{(\theta,-\theta)}$ ($\theta \in (0,\pi)$), and $B$ has the form (3.10) with parameters $\alpha, b$, where $\alpha \in (0,\pi), b \in \mathbb{C}$ and $0 \le |b| \le 1$.

When $b \neq 0$, we put $z_1 = -i\overline{b}e^{i\theta}$, $z_2 = i/\overline{b}$ and $z_3 = b(\pm\sqrt{1-|b|^2}\sin\alpha + i\cos\alpha)$.

Therefore,

$$A = I_1 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & z_1 \\ 0 & \overline{z}_1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \overline{z}_2 \\ 0 & z_2 & 0 \end{bmatrix},$$

and

$$B = I_3 I_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{1-|z_3|^2} & \overline{z}_3 \\ 0 & z_3 & \sqrt{1-|z_3|^2} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & \overline{z}_2 \\ 0 & z_2 & 0 \end{bmatrix}.$$

The polar vectors to the complex lines corresponding to $I_k$ are $\mathbf{n}_k$, where

$$\mathbf{n}_1 = \begin{bmatrix} 0 \\ \frac{\overline{b}}{\sqrt{2}} \\ \frac{ie^{-i\theta}}{\sqrt{2}} \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} 0 \\ \frac{-i}{\sqrt{2}} \\ \frac{1}{\overline{b}\sqrt{2}} \end{bmatrix}, \mathbf{n}_3 = \begin{bmatrix} 0 \\ \sqrt{\frac{1-|b|\sin\alpha}{2}} \\ \frac{b(\pm\sqrt{1-|b|^2}\sin\alpha + i\cos\alpha)}{\sqrt{2(1-|b|\sin\alpha)}} \end{bmatrix}.$$

It is clear that $(A, B)$ is $\mathbb{C}$-decomposable.

When $b = 0$, it is apparent from Remark 3.1 that $(A, B)$ is $\mathbb{C}$-decomposable.

(iii). If one is a regular elliptic element and the other one is a complex reflection in a point, without loss of generality, we suppose $A$ is a regular elliptic and $B$ is a complex reflection in a point. Since $A, B$ have the same fix point in $\mathbf{H}^2_{\mathbb{C}}$, we set $A = E_{(\theta,-\theta)}$ ($\theta \in (0,\pi)$), $B = E_{(\pi,-\pi)}$. From Remark 3.1, then $(A, B)$ is $\mathbb{C}$-decomposable. □

**Proposition 4.2.** *Let $A, B \in \mathbf{SU}(2,1)$ have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$.*

(i) *If $A$ and $B$ are both loxodromic elements, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible and* $\mathrm{fix}(A) = \mathrm{fix}(B)$.

(ii) *If $A$ or $B$ is a loxodromic element and the other one is a* 3*-step unipotent parabolic element, then $(A, B)$ is not $\mathbb{C}$-decomposable.*

(iii) *If $A$ and $B$ are both* 3*-step unipotent parabolic elements, then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ don't commute or $A, B$ have the same invariant fan.*

Note that the 3 parts of Proposition 4.2 cover all cases where $A$ and $B$ have a common fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$, because an elliptic element which has a fixed point on $\partial\mathbf{H}^2_{\mathbb{C}}$ is not $\mathbb{C}$-strongly reversible and a parabolic element which is not 3-step unipotent is not $\mathbb{C}$-strongly reversible too.

We are now turning to the proof of Proposition 4.2.

Proof. (i) ($\Rightarrow$) $A$ and $B$ are both loxodromic elements and $(A, B)$ is $\mathbb{C}$-decomposable. Then $A = I_1I_2, B = I_3I_2$, where $I_k(k = 1, 2, 3)$ is complex symmetry. Suppose $A$ fixes the points $p, q$ and $B$ fixes the points $p, q'$. From Proposition 2.2, we get $I_2(p) = q = q'$. Thus $\mathrm{fix}(A) = \mathrm{fix}(B)$.

($\Leftarrow$) $A, B$ are both $\mathbb{C}$-strongly reversible and $\mathrm{fix}(A) = \mathrm{fix}(B)$. Without loss of generality, we set the two fixed points are 0 and $\infty$. By Theorem 1, $A, B$ are conjugate to

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1/\lambda \\ 0 & -1 & 0 \\ \lambda & 0 & 0 \end{bmatrix} \ (\lambda > 1).$$

Therefore $(A, B)$ is $\mathbb{C}$-decomposable.

(ii) Assume that $A$ is a loxodromic element and $B$ is a 3-step unipotent parabolic element. The fixed points of $A$ are $p, q$, the fixed point of $B$ is $p$. If there exist three complex symmetries $I_1, I_2, I_3$ such that $A = I_1I_2$ and $B = I_3I_2$. Then from Proposition 2.2 and 3.1, we get $p = q$. This is a contradiction to $p \neq q$. So $(A, B)$ is not $\mathbb{C}$-decomposable.

(iii) Let $(A, B)$ be a pair of 3-step unipotent parabolic elements of $\mathbf{SU}(2, 1)$ and $A, B$ have the same fixed point.

($\Rightarrow$) If $(A, B)$ is $\mathbb{C}$-decomposable, we can assume $A = I_1I_2$ and $B = I_3I_2$. From Proposition 3.1, we know that $I_2$ must fix a complex line in the invariant fan of $A$ and one in the invariant fan of $B$. Hence, these two fans must intersect in (at least) a complex line. Therefore they are either the same or non-parallel.

($\Leftarrow$) If $A$ and $B$ either do not commute or have the same invariant fan, there exists a complex line $L$ contained in both of their invariant fans. Writing $I_2$ for the complex symmetry fixing $L$, it is easy to check $AI_2$ and $BI_2$ are both complex symmetries. Thus $(A, B)$ is $\mathbb{C}$-decomposable.

This completes the proof of Proposition 4.2. □

**4.3. The $\mathbb{C}$-decomposability of one loxodromic and one parabolic.** In this subsection, we consider the case that $A$ is a loxodromic element and $B$ is a parabolic element. Now we prove the following theorem.

**Theorem 17.** *Let $(A, B)$ be a pair of elements of $\mathbf{SU}(2, 1)$, where $A$ is a loxodromic element and $B$ is a parabolic element. Then $(A, B)$ is $\mathbb{C}$-decomposable if and only if $A, B$ are both $\mathbb{C}$-strongly reversible, $\mathrm{tr}(AB) \in \mathbb{R}$, $\mathrm{tr}(BA^{-1}) \in \mathbb{R}$, and $A, B$ have distinct fixed points.*

Proof. ($\Rightarrow$) If the pair $(A, B)$ is $\mathbb{C}$-decomposable, we can normalise the parabolic element $B = T_{(1,0)}$ and have the decomposition given in equation (3.8). Then

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

should conjugate $A$ to its inverse. Hence $A$ has the form

$$A = \begin{bmatrix} a & b & c \\ d & e & -\overline{b} \\ g & -\overline{d} & \overline{a} \end{bmatrix}$$

where $a, b, d \in \mathbb{C}$, $c, e, g \in \mathbb{R}$. This immediately implies

$$\operatorname{tr}(AB) = 2\Re(a) - 2\Re(d) + e - g/2, \quad \operatorname{tr}(B^{-1}A) = 2\Re(a) + 2\Re(d) + e - g/2$$

are real. Moreover, to show $A$ and $B$ have distinct fixed points, it suffices to show $g \neq 0$. If $g = 0$, since $2\Re(a)g + |d|^2 = 0$, then $d = 0$ and so $a^2 - bd + cg = 1$ implies $a^2 = 1$; $\det(A) = |a|^2 e = 1$ implies $e = 1$. Thus $\operatorname{tr}(A) = 3$ or $-1$, which contradicts the assumption $A$ is loxodromic.

($\Leftarrow$) If $A, B$ are both $\mathbb{C}$-strongly reversible and $\operatorname{fix}(A) \cap \operatorname{fix}(B) = \emptyset$, we may assume $A = D_r$ $(r > 1)$ by Theorem 1. Without loss of generality, the fixed point of $B$ is $q = (x, t, 0) \neq 0, \infty$ $(x, t \in \mathbb{R})$. The standard lift of $q$ is $\mathbf{q} = \begin{bmatrix} (-x^2 + it)/2 & x & 1 \end{bmatrix}^T$. $B$ is conjugate to $T_{(1,0)}$ by Theorem 14, then we can denote $B$ by

$$\begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}, \tag{4.12}$$

where $\lambda_{11} = 1 + \overline{f}g - (\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2)$, $\lambda_{12} = \overline{f}gx - (\frac{-x^2+it}{2})(e\overline{g} + \frac{1}{2}|g|^2 x)$, $\lambda_{13} = i\Im[(x^2+it)\overline{f}g] - \frac{x^4+t^2}{8}|g|^2$, $\lambda_{21} = \overline{e}g - \overline{g}dx - \frac{1}{2}|g|^2 x$, $\lambda_{22} = 1 - \frac{1}{2}|g|^2 x^2 + 2xi\Im(\overline{e}g)$, $\lambda_{23} = (\frac{-x^2-it}{2})(\overline{e}g - \frac{1}{2}|g|^2 x) - \overline{g}fx$, $\lambda_{31} = 2i\Im(\overline{d}g) - \frac{1}{2}|g|^2$, $\lambda_{32} = \overline{d}gx - e\overline{g} - \frac{1}{2}|g|^2 x$, $\lambda_{33} = 1 - f\overline{g} + (\frac{-x^2-it}{2})(\overline{d}g - \frac{1}{2}|g|^2)$, $d, e, f, g \in \mathbb{C}$, $g \neq 0$ and $2\Re(d\overline{f}) + |e|^2 = 1$, $e\overline{g}x = \frac{x^2-it}{2}d\overline{g} - f\overline{g}$.

As $\operatorname{tr}(AB) \in \mathbb{R}$ and $\operatorname{tr}(BA^{-1}) \in \mathbb{R}$, a simple manipulation yields

$$r\overline{f}g - r(\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2) + 2xi\Im(\overline{e}g) - \frac{1}{r}f\overline{g} - \frac{1}{r}(\frac{x^2+it}{2})(\overline{d}g - \frac{|g|^2}{2}) \in \mathbb{R}, \tag{4.13}$$

and

$$\frac{1}{r}\overline{f}g - \frac{1}{r}(\frac{-x^2+it}{2})(d\overline{g} + \frac{1}{2}|g|^2) + 2xi\Im(\overline{e}g) - rf\overline{g} - r(\frac{x^2+it}{2})(\overline{d}g - \frac{|g|^2}{2}) \in \mathbb{R}. \tag{4.14}$$

(4.13) minus (4.14), we assert $t = 0$, then $x \neq 0$. Substitute $t = 0$ into formula (4.13), we find

$$2xi\Im(\overline{e}g) + r\overline{f}g - \frac{f\overline{g}}{r} + \frac{x^2}{2}(rd\overline{g} - \frac{\overline{d}g}{r}) \in \mathbb{R},$$

then $\Im(\overline{e}g) = 0$.

Set $L$ be a complex line spanned by 0 and $\infty$. Let $L_2$ be a complex line through $q$ orthogonal to $L$, and $I_2$ is the complex symmetry fixing $L_2$. In the case above,

$$I_2 = \begin{bmatrix} 0 & 0 & \frac{x^2}{2} \\ 0 & -1 & 0 \\ \frac{2}{x^2} & 0 & 0 \end{bmatrix}.$$

By a simple calculation, we can derive that $AI_2$ and $BI_2$ are both complex symmetries. Therefore, we claim that $(A, B)$ is $\mathbb{C}$-decomposable. $\square$

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Xue-Jing Ren
School of Mathematical Sciences and Chemical Engineering
Changzhou Institute of Technology
Changzhou, 213000
China
e-mail: rxj@hnu.edu.cn

Bao-Hua Xie
College of Mathematics and Econometrics
Hunan University
Changsha, 410082
China
e-mail: xiebaohua82@163.com

Yue-Ping Jiang
College of Mathematics and Econometrics
Hunan University
Changsha, 410082
China
e-mail: ypjiang@hnu.edu.cn