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LIE ALGEBRAS CONSTRUCTED WITH LIE MODULES AND THEIR POSITIVELY AND NEGATIVELY GRADED MODULES

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Abstract

In this paper, we shall give a way to construct a graded Lie algebra $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ from a standard pentad $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ which consists of a Lie algebra $\mathfrak{g}$ which has a non-degenerate invariant bilinear form B_0 and $\mathfrak{g}$-modules (ρ,V) and $\mathcal{V}\subset \mathrm{Hom}(V,F)$ all defined over a field F with characteristic 0. In general, we do not assume that these objects are finite-dimensional. We can embed the objects $\mathfrak{g},\rho,V,\mathcal{V}$ into $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$. Moreover, we construct specific positively and negatively graded modules of $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$. Finally, we give a chain rule on the embedding rules of standard pentads.

1. Introduction

A standard quadruplet is a quadruplet of the form $(\mathfrak{g},\rho,V,B_0)$, where $\mathfrak{g}$ is a finite-dimensional reductive Lie algebra, (ρ,V) a finite-dimensional representation of $\mathfrak{g}$ and B_0 a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ all defined over the complex number field $\mathbb{C}$, which satisfies the conditions that ρ is faithful and completely reducible and that V does not have a non-zero invariant element. In [8], the author proved that any standard quadruplet $(\mathfrak{g},\rho,V,B_0)$ has a graded Lie algebra, denoted by $L(\mathfrak{g},\rho,V,B_0)=\bigoplus_{n\in\mathbb{Z}}V_n$, such that $V_0\simeq\mathfrak{g}$, $V_1\simeq V$ and $V_{-1}\simeq\mathrm{Hom}(V,\mathbb{C})$ (see [8, Theorem 2.11]). That is, any finite-dimensional reductive Lie algebra and its finite-dimensional faithful and completely reducible representation can be embedded into some (finite or infinite-dimensional) graded Lie algebra. We call a graded Lie algebra of the form $L(\mathfrak{g},\rho,V,B_0)$ the Lie algebra associated with a standard quadruplet. Some well-known Lie algebras correspond to some standard quadruplet, for example, finite-dimensional semisimple Lie algebras and loop algebras. Moreover, the bilinear form B_0 can be also embedded into $L(\mathfrak{g},\rho,V,B_0)$, i.e. there exists a non-degenerate symmetric invariant bilinear form on $L(\mathfrak{g},\rho,V,B_0)$ whose restriction to $V_0\times V_0$ coincides with B_0 (see [8, Proposition 3.2]). By the way, H. Rubenthaler obtained some similar results in [7] using the Kac theory in [2].

The first purpose of this paper is to extend the theory of standard quadruplets to the cases where the objects are infinite-dimensional. For this, we need to consider pentads $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ instead of quadruplets, where $\mathfrak{g}$ is a finite or infinite-dimensional Lie algebra, $\rho:\mathfrak{g}\otimes V\to V$ a representation of $\mathfrak{g}$ on a finite or infinite-dimensional vector space V, $\mathcal{V}$ a $\mathfrak{g}$-submodule of $\mathrm{Hom}(V,F)$, B_0 a non-degenerate invariant bilinear form on $\mathfrak{g}$ all defined over a field F with characteristic 0. In general, we do not assume that B_0 is symmetric. We define the notion of *standard pentads* by the existence of a linear map

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$\Phi_\rho : V \otimes \mathcal{V} \to \mathfrak{g}$ satisfying $B_0(a, \Phi_\rho(v \otimes \phi)) = \langle \rho(a \otimes v), \phi \rangle$ for any $a \in \mathfrak{g}$, $v \in V$ and $\phi \in \mathcal{V}$. A standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$ can be naturally regarded as a standard pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B_0)$, and, thus, we can say that the notion of standard pentads is an extension of the notion of standard quadruplets. Then, by a similar argument to the argument in [8], we can construct a graded Lie algebra from an arbitrary standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ denoted by $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$ such that the objects $\mathfrak{g}, \rho, V, \mathcal{V}$ can be embedded into it. We call such a graded Lie algebra *a Lie algebra associated with a standard pentad*. This is the first main result of this paper. Of course, the graded Lie algebra associated with a standard quadruplet $(\mathfrak{g}, \rho, V, B_0)$ is isomorphic to the graded Lie algebra associated with a standard pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B_0)$. Moreover, if the bilinear form B_0 of $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is symmetric, then B_0 can be also embedded into $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, i.e. there exists a non-degenerate symmetric invariant bilinear form B_L on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ whose restriction to $V_0 \times V_0$ coincides with B_0.

When B_0 is symmetric, we can expect that a Lie algebra of the form $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ (not necessary finite-dimensional) and its representation can be embedded into some graded Lie algebra using B_L. The second purpose is to construct positively graded modules and negatively graded modules of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ which can be embedded into some graded Lie algebra under some assumptions. In general, it is known that for any graded Lie algebra $\mathfrak{l} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{l}_n$ and $\mathfrak{l}_0$-module U, there exists a positively (respectively negatively) graded $\mathfrak{l}$-module such that the base space (respectively top space) is the given $\mathfrak{l}_0$-module U (see [9, Theorem 1.2]). In this paper, we shall try to construct such $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-modules from a $\mathfrak{g}$-module (π, U) using a similar way to the construction of a Lie algebra associated with a standard pentad. Precisely, we inductively construct a positively (respectively negatively) graded $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module $(\tilde{\pi}^+, \tilde{U}^+)$, $\tilde{U}^+ = \bigoplus_{m \geq 0} U_m^+$ (respectively $(\tilde{\pi}^-, \tilde{U}^-)$, $\tilde{U}^- = \bigoplus_{m \leq 0} U_m^-$) such that the "base space" U_0^+ (respectively the "top space" U_0^-) is the given $\mathfrak{g}$-module U. In general, the modules $\tilde{U}^+$ and $\tilde{U}^-$ are infinite-dimensional. We shall try to embed $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and its module of the form $\tilde{U}^+$ into some graded Lie algebra. If we assume that B_0 is symmetric and that U has a $\mathfrak{g}$-submodule $\mathcal{U}$ of $\mathrm{Hom}(U, F)$ such that $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ is a standard pentad, then we can embed the objects $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $\tilde{U}^+$ into some graded Lie algebra. Precisely, under these assumptions, we have that a pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is also standard, and, thus, we can embed the objects $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, $\tilde{U}^+$, $\tilde{\mathcal{U}}^-$ into the graded Lie algebra $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$. In this situation, we have a "chain rule" of the Lie algebras associated with a standard pentad. This is the second main result of this paper.

This paper consists of three sections.

In section 2, we shall study the Lie algebras associated with a standard pentad. First, in section 2.1, we define the notion of standard pentads (see Definition 2.2) and construct a graded Lie algebra from a standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, which is denoted by $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$ (see Theorem 2.15). In section 2.2, we consider some properties of Lie algebras of the form $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ such that B_0 is symmetric. In these cases, we can also embed the bilinear form B_0 into $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, i.e. we can obtain a non-degenerate symmetric invariant bilinear form on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ whose restriction to $V_0 \times V_0$ coincides with B_0 (see Proposition 2.18). Moreover, the Lie algebra $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ can be characterized by the transitivity and the existence of such a bilinear form (see Theorem 2.20). Finally, we give two lemmas on derivations on $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ (see Lemmas 2.37 and 2.38).

In section 3, we shall study positively and negatively graded modules of a Lie algebra of the form $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$. First, in sections 3.1 and 3.2, we shall construct positively graded $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$-module and negatively graded $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$-module from a $\mathfrak{g}$-module (π,U), i.e. we shall give another proof of [9, Theorem 1.2] in the special cases where the graded Lie algebra is of the form $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$. In section 3.1, we construct a family of $\mathfrak{g}$-modules $\{U^+_m\}_{m\geq 0}$ (respectively $\{U^-_m\}_{m\leq 0}$) from the pentad $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ and the $\mathfrak{g}$-module (π,U) by induction. In section 3.2, we define a structure of positively (respectively negatively) graded $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$-module on $\tilde{U}^+ := \bigoplus_{m\geq 0} U^+_m$ (respectively $\tilde{U}^- := \bigoplus_{m\leq 0} U^-_m$). We call this positively (respectively negatively) graded module of $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ the *positive extension* (respectively *negative extension*) of U with respect to $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ (see Theorems 3.12 and 3.14). These modules are transitive and characterized by their transitivity (see Theorem 3.17). In sections 3.3 and 3.4, we try to construct a standard pentad which contains a Lie algebra of the form $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ and its module of the form $\tilde{U}^+$. For this, we need to assume that B_0 is symmetric and that U is embedded into some standard pentad $(\mathfrak{g},\pi,U,\mathcal{U},B_0)$. In section 3.3, for the $\mathfrak{g}$-submodule $\mathcal{U}$ of $\mathrm{Hom}(U,F)$, we shall extend the canonical pairing $U\times\mathcal{U}$ to $\tilde{U}^+\times\tilde{\mathcal{U}}^-$. Moreover, in section 3.4, we shall construct the Φ-map of $(L(\mathfrak{g},\rho,V,\mathcal{V},B_0),\tilde{\pi}^+,\tilde{U}^+,\tilde{\mathcal{U}}^-,B_L)$ from the Φ-map of the pentad $(\mathfrak{g},\pi,U,\mathcal{U},B_0)$ inductively. Consequently, under the assumptions that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ and $(\mathfrak{g},\pi,U,\mathcal{U},B_0)$ are standard pentads and that their bilinear form B_0 is symmetric, we can embed the Lie algebra $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ and its module $\tilde{U}^+$ into a standard pentad $(L(\mathfrak{g},\rho,V,\mathcal{V},B_0),\tilde{\pi}^+,\tilde{U}^+,\tilde{\mathcal{U}}^-,B_L)$. Finally, in section 3.5, we consider the graded Lie algebra $L(L(\mathfrak{g},\rho,V,\mathcal{V},B_0),\tilde{\pi}^+,\tilde{U}^+,\tilde{\mathcal{U}}^-,B_L)$ under the situation of sections 3.3 and 3.4. From the constructions of $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$, $\tilde{U}^+$ and $\tilde{\mathcal{U}}^-$, we can expect that this graded Lie algebra is written using the data $\mathfrak{g},\rho,V,\mathcal{V},B_0$ and $U,\mathcal{U}$. Indeed, we have the following result on the structures of Lie algebras:

$$L(L(\mathfrak{g},\rho,V,\mathcal{V},B_0),\tilde{\pi}^+,\tilde{U}^+,\tilde{\mathcal{U}}^-,B_L)\simeq L(\mathfrak{g},\rho\oplus\pi,V\oplus U,\mathcal{V}\oplus\mathcal{U},B_0)$$

up to grading. This is a chain rule in the theory of standard pentads (see Theorem 3.26).

Notation 1.1. In this paper, we regard a representation ρ of a Lie algebra $\mathfrak{l}$ on V as a linear map $\rho:\mathfrak{l}\otimes V\to V$ which satisfies that

$$\rho([a,b]\otimes v)=\rho(a\otimes\rho(b\otimes v))-\rho(b\otimes\rho(a\otimes v))$$

for any $a,b\in\mathfrak{l}$ and $v\in V$.

Definition 1.2. In this paper, we say that a Lie algebra $\mathfrak{l}$ is a $\mathbb{Z}$-*graded Lie algebra* or simply a *graded Lie algebra* if and only if there exist vector subspaces $\mathfrak{l}_n$ of $\mathfrak{l}$ for all $n\in\mathbb{Z}$ such that:

- $\mathfrak{l}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{l}_n$ and $[\mathfrak{l}_n,\mathfrak{l}_m]\subset\mathfrak{l}_{n+m}$ for any $n,m\in\mathbb{Z}$,
- $\mathfrak{l}$ is generated by $\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$.

In general, we do not assume that each $\mathfrak{l}_n$ is finite-dimensional (cf. [2, Definition 1]).

Moreover, if $\mathfrak{l}$ satisfies the following two conditions, we say that $\mathfrak{l}$ is transitive (see [2, Definition 2]):

- for $x\in\mathfrak{l}_i$, $i\geq 0$, $[x,\mathfrak{l}_{-1}]=\{0\}$ implies $x=0$,

- for $x \in \mathfrak{l}_i$, $i \leq 0$, $[x, \mathfrak{l}_1] = \{0\}$ implies $x = 0$.

DEFINITION 1.3. In this paper, we say that a module (ϖ^+, W), $W = \bigoplus_{m\geq 0} W_m$ (respectively (ϖ^-, W), $W = \bigoplus_{m\leq 0} W_m$) of a graded Lie algebra $\bigoplus_{n\in\mathbb{Z}} \mathfrak{l}_n$ is positively graded (respectively negatively graded) when $\varpi^+(\mathfrak{l}_n \otimes W_m) \subset W_{n+m}$ (respectively $\varpi^-(\mathfrak{l}_n \otimes W_m) \subset W_{n+m}$) for any n, m (cf. [9, Definition 0.1]), and, moreover, we say that a positively graded module (ϖ^+, W) (respectively a negatively graded module (ϖ^-, W)) is transitive when the following condition holds (cf. [9, Definition 1.1]):

$$\text{for } w \in W_m,\ m \geq 1,\ \varpi^+(V_{-1} \otimes w) = \{0\} \text{ implies } w = 0$$
$$(\text{respectively} \quad \text{for } w \in W_m,\ m \leq -1,\ \varpi^-(V_1 \otimes w) = \{0\} \text{ implies } w = 0).$$

NOTATION 1.4. In this paper, we denote the set of all natural numbers, integers and complex numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$ respectively. We denote the set of matrices of size $n \times m$ ($n, m \in \mathbb{N}$) whose entries are belong to a ring R by $M(n, m; R)$, the unit matrix and the zero matrix of size n by I_n and O_n respectively. Moreover, δ_{kl} stands for the Kronecker delta, $\mathrm{Tr}(A)$ stands for the trace of a square matrix A.

2. Standard pentads and corresponding Lie algebras

2.1. Standard pentads. Let us start with the definitions of Φ-map and standard pentads.

DEFINITION 2.1 (Φ-map, cf. [8, Definition 1.9]). Let F be a field with characteristic 0. Let $\mathfrak{g}$ be a Lie algebra with non-degenerate invariant bilinear form B_0, $\rho : \mathfrak{g} \otimes V \to V$ a representation of $\mathfrak{g}$ on a vector space V and $\mathcal{V}$ a $\mathfrak{g}$-submodule of $\mathrm{Hom}(V, F)$ all defined over F. We denote the canonical pairing between V and $\mathrm{Hom}(V, F)$ by $\langle \cdot, \cdot \rangle$ and the canonical representation of $\mathfrak{g}$ on $\mathcal{V}$ by ϱ. Then, if a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a linear map $\Phi_\rho : V \otimes \mathcal{V} \to \mathfrak{g}$ which satisfies an equation

$$B_0(a, \Phi_\rho(v \otimes \phi)) = \langle \rho(a \otimes v), \phi \rangle = -\langle v, \varrho(a \otimes \phi) \rangle \tag{2.1}$$

for any $a \in \mathfrak{g}$, $v \in V$ and $\phi \in \mathcal{V}$, we call it a Φ-map of the pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. Moreover, when a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a Φ-map, we define a linear map $\Psi_\rho : \mathcal{V} \otimes V \to \mathfrak{g}$ by:

$$B_0(a, \Psi_\rho(\phi \otimes v)) = \langle v, \varrho(a \otimes \phi) \rangle = -\langle \rho(a \otimes v), \phi \rangle. \tag{2.2}$$

We call this map Ψ_ρ a Ψ-map of $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

In general, a pentad might not have a Φ-map. If a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has a Φ-map, then the equation (2.1) determines the linear map Φ_ρ uniquely. Moreover, we have an equation

$$\Phi_\rho(v \otimes \phi) + \Psi_\rho(\phi \otimes v) = 0$$

for any $v \in V$ and $\phi \in \mathcal{V}$.

DEFINITION 2.2 (Standard pentads). We retain to use the notation of Definition 2.1. If a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ satisfies the following conditions, we call it a *standard pentad*:

$$\text{the restriction of } \langle \cdot, \cdot \rangle \text{ to } V \times \mathcal{V} \text{ is non-degenerate,} \tag{2.3}$$

$$\text{there exists a Φ-map from } V \otimes \mathcal{V} \text{ to } \mathfrak{g}. \tag{2.4}$$

Lemma 2.3. *Under the notation of* Definitions 2.1 *and* 2.2*, we have the following claims:*

(2.5) *if V is finite-dimensional, then a vector space $\mathcal{V}$ satisfying* (2.3) *coincides with* $\mathrm{Hom}(V, F)$*,*

(2.6) *if $\mathfrak{g}$ is finite-dimensional, then any pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ satisfies the condition* (2.4).

In particular, if both $\mathfrak{g}$ and V are finite-dimensional, then any quadruplet $(\mathfrak{g}, \rho, V, B_0)$ can be naturally regarded as a standard pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, F), B_0)$.

Proof. The claim (2.5) is clear. Let us show the claim (2.6). If $\mathfrak{g}$ is finite-dimensional, then the dual space of $\mathfrak{g}$ can be identified with $\mathfrak{g}$. Precisely, if $\mathfrak{g}$ is finite-dimensional, then any linear map $f : \mathfrak{g} \to F$ corresponds to some element $A \in \mathfrak{g}$ such that

$$f(a) = B_0(a, A)$$

for any $a \in \mathfrak{g}$. Thus, for any $v \in V$ and $\phi \in \mathcal{V}$, there exists an element of $\mathfrak{g}$ which corresponds to a linear map $\mathfrak{g} \to F$ defined by

$$a \mapsto \langle \rho(a \otimes v), \phi \rangle.$$

It means that the pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ has the Φ-map. □

REMARK 2.4. If V is infinite-dimensional, then a submodule $\mathcal{V}$ of $\mathrm{Hom}(V, F)$ satisfying the condition (2.3) does not necessary coincide with $\mathrm{Hom}(V, F)$.

REMARK 2.5. In general, a Lie algebra $\mathfrak{g}$ and its module (ρ, V) might not have a $\mathfrak{g}$-submodule $\mathcal{V} \subset \mathrm{Hom}(V, F)$ and a bilinear form B_0 such that a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is standard.

EXAMPLE 2.6. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, K be the Killing form on $\mathfrak{g}$ and $\mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ be the loop algebra (see [3, Ch.7]). Let $K_{\mathcal{L}}$ be a bilinear form on $\mathcal{L}(\mathfrak{g})$ defined by:

$$K_{\mathcal{L}}(t^n \otimes X, t^m \otimes Y) := \delta_{n+m,0} K(X, Y).$$

Clearly, the bilinear form $K_{\mathcal{L}}$ is non-degenerate and invariant. Thus, we can regard $\mathcal{L}(\mathfrak{g})$ itself as a $\mathcal{L}(\mathfrak{g})$-submodule of $\mathrm{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C})$ via the non-degenerate invariant bilinear form $K_{\mathcal{L}}$. Then, a pentad $(\mathcal{L}(\mathfrak{g}), \mathrm{ad}, \mathcal{L}(\mathfrak{g}), \mathcal{L}(\mathfrak{g}), K_{\mathcal{L}})$, where ad stands for the adjoint representation, is standard. In fact, we have the condition (2.3) clearly, and, we can identify the bracket product $\mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}(\mathfrak{g})$ with the Φ-map of $(\mathcal{L}(\mathfrak{g}), \mathrm{ad}, \mathcal{L}(\mathfrak{g}), \mathcal{L}(\mathfrak{g}), K_{\mathcal{L}})$, denoted by Φ^1_{ad}.

However, a pentad $(\mathcal{L}(\mathfrak{g}), \mathrm{ad}, \mathcal{L}(\mathfrak{g}), \mathrm{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), K_{\mathcal{L}})$ is not standard since it does not have the Φ-map. In fact, if we assume that this pentad might have the Φ-map, denoted by Φ^2_{ad}, and put

$$H_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_0 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g},$$
$$\phi_{Y_0} \in \mathrm{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), \quad \langle t^n \otimes X, \phi_{Y_0} \rangle := K(Y_0, X),$$

then an element $\Phi^2_{\mathrm{ad}}((1 \otimes X_0) \otimes \phi_{Y_0}) \in \mathcal{L}(\mathfrak{g})$ satisfies the equation

$$K_{\mathcal{L}}(t^n \otimes H_0, \Phi^2_{\mathrm{ad}}((1 \otimes X_0) \otimes \phi_{Y_0})) = \langle [t^n \otimes H_0, 1 \otimes X_0], \phi_{Y_0} \rangle \\ = \langle t^n \otimes 2X_0, \phi_{Y_0} \rangle \\ = K(Y_0, 2X_0) \\ = 8 \tag{2.7}$$

for any $n \in \mathbb{Z}$. The Lie algebra $\mathcal{L}(\mathfrak{g})$ does not have an element satisfying (2.7) for any $n \in \mathbb{Z}$, and, thus, the pentad $(\mathcal{L}(\mathfrak{g}), \mathrm{ad}, \mathcal{L}(\mathfrak{g}), \mathrm{Hom}(\mathcal{L}(\mathfrak{g}), \mathbb{C}), K_{\mathcal{L}})$ does not have the Φ-map.

On the Φ-map and Ψ-map of a standard pentad, we have similar properties to ones of the Φ-map and Ψ-map of a standard quadruplet (see [8]).

Proposition 2.7. *The Φ-map and the Ψ-map of a standard quadruplet* $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *are homomorphisms of Lie modules.* (*cf.* [8, Proposition 1.3]).

Proof. We can prove it by the same way to [8, Proposition 1.3]. □

Definition 2.8. Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad. For each element $v \in V$ and $\phi \in \mathcal{V}$, we define linear maps $\Phi_{\rho,v} \in \mathrm{Hom}(\mathcal{V}, \mathfrak{g})$ and $\Psi_{\rho,\phi} \in \mathrm{Hom}(V, \mathfrak{g})$ by:

$$\Phi_{\rho,v}(\psi) := \Phi_\rho(v \otimes \psi), \quad \Psi_{\rho,\phi}(u) := \Psi_\rho(\phi \otimes u)$$

for any $u \in V$ and $\psi \in \mathcal{V}$. Moreover, we define the following linear maps:

$$\Phi^\circ_\rho : V \to \mathrm{Hom}(\mathcal{V}, \mathfrak{g}) \qquad \Psi^\circ_\rho : \mathcal{V} \to \mathrm{Hom}(V, \mathfrak{g})$$
$$v \mapsto \Phi_{\rho,v}, \qquad \phi \mapsto \Psi_{\rho,\phi}.$$

To simplify, we denote $\Phi_{\rho,v}(\psi)$ and $\Psi_{\rho,\phi}(u)$ by $v(\psi)$ and $\phi(u)$ respectively.

Definition 2.9. Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad. Put $V_0 := \mathfrak{g}$, $V_1 := V$ and $V_{-1} := \mathcal{V}$ and denote the canonical representations of $\mathfrak{g}$ on V_0 and $V_{\pm 1}$ by ρ_0 and $\rho_{\pm 1}$. We define homomorphisms of $\mathfrak{g}$-modules p_0 and q_0 by:

$$p_0 : V_1 \otimes V_0 \to V_1$$
$$v_1 \otimes a \mapsto -\rho_1(a \otimes v_1),$$
$$q_0 : V_{-1} \otimes V_0 \to V_{-1}$$
$$\phi_{-1} \otimes b \mapsto -\rho_{-1}(b \otimes \phi_{-1}).$$

Moreover, we define homomorphisms of $\mathfrak{g}$-modules p_1 and q_{-1} by:

$$p_1 : V_1 \otimes V_1 \to \mathrm{Hom}(V_{-1}, V_1)$$
$$v_1 \otimes u_1 \mapsto (\eta_{-1} \mapsto \rho_1(v_1(\eta_{-1}) \otimes u_1) - \rho_1(u_1(\eta_{-1}) \otimes v_1)),$$
$$q_{-1} : V_{-1} \otimes V_{-1} \to \mathrm{Hom}(V_1, V_{-1})$$
$$\phi_{-1} \otimes \psi_{-1} \mapsto (\xi_1 \mapsto \rho_{-1}(\phi_{-1}(\xi_1) \otimes \psi_{-1}) - \rho_{-1}(\psi_{-1}(\xi_1) \otimes \phi_{-1})),$$

where $v_1(\eta_{-1}) \in V_0$ and $\phi_{-1}(\xi_1) \in V_0$ stand for $\Phi_{\rho,v_1}(\eta_{-1})$ and $\Psi_{\rho,\phi_{-1}}(\xi_1)$ respectively.

Moreover, suppose that $i \geq 2$ and there exist $\mathfrak{g}$-modules (ρ_{i-1}, V_{i-1}) and (ρ_{-i+1}, V_{-i+1}) and homomorphisms of $\mathfrak{g}$-modules $p_{i-1} : V_1 \otimes V_{i-1} \to \mathrm{Hom}(V_{-1}, V_{i-1})$ and $q_{-i+1} : V_{-1} \otimes V_{-i+1} \to \mathrm{Hom}(V_1, V_{-i+1})$. Then, we put $V_i := \mathrm{Im}\ p_{i-1}$, $V_{-i} := \mathrm{Im}\ q_{-i+1}$ and define linear maps p_i, q_{-i} by:

$$p_i : V_1 \otimes V_i \to \mathrm{Hom}(V_{-1}, V_i)$$
$$v_1 \otimes u_i \mapsto (\eta_{-1} \mapsto \rho_i(v_1(\eta_{-1}) \otimes u_i) + p_{i-1}(v_1 \otimes u_i(\eta_{-1}))),$$
$$q_{-i} : V_{-1} \otimes V_{-i} \to \mathrm{Hom}(V_1, V_{-i})$$
$$\phi_{-1} \otimes \psi_{-i} \mapsto (\xi_1 \mapsto \rho_{-i}(\phi_{-1}(\xi_1) \otimes \psi_{-i}) + q_{-i+1}(\phi_{-1} \otimes \psi_{-i}(\xi_1))),$$

where $u_i(\eta_{-1}) \in V_{i-1}$ and $\psi_{-i}(\xi_1) \in V_{-i+1}$ are the images of η_{-1} and ξ_1 via u_i and ψ_{-i} respectively. Then, the linear maps p_i and q_{-i} are homomorphisms of $\mathfrak{g}$-modules (cf. [8, Proposition 1.10]). We denote the images of p_i and q_{-i} by V_{i+1} and V_{-i-1} and the canonical representations of $\mathfrak{g}$ on V_{i+1} and V_{-i-1} by ρ_{i+1} and ρ_{-i-1} respectively. Thus, inductively, we obtain $\mathfrak{g}$-modules V_n and representations ρ_n of $\mathfrak{g}$ on V_n for all $n \in \mathbb{Z}$. We call V_n the *n-graduation* of $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

Remark 2.10. For any $v_1 \in V_1$ and $\phi_{-1} \in V_{-1}$, we have

$$p_1(v_1 \otimes v_1)(\eta_{-1}) = \rho_1(v_1(\eta_{-1}) \otimes v_1) - \rho_1(v_1(\eta_{-1}) \otimes v_1) = 0,$$
$$q_{-1}(\phi_{-1} \otimes \phi_{-1})(\xi_1) = \rho_{-1}(\phi_{-1}(\xi_1) \otimes \phi_{-1}) - \rho_{-1}(\phi_{-1}(\xi_1) \otimes \phi_{-1}) = 0.$$

In general, we do not assume that ρ and ϱ are surjective, i.e. we do not assume that $V_1 = \mathrm{Im}\, p_0$ and $V_{-1} = \mathrm{Im}\, q_0$. In particular cases where these linear maps are surjective, we have the following proposition.

Proposition 2.11. *If $\rho : \mathfrak{g} \otimes V \to V$ and $\varrho : \mathfrak{g} \otimes \mathcal{V} \to \mathcal{V}$ are surjective, then Φ_ρ° and Ψ_ρ° are injective, and, thus, V and $\mathcal{V}$ can be regarded as $\mathfrak{g}$-submodules of* $\mathrm{Hom}(V_{-1}, V_0)$ *and* $\mathrm{Hom}(V_1, V_0)$ *respectively.*

Proof. To show this proposition, we use the condition (2.3). Let us show that the linear map Φ_ρ° is injective. We take an arbitrary element $v \in V$ which satisfies that $\Phi_{\rho,v} = 0$. Then we have

$$0 = B_0(a, \Phi_{\rho,v}(\phi)) = \langle \rho(a \otimes v), \phi \rangle = -\langle v, \varrho(a \otimes \phi) \rangle \tag{2.8}$$

for all $a \in \mathfrak{g}$ and $\phi \in \mathcal{V}$. By the condition (2.3) and the assumption that ϱ is surjective, we have that $v = 0$. Therefore, we obtain that Φ_ρ° is injective. Similarly, we can show that Ψ_ρ° is injective. □

Definition 2.12. We define the following bilinear maps

$$[\cdot,\cdot]_n^0 : V_0 \times V_n \to V_n, \quad [\cdot,\cdot]_n^1 : V_1 \times V_n \to V_{n+1}, \quad [\cdot,\cdot]_n^{-1} : V_{-1} \times V_n \to V_{n-1}$$

by:

$$[a_0, z_n]_n^0 := \rho_n(a_0 \otimes z_n),$$

$$[x_1, z_n]_n^1 := \begin{cases} p_n(x_1 \otimes z_n) & (n \geq 0) \\ -z_n(x_1) & (n \leq -1) \end{cases},$$

$$[y_{-1}, z_n]_n^{-1} := \begin{cases} -z_n(y_{-1}) & (n \geq 1) \\ q_n(y_{-1} \otimes z_n) & (n \leq 0) \end{cases}$$

where $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$ and $z_n \in V_n$. Note that $z_n(x_1)$ stands for $\Psi_{\rho,z_{-1}}(x_1)$ when $n = -1$ and the image of x_1 via $z_n \in \mathrm{Hom}(V_1, V_{n+1})$ when $n \leq -2$. Moreover, for $i \geq 1$, we define the following bilinear maps

$$[\cdot,\cdot]^{i+1}_n : V_{i+1} \times V_n \to V_{i+n+1}, \quad [\cdot,\cdot]^{-i-1}_n : V_{-i-1} \times V_n \to V_{-i+n-1}$$

by:

$$[p_i(x_1 \otimes z_i), w_n]^{i+1}_n := [x_1, [z_i, w_n]^i_n]^1_{i+n} - [z_i, [x_1, w_n]^1_n]^i_{n+1} \quad (x_1 \in V_1, z_i \in V_i, w_n \in V_n) \tag{2.9}$$

and

$$[q_{-i}(y_{-1} \otimes \omega_{-i}), w_n]^{-i-1}_n := [y_{-1}, [\omega_{-i}, w_n]^{-i}_n]^{-1}_{-i+n} - [\omega_{-i}, [y_{-1}, w_n]^{-1}_n]^{-i}_{n-1} \quad (y_{-1} \in V_{-1}, \omega_{-i} \in V_{-i}, w_n \in V_n) \tag{2.10}$$

inductively. Then the bilinear maps (2.9) and (2.10) are well-defined. It can be shown by the same argument to the argument of [8, Propositions 2.5 and 2.6]. Consequently, we can define a bilinear map $[\cdot,\cdot]^n_m : V_n \times V_m \to V_{n+m}$ for any $n, m \in \mathbb{Z}$.

Definition 2.13. For a standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, we denote a direct sum of its n-graduations by $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, i.e.

$$L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) := \bigoplus_{n \in \mathbb{Z}} V_n.$$

Moreover, we define a bilinear map $[\cdot,\cdot] : L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \times L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \to L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ by

$$[x_n, y_m] := [x_n, y_m]^n_m \tag{2.11}$$

for any $n, m \in \mathbb{Z}$, $x_n \in V_n$ and $y_m \in V_m$.

Proposition 2.14. *This bilinear map* $[\cdot,\cdot]$ *satisfies the following equations*

$$[x, y] + [y, x] = 0, \tag{2.12}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \tag{2.13}$$

for any $x, y, z \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

Proof. We can prove it by the same argument to the argument of [8, Propositions 2.9 and 2.10]. □

As a corollary of Proposition 2.14, we have the following theorem immediately.

Theorem 2.15 (Lie algebra associated with a standard pentad)**.** *Let* $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *be a standard pentad over a field F with characteristic* 0*. Then the vector space* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$ *is a graded Lie algebra with a bracket product* $[\cdot,\cdot]$ *defined in* Definition 2.13*. We call this graded Lie algebra* the Lie algebra associated with $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ (*cf.* [8, Theorem 2.11]).

Remark 2.16. Note that we can prove Theorem 2.15 without the assumption that the bilinear form B_0 is symmetric.

Note that $V_0 = \mathfrak{g}$ and that the V_0-modules V_0, V_1, V_{-1} are isomorphic to $\mathfrak{g}$, V, $\mathcal{V}$ respectively. In this sense, we can say that the objects $\mathfrak{g}$, (ρ, V), $(\varrho, \mathcal{V})$ can be embedded into $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

In particular, when ρ and ϱ are faithful and surjective, we have a similar result on the structure of a graded Lie algebra of the form $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ to the result which is obtained by H. Rubenthaler in [7, Proposition 3.4.2]. We can show the following proposition by Proposition 2.11 immediately.

Proposition 2.17. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad. If both $\rho : \mathfrak{g} \otimes V \to V$ and $\varrho : \mathfrak{g} \otimes \mathcal{V} \to \mathcal{V}$ are faithful and surjective, then the graded Lie algebra $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is transitive.*

2.2. Standard pentads with a symmetric bilinear form. In the previous section, we proved that for any standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, there exists a graded Lie algebra such that $\mathfrak{g}$, ρ, V and $\mathcal{V}$ can be embedded into it. In this section, we discuss cases where B_0 is symmetric. In these cases, we can also embed B_0 into $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and we can obtain some useful properties.

Proposition 2.18. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad such that B_0 is symmetric. We define a symmetric bilinear form B_L on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ inductively as follows:*

$$\begin{cases} B_L(a, b) = B_0(a, b), \\ B_L(v, \phi) = \langle v, \phi \rangle, \\ B_L(p_i(v_1 \otimes u_i), q_{-i}(\phi_{-1} \otimes \psi_{-i})) = B_L(u_i, [q_{-i}(\phi_{-1} \otimes \psi_{-i}), v_1]), \\ B_L(x_n, y_m) = 0 \end{cases}$$

for any $a, b \in V_0$, $v \in V$, $\phi \in \mathcal{V}$, $i \geq 1$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$, $u_i \in V_i$, $\psi_{-i} \in V_{-i}$, $n, m \in \mathbb{Z}$, $n + m \neq 0$, $x_n \in V_n$ and $y_m \in V_m$. Then B_L is a non-degenerate symmetric invariant bilinear form on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ (cf. [8, Proposition 3.2]).

Proof. Note that it is clear that the restriction of B_L to $V_0 \times V_0$ and $V_1 \times V_{-1}$ is well-defined. Let us show the well-definedness of B_L on $V_2 \times V_{-2}$. For any $v_1, u_1 \in V_1$ and $\phi_{-1}, \psi_{-1} \in V_{-1}$, we have

$$\begin{aligned} B_L(u_1, [q_{-1}(\phi_{-1} \otimes \psi_{-1}), v_1]) &= B_L(u_1, [[\phi_{-1}, v_1], \psi_{-1}] + [\phi_{-1}, [\psi_{-1}, v_1]]) \\ &= \langle u_1, [[\phi_{-1}, v_1], \psi_{-1}] + [\phi_{-1}, [\psi_{-1}, v_1]] \rangle \\ &= B_0([\phi_{-1}, v_1], \psi_{-1}(u_1)) - B_0([\psi_{-1}, v_1], \phi_{-1}(u_1)) \\ &= B_0([\phi_{-1}, v_1], \psi_{-1}(u_1)) - B_0(\phi_{-1}(u_1), [\psi_{-1}, v_1]) \\ &\qquad \text{(by the assumption that } B_0 \text{ is symmetric)} \\ &= B_0([v_1, \phi_{-1}], u_1(\psi_{-1})) - B_0(u_1(\phi_{-1}), [v_1, \psi_{-1}]) \\ &= \langle [[v_1, \phi_{-1}], u_1] + [v_1, [u_1, \phi_{-1}]], \psi_{-1} \rangle \\ &= B_L([p_1(v_1 \otimes u_1), \phi_{-1}], \psi_{-1}). \end{aligned} \tag{2.14}$$

Thus, if $v_1^1, \ldots, v_1^l, u_1^1, \ldots, u_1^l \in V_1$ and $\phi_{-1}^1, \ldots, \phi_{-1}^k, \psi_{-1}^1, \ldots, \psi_{-1}^k \in V_{-1}$ satisfy equations

$$\sum_{s=1}^{l} p_1(v_1^s \otimes u_1^s) = 0, \quad \sum_{t=1}^{k} q_{-1}(\phi_{-1}^t \otimes \psi_{-1}^t) = 0,$$

then

$$\sum_{s=1}^{l} B_L(u_1^s, [q_{-1}(\phi_{-1} \otimes \psi_{-1}), v_1^s]) = \sum_{s=1}^{l} B_L([p_1(v_1^s \otimes u_1^s), \phi_{-1}], \psi_{-1}) = 0,$$

$$\sum_{t=1}^{k} B_L(u_1, [q_{-1}(\phi_{-1}^t \otimes \psi_{-1}^t), v_1]) = 0$$

for any $v_1, u_1 \in V_1$ and $\phi_{-1}, \psi_{-1} \in V_{-1}$, that is, we have the well-definedness of B_L on $V_2 \times V_{-2}$. This $B_L |_{V_2 \times V_{-2}}$ is $\mathfrak{g}$-invariant. Moreover, by a similar argument, we have the well-definedness of B_L on $V_i \times V_{-i}$ for each $i \geq 3$ by induction (see [8, section 1.2]). Consequently, we can show the well-definedness of B_L on the whole $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and that B_L is non-degenerate symmetric invariant by the same argument as the argument in [8, section 1.2 and Proposition 3.2]. □

Remark 2.19. We need the assumption that B_0 is symmetric to show that the bilinear form B_L is $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-invariant. Precisely, we need this assumption to show an equation

$$B_L(v_1, [\phi_{-1}, a]) = B_L([v_1, \phi_{-1}], a)$$

for any $a \in V_0$, $v_1 \in V_1$, $\phi_{-1} \in V_{-1}$.

Under the assumption that B_0 is symmetric, the graded Lie algebra is characterized by the existence of such a bilinear form. The following is a proposition concerning the "universality" and "uniqueness" of Lie algebras associated with a standard pentad with a symmetric bilinear form.

Theorem 2.20. *Let $\mathfrak{L} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{L}_n$ be a graded Lie algebra which has a non-degenerate symmetric invariant bilinear form $B_{\mathfrak{L}}$. If $\mathfrak{L}$ and $B_{\mathfrak{L}}$ satisfy the following conditions, then a pentad $(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ is standard and $\mathfrak{L}$ is isomorphic to $L(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$:*

(2.15) $\mathfrak{L}_{i+1} = [\mathfrak{L}_1, \mathfrak{L}_i]$, $\mathfrak{L}_{-i-1} = [\mathfrak{L}_{-1}, \mathfrak{L}_{-i}]$ *for all $i \geq 1$,*

(2.16) *the restriction of $B_{\mathfrak{L}}$ to $\mathfrak{L}_i \times \mathfrak{L}_{-i}$ is non-degenerate for any $i \geq 0$,*

where ad *stands for the adjoint representation of $\mathfrak{L}$ on itself* (*cf.* [8, Proposition 3.3]).

Proof. First of all, let us check that the pentad $(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ is standard. By (2.16), we can obtain that $B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0}$ is non-degenerate and that $\mathfrak{L}_1$ and $\mathfrak{L}_{-1}$ satisfy the condition (2.3). It is easy to show that we can identify the restriction of the bracket product $[\cdot, \cdot]$ of $\mathfrak{L}$ to $\mathfrak{L}_1 \times \mathfrak{L}_{-1} \to \mathfrak{L}_0$ with the Φ-map of the pentad $(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$. Thus, the condition (2.4) holds.

We denote the n-graduation of $(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ by $(\mathfrak{L})_n$ for any $n \in \mathbb{Z}$ and a bilinear form on $L(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$ obtained in Proposition 2.18 by $(B)_{\mathfrak{L}}$. Let $\sigma_0 : (\mathfrak{L})_0 \to \mathfrak{L}_0$ and $\sigma_{\pm 1} : (\mathfrak{L})_{\pm 1} \to \mathfrak{L}_{\pm 1}$ be the identity maps respectively. Then the linear maps σ_0 and $\sigma_{\pm 1}$ satisfy the following equations:

(2.17) $$[\sigma_0(a), \sigma_{\pm 1}(x_{\pm 1})] = \sigma_{\pm 1}([a, x_{\pm 1}]),$$

(2.18) $$[\sigma_1(x_1), \sigma_{-1}(x_{-1})] = \sigma_0([x_1, x_{-1}])$$

for any $a \in (\mathfrak{L})_0$ and $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$. Indeed, the equation (2.17) is clear, and, we have

(2.19)
$$B_{\mathfrak{L}}(\sigma_0(b), [\sigma_1(x_1), \sigma_{-1}(x_{-1})]) = B_{\mathfrak{L}}([\sigma_0(b), \sigma_1(x_1)], \sigma_{-1}(x_{-1})) = B_{\mathfrak{L}}(\sigma_1([b, x_1]), \sigma_{-1}(x_{-1}))$$
$$= (B)_{\mathfrak{L}}([b, x_1], x_{-1}) = (B)_{\mathfrak{L}}(b, [x_1, x_{-1}]) = B_{\mathfrak{L}}(\sigma_0(b), \sigma_0([x_1, x_{-1}]))$$

for any $b \in (\mathfrak{L})_0$. Thus, we can obtain the equation (2.18).

For each $i \geq 1$, we define linear maps $\sigma_{i+1} : (\mathfrak{L})_{i+1} \to \mathfrak{L}_{i+1}$ and $\sigma_{-i-1} : (\mathfrak{L})_{-i-1} \to \mathfrak{L}_{-i-1}$ by:

$$\sigma_{i+1} : p_i(x_1 \otimes z_i) \mapsto [\sigma_1(x_1), \sigma_i(z_i)], \tag{2.20}$$
$$\sigma_{-i-1} : q_{-i}(x_{-1} \otimes z_{-i}) \mapsto [\sigma_{-1}(x_{-1}), \sigma_{-i}(z_{-i})] \tag{2.21}$$

for any $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$ and $z_{\pm i} \in (\mathfrak{L})_{\pm i}$ inductively. Note that it follows from (2.17) that the linear maps σ_1 and σ_{-1} on $\rho(\mathfrak{g} \otimes V)$ and $\varrho(\mathfrak{g} \otimes \mathcal{V})$ defined by the same equations as (2.20) and (2.21) where $i = 0$ coincide with the identity maps respectively. We can prove that the linear maps σ_n ($n \in \mathbb{Z}$) are well-defined and satisfy

$$[\sigma_0(a), \sigma_n(z_n)] = \sigma_n([a, z_n]), \tag{2.22}$$
$$[\sigma_{\pm 1}(x_{\pm 1}), \sigma_n(z_n)] = \sigma_{n\pm 1}([x_{\pm 1}, z_n]) \tag{2.23}$$

for any $n \in \mathbb{Z}$, $a \in (\mathfrak{L})_0$, $x_{\pm 1} \in (\mathfrak{L})_{\pm 1}$ and $z_n \in (\mathfrak{L})_n$ by a similar argument to the argument of [8, Proposition 3.3]. Then a linear map $\sigma : L(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0}) \to \mathfrak{L}$ defined by

$$\sigma(z_n) := \sigma_n(z_n), \tag{2.24}$$

where $n \in \mathbb{Z}$ and $z_n \in (\mathfrak{L})_n \subset L(\mathfrak{L}_0, \mathrm{ad}, \mathfrak{L}_1, \mathfrak{L}_{-1}, B_{\mathfrak{L}} |_{\mathfrak{L}_0 \times \mathfrak{L}_0})$, is an isomorphism of Lie algebras. We can also prove this by a similar argument to the argument of [8, Proposition 3.3]. □

As a corollary of Theorem 2.20, we can say that the theory of standard pentads is an extension of the theory of standard quadruplets.

Proposition 2.21. *Let* $(\mathfrak{g}, \rho, V, B_0)$ *be a standard quadruplet* (*see* [8, Definition 1.9]). *Then the Lie algebra* $L(\mathfrak{g}, \rho, V, B_0)$ *associated with* $(\mathfrak{g}, \rho, V, B_0)$ (*see* [8, Theorem 2.11]) *is isomorphic to the Lie algebra* $L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B_0)$.

Definition 2.22. Let $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ be standard pentads. We say that these pentads are *equivalent* if and only if there exists an isomorphism of Lie algebras $\tau : \mathfrak{g}^1 \to \mathfrak{g}^2$, linear isomorphisms $\sigma : V^1 \to V^2$, $\varsigma : \mathcal{V}^1 \to \mathcal{V}^2$ and a non-zero element $c \in F$ such that

$$\sigma(\rho^1(a^1 \otimes x^1)) = \rho^2(\tau(a^1) \otimes \sigma(x^1)), \tag{2.25}$$
$$\varsigma(\varrho^1(a^1 \otimes y^1)) = \varrho^2(\tau(a^1) \otimes \varsigma(y^1)), \tag{2.26}$$
$$\langle x^1, y^1 \rangle^1 = \langle \sigma(x^1), \varsigma(y^1) \rangle^2, \tag{2.27}$$
$$B_0^1(a^1, b^1) = cB_0^2(\tau(a^1), \tau(b^1)) \tag{2.28}$$

where $a^1, b^1 \in \mathfrak{g}^1$, $x^1 \in V^1$, $y^1 \in \mathcal{V}^1$ and $\langle \cdot, \cdot \rangle^i$ stands for the pairing between V^i and $\mathcal{V}^i$ ($i = 1, 2$). We denote this equivalence relation by

$$(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \simeq (\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2). \tag{2.29}$$

REMARK 2.23. Note that if V is finite-dimensional, then linear isomorphisms τ, σ satisfying (2.25) induce a linear isomorphism from $\mathcal{V}^1 = \mathrm{Hom}(V^1, F)$ to $\mathcal{V}^2 = \mathrm{Hom}(V^2, F)$ satisfying (2.26) and (2.27).

Proposition 2.24. *If standard pentads* $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ *and* $(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ *are equivalent, then the Lie algebras associated with them are isomorphic, i.e. we have*

$$L(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \simeq L(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2) \tag{2.30}$$

(*cf.* [8, Proposition 3.6]).

Proof. We denote the n-graduation of $(\mathfrak{g}^i, \rho^i, V^i, \mathcal{V}^i, B_0^i)$ by V_n^i $(i = 1, 2)$ for all $n \in \mathbb{Z}$ and the bilinear forms on $L(\mathfrak{g}^i, \rho^i, V^i, \mathcal{V}^i, B_0^i)$ defined in Proposition 2.18 by B_L^i $(i = 1, 2)$. Under the notation of Definition 2.22, we define linear maps $\sigma_0 := \tau : V_0^1 \to V_0^2$, $\sigma_1 := \frac{1}{c}\sigma : V_1^1 \to V_1^2$ and $\sigma_{-1} := \varsigma : V_{-1}^1 \to V_{-1}^2$. Then, these linear maps σ_0 and $\sigma_{\pm 1}$ satisfy the same equations as (2.17) and (2.18). In fact, the equation (2.17) is clear, and, we have

$$\begin{aligned} &B_0^2(\sigma_0(a_0^1), [\sigma_1(x_1^1), \sigma_{-1}(y_{-1}^1)]) = B_L^2(\sigma_1([a_0^1, x_1^1]), \sigma_{-1}(y_{-1}^1)) \\ &= \frac{1}{c} B_L^1([a_0^1, x_1^1], y_{-1}^1) = \frac{1}{c} B_0^1(a_0^1, [x_1^1, y_{-1}^1]) = B_0^2(\sigma_0(a_0^1), \sigma_0([x_1^1, y_{-1}^1])) \end{aligned}$$

for any $a_0^1 \in V_0^1$, $x_1^1 \in V_1^1$ and $y_{-1}^1 \in V_{-1}^1$. Thus, we have the equation (2.18). Then, by the same argument as the argument in proof of Theorem 2.20, we can construct an isomorphism of Lie algebras from $L(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ to $L(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$. □

REMARK 2.25. The converse of Proposition 2.22 is not true. In fact, we have an example of two non-equivalent pentads such that the corresponding Lie algebras are isomorphic (see [8, pp. 398–399]).

DEFINITION 2.26. Let $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ be standard pentads. Let $\rho^1 \boxplus \rho^2$ and $\varrho^1 \boxplus \varrho^2$ be representations of $\mathfrak{g}^1 \oplus \mathfrak{g}^2$ on $V^1 \oplus V^2$ and $\mathcal{V}^1 \oplus \mathcal{V}^2$ defined by:

$$\begin{aligned} &(\rho^1 \boxplus \rho^2)((a^1, a^2) \otimes (v^1, v^2)) := (\rho^1(a^1 \otimes v^1), \rho^2(a^2 \otimes v^2)), \\ &(\varrho^1 \boxplus \varrho^2)((b^1, b^2) \otimes (\phi^1, \phi^2)) := (\varrho^1(b^1 \otimes \phi^1), \varrho^2(b^2 \otimes \phi^2)) \end{aligned}$$

where $a^i, b^i \in \mathfrak{g}^i$, $v^i \in V^i$, $\phi^i \in \mathcal{V}^i$ $(i = 1, 2)$. Let $B_0^1 \oplus B_0^2$ be a bilinear form on $\mathfrak{g}^1 \oplus \mathfrak{g}^2$ defined by:

$$(B_0^1 \oplus B_0^2)((a^1, a^2), (b^1, b^2)) := B_0^1(a^1, b^1) + B_0^2(a^2, b^2) \tag{2.31}$$

where $a^i, b^i \in \mathfrak{g}^i$ $(i = 1, 2)$. Then, clearly, a pentad $(\mathfrak{g}^1 \oplus \mathfrak{g}^2, \rho^1 \boxplus \rho^2, V^1 \oplus V^2, \mathcal{V}^1 \oplus \mathcal{V}^2, B_0^1 \oplus B_0^2)$ is also a standard pentad. We call it a *direct sum* of $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ and $(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ and denote it by $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$.

Proposition 2.27. *Let* $(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1)$ *and* $(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ *be standard pentads. Then the Lie algebra* $L((\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \oplus (\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2))$ *is isomorphic to* $L(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \oplus L(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$ (*cf.* [8, Proposition 3.9]).

Proof. We retain to use the notation of Proposition 2.24. Then, we have the following $\mathbb{Z}$-grading of $L(\mathfrak{g}^1, \rho^1, V^1, \mathcal{V}^1, B_0^1) \oplus L(\mathfrak{g}^2, \rho^2, V^2, \mathcal{V}^2, B_0^2)$:

$$L(\mathfrak{g}^1,\rho^1,V^1,\mathcal{V}^1,B_0^1)\oplus L(\mathfrak{g}^2,\rho^2,V^2,\mathcal{V}^2,B_0^2)=\bigoplus_{n\in\mathbb{Z}}(V_n^1\oplus V_n^2). \tag{2.32}$$

By Theorem 2.20, we have our claim. □

DEFINITION 2.28. Let $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ be a standard pentad. We say that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is *decomposable* if and only if there exist standard pentads $(\mathfrak{a},\rho_{\mathfrak{a}},V_{\mathfrak{a}},\mathcal{V}_{\mathfrak{a}},B_{0,\mathfrak{a}})$ and $(\mathfrak{b},\rho_{\mathfrak{b}},V_{\mathfrak{b}},\mathcal{V}_{\mathfrak{b}},B_{0,\mathfrak{b}})$ such that

$$(\dim\mathfrak{a}+\dim V_{\mathfrak{a}})(\dim\mathfrak{b}+\dim V_{\mathfrak{b}})\neq 0, \tag{2.33}$$

$$(\mathfrak{g},\rho,V,\mathcal{V},B_0)\simeq(\mathfrak{a},\rho_{\mathfrak{a}},V_{\mathfrak{a}},\mathcal{V}_{\mathfrak{a}},B_{0,\mathfrak{a}})\oplus(\mathfrak{b},\rho_{\mathfrak{b}},V_{\mathfrak{b}},\mathcal{V}_{\mathfrak{b}},B_{0,\mathfrak{b}}). \tag{2.34}$$

If $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is not decomposable, we say that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is *indecomposable*.

DEFINITION 2.29. Let $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ be a standard pentad. We say that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is *reducible* if and only if there exist an ideal $\mathfrak{a}$ of $\mathfrak{g}$ and $\mathfrak{g}$-submodules $V_{\mathfrak{a}}$ and $\mathcal{V}_{\mathfrak{a}}$ of V and $\mathcal{V}$ satisfying that:

$$\{0\}\neq\mathcal{V}_{\mathfrak{a}}\oplus\mathfrak{a}\oplus V_{\mathfrak{a}}\subsetneq\mathcal{V}\oplus\mathfrak{a}\oplus V, \tag{2.35}$$

$$\rho(\mathfrak{a}\otimes V),\rho(\mathfrak{g}\otimes V_{\mathfrak{a}})\subset V_{\mathfrak{a}}\text{ and }\varrho(\mathfrak{a}\otimes\mathcal{V}),\varrho(\mathfrak{g}\otimes\mathcal{V}_{\mathfrak{a}})\subset\mathcal{V}_{\mathfrak{a}}, \tag{2.36}$$

$$\Phi_\rho(V_{\mathfrak{a}}\otimes\mathcal{V}),\Phi_\rho(V\otimes\mathcal{V}_{\mathfrak{a}})\subset\mathfrak{a}. \tag{2.37}$$

And, we say that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is *irreducible* if and only if it is not reducible.

REMARK 2.30. If a standard pentad is irreducible, then it is indecomposable.

Proposition 2.31. *Let $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ be an irreducible standard pentad. Then the representations $\rho:\mathfrak{g}\otimes V\to V$, $\varrho:\mathfrak{g}\otimes\mathcal{V}\to\mathcal{V}$ and the Φ-map $\Phi_\rho:V\otimes\mathcal{V}\to\mathfrak{g}$ are surjective.*

Proof. If $\varrho(\mathfrak{g}\otimes\mathcal{V})\oplus\Phi_\rho(V\otimes\mathcal{V})\oplus\rho(\mathfrak{g}\otimes V)=\{0\}$, it follows that $\dim\mathcal{V}=\dim\mathfrak{g}=\dim V=0$ from the assumption that $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ is irreducible. In particular, we have $\varrho(\mathfrak{g}\otimes\mathcal{V})=\mathcal{V}=\{0\}$ and $\rho(\mathfrak{g}\otimes V)=V=\{0\}$. If $\varrho(\mathfrak{g}\otimes\mathcal{V})\oplus\Phi_\rho(V\otimes\mathcal{V})\oplus\rho(\mathfrak{g}\otimes V)\neq\{0\}$, since it satisfies the conditions (2.36) and (2.37), we have $\varrho(\mathfrak{g}\otimes\mathcal{V})\oplus\Phi_\rho(V\otimes\mathcal{V})\oplus\rho(\mathfrak{g}\otimes V)=\mathcal{V}\oplus\mathfrak{g}\oplus V$. □

Proposition 2.32. *Let $(\mathfrak{g},\rho,V,\mathcal{V},B_0)$ be an irreducible standard pentad whose representation ρ is faithful and denote the Lie algebra associated with it by $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)=\bigoplus_{n\in\mathbb{Z}}V_n$. Let N (respectively M) be an integer such that V_{N+1} is not $\{0\}$ (respectively V_{-M-1} is not $\{0\}$). Then for any non-zero element $z_N\in V_N$ (respectively $\omega_{-M}\in V_{-M}$), there exists an element $x_1\in V_1$ such that $[x_1,z_N]\neq 0$ (respectively $y_{-1}\in V_{-1}$ such that $[y_{-1},\omega_{-M}]\neq 0$) (cf. [8, Proposition 3.11]).*

Proof. When $N\leq -1$, we have our claim by Propositions 2.11, 2.17 and 2.31. When $N=0$, we have our claim by the assumption that ρ is faithful. Assume that $N\geq 1$, $V_{N+1}\neq\{0\}$ and put $\mathfrak{a}_N:=\{a_N\in V_N\mid[x_1,a_N]=0\text{ for any }x_1\in V_1\}$ and $\mathfrak{a}_n:=\{a_n\in V_n\mid[x_1,a_n]\in\mathfrak{a}_{n+1}$ for any $x_1\in V_1\}$ for $n\leq N-1$ inductively. Then $\mathfrak{a}_n$ is a V_0-submodule of V_n for each n, i.e. $[V_0,\mathfrak{a}_n]\subset\mathfrak{a}_n$, and, we have that $[V_{\pm1},\mathfrak{a}_n]\subset\mathfrak{a}_{n\pm1}$ for any $n\in\mathbb{Z}$ (see [8, the proof of Proposition 3.11]). In particular, $\mathfrak{a}_{-1}\oplus\mathfrak{a}_0\oplus\mathfrak{a}_1$ satisfies the conditions (2.36) and (2.37). If $\mathfrak{a}_{-1}\oplus\mathfrak{a}_0\oplus\mathfrak{a}_1=\mathcal{V}\oplus\mathfrak{g}\oplus V$, then we have $\mathfrak{a}_N=V_N$ and a contradiction to the assumption that $V_{N+1}\neq\{0\}$. Thus we have $\mathfrak{a}_1=\{0\}$, and, thus, $\mathfrak{a}_2=\{0\},\dots,\mathfrak{a}_N=\{0\}$ by the transitivity of $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$. Similarly, we have our result for M such that $V_{-M-1}\neq\{0\}$. □

Proposition 2.33. *Let* $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *be an irreducible standard pentad whose representation* ρ *is faithful. If the Lie algebra* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *is finite-dimensional, then* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *is simple* (*cf.* [8, Proposition 3.12]). *Moreover, if* $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *is defined over* $\mathbb{C}$ *and* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ *is a finite-dimensional simple Lie algebra, then a triplet* $(\mathfrak{g}, \rho, V)$ *corresponds to some prehomogeneous vector space of parabolic type* (*see* [8, Theorem 3.13]).

Proof. We can show this by Proposition 2.32 and the same argument to the argument of [8, Proposition 3.12 and Theorem 3.13]. □

A prehomogeneous vector space of parabolic type (abbrev. a PV of parabolic type) is a PV which can be obtained from a $\mathbb{Z}$-graded finite-dimensional semisimple Lie algebra. PVs of parabolic type are classified by H. Rubenthaler (see [4, 5, 6]).

Example 2.34. Let $m \geq 2$ and $\mathfrak{g} = \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{sl}_m(\mathbb{C})$, $\rho = \Lambda_1$ a representation of $\mathfrak{g}$ on $\mathbb{C}^m$ defined by

$$\Lambda_1((a, A) \otimes v) := av + Av \quad (a \in \mathfrak{gl}_1, A \in \mathfrak{sl}_m, v \in V),$$

$B_0 = \kappa_m$ a bilinear form on $\mathfrak{g}$ defined by

$$\kappa_m((a, A), (a', A')) := \frac{m}{m+1} aa' + \mathrm{Tr}(AA') \quad (a, a' \in \mathfrak{gl}_1, A, A' \in \mathfrak{sl}_m).$$

Then, a pentad $(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B_0) = (\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m)$ is a standard pentad which has a (m^2+2m)-dimensional graded simple Lie algebra $L(\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m) = V_{-1} \oplus V_0 \oplus V_1$ (see [8, Example 1.14]). This Lie algebra $L(\mathfrak{gl}_1 \oplus \mathfrak{sl}_m, \Lambda_1, \mathbb{C}^m, \mathbb{C}^m, \kappa_m)$ is isomorphic to $\mathfrak{sl}_{m+1}$. Indeed, from the classification of PVs of parabolic type (see [4, 5, 6]) and the dimension of $L(\mathfrak{g}, \rho, V, \mathrm{Hom}(V, \mathbb{C}), B_0)$, it is isomorphic to $\mathfrak{sl}_{m+1}$.

Example 2.35. Put $\mathfrak{g} := \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, $V := \mathbb{C}^2 = M(2, 1; \mathbb{C})$, $\mathcal{V} := \mathbb{C}^2$ and define representations $\rho : \mathfrak{g} \otimes V \to V$, $\varrho : \mathfrak{g} \otimes \mathcal{V} \to \mathcal{V}$ by:

$$\rho((a, b, A) \otimes v) := bv + Av, \quad \varrho((a, b, A) \otimes \phi) := -b\phi - {}^tA\phi$$

for any $(a, b, A) \in \mathfrak{g}$, $v \in V$, $\phi \in \mathcal{V}$. We can identify $\mathcal{V}$ with $\mathrm{Hom}(V, \mathbb{C})$ via the following bilinear map $\langle \cdot, \cdot \rangle_V : V \times \mathcal{V} \to \mathbb{C}$ defined by:

$$\langle v, \phi \rangle_V := {}^tv\phi.$$

Let B_0 be a bilinear form on $\mathfrak{g}$ defined by:

$$B_0((a, b, A), (a', b', A')) := \frac{3}{4} aa' + bb' + \frac{1}{2}(ab' + a'b) + \mathrm{Tr}(AA').$$

Then, a pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is a standard pentad whose Φ-map is given by:

$$\Phi_\rho(v \otimes \phi) = (-{}^tv\phi, \frac{3}{2}{}^tv\phi, v{}^t\phi - \frac{1}{2}{}^tv\phi I_2).$$

The Lie algebra $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is isomorphic to $\mathfrak{gl}_1 \oplus \mathfrak{sl}_3$. Indeed, if we put $\mathfrak{g}_V^1 := \mathbb{C} \cdot (1, 0, O_2)$, $\mathfrak{g}_V^2 := \mathbb{C} \cdot (-\frac{2}{3}, 1, O_2) \oplus \mathfrak{sl}_2$, then we have

$$\begin{aligned} L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) &\simeq L((\mathfrak{g}_V^1, \rho|_{\mathfrak{g}_V^1}, \{0\}, \{0\}, B_0|_{\mathfrak{g}_V^1 \times \mathfrak{g}_V^1}) \oplus (\mathfrak{g}_V^2, \rho|_{\mathfrak{g}_V^2}, V, \mathcal{V}, B_0|_{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2})) \\ &\simeq \mathfrak{g}_V^1 \oplus L(\mathfrak{g}_V^2, \rho|_{\mathfrak{g}_V^2}, V, \mathcal{V}, B_0|_{\mathfrak{g}_V^2 \times \mathfrak{g}_V^2}) \simeq \mathfrak{g}_V^1 \oplus \mathcal{V} \oplus \mathfrak{g}_V^2 \oplus V \\ &\simeq \mathfrak{gl}_1 \oplus \mathfrak{sl}_3 \end{aligned}$$

from Example 2.34. Moreover, under this identification, the bilinear form B_L on $L(\mathfrak{g}_V^2, \rho|_{\mathfrak{g}_V^2}, V, \mathcal{V}, B_0|_{\mathfrak{g}_V^2\times\mathfrak{g}_V^2})$ is given by $B_L(\hat{A}, \hat{A}') = \mathrm{Tr}(\hat{A}\hat{A}')$ $(\hat{A}, \hat{A}' \in \mathfrak{sl}_3)$. In fact, if we put

$$h := (0, 0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \in \mathfrak{g}_V^2,$$

then $B_0(h, h) = 2$. On the other hand, we can obtain $\mathrm{Tr}(\mathrm{ad}\, h\, \mathrm{ad}\, h) = 12$, where ad stands for the adjoint representation of $L(\mathfrak{g}_V^2, \rho|_{\mathfrak{g}_V^2}, V, \mathcal{V}, B_0|_{\mathfrak{g}_V^2\times\mathfrak{g}_V^2})$, by a direct calculation. Since any non-degenerate invariant bilinear form on $\mathfrak{sl}_3$ is a scalar multiple of the Killing form, we can obtain that B_L is $1/6$ times the Killing form of $\mathfrak{sl}_3$, i.e. $B_L(\hat{A}, \hat{A}') = \mathrm{Tr}(\hat{A}\hat{A}')$.

Proposition 2.36. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad whose representation ρ is faithful. Under this assumption, the pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is irreducible if and only if the Lie algebra $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ does not have a non-zero proper graded ideal.*

Proof. Assume that $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is reducible. Under the notation of Definition 2.29, we put $\mathfrak{a}_{-1} := \mathcal{V}_\mathfrak{a}$, $\mathfrak{a}_0 := \mathfrak{a}$, $\mathfrak{a}_1 := V_\mathfrak{a}$. Moreover, we put $\mathfrak{a}_n := [V_1, \mathfrak{a}_{n-1}]$ for all $n \geq 2$ and $\mathfrak{a}_m := [V_{-1}, \mathfrak{a}_{m+1}]$ for all $m \leq -2$ inductively. Then a direct sum $\mathfrak{A} := \bigoplus_{n\in\mathbb{Z}} \mathfrak{a}_n$ is a non-zero proper graded ideal of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. In fact, by the assumption that $[V_i, \mathfrak{a}_j] \subset \mathfrak{a}_{i+j}$ for any $-1 \leq i, j, i+j \leq 1$, we can easily show that $[V_0, \mathfrak{A}], [V_{\pm 1}, \mathfrak{A}] \subset \mathfrak{A}$ by induction. Since $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is generated by V_0 and $V_{\pm 1}$, we have $[L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \mathfrak{A}] \subset \mathfrak{A}$. Thus, $\mathfrak{A}$ is a graded ideal. Since $\{0\} \neq \mathfrak{a}_{-1} \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_1 \subsetneq \mathcal{V} \oplus \mathfrak{g} \oplus V$, we have $\{0\} \neq \mathfrak{A} \subsetneq L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

Conversely, assume that $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is irreducible. Let $\mathfrak{b} = \sum_{n\in\mathbb{Z}}(\mathfrak{b} \cap V_n)$ be a non-zero graded ideal of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and put $\mathfrak{b}_n := \mathfrak{b} \cap V_n$. Then, by Proposition 2.32, we can obtain that $\mathfrak{b}_0 \neq \{0\}$. In fact, since $\mathfrak{b} \neq \{0\}$, there exists an integer $n \in \mathbb{Z}$ and a non-zero element $z_n \in \mathfrak{b}_n$. For example, if $n \geq 1$, then there exist n elements $y_{-1}^1, \ldots, y_{-1}^n \in V_{-1}$ such that $[y_{-1}^n, [\cdots, [y_{-1}^1, z_n]\cdots]] \in \mathfrak{b}_0 \setminus \{0\}$. Since $\mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ satisfies the conditions (2.36) and (2.37), it coincides with $V_{-1} \oplus V_0 \oplus V_1$, and, thus, $\mathfrak{b} = L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. □

The following lemmas are to construct a derivation on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. They are used in Theorem 3.26.

Lemma 2.37. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad, $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n\in\mathbb{Z}} V_n$ be the Lie algebra associated with it. Let $\alpha_i : V_i \to V_i$ $(i = 0, \pm 1)$ be linear maps which satisfy*

$$\alpha_{i+j}([a_i, b_j]) = [\alpha_i(a_i), b_j] + [a_i, \alpha_j(b_j)] \tag{2.38}$$

for any $-1 \leq i, j, i+j \leq 1$ and elements $a_i \in V_i$, $b_j \in V_j$. Then, there exists a linear map $\alpha : L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \to L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ such that α is a derivation on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and its restriction to V_i $(i = 0, \pm 1)$ coincides with α_i.

Proof. First, let us construct linear maps $\alpha_i : V_i \to V_i$ for all $i \in \mathbb{Z}$ by induction. Let $i \geq 1$ and assume that the integer i satisfies the condition that we have linear maps $\alpha_j : V_j \to V_j$ for all $0 \leq j \leq i$ which satisfy the following equations:

$$\begin{aligned} &\alpha_j([a_0, b_j]) = [\alpha_0(a_0), b_j] + [a_0, \alpha_j(b_j)], \\ &\alpha_j([x_1, b_{j-1}]) = [\alpha_1(x_1), b_{j-1}] + [x_1, \alpha_{j-1}(b_{j-1})], \\ &\alpha_{j-1}([y_{-1}, b_j]) = [\alpha_{-1}(y_{-1}), b_j] + [y_{-1}, \alpha_j(b_j)] \end{aligned}$$

for any $0 \leq j \leq i$, $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $b_j \in V_j$ and $b_{j-1} \in V_{j-1}$. By the assumption

(2.38), when $i=1$ the given linear maps α_0, $\alpha_{\pm 1}$ satisfy these equations. Then we define a linear map $\alpha_{i+1}: V_{i+1} \to V_{i+1}$ by:

$$\alpha_{i+1}([x_1, b_i]) := [\alpha_1(x_1), b_i] + [x_1, \alpha_i(b_i)] \tag{2.39}$$

for any $x_1 \in V_1$ and $b_i \in V_i$. Let us check the well-definedness of α_{i+1}. In fact, for any $y_{-1} \in V_{-1}$, $x_1 \in V_1$ and $b_i \in V_i$, we have

$$\begin{aligned}
&[y_{-1}, [\alpha_1(x_1), b_i] + [x_1, \alpha_i(b_i)]] = [y_{-1}, [\alpha_1(x_1), b_i]] + [y_{-1}, [x_1, \alpha_i(b_i)]] \\
&\quad = [[y_{-1}, \alpha_1(x_1)], b_i] + [\alpha_1(x_1), [y_{-1}, b_i]] + [[y_{-1}, x_1], \alpha_i(b_i)] + [x_1, [y_{-1}, \alpha_i(b_i)]] \\
&\quad = [\alpha_0([y_{-1}, x_1]), b_i] - [[\alpha_{-1}(y_{-1}), x_1], b_i] + [\alpha_1(x_1), [y_{-1}, b_i]] \\
&\qquad + [[y_{-1}, x_1], \alpha_i(b_i)] + [x_1, \alpha_{i-1}([y_{-1}, b_i])] - [x_1, [\alpha_{-1}(y_{-1}), b_i]] \\
&\quad = \alpha_i([[y_{-1}, x_1], b_i]) + \alpha_i([x_1, [y_{-1}, b_i]]) - [\alpha_{-1}(y_{-1}), [x_1, b_i]] \\
&\quad = \alpha_i([y_{-1}, [x_1, b_i]]) - [\alpha_{-1}(y_{-1}), [x_1, b_i]].
\end{aligned} \tag{2.40}$$

Thus, if $x_1^1, \ldots, x_1^l \in V_1$ and $b_i^1, \ldots, b_i^l \in V_i$ satisfy $\sum_{s=1}^{l} [x_1^s, b_i^s] = 0$, then we have

$$\sum_{s=1}^{l} [y_{-1}, [\alpha_1(x_1^s), b_i^s] + [x_1^s, \alpha_i(b_i^s)]] = 0$$

for any $y_{-1} \in V_{-1}$. Therefore, we have $\sum_{s=1}^{l}([\alpha_1(x_1^s), b_i^s] + [x_1^s, \alpha_i(b_i^s)]) = 0$ and the well-definedness of α_{i+1}. Moreover, α_{i+1} satisfies the following equations:

$$\alpha_{i+1}([a_0, b_{i+1}]) = [\alpha_0(a_0), b_{i+1}] + [a_0, \alpha_{i+1}(b_{i+1})], \tag{2.41}$$

$$\alpha_{i+1}([x_1, b_i]) = [\alpha_1(x_1), b_i] + [x_1, \alpha_i(b_i)], \tag{2.42}$$

$$\alpha_i([y_{-1}, b_{i+1}]) = [\alpha_{-1}(y_{-1}), b_{i+1}] + [y_{-1}, \alpha_{i+1}(b_{i+1})] \tag{2.43}$$

for any $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $b_i \in V_i$ and $b_{i+1} \in V_{i+1}$. In fact, for any $a_0 \in V_0$, $x_1 \in V_1$, and $b_i \in V_i$, we have

$$\begin{aligned}
&\alpha_{i+1}([a_0, [x_1, b_i]]) = \alpha_{i+1}([[a_0, x_1], b_i]) + \alpha_{i+1}([x_1, [a_0, b_i]]) \\
&\quad = [\alpha_1([a_0, x_1]), b_i] + [[a_0, x_1], \alpha_i(b_i)] + [\alpha_1(x_1), [a_0, b_i]] + [x_1, \alpha_i([a_0, b_i])] \\
&\quad = [[\alpha_0(a_0), x_1], b_i] + [[a_0, \alpha_1(x_1)], b_i] + [[a_0, x_1], \alpha_i(b_i)] \\
&\qquad + [\alpha_1(x_1), [a_0, b_i]] + [x_1, [\alpha_0(a_0), b_i]] + [x_1, [a_0, \alpha_i(b_i)]] \\
&\quad = [\alpha_0(a_0), [x_1, b_i]] + [a_0, [\alpha_1(x_1), b_i]] + [a_0, [x_1, \alpha_i(b_i)]] \\
&\quad = [\alpha_0(a_0), [x_1, b_i]] + [a_0, \alpha_{i+1}([x_1, b_i])].
\end{aligned} \tag{2.44}$$

Thus, we can obtain the equation (2.41). The equation (2.42) is clear. The equation (2.43) follows from (2.40). Thus, inductively, we can obtain linear maps α_i for all $i \geq 0$, and, similarly, we can construct linear maps $\alpha_{-i}: V_{-i} \to V_{-i}$ for all $i \geq 0$. Consequently, we have linear maps $\alpha_n: V_n \to V_n$ for all $n \in \mathbb{Z}$ which satisfy

$$\alpha_n([a_0, b_n]) = [\alpha_0(a_0), b_n] + [a_0, \alpha_n(b_n)], \tag{2.45}$$

$$\alpha_{n+1}([x_1, b_n]) = [\alpha_1(x_1), b_n] + [x_1, \alpha_n(b_n)], \tag{2.46}$$

$$\alpha_{n-1}([y_{-1}, b_n]) = [\alpha_{-1}(y_{-1}), b_n] + [y_{-1}, \alpha_n(b_n)] \tag{2.47}$$

for any $a_0 \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$ and $b_n \in V_n$.

We define a linear map $\alpha : L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) \to L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ by:

$$\alpha(a_n) := \alpha_n(a_n) \tag{2.48}$$

for any $n \in \mathbb{Z}$ and $a_n \in V_n$. Then α is a derivation of Lie algebras. In fact, we can show the following equation

$$\alpha([a_n, b_m]) = [\alpha(a_n), b_m] + [a_n, \alpha(b_m)] \tag{2.49}$$

for any $n, m \in \mathbb{Z}$, $a_n \in V_n$ and $b_m \in V_m$ by the equations (2.45), (2.46), (2.47) inductively. □

Lemma 2.38. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad and α be a derivation on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. If α satisfies the equation*

$$B_L(\alpha(z), \omega) = -B_L(z, \alpha(\omega)) \tag{2.50}$$

for any $z = z_n \in V_n$ $(n = 0, \pm 1)$ and $\omega \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, then we have the same equation for any $z, \omega \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

Proof. We argue our claim in the cases where $z = z_n \in V_n$ for some n and prove it by induction on n. Suppose that $n \geq 0$. If $n = 0, 1$, then our claim follows from the assumption. Suppose that $n \geq 2$. Then, by the induction hypothesis, we have

$$\begin{aligned} B_L(\alpha([x_1, z_{n-1}]), \omega) &= B_L([\alpha(x_1), z_{n-1}], \omega) + B_L([x_1, \alpha(z_{n-1})], \omega) \\ &= -B_L(z_{n-1}, [\alpha(x_1), \omega]) - B_L(\alpha(z_{n-1}), [x_1, \omega]) \\ &= -B_L(z_{n-1}, [\alpha(x_1), \omega]) + B_L(z_{n-1}, \alpha([x_1, \omega])) \\ &= B_L(z_{n-1}, [x_1, \alpha(\omega)]) \\ &= -B_L([x_1, z_{n-1}], \alpha(\omega)) \end{aligned}$$

for any $x_1 \in V_1$, $z_{n-1} \in V_{n-1}$. Since $V_n = [V_1, V_{n-1}]$, we have our claim for n. Thus, by induction, we have our claim for all $n \geq 0$. Similarly, we can show our claim for $n \leq -1$. □

3. Graded modules of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$

3.1. A construction of vector spaces $\tilde{U}^+$ and $\tilde{U}^-$. As mentioned in section 1, the purpose of this and the next section is to construct a positively graded module and a negatively graded module of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ from a given $\mathfrak{g}$-module U, which will be denoted by $\tilde{U}^+$ and $\tilde{U}^-$. First, we construct $\tilde{U}^+$ and $\tilde{U}^-$ as vector spaces by induction.

Definition 3.1. Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad and $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n \in \mathbb{Z}} V_n$ be the Lie algebra associated with it. Let $\pi : \mathfrak{g} \otimes U \to U$ be a representation of $\mathfrak{g} = V_0$ on a vector space U over F. We put $U_0^+ = U_0^- := U$, $\pi_0^+ = \pi_0^- := \pi$ and define linear maps $r_0^+ : V_1 \otimes U_0^+ \to \mathrm{Hom}(V_{-1}, U_0^+)$ and $r_0^- : V_{-1} \otimes U_0^- \to \mathrm{Hom}(V_1, U_0^-)$ by:

$$\begin{aligned} r_0^+ : V_1 \otimes U_0^+ &\to \mathrm{Hom}(V_{-1}, U_0^+) \\ x_1 \otimes u_0 &\mapsto (\eta_{-1} \mapsto \pi_0^+([\eta_{-1}, x_1] \otimes u_0)), \end{aligned} \tag{3.1}$$

$$(3.2) \qquad r_0^- : V_{-1} \otimes U_0^- \to \mathrm{Hom}(V_1, U_0^-)$$
$$y_{-1} \otimes u_0 \mapsto (\xi_1 \mapsto \pi_0^-([\xi_1, y_{-1}] \otimes u_0)).$$

Proposition 3.2. *The maps r_0^+ and r_0^- are homomorphisms of $\mathfrak{g}$-modules.*

Proof. We prove for r_0^+. For any elements $a \in \mathfrak{g}$, $x_1 \in V_1$, $\eta_{-1} \in V_{-1}$ and $u_0 \in U_0^+$, we have

$$\begin{aligned} r_0^+([a, x_1] \otimes u_0 + x_1 \otimes \pi_0^+(a \otimes u_0))(\eta_{-1}) &= \pi_0^+([\eta_{-1}, [a, x_1]] \otimes u_0) + \pi_0^+([\eta_{-1}, x_1] \otimes \pi_0^+(a \otimes u_0)) \\ &= \pi_0^+([a, [\eta_{-1}, x_1]] \otimes u_0) - \pi_0^+([[a, \eta_{-1}], x_1]] \otimes u_0) + \pi_0^+([\eta_{-1}, x_1] \otimes \pi_0^+(a \otimes u_0)) \\ &= \pi_0^+(a \otimes \pi_0^+([\eta_{-1}, x_1] \otimes u_0)) - \pi_0^+([[a, \eta_{-1}], x_1]] \otimes u_0) \\ &= \pi_0^+(a \otimes (r_0^+(x_1 \otimes u_0)(\eta_{-1}))) - r_0^+(x_1 \otimes u_0)([a, \eta_{-1}]). \end{aligned}$$

Thus r_0^+ is a homomorphism of $\mathfrak{g}$-modules. Similarly, we can prove that r_0^- is a homomorphism of $\mathfrak{g}$-modules. □

It follows from Proposition 3.2 that the linear spaces $U_1^+ := \mathrm{Im}\ r_0^+$ and $U_{-1}^- := \mathrm{Im}\ r_0^-$ have the canonical $\mathfrak{g}$-module structures. We denote these canonical representations by π_1^+ and π_{-1}^- respectively. Moreover, we inductively construct $\mathfrak{g}$-modules $U_2^+, U_3^+, \ldots$ by using the following proposition.

Proposition 3.3. *Assume that there exist $\mathfrak{g}$-modules (ϖ^+, W^+), (ϖ^-, W^-) and $\mathfrak{g}$-module homomorphisms $\lambda^+ : V_1 \otimes W^+ \to \mathrm{Hom}(V_{-1}, W^+)$ and $\lambda^- : V_{-1} \otimes W^- \to \mathrm{Hom}(V_1, W^-)$. We put $\hat{W}^+ := \mathrm{Im}\ \lambda^+$, $\hat{W}^+ := \mathrm{Im}\ \lambda^-$ and denote the canonical representations of $\mathfrak{g}$ on them by $\hat{\varpi}^+$ and $\hat{\varpi}^-$ respectively. Then the following linear maps are $\mathfrak{g}$-module homomorphisms:*

$$(3.3) \qquad \hat{\lambda}^+ : V_1 \otimes \hat{W}^+ \to \mathrm{Hom}(V_{-1}, \hat{W}^+)$$
$$x_1 \otimes \hat{w}^+ \mapsto (\eta_{-1} \mapsto \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{w}^+) + \lambda^+(x_1 \otimes \hat{w}^+(\eta_{-1}))),$$
$$(3.4) \qquad \hat{\lambda}^- : V_{-1} \otimes \hat{W}^- \to \mathrm{Hom}(V_1, \hat{W}^-)$$
$$y_{-1} \otimes \hat{w}^- \mapsto (\xi_1 \mapsto \hat{\varpi}^-([\xi_1, y_{-1}] \otimes \hat{w}^-) + \lambda^-(y_{-1} \otimes \hat{w}^-(\xi_1))).$$

Proof. We can prove it by a similar argument to the argument of [8, Proposition 1.10]. Take any elements $a \in \mathfrak{g}$, $x_1 \in V_1$, $\eta_{-1} \in V_{-1}$ and $\hat{w}^+ \in \hat{W}^+$. Then we have

$$\begin{aligned} &(\hat{\lambda}^+([a, x_1] \otimes \hat{w}^+) + \hat{\lambda}^+(x_1 \otimes \hat{\varpi}^+(a \otimes \hat{w}^+)))(\eta_{-1}) \\ &\quad = \hat{\varpi}^+([\eta_{-1}, [a, x_1]] \otimes \hat{w}^+) + \lambda^+([a, x_1] \otimes \hat{w}^+(\eta_{-1})) \\ &\qquad + \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{\varpi}^+(a \otimes \hat{w}^+)) + \lambda^+(x_1 \otimes (\hat{\varpi}^+(a \otimes \hat{w}^+)(\eta_{-1}))) \\ &\quad = \hat{\varpi}^+([a, [\eta_{-1}, x_1]] \otimes \hat{w}^+) + \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{\varpi}^+(a \otimes \hat{w}^+)) + \lambda^+([a, x_1] \otimes \hat{w}^+(\eta_{-1})) \\ &\qquad + \lambda^+(x_1 \otimes \varpi^+(a \otimes \hat{w}^+(\eta_{-1}))) - \hat{\varpi}^+([[a, \eta_{-1}], x_1] \otimes \hat{w}^+) - \lambda^+(x_1 \otimes \hat{w}^+([a, \eta_{-1}])) \\ &\quad = \hat{\varpi}^+(a \otimes \hat{\varpi}^+([\eta_{-1}, x_1] \otimes \hat{w}^+)) + \hat{\varpi}^+(a \otimes \lambda^+(x_1 \otimes \hat{w}^+(\eta_{-1}))) \\ &\qquad - \hat{\varpi}^+([[a, \eta_{-1}], x_1] \otimes \hat{w}^+) - \lambda^+(x_1 \otimes \hat{w}^+([a, \eta_{-1}])) \\ &\quad = \hat{\varpi}^+(a \otimes \hat{\lambda}^+(x_1 \otimes \hat{w}^+)(\eta_{-1})) - \hat{\lambda}^+(x_1 \otimes \hat{w}^+)([a, \eta_{-1}]). \end{aligned}$$

Thus, $\hat{\lambda}^+$ is a homomorphism of $\mathfrak{g}$-modules. By the same way, we can prove that $\hat{\lambda}^-$ is also a $\mathfrak{g}$-module homomorphism. □

DEFINITION 3.4. Suppose that $j \geq 1$ and there exist $\mathfrak{g}$-modules (π_{j-1}^+, U_{j-1}^+), $(\pi_{-j+1}^-, U_{-j+1}^-)$ and homomorphisms of $\mathfrak{g}$-modules $r_{j-1}^+ : V_1 \otimes U_{j-1}^+ \to \mathrm{Hom}(V_{-1}, U_{j-1}^+)$ and $r_{-j+1}^- : V_{-1} \otimes$

$U^-_{-j+1} \to \mathrm{Hom}(V_1, U^-_{-j+1})$. Put $U^+_j := \mathrm{Im}\ r^+_{j-1}$ and $U^-_{-j} := \mathrm{Im}\ r^-_{-j+1}$. Then we define linear maps r^+_j and r^-_{-j} by:

$$r^+_j : V_1 \otimes U^+_j \to \mathrm{Hom}(V_1, U^+_j) \tag{3.5}$$
$$x_1 \otimes u^+_j \mapsto (\eta_1 \mapsto \pi^+_j([\eta_{-1}, x_1] \otimes u^+_j) + r^+_{j-1}(x_1 \otimes u^+_j(\eta_{-1}))),$$

$$r^-_{-j} : V_{-1} \otimes U^-_{-j} \to \mathrm{Hom}(V_1, U^-_{-j}) \tag{3.6}$$
$$y_{-1} \otimes u^-_{-j} \mapsto (\xi_1 \mapsto \pi^-_{-j}([\xi_1, y_{-1}] \otimes u^-_{-j}) + r^-_{-j+1}(y_{-1} \otimes u^-_{-j}(\xi_1))).$$

Then, by Proposition 3.3, r^+_j and r^-_{-j} are homomorphisms of $\mathfrak{g}$-modules. We denote by U^+_{j+1} and U^-_{-j-1} the images of r^+_j and r^-_{-j} and the canonical representations of $\mathfrak{g}$ on U^+_{j+1} and U^-_{-j-1} by π^+_{j+1} and π^-_{-j-1} respectively. Moreover, we put

$$U^+_{-j} := \{0\}, \quad U^-_j := \{0\}$$

for $j \geq 1$. We denote the zero representations of $\mathfrak{g}$ on U^+_{-j} and U^-_j by π^+_{-j} and π^-_j for all $j \geq 1$. Thus, inductively, we obtain $\mathfrak{g}$-modules (π^+_m, U^+_m), (π^-_m, U^-_m) for all $m \in \mathbb{Z}$. Under these notation, we define linear spaces $\tilde{U}^+$ and $\tilde{U}^-$ by:

$$\tilde{U}^+ := \bigoplus_{m\in\mathbb{Z}} U^+_n, \quad \tilde{U}^- := \bigoplus_{m\in\mathbb{Z}} U^-_n. \tag{3.7}$$

Throughout this paper, we use these notation.

3.2. A construction of representations of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ on $\tilde{U}^+$ and $\tilde{U}^-$. In this section, we define $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module structures on vector spaces $\tilde{U}^+$ and $\tilde{U}^-$ constructed in (3.7). For this, we start with the following definition.

Definition 3.5. We define the following linear maps:

$$\pi^+_{0,m} : V_0 \otimes U^+_m \to U^+_m, \quad \pi^+_{1,m} : V_1 \otimes U^+_m \to U^+_{m+1}, \quad \pi^+_{-1,m} : V_{-1} \otimes U^+_m \to U^+_{m-1}$$

by:

$$\pi^+_{0,m}(a \otimes u^+_m) := \pi^+_m(a \otimes u^+_m) \quad (m \in \mathbb{Z}), \tag{3.8}$$

$$\pi^+_{1,m}(x_1 \otimes u^+_m) := \begin{cases} r^+_m(x_1 \otimes u^+_m) & (m \geq 0) \\ 0 & (m \leq -1) \end{cases}, \tag{3.9}$$

$$\pi^+_{-1,m}(y_{-1} \otimes u^+_m) := \begin{cases} u^+_m(y_{-1}) & (m \geq 1) \\ 0 & (m \leq 0) \end{cases} \tag{3.10}$$

where $a \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$ and $u^+_m \in U^+_m$.

By the above definition, we can obtain the following proposition.

Proposition 3.6. *Under the above notation, we have the following equations:*

$$\pi^+_{1,m}([x_1, a] \otimes u^+_m) = \pi^+_{1,m}(x_1 \otimes \pi^+_{0,m}(a \otimes u^+_m)) - \pi^+_{0,m+1}(a \otimes \pi^+_{1,m}(x_1 \otimes u^+_m)), \tag{3.11}$$

$$\pi^+_{-1,m}([y_{-1}, a] \otimes u^+_m) = \pi^+_{-1,m}(y_{-1} \otimes \pi^+_{0,m}(a \otimes u^+_m)) - \pi^+_{0,m-1}(a \otimes \pi^+_{-1,m}(y_{-1} \otimes u^+_m)), \tag{3.12}$$

$$\pi^+_{1,m-1}(x_1 \otimes \pi^+_{-1,m}(y_{-1} \otimes u^+_m)) = \pi^+_{0,m}([x_1, y_{-1}] \otimes u_m) + \pi^+_{-1,m+1}(y_{-1} \otimes \pi^+_{1,m}(x_1 \otimes u_m)). \tag{3.13}$$

Proof. Let us show (3.13). The equations (3.11) and (3.12) can be shown similarly. If $m \leq -1$, then (3.13) is clear. If $m = 0$, then the left hand side equals to 0. For the right hand side, we have

$$\pi^+_{0,0}([x_1, y_{-1}] \otimes u_0) + \pi^+_{-1,1}(y_{-1} \otimes \pi^+_{1,0}(x_1 \otimes u_0))$$
$$= \pi^+_0([x_1, y_{-1}] \otimes u^+_0) + r^+_0(x_1 \otimes u^+_0)(y_{-1}) = \pi^+_0([x_1, y_{-1}] \otimes u^+_0) + \pi^+_0([y_{-1}, x_1] \otimes u^+_0) = 0.$$

Thus we have (3.13) when $m = 0$. For $m \geq 1$, the equation (3.13) follows from definition. □

Definition 3.7. We define the following linear maps for $i \geq 1$ inductively:

(3.14) $$\pi^+_{i+1,m} : V_{i+1} \otimes U^+_m \to U^+_{i+m+1}$$
$$p_i(x_1 \otimes z_i) \otimes u^+_m \mapsto \pi^+_{1,i+m}(x_1 \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) - \pi^+_{i,m+1}(z_i \otimes \pi^+_{1,m}(x_1 \otimes u^+_m)),$$

(3.15)
$$\pi^+_{-i-1,m} : V_{-i-1} \otimes U^+_m \to U^+_{-i+m-1}$$
$$q_{-i}(y_{-1} \otimes \omega_{-i}) \otimes u^+_m \mapsto \pi^+_{-1,-i+m}(y_{-1} \otimes \pi^+_{-i,m}(\omega_{-i} \otimes u^+_m))$$
$$- \pi^+_{-i,m-1}(\omega_{-i} \otimes \pi^+_{-1,m}(y_{-1} \otimes u^+_m)).$$

Note that the linear maps $\pi^+_{0,m}$, $\pi^+_{\pm 1,m}$ defined in Definition 3.5 satisfy the same equations as (3.14) and (3.15) in the cases where $i = 0$ by Proposition 3.6. For $i \geq 1$, we must show the well-definedness of Definition 3.7. To prove it, let us show the following two propositions.

Proposition 3.8. (*The well-definedness of* $\pi^+_{i+1,m}$ *given in* (3.14)) *Suppose that* $i \geq 0$. *Suppose that the linear map* $\pi^+_{i,m}$ *defined in* (3.14) *is well-defined for any* $m \in \mathbb{Z}$ *and satisfies the following equations:*

(3.16) $$\pi^+_{0,i+m}(a \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) = \pi^+_{i,m}([a, z_i] \otimes u^+_m) + \pi^+_{i,m}(z_i \otimes \pi^+_{0,m}(a \otimes u^+_m)),$$

(3.17) $$\pi^+_{i,m-1}(z_i \otimes \pi^+_{-1,m}(y_{-1} \otimes u^+_m)) = \pi^+_{i-1,m}([z_i, y_{-1}] \otimes u^+_m) + \pi^+_{-1,i+m}(y_{-1} \otimes \pi^+_{i,m}(z_i \otimes u^+_m)).$$

If $x^1_1, \ldots, x^l_1 \in V_1$ *and* $z^1_i, \ldots, z^l_i \in V_i$ *satisfy* $\sum_{s=1}^l p_i(x^s_1 \otimes z^s_i) = 0$, *then we have*

(3.18) $$\sum_{s=1}^{l} (\pi^+_{1,i+m}(x^s_1 \otimes \pi^+_{i,m}(z^s_i \otimes u^+_m)) - \pi^+_{i,m+1}(z^s_i \otimes \pi^+_{1,m}(x^s_1 \otimes u^+_m))) = 0$$

for all $m \in \mathbb{Z}$ *and* $u^+_m \in U^+_m$. *In particular, we can obtain the well-definedness of the linear map* $\pi^+_{i+1,m}$ *defined in* (3.14) *for any* $m \in \mathbb{Z}$. *Moreover, the linear maps* $\pi^+_{i+1,m}$ ($m \in \mathbb{Z}$) *satisfy the following equations:*

(3.19) $$\pi^+_{0,i+m+1}(a \otimes \pi^+_{i+1,m}(z_{i+1} \otimes u^+_m))$$
$$= \pi^+_{i+1,m}([a, z_{i+1}] \otimes u^+_m) + \pi^+_{i+1,m}(z_{i+1} \otimes \pi^+_{0,m}(a \otimes u^+_m)),$$

(3.20) $$\pi^+_{i+1,m-1}(z_{i+1} \otimes \pi^+_{-1,m}(y_{-1} \otimes u^+_m))$$
$$= \pi^+_{i,m}([z_{i+1}, y_{-1}] \otimes u^+_m) + \pi^+_{-1,i+m+1}(y_{-1} \otimes \pi^+_{i+1,m}(z_{i+1} \otimes u^+_m)).$$

Proof. We argue by induction on i. For $i = 0$, our claim follows from Proposition 3.6. Suppose that $i \geq 1$. We fix i and argue (3.18) by induction on m. First, if $m \leq -1$, then the equation (3.18) is clear. If $m \geq 0$, then we have

(3.21)

$$\begin{aligned}
&\pi^+_{-1,i+m+1}(y_{-1}\otimes\pi^+_{1,i+m}(x_1\otimes\pi^+_{i,m}(z_i\otimes u^+_m))-\pi^+_{i,m+1}(z_i\otimes\pi^+_{1,m}(x_1\otimes u^+_m)))\\
&\quad=\pi^+_{0,i+m}([y_{-1},x_1]\otimes\pi^+_{i,m}(z_i\otimes u^+_m))+\pi^+_{1,i+m-1}(x_1\otimes\pi^+_{-1,i+m}(y_{-1}\otimes\pi^+_{i,m}(z_i\otimes u^+_m)))\\
&\qquad-\pi^+_{i-1,m+1}([y_{-1},z_i]\otimes\pi^+_{1,m}(x_1\otimes u^+_m))-\pi^+_{i,m}(z_i\otimes\pi^+_{-1,m+1}(y_{-1}\otimes\pi^+_{1,m}(x_1\otimes u^+_m)))\\
&\quad=\pi^+_{0,i+m}([y_{-1},x_1]\otimes\pi^+_{i,m}(z_i\otimes u^+_m))+\pi^+_{1,i+m-1}(x_1\otimes\pi^+_{i-1,m}([y_{-1},z_i]\otimes u^+_m))\\
&\qquad+\pi^+_{1,i+m-1}(x_1\otimes\pi^+_{i,m-1}(z_i\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))-\pi^+_{i-1,m+1}([y_{-1},z_i]\otimes\pi^+_{1,m}(x_1\otimes u^+_m))\\
&\qquad-\pi^+_{i,m}(z_i\otimes\pi^+_{0,m}([y_{-1},x_1]\otimes u^+_m))-\pi^+_{i,m}(z_i\otimes\pi^+_{1,m-1}(x_1\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))\\
&\quad=\pi^+_{i,m}([[y_{-1},x_1],z_i]\otimes u^+_m)+\pi^+_{i,m}([x_1,[y_{-1},z_i]]\otimes u^+_m)\\
&\qquad+\pi^+_{1,i+m-1}(x_1\otimes\pi^+_{i,m-1}(z_i\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))-\pi^+_{i,m}(z_i\otimes\pi^+_{1,m-1}(x_1\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))\\
&\quad=\pi^+_{i,m}([y_{-1},[x_1,z_i]]\otimes u^+_m)\\
&\qquad+\pi^+_{1,i+m-1}(x_1\otimes\pi^+_{i,m-1}(z_i\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))-\pi^+_{i,m}(z_i\otimes\pi^+_{1,m-1}(x_1\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))
\end{aligned}$$

for any $x_1\in V_1$, $z_i\in V_i$, $y_{-1}\in V_{-1}$ and $u^+_m\in U^+_m$. By the induction hypotheses on i and m, if we take elements $x^1_1,\dots,x^l_1\in V_1$ and $z^1_i,\dots,z^l_i\in V_i$ satisfying $\sum_{s=1}^l p_i(x^s_1\otimes z^s_i)=0$, then we have

(3.22)

$$\sum_{s=1}^{l}\pi^+_{i,m}([[x^s_1,z^s_i],y_{-1}]\otimes u^+_m)=0\quad\text{(by the induction hypothesis on } i\text{)},$$

(3.23)

$$\begin{aligned}
&\sum_{s=1}^{l}(\pi^+_{1,i+m-1}(x^s_1\otimes\pi^+_{i,m-1}(z^s_i\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m)))-\pi^+_{i,m}(z^s_i\otimes\pi^+_{1,m-1}(x^s_1\otimes\pi^+_{-1,m}(y_{-1}\otimes u^+_m))))\\
&\quad=0\quad\text{(by the induction hypothesis on } m\text{)}.
\end{aligned}$$

Thus, we have

$$\sum_{s=1}^{l}\pi^+_{-1,i+m+1}(y_{-1}\otimes\pi^+_{1,i+m}(x^s_1\otimes\pi^+_{i,m}(z^s_i\otimes u^+_m))-\pi^+_{i,m+1}(z^s_i\otimes\pi^+_{1,m}(x^s_1\otimes u^+_m)))=0 \tag{3.24}$$

from (3.21). Since $i+m+1\geq 1$, we can obtain that

(3.25)

$$\begin{aligned}
&\sum_{s=1}^{l}(\pi^+_{1,i+m}(x^s_1\otimes\pi^+_{i,m}(z^s_i\otimes r_{m+1}(y_{-1}\otimes u^+_{m+1})))-\pi^+_{i,m+1}(z^s_i\otimes\pi^+_{1,m}(x_1\otimes r_{m+1}(y_{-1}\otimes u^+_{m+1}))))\\
&\quad=0\in U^+_{i+m+1}\subset\mathrm{Hom}(V_{-1},U^+_{i+m}).
\end{aligned}$$

Therefore we can obtain the well-definedness of the linear map $\pi^{i+1}_m: V_{i+1}\otimes U^+_m\to U^+_{i+m+1}$ given in (3.14) for any m.

In order to complete the proof, we must show the equations (3.20) and (3.19). Let us show (3.20). Under the above notation, for any $m\in\mathbb{Z}$, we have

$$\begin{aligned}
&\pi^+_{0,i+m+1}(a\otimes\pi^+_{i+1,m}(p_i(x_1\otimes z_i)\otimes u^+_m))\\
&\quad=\pi^+_{0,i+m+1}(a\otimes\pi^+_{1,i+m}(x_1\otimes\pi^+_{i,m}(z_i\otimes u^+_m)))-\pi^+_{0,i+m+1}(a\otimes\pi^+_{i,m+1}(z_i\otimes\pi^+_{1,m}(x_1\otimes u^+_m)))\\
&\quad=\pi^+_{1,i+m}([a,x_1]\otimes\pi^+_{i,m}(z_i\otimes u^+_m))+\pi^+_{1,i+m}(x_1\otimes\pi^+_{0,i+m}(a\otimes\pi^+_{i,m}(z_i\otimes u^+_m)))
\end{aligned}$$

$$
\begin{aligned}
&- \pi^+_{i,m+1}([a, z_i] \otimes \pi^+_{1,m}(x_1 \otimes u^+_m)) - \pi^+_{i,m+1}(z_i \otimes \pi^+_{0,m+1}(a \otimes \pi^+_{1,m}(x_1 \otimes u^+_m)))\\
&= \pi^+_{1,i+m}([a, x_1] \otimes \pi^+_{i,m}(z_i \otimes u^+_m)) + \pi^+_{1,i+m}(x_1 \otimes \pi^+_{i,m}([a, z_i] \otimes u^+_m))\\
&\quad + \pi^+_{1,i+m}(x_1 \otimes \pi^+_{i,m}(z_i \otimes \pi^+_{0,m}(a \otimes u^+_m))) - \pi^+_{i,m+1}([a, z_i] \otimes \pi^+_{1,m}(x_1 \otimes u^+_m))\\
&\quad - \pi^+_{i,m+1}(z_i \otimes \pi^+_{1,m}([a, x_1] \otimes u^+_m)) - \pi^+_{i,m+1}(z_i \otimes \pi^+_{1,m}(x_1 \otimes \pi^+_{0,m}(a \otimes u^+_m)))\\
&= \pi^+_{i+1,m}([[a, x_1], z_i] \otimes u^+_m) + \pi^+_{i+1,m}([x_1, [a, z_i]] \otimes u^+_m) + \pi^+_{i+1,m}([x_1, z_i] \otimes \pi^+_{0,m}(a \otimes u^+_m))\\
&= \pi^+_{i+1,m}([a, p_i(x_1 \otimes z_i)] \otimes u^+_m) + \pi^+_{i+1,m}(p_i(x_1 \otimes z_i) \otimes \pi^+_{0,m}(a \otimes u^+_m)).
\end{aligned}
$$

Thus we have (3.20). The equation (3.19) follows from (3.21). This completes the proof. □

Proposition 3.9. (*The well-definedness of* $\pi^+_{-i-1,m}$ *given in* (3.15)) *Suppose that* $i \geq 0$. *Suppose that the linear map* $\pi^+_{-i,m}$ *defined in* (3.15) *is well-defined for any* $m \in \mathbb{Z}$ *and satisfies the following equations:*

(3.26)
$$\pi^+_{0,-i+m}(a \otimes \pi^+_{-i,m}(\omega_{-i} \otimes u^+_m)) = \pi^+_{-i,m}([a, \omega_{-i}] \otimes u^+_m) + \pi^+_{-i,m}(\omega_{-i} \otimes \pi^+_{0,m}(a \otimes u^+_m)),$$

(3.27)
$$\pi^+_{-i,m+1}(\omega_{-i} \otimes \pi^+_{1,m}(x_1 \otimes u^+_m)) = \pi^+_{-i+1,m}([\omega_{-i}, x_1] \otimes u^+_m) + \pi^+_{1,-i+m}(x_1 \otimes \pi^+_{-i,m}(\omega_{-i} \otimes u^+_m)).$$

If $y^1_{-1}, \ldots, y^l_{-1} \in V_{-1}$ *and* $\omega^1_{-i}, \ldots, \omega^l_{-i} \in V_{-i}$ *satisfy* $\sum_{s=1}^l q_{-i}(y^s_{-1} \otimes \omega^s_{-i}) = 0$, *then we have*

$$\sum_{s=1}^{l}(\pi^+_{-1,-i+m}(y^s_{-1} \otimes \pi^+_{-i,m}(\omega^s_{-i} \otimes u^+_m)) - \pi^+_{-i,m-1}(\omega^s_{-i} \otimes \pi^+_{-1,m}(y^s_{-1} \otimes u^+_m))) = 0 \tag{3.28}$$

for all $m \in \mathbb{Z}$ *and* $u^+_m \in U^+_m$. *In particular, we can obtain the well-definedness of the linear map* $\pi^+_{-i-1,m}$ *defined in* (3.15) *for any* $m \in \mathbb{Z}$. *Moreover, the maps* $\pi^+_{-i-1,m}$ ($m \in \mathbb{Z}$) *satisfy the following equations:*

$$
\begin{aligned}
&\pi^+_{0,-i+m-1}(a \otimes \pi^+_{-i-1,m}(\omega_{-i-1} \otimes u^+_m))\\
&\qquad = \pi^+_{-i-1,m}([a, \omega_{-i-1}] \otimes u^+_m) + \pi^+_{-i-1,m}(\omega_{-i-1} \otimes \pi^+_{0,m}(a \otimes u^+_m)),
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
&\pi^+_{-i-1,m+1}(\omega_{-i-1} \otimes \pi^+_{1,m}(x_1 \otimes u^+_m))\\
&\qquad = \pi^+_{-i,m}([\omega_{-i-1}, x_1] \otimes u^+_m) + \pi^+_{1,-i+m-1}(x_1 \otimes \pi^+_{-i-1,m}(\omega_{-i-1} \otimes u^+_m)).
\end{aligned} \tag{3.30}
$$

Proof. If $i = 0$, then our claim immediately follows from the definition. Suppose that $i \geq 1$. We fix i and discuss by induction on m. If $m \leq 0$, the equation (3.28) is clear. Suppose that $m \geq 1$. Then, for any $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $\omega_{-i} \in V_{-i}$ and $u^+_{m-1} \in U^+_{m-1}$, we have

(3.31)
$$
\begin{aligned}
&\pi^+_{-1,-i+m}(y_{-1} \otimes \pi^+_{-i,m}(\omega_{-i} \otimes \pi^+_{1,m-1}(x_1 \otimes u^+_{m-1})))\\
&\quad - \pi^+_{-i,m-1}(\omega_{-i} \otimes \pi^+_{-1,m}(y_{-1} \otimes \pi^+_{1,m-1}(x_1 \otimes u^+_{m-1})))\\
&= \pi^+_{-1,-i+m}(y_{-1} \otimes \pi^+_{-i+1,m-1}([\omega_{-i}, x_1] \otimes u^+_{m-1}))\\
&\quad + \pi^+_{-1,-i+m}(y_{-1} \otimes \pi^+_{1,-i+m-1}(x_1 \otimes \pi^+_{-i,m-1}(\omega_{-i} \otimes u^+_{m-1})))\\
&\quad - \pi^+_{-i,m-1}(\omega_{-i} \otimes \pi^+_{0,m-1}([y_{-1}, x_1] \otimes u^+_{m-1}))\\
&\quad - \pi^+_{-i,m-1}(\omega_{-i} \otimes \pi^+_{1,m-2}(x_1 \otimes \pi^+_{-1,m-1}(y_{-1} \otimes u^+_{m-1})))\\
&= \pi^+_{-1,-i+m}(y_{-1} \otimes \pi^+_{-i+1,m-1}([\omega_{-i}, x_1] \otimes u^+_{m-1})) - \pi^+_{-i,m-1}(\omega_{-i} \otimes \pi^+_{0,m-1}([y_{-1}, x_1] \otimes u^+_{m-1}))
\end{aligned}
$$

$$
\begin{aligned}
&+\pi^+_{0,-i+m-1}([y_{-1},x_1]\otimes\pi^+_{-i,m-1}(\omega_{-i}\otimes u^+_{m-1}))\\
&+\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-1,-i+m-1}(y_{-1}\otimes\pi^+_{-i,m-1}(\omega_{-i}\otimes u^+_{m-1})))\\
&-\pi^+_{-i+1,m-2}([\omega_{-i},x_1]\otimes\pi^+_{-1,m-1}(y_{-1}\otimes u^+_{m-1}))\\
&-\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-i,m-2}(\omega_{-i}\otimes\pi^+_{-1,m-1}(y_{-1}\otimes u^+_{m-1})))\\
=&-\pi^+_{-i,m-1}([[\omega_{-i},x_1],y_{-1}]\otimes u^+_{m-1})+\pi^+_{-i,m-1}([[y_{-1},x_1],\omega_{-i}]\otimes u^+_{m-1})\\
&+\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-1,-i+m-1}(y_{-1}\otimes\pi^+_{-i,m-1}(\omega_{-i}\otimes u^+_{m-1})))\\
&-\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-i,m-2}(\omega_{-i}\otimes\pi^+_{-1,m-1}(y_{-1}\otimes u^+_{m-1})))\\
=&\ \pi^+_{-i,m-1}([[y_{-1},\omega_{-i}],x_1]\otimes u^+_{m-1})\\
&+\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-1,-i+m-1}(y_{-1}\otimes\pi^+_{-i,m-1}(\omega_{-i}\otimes u^+_{m-1})))\\
&-\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-i,m-2}(\omega_{-i}\otimes\pi^+_{-1,m-1}(y_{-1}\otimes u^+_{m-1}))).
\end{aligned}
$$

By the induction hypotheses on i and m, if we take elements $y^1_{-1},\dots,y^l_{-1}\in V_{-1}$ and $\omega^1_{-i},\dots,\omega^l_{-i}\in V_{-i}$ satisfying $\sum_{s=1}^l q_{-i}(y^s_{-1}\otimes\omega^s_{-i})=0$, then we have

$$
(3.32)\qquad \sum_{s=1}^{l}\pi^+_{-i,m-1}([[y^s_{-1},\omega^s_{-i}],x_1]\otimes u^+_{m-1})=0\quad\text{(by the induction hypothesis on } i\text{)},
$$

$$
\begin{aligned}
(3.33)\qquad &\sum_{s=1}^{l}(\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-1,-i+m-1}(y_{-1}\otimes\pi^+_{-i,m-1}(\omega_{-i}\otimes u^+_{m-1})))\\
&\quad-\pi^+_{1,-i+m-2}(x_1\otimes\pi^+_{-i,m-2}(\omega_{-i}\otimes\pi^+_{-1,m-1}(y_{-1}\otimes u^+_{m-1}))))=0\\
&\qquad\qquad\text{(by the induction hypothesis on } m\text{)}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(3.34)\qquad &\sum_{s=1}^{l}(\pi^+_{-1,-i+m}(y^s_{-1}\otimes\pi^+_{-i,m}(\omega^s_{-i}\otimes\pi^+_{1,m-1}(x_1\otimes u^+_{m-1})))\\
&\quad-\pi^+_{-i,m-1}(\omega^s_{-i}\otimes\pi^+_{-1,m}(y^s_{-1}\otimes\pi^+_{1,m-1}(x_1\otimes u^+_{m-1}))))=0
\end{aligned}
$$

from (3.31). Since $\pi^+_{1,m-1}:V_1\otimes U^+_{m-1}\to U^+_m$ is surjective, we can obtain the equation (3.28). Therefore we can obtain the well-definedness of the linear map $\pi^+_{-i-1,m}:V_{-i-1}\otimes U^+_m\to U^+_{-i+m-1}$ given in (3.15) for any m.

The equation (3.29) can be shown by a similar way to the proof of Proposition 3.8. Moreover, the equation (3.30) follows from (3.31). □

Definition 3.10. By the above propositions, Propositions 3.8 and 3.9, we define a linear map $\tilde{\pi}^+:L(\mathfrak{g},\rho,V,\mathcal{V},B_0)\otimes\tilde{U}^+\to\tilde{U}^+$ by:

$$
\tilde{\pi}^+(z_n\otimes u^+_m):=\pi^+_{n,m}(z_n\otimes u^+_m)
$$

where $n,m\in\mathbb{Z}$, $z_n\in V_n$ and $u^+_m\in U^+_m$.

This linear map $\tilde{\pi}^+$ satisfies the following equations:

$$
\begin{aligned}
&\tilde{\pi}^+([a,z_n]\otimes u^+_m)=\tilde{\pi}^+(a\otimes\tilde{\pi}^+(z_n\otimes u^+_m))-\tilde{\pi}^+(z_n\otimes\tilde{\pi}^+(a\otimes u^+_m)),\\
&\tilde{\pi}^+([x_1,z_n]\otimes u^+_m)=\tilde{\pi}^+(x_1\otimes\tilde{\pi}^+(z_n\otimes u^+_m))-\tilde{\pi}^+(z_n\otimes\tilde{\pi}^+(x_1\otimes u^+_m)),\\
&\tilde{\pi}^+([y_{-1},z_n]\otimes u^+_m)=\tilde{\pi}^+(y_{-1}\otimes\tilde{\pi}^+(z_n\otimes u^+_m))-\tilde{\pi}^+(z_n\otimes\tilde{\pi}^+(y_{-1}\otimes u^+_m))
\end{aligned}
$$

for any $n, m \in \mathbb{Z}$, $a \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $z_n \in V_n$ and $u^+_m \in U^+_m$. Moreover, we have the following proposition on $\tilde{\pi}^+$.

Proposition 3.11. *The map $\tilde{\pi}^+$ satisfies the following equation:*

$$\tilde{\pi}^+([x, y] \otimes u) = \tilde{\pi}^+(x \otimes \tilde{\pi}^+(y \otimes u)) - \tilde{\pi}^+(y \otimes \tilde{\pi}^+(x \otimes u)) \tag{3.35}$$

for any $x, y \in L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $u^+ \in \tilde{U}^+$.

Proof. To prove our claim, it is sufficient to show the case where $x = z_n \in V_n$ for some $n \in \mathbb{Z}$. We argue by induction on n.

Assume that $n \geq 0$. For $n = 0, 1$, our result has been shown. For $n \geq 2$. We can assume that $z_n = p_{n-1}(x_1 \otimes z_{n-1})$ for some $x_1 \in V_1$ and $z_{n-1} \in V_{n-1}$ without loss of generality. Then, by the induction hypothesis, we have

$$\begin{aligned}
&\tilde{\pi}^+([p_{n-1}(x_1 \otimes z_{n-1}), y] \otimes u^+) = \tilde{\pi}^+([x_1, [z_{n-1}, y]] \otimes u^+) - \tilde{\pi}^+([z_{n-1}, [x_1, y]] \otimes u^+) \\
&\quad = \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+([z_{n-1}, y] \otimes u^+)) - \tilde{\pi}^+([z_{n-1}, y] \otimes \tilde{\pi}^+(x_1 \otimes u^+)) \\
&\qquad - \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+([x_1, y] \otimes u^+)) + \tilde{\pi}^+([x_1, y] \otimes \tilde{\pi}^+(z_{n-1} \otimes u^+)) \\
&\quad = \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(y \otimes u^+))) - \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y \otimes \tilde{\pi}^+(z_{n-1} \otimes u^+))) \\
&\qquad - \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(y \otimes \tilde{\pi}^+(x_1 \otimes u^+))) + \tilde{\pi}^+(y \otimes \tilde{\pi}(z_{n-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+))) \\
&\qquad - \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y \otimes u^+))) + \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(y \otimes \tilde{\pi}^+(x_1 \otimes u^+))) \\
&\qquad + \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y \otimes \tilde{\pi}^+(z_{n-1} \otimes u^+))) - \tilde{\pi}^+(y \otimes \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(z_{n-1} \otimes u^+))) \\
&\quad = \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(y \otimes u^+))) - \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y \otimes u^+))) \\
&\qquad - \tilde{\pi}^+(y \otimes \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(z_{n-1} \otimes u^+))) + \tilde{\pi}^+(y \otimes \tilde{\pi}^+(z_{n-1} \otimes \tilde{\pi}^+(x_1 \otimes u^+))) \\
&\quad = \tilde{\pi}^+([x_1, z_{n-1}] \otimes \tilde{\pi}^+(y \otimes u^+)) - \tilde{\pi}^+(y \otimes \tilde{\pi}^+([x_1, z_{n-1}] \otimes u^+)) \\
&\quad = \tilde{\pi}^+(p_{n-1}(x_1 \otimes z_{n-1}) \otimes \tilde{\pi}^+(y \otimes u^+)) - \tilde{\pi}^+(y \otimes \tilde{\pi}^+(p_{n-1}(x_1 \otimes z_{n-1}) \otimes u^+)).
\end{aligned}$$

Thus, we have our result for any $n \geq 0$.

Similarly, we can obtain our result for any $n \leq -1$. This completes the proof. □

From Proposition 3.11, we have the following theorem.

Theorem 3.12. *The vector space $\tilde{U}^+ = \bigoplus_{m \in \mathbb{Z}} U^+_m = \bigoplus_{m \geq 0} U^+_m$ has a structure of a positively graded $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module whose representation is $\tilde{\pi}^+$. We call the module $(\tilde{\pi}^+, \tilde{U}^+)$* the positive extension of U with respect to a standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. (*This is a special case of* [9, Theorem 1.2].)

By the same argument, we can obtain a negatively graded Lie module of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$.

Definition 3.13. We define the following linear maps:

$$\pi^-_{0,m} : V_0 \otimes U^-_m \to U^-_m, \quad \pi^-_{1,m} : V_1 \otimes U^-_m \to U^-_{m+1}, \quad \pi^-_{-1,m} : V_{-1} \otimes U^-_m \to U^-_{m-1}$$

by:

$$\pi^-_{0,m}(a \otimes u^-_m) := \pi^-_m(a \otimes u^-_m), \tag{3.36}$$

$$\pi^-_{1,m}(x_1 \otimes u^-_m) := \begin{cases} 0 & (m \geq 0) \\ u^-_m(x_1) & (m \leq -1) \end{cases}, \tag{3.37}$$

$$\pi^-_{-1,m}(y_{-1} \otimes u^-_m) := \begin{cases} 0 & (m \geq 1) \\ r^-_m(y_{-1} \otimes u^-_m) & (m \leq 0) \end{cases} \tag{3.38}$$

where $m \in \mathbb{Z}$, $a \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$ and $u^-_m \in U^-_m$.

Theorem 3.14. *The vector space* $\tilde{U}^- = \bigoplus_{m \in \mathbb{Z}} U^-_m = \bigoplus_{m \leq 0} U^-_m$ *has a structure of a negatively graded* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$*-module whose representation is* $\tilde{\pi}^-$*. We call the module* $(\tilde{\pi}^-, \tilde{U}^-)$ the negative extension of U with respect to a standard pentad $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. (*This is a special case of* [9, Theorem 1.2].)

Note that an arbitrary module of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is not necessary written in the form of $\tilde{U}^+$ or $\tilde{U}^-$. For example, the adjoint representation of a loop algebra $L(\mathfrak{sl}_2, \mathrm{ad}, \mathfrak{sl}_2, \mathfrak{sl}_2, K_{\mathfrak{sl}_2}) = \mathcal{L}(\mathfrak{sl}_2(\mathbb{C})) = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{sl}_2(\mathbb{C}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}t^n \otimes \mathfrak{sl}_2$, where $K_{\mathfrak{sl}_2}$ is the Killing form of $\mathfrak{sl}_2$, cannot be written in the form of positively or negatively graded module. Indeed, $\mathcal{L}(\mathfrak{sl}_2(\mathbb{C}))$ does not have a non-zero element which commutes with any element of the form $t \otimes X$ or $t^{-1} \otimes X$ $(X \in \mathfrak{sl}_2)$.

Proposition 3.15. *Under the notation of* Theorems 3.12 *and* 3.14*,* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$*-modules* $\tilde{U}^+$ *and* $\tilde{U}^-$ *have the following properties:*

(3.39) $\tilde{U}^+$ *and* $\tilde{U}^-$ *are transitive,*

(3.40) $\tilde{U}^+$ *and* $\tilde{U}^-$ *are* $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$*-irreducible if and only if* $U = U^+_0 = U^-_0$ *is* $\mathfrak{g}$*-irreducible.*

(*This is a special case of* [9, Theorem 1.1].)

Proof. By the definition, we can show (3.39) immediately.

Let us show (3.40). Assume that U is an irreducible $\mathfrak{g}$-module. Let $\underline{W}$ be an arbitrary non-zero $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-submodule of $\tilde{U}^+$. Then we have that $\underline{W} \cap U^+_0 \neq \{0\}$ (cf. [9, Corollary 1.2]). In fact, take a non-zero element $\underline{w} \in \underline{W}$. Then there exist integers $0 \leq m_1 < \cdots < m_k$ and $\underline{w}_{m_1} \in \underline{W} \cap U^+_{m_1}, \ldots, \underline{w}_{m_k} \in \underline{W} \cap U^+_{m_k}$ such that $\underline{w} = \underline{w}_{m_1} + \cdots + \underline{w}_{m_k}$. Since $\tilde{U}^+$ is transitive, we can take $y^1_{-1}, \ldots, y^{m_k}_{-1} \in V_{-1}$ such that $0 \neq \tilde{\pi}^+(y^1_{-1} \otimes \cdots \otimes \tilde{\pi}^+(y^{m_k}_{-1} \otimes w) \cdots) \in \underline{W} \cap U^+_0$. By the assumption that U is irreducible, we have $\underline{W} \cap U^+_0 = U$. Since $\tilde{U}^+$ is generated by $U = U^+_0$ and V_0, V_1, we have that $\underline{W}$ coincides with $\tilde{U}^+$.

Conversely, assume that $\tilde{U}^+$ is an irreducible $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module. Take a non-zero $\mathfrak{g}$-submodule W of U. Then a submodule $\underline{W}$ of $\tilde{U}^+$ which is generated by V_0, V_1, W is a non-zero $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-submodule of $\tilde{U}^+$. Thus, $\underline{W} = \tilde{U}^+$, and, in particular, $W = \underline{W} \cap U^+_0 = U$. Similarly, we can show (3.40) for the negative extension $\tilde{U}^-$. □

Example 3.16. We retain to use the notations of Example 2.35. Put $U := \mathbb{C}$ and define a representation $\pi : \mathfrak{g} \otimes U \to U$ by:

$$\pi((a, b, A) \otimes u) := au$$

for any $u \in U$. Then, the positive extension $\tilde{U}^+$ of U with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is 3-dimensional irreducible representation of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = V_{-1} \oplus V_0 \oplus V_1 \simeq \mathfrak{gl}_1 \oplus \mathfrak{sl}_3$. In fact, for any $v \in V_1 = V$, $\phi \in V_{-1} = \mathcal{V}$ and $u \in U$, we have

$$\tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v \otimes u)) = -\tilde{\pi}^+(\Phi_\rho(v \otimes \phi) \otimes u) = -\pi((-{}^t v\phi, \frac{3}{2}{}^t v\phi, v{}^t\phi - \frac{1}{2}{}^t v\phi I_2) \otimes u) = {}^t v\phi u.$$

Thus, the element $\tilde{\pi}^+(v \otimes u)$ can be identified with $uv \in V_1 = V$ via $\langle \cdot, \cdot \rangle_V$, in particular, U_1^+ is 2-dimensional. Moreover, we have

$$\begin{aligned}
&\tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v' \otimes \tilde{\pi}^+(v \otimes u))) \\
&\quad = -\tilde{\pi}^+((-{}^t v'\phi, \frac{3}{2}{}^t v'\phi, v'{}^t\phi - \frac{1}{2}{}^t v'\phi I_2) \otimes \tilde{\pi}^+(v \otimes u)) + \tilde{\pi}^+(v' \otimes \tilde{\pi}^+(\phi \otimes \tilde{\pi}^+(v \otimes u))) \\
&\quad = -\tilde{\pi}^+({}^t v'\phi \cdot v \otimes u) - \tilde{\pi}^+({}^t v\phi \cdot v' \otimes u) + \tilde{\pi}^+(v \otimes {}^t v'\phi u) + \tilde{\pi}^+(v' \otimes {}^t v\phi u) \\
&\quad = 0
\end{aligned}$$

for any $v, v' \in V_1$, $\phi \in V_{-1}$ and $u \in U$. Therefore, the positive extension $\tilde{U}^+ = U_0^+ \oplus U_1^+$ is a 3-dimensional irreducible representation (see Proposition 3.15).

The positive and negative extensions of U are characterized by the transitivity.

Theorem 3.17. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad. Let $(\underline{\pi}, \underline{U}) = (\underline{\pi}, \bigoplus_{m\geq 0} \underline{U}_m)$ (respectively $(\underline{\varpi}, \underline{\mathcal{U}}) = (\underline{\varpi}, \bigoplus_{m\leq 0} \underline{\mathcal{U}}_m)$) be a positively graded Lie module (respectively a negatively graded Lie module) of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$. If the $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-module $(\underline{\pi}, \underline{U})$ (respectively $(\underline{\varpi}, \underline{\mathcal{U}})$) is transitive and generated by V_0, V_1 and $\underline{U}_0$ (respectively generated by V_0, V_{-1} and $\underline{\mathcal{U}}_0$), then $\underline{U}$ is isomorphic to the positive extension of $\underline{U}_0$ with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ (respectively $\underline{\mathcal{U}}$ is isomorphic to the negative extension of $\underline{\mathcal{U}}_0$ with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$). (This is a special case of* [9, Theorem 1.2].)

Proof. We denote the positive extension of $\underline{U}_0$ with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ by

$$\widetilde{\underline{U}_0}^+ = \bigoplus_{m\geq 0} (\underline{U})_m^+$$

and the canonical representation of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ on $\widetilde{\underline{U}_0}^+$ by $\widetilde{\underline{\pi}}^+$. Note that $(\underline{U})_0^+ = \underline{U}_0$. We let $\tau_0 : (\underline{U})_0^+ \to \underline{U}_0$ be the identity map on $(\underline{U})_0^+ = \underline{U}_0$ and define linear maps $\tau_i : (\underline{U})_i^+ \to \underline{U}_i$ by

$$\tau_i(r_{i-1}^+(x_1 \otimes \underline{u}_{i-1}^+)) := \underline{\pi}(x_1 \otimes \tau_{i-1}(\underline{u}_{i-1}^+))$$

for $i \geq 1$ and any $x_1 \in V_1$ and $\underline{u}_{i-1}^+ \in (\underline{U})_{i-1}^+$ inductively. These τ_i's are well-defined and satisfy the following equation:

$$\tau_{i+j}(\widetilde{\underline{\pi}}^+(a_j \otimes \underline{u}_i^+)) = \underline{\pi}(a_j \otimes \tau_i(\underline{u}_i^+)) \tag{3.41}$$

for $j = 0, \pm 1$ and any $a_j \in V_j$, $\underline{u}_i^+ \in (\underline{U})_i^+$. Let us show it by induction on i. It is clear that the equation (3.41) holds when $i = 0$ and $j = 0, -1$. In order to show the equation (3.41) for $i = 0$ and $j = 1$, let us show that τ_1 is well-defined. Take an arbitrary element $y_{-1} \in V_{-1}$, then we have

$$\begin{aligned}
\underline{\pi}(y_{-1} \otimes \underline{\pi}(x_1 \otimes \tau_0(\underline{u}_0^+))) &= \underline{\pi}([y_{-1}, x_1] \otimes \tau_0(\underline{u}_0^+)) + \underline{\pi}(x_1 \otimes \underline{\pi}(y_{-1} \otimes \tau_0(\underline{u}_0^+))) \\
&= \underline{\pi}([y_{-1}, x_1] \otimes \tau_0(\underline{u}_0^+)) = \tau_0(\widetilde{\underline{\pi}}^+([y_{-1}, x_1] \otimes \underline{u}_0^+)) = \tau_0(\widetilde{\underline{\pi}}^+(y_{-1} \otimes r_0^+(x_1 \otimes \underline{u}_0^+))).
\end{aligned} \tag{3.42}$$

Thus, if $x_1^1, \ldots, x_1^l \in V_1$ and $\underline{u}_0^{+1}, \ldots, \underline{u}_0^{+l} \in (\underline{U})_0^+$ satisfy $\sum_{s=1}^l r_0^+(x_1^s \otimes \underline{u}_0^{+s}) = 0$, then we have

$$\sum_{s=1}^{l} \underline{\pi}(y_{-1} \otimes \underline{\pi}(x_1^s \otimes \tau_0(\underline{u}_0^{+s}))) = 0$$

for any $y_{-1} \in V_{-1}$. Since $(\underline{\pi}, \underline{U})$ is transitive, it follows that $\sum_{s=1}^{l} \underline{\pi}(x_1^s \otimes \tau_0(\underline{u}_0^{+s})) = 0$, and, thus, we have the well-definedness of τ_1. By the equation (3.42), we can obtain the equation (3.41) where $i = 0$ and $j = 1$.

Let $i \geq 1$ and assume that $\tau_0, \ldots, \tau_i$ are well-defined and that τ_i satisfies the equation (3.41) for $j = 0, -1$. Then for any $y_{-1} \in V_{-1}$, we have

$$\begin{aligned} \underline{\pi}(y_{-1} \otimes \underline{\pi}(x_1 \otimes \tau_i(\underline{u}_i^+))) &= \underline{\pi}([y_{-1}, x_1] \otimes \tau_i(\underline{u}_i^+)) + \underline{\pi}(x_1 \otimes \underline{\pi}(y_{-1} \otimes \tau_i(\underline{u}_i^+))) \\ &= \tau_i(\widetilde{\underline{\pi}}^+([y_{-1}, x_1] \otimes \underline{u}_i^+)) + \tau_i(\widetilde{\underline{\pi}}^+(x_1 \otimes \widetilde{\underline{\pi}}^+(y_{-1} \otimes \underline{u}_i^+))) \\ &= \tau_i(\widetilde{\underline{\pi}}^+(y_{-1} \otimes \widetilde{\underline{\pi}}^+(x_1 \otimes \underline{u}_i^+))) = \tau_i(\widetilde{\underline{\pi}}^+(y_{-1} \otimes r_0^+(x_1 \otimes \underline{u}_i^+))). \end{aligned} \tag{3.43}$$

Thus, by the same argument to the argument of the case where $i = 0$ and $j = 1$, we have the well-definedness of τ_{i+1}, i.e. τ_i satisfies satisfies the equation (3.41) for $j = 1$, and that τ_{i+1} satisfies the equation (3.41) for $j = -1$. Moreover, by a similar argument to the argument of (3.43), we have that τ_{i+1} satisfies the equation (3.41) for $j = 0$. Therefore, by induction on i, we can obtain the well-definedness of τ_i and the equation (3.41) for all $i \geq 0$ and $j = 0, \pm 1$.

We define a linear map $\tau : \widetilde{\underline{U}_0}^+ \to \underline{U}$ by

$$\tau(\underline{u}_i^+) := \tau_i(\underline{u}_i^+) \tag{3.44}$$

for any $i \geq 0$ and $\underline{u}_i^+ \in (\underline{U})_i^+$. This τ is an isomorphism of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-modules. In fact, by the assumption that $\underline{U}$ is generated by V_1 and $\underline{U}_0$, we have the surjectivity of τ. Moreover, by the equation (3.41) in the cases where $i \geq 1$ and $j = -1$ and the definition of τ_0 and the transitivity of the positive extension of $\widetilde{\underline{U}_0}^+$, we have the injectivity of τ. Thus, τ is bijective. Moreover, since $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is generated by V_0, $V_{\pm 1}$, it follows that τ is a homomorphism of $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-modules from the equation (3.41). Therefore $\underline{U}$ is isomorphic to $\widetilde{\underline{U}_0}^+$ as $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-modules.

By the same argument, we can prove our claim for $(\underline{\varpi}, \underline{\mathcal{U}})$. □

As an application of Theorem 3.17, we have the following proposition.

Proposition 3.18. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ be a standard pentad and U, W (respectively $\mathcal{U}, \mathcal{W}$) be $\mathfrak{g}$-modules. Then the positive extension of $U \oplus W$ (respectively the negative extension of $\mathcal{U} \oplus \mathcal{W}$) with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ is isomorphic to a direct sum of positive extensions of U and W (respectively negative extensions of $\mathcal{U}$ and $\mathcal{W}$) with respect to $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$, i.e.*

$$\widetilde{(U \oplus W)}^+ \simeq \tilde{U}^+ \oplus \tilde{W}^+ \quad (\textit{respectively } \widetilde{(\mathcal{U} \oplus \mathcal{W})}^- \simeq \tilde{\mathcal{U}}^- \oplus \tilde{\mathcal{W}}^-).$$

3.3. A pairing between $(\tilde{\pi}^+, \tilde{U}^+)$ and $(\tilde{\varpi}^-, \tilde{\mathcal{U}}^-)$. In the previous section, we constructed positively and negatively graded $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-modules. Next, let us try to embed these modules into some graded Lie algebra. For this, we need to embed $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\tilde{\pi}^+, \tilde{U}^+)$ into some standard pentad. However, as mentioned in Remark 2.5, the objects $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $\tilde{U}^+$ might not have a submodule of $\mathrm{Hom}(\tilde{U}^+, F)$ and a bilinear form on $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ satisfying the conditions (2.3) and (2.4). In the present and the next sections, we only consider the cases where B_0 is symmetric and U has a submodule $\mathcal{U} \subset \mathrm{Hom}(U, F)$ such that $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ is standard. Then, we can show that a pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is standard. First, in this section, we consider the negative extension $\tilde{\mathcal{U}}^-$ of $\mathcal{U}$ and construct a non-degenerate invariant bilinear form $\tilde{U}^+ \times \tilde{\mathcal{U}}^- \to F$ under the assumption (2.3) inductively (cf. [9, Remark 1.4]). In the next section, we shall construct the Φ-map of the pentad $(L(\mathfrak{g},\rho,V,\mathcal{V},B_0),\tilde{\pi}^+,\tilde{U}^+,\tilde{\mathcal{U}}^-,B_L)$.

DEFINITION 3.19. Let $(\tilde{\pi}^+,\tilde{U}^+)$ and $(\tilde{\varpi}^-,\tilde{\mathcal{U}}^-)$, $\mathcal{U}\subset\mathrm{Hom}(U,F)$ be $\mathfrak{g}$-modules such that the restriction of the canonical pairing $\langle\cdot,\cdot\rangle_0 : U\times\mathcal{U}\to F$ is non-degenerate, and, let $\tilde{U}^+$ and $\tilde{\mathcal{U}}^-$ be the positive and negative extensions of U and $\mathcal{U}$ respectively. We define a bilinear map $\langle\cdot,\cdot\rangle_0^0$ by:

$$\begin{aligned}\langle\cdot,\cdot\rangle_0^0 :& U_0^+\times\mathcal{U}_0^-\to F\\ &(u_0^+,w_0^-)\mapsto\langle u_0^+,w_0^-\rangle_0.\end{aligned}\tag{3.45}$$

Moreover, for $i\geq 1$, we define a bilinear map $\langle\cdot,\cdot\rangle_{-i}^i$ by:

(3.46)
$$\begin{aligned}&\langle\cdot,\cdot\rangle_{-i}^i : U_i^+\times\mathcal{U}_{-i}^-\to F\\ &\quad(r_{i-1}^+(x_1\otimes u_{i-1}^+),r_{-i+1}^-(y_{-1}\otimes w_{-i+1}^-))\mapsto-\langle\tilde{\pi}^+(y_{-1}\otimes r_{i-1}^+(x_1\otimes u_{i-1}^+)),w_{-i+1}^-\rangle_{-i+1}^{i-1}\end{aligned}$$

inductively.

The well-definedness of Definition 3.19 can be obtained by the following proposition.

Proposition 3.20. *Let $j\geq 0$. Assume that the bilinear map $\langle\cdot,\cdot\rangle_{-j}^j$ defined in* (3.46) *is well-defined and satisfies the following equations:*

$$\langle\tilde{\pi}^+(a\otimes u_j^+),w_{-j}^-\rangle_{-j}^j+\langle u_j^+,\tilde{\pi}^-(a\otimes w_{-j}^-)\rangle_{-j}^j=0,\tag{3.47}$$

$$\langle\tilde{\pi}^+(y_{-1}\otimes\tilde{\pi}^+(x_1\otimes u_j^+)),w_{-j}^-\rangle_{-j}^j=\langle u_j^+,\tilde{\pi}^-(x_1\otimes\tilde{\pi}^-(y_{-1}\otimes w_{-j}^-))\rangle_{-j}^j\tag{3.48}$$

$$\langle\tilde{\pi}^+(x_1\otimes u_{j-1}^+),w_{-j}^-\rangle_{-j}^j=\begin{cases}-\langle u_{j-1}^+,\tilde{\pi}^-(x_1\otimes w_{-j}^-)\rangle_{-j+1}^{j-1} & (j\geq 1)\\ 0 & (j=0)\end{cases}\tag{3.49}$$

for any $a\in\mathfrak{g}=V_0\subset L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$, $x_1\in V_1$, $y_{-1}\in V_{-1}$, $u_{j-1}^+\in U_{j-1}^+$, $u_j^+\in U_j^+$ and $w_{-j}^-\in\mathcal{U}_{-j}^-$. Then the bilinear map $\langle\cdot,\cdot\rangle_{-j-1}^{j+1}$ defined in (3.46) *is also well-defined and satisfies the following equations:*

$$\langle\tilde{\pi}^+(a\otimes u_{j+1}^+),w_{-j-1}^-\rangle_{-j-1}^{j+1}+\langle u_{j+1}^+,\tilde{\pi}^-(a\otimes w_{-j-1}^-)\rangle_{-j-1}^{j+1}=0,\tag{3.50}$$

$$\langle\tilde{\pi}^+(y_{-1}\otimes\tilde{\pi}^+(x_1\otimes u_{j+1}^+)),w_{-j-1}^-\rangle_{-j-1}^{j+1}=\langle u_{j+1}^+,\tilde{\pi}^-(x_1\otimes\tilde{\pi}^-(y_{-1}\otimes w_{-j-1}^-))\rangle_{-j-1}^{j+1}\tag{3.51}$$

$$\langle\tilde{\pi}^+(x_1\otimes u_j^+),w_{-j-1}^-\rangle_{-j-1}^{j+1}=-\langle u_j^+,\tilde{\pi}^-(x_1\otimes w_{-j-1}^-)\rangle_{-j}^j\tag{3.52}$$

for any $a\in\mathfrak{g}=V_0\subset L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$, $x_1\in V_1$, $y_{-1}\in V_{-1}$, $u_j^+\in U_j^+$, $u_{j+1}^+\in U_{j+1}^+$ and $w_{-j-1}^-\in\mathcal{U}_{-j-1}^-$.

Proof. First, we let $j=0$. It is clear that $\langle\cdot,\cdot\rangle_0^0$ satisfies (3.47) and (3.49). Let us show that $\langle\cdot,\cdot\rangle_0^0$ satisfies (3.48). Indeed, under the above notation, we have

$$\begin{aligned}\langle\tilde{\pi}^+(y_{-1}\otimes\tilde{\pi}^+(x_1\otimes u_0^+)),w_0\rangle_0^0&=\langle\tilde{\pi}^+([y_{-1},x_1]\otimes u_0^+),w_0\rangle_0^0=\langle u_0^+,\tilde{\pi}^-([x_1,y_{-1}]\otimes w_0)\rangle_0^0\\&=\langle u_0^+,\tilde{\pi}^-(x_1\otimes\tilde{\pi}^-(y_{-1}\otimes w_0))\rangle_0^0.\end{aligned}\tag{3.53}$$

Thus, the bilinear map $\langle\cdot,\cdot\rangle_0^0$ satisfies the assumptions of Proposition 3.20.

Next, let us show that the bilinear map $\langle\cdot,\cdot\rangle_{-1}^1$ is well-defined. Take arbitrary natural numbers $\nu,\mu\in\mathbb{N}$ and elements $x_1^1,\ldots,x_1^\nu\in V_1$, $u_0^{+,1},\ldots,u_0^{+,\nu}\in U_0^+$, $y_{-1}^1,\ldots,y_{-1}^\mu\in V_{-1}$,

$w_0^{-,1}, \dots, w_0^{-,\mu} \in \mathcal{U}_0^-$ satisfying

$$\sum_{s=1}^{\nu} r_0^+(x_1^s \otimes u_0^{+,s}) = 0, \quad \sum_{t=1}^{\mu} r_0^-(y_{-1}^t \otimes w_0^{-,t}) = 0.$$

Then, for any $y_{-1} \in V_{-1}$, $w_0^- \in \mathcal{U}_0^-$, $x_1 \in V_1$ and $u_0^+ \in U_0^+$, we have

$$\langle \sum_{s=1}^{\nu} r_0^+(x_1^s \otimes u_0^{+,s})(y_{-1}), w_0^- \rangle_0^0 = 0, \tag{3.54}$$

and, by the equation (3.53), we have

$$\sum_{t=1}^{\mu} \langle r_0^+(x_1 \otimes u_0^+)(y_{-1}^t), w_0^{-,t} \rangle_0^0 = \sum_{t=1}^{\mu} \langle u_0^+, r_0^-(y_{-1}^t \otimes w_0^{-,t})(x_1) \rangle_0^0 = 0. \tag{3.55}$$

By (3.54) and (3.55), we can obtain that $\langle \cdot, \cdot \rangle_{-1}^1$ is well-defined.

Let us consider properties of $\langle \cdot, \cdot \rangle_{-1}^1$. By (3.53), we have that $\langle \cdot, \cdot \rangle_{-1}^1$ satisfies

$$\langle \tilde{\pi}^+(x_1 \otimes u_0^+), w_{-1}^- \rangle_{-1}^1 = -\langle u_0^+, \tilde{\pi}^-(x_1 \otimes w_{-1}^-) \rangle_{-1}^1 \tag{3.56}$$

for any $x_1 \in V_1$, $u_0^+ \in U_0^+$ and $w_{-1}^- \in \mathcal{U}_{-1}^-$, i.e. $\langle \cdot, \cdot \rangle_{-1}^1$ satisfies the equation (3.52). Moreover, $\langle \cdot, \cdot \rangle_{-1}^1$ satisfies the equations (3.50) and (3.51). In fact, for all $a \in V_0$, $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $u_0^+ \in U_0^+$ and $w_0^- \in \mathcal{U}_0^-$, we have

$$\begin{aligned} \langle \tilde{\pi}^+(a \otimes r_0^+(x_1 \otimes u_0^+)), r_0^-(y_{-1} \otimes w_0^-) \rangle_{-1}^1 &= -\langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(a \otimes \tilde{\pi}^+(x_1 \otimes u_0^+))), w_0^- \rangle_0^0 \\ &= -\langle \tilde{\pi}^+(a \otimes \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u_0^+))), w_0^- \rangle_0^0 + \langle \tilde{\pi}^+([a, y_{-1}] \otimes \tilde{\pi}^+(x_1 \otimes u_0^+)), w_0^- \rangle_0^0 \\ &= \langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u_0^+)), \tilde{\pi}^-(a \otimes w_0^-) \rangle_0^0 - \langle \tilde{\pi}^+(x_1 \otimes u_0^+), \tilde{\pi}^-([a, y_{-1}] \otimes w_0^-) \rangle_{-1}^1 \\ &= -\langle \tilde{\pi}^+(x_1 \otimes u_0^+), \tilde{\pi}^-(y_{-1} \otimes \tilde{\pi}^-(a \otimes w_0^-)) \rangle_{-1}^1 - \langle \tilde{\pi}^+(x_1 \otimes u_0^+), \tilde{\pi}^-([a, y_{-1}] \otimes w_0^-) \rangle_{-1}^1 \\ &= -\langle r_0^+(x_1 \otimes u_0^+), \tilde{\pi}^-(a \otimes r_0^-(y_{-1} \otimes w_0^-)) \rangle_{-1}^1. \end{aligned} \tag{3.57}$$

Thus $\langle \cdot, \cdot \rangle_{-1}^1$ satisfies (3.50). And, from (3.56) and (3.57), we have

$$\begin{aligned} &\langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(x_1 \otimes u_1^+)), w_{-1}^- \rangle_{-1}^1 \\ &= \langle \tilde{\pi}^+([y_{-1}, x_1] \otimes u_1^+), w_{-1}^- \rangle_{-1}^1 + \langle \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(y_{-1} \otimes u_1^+)), w_{-1}^- \rangle_{-1}^1 \\ &= -\langle u_1^+, \tilde{\pi}^-([y_{-1}, x_1] \otimes w_{-1}^-) \rangle_{-1}^1 - \langle \tilde{\pi}^+(y_{-1} \otimes u_1^+), \tilde{\pi}^-(x_1 \otimes w_{-1}^-) \rangle_0^0 \\ &= \langle u_1^+, \tilde{\pi}^-([x_1, y_{-1}] \otimes w_{-1}^-) \rangle_{-1}^1 + \langle u_1^+, \tilde{\pi}^-(y_{-1} \otimes \tilde{\pi}^-(x_1 \otimes w_{-1}^-)) \rangle_{-1}^1 \\ &= \langle u_1^+, \tilde{\pi}^-(x_1 \otimes \tilde{\pi}^-(y_{-1} \otimes w_{-1}^-)) \rangle_{-1}^1 \end{aligned} \tag{3.58}$$

for any $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $u_1^+ \in U_1^+$ and $w_{-1}^- \in \mathcal{U}_{-1}^-$. Thus $\langle \cdot, \cdot \rangle_{-1}^1$ satisfies (3.51).

We let $j \geq 1$. Suppose that the bilinear map $\langle \cdot, \cdot \rangle_{-j}^j$ is well-defined and satisfies the equations (3.47), (3.48) and (3.49). Let us show the well-definedness of $\langle \cdot, \cdot \rangle_{-j-1}^{j+1}$. Take arbitrary natural numbers $\nu, \mu \in \mathbb{N}$ and elements $x_1^1, \dots, x_1^\nu \in V_1$, $u_j^{+,1}, \dots, u_j^{+,\nu} \in U_0^+$, $y_{-1}^1, \dots, y_{-1}^\mu \in V_{-1}$, $w_{-j}^{-,1}, \dots, w_{-j}^{-,\mu} \in \mathcal{U}_0^-$ satisfying

$$\sum_{s=1}^{\nu} r_j^+(x_1^s \otimes u_j^{+,s}) = 0, \quad \sum_{t=1}^{\mu} r_{-j}^-(y_{-1}^t \otimes w_{-j}^{-,t}) = 0. \tag{3.59}$$

Then, by the equation (3.48) and the same argument to the argument of (3.54) and (3.55), we have the following equations:

$$(3.60)\qquad \langle\sum_{s=1}^{\nu} r_j^+(x_1^s\otimes u_j^{+,s})(y_{-1}),w_{-j}^-\rangle_{-j}^j=0,\quad \sum_{t=1}^{\mu}\langle r_j^+(x_1\otimes u_j^+)(y_{-1}^t),w_{-j}^{-,t}\rangle_{-j}^j=0.$$

Thus, we have that the bilinear map $\langle\cdot,\cdot\rangle_{-j-1}^{j+1}$ is well-defined.

From the equation (3.48), we have

$$(3.61)\qquad \langle\tilde{\pi}^+(x_1\otimes u_j^+),w_{-j-1}^-\rangle_{-j-1}^{j+1}=-\langle u_j^+,\tilde{\pi}^-(x_1\otimes w_{-j-1}^-)\rangle_{-j}^j$$

for any $x_1\in V_1$, $u_j^+\in U_j^+$ and $w_{-j-1}\in\mathcal{U}_{-j-1}^-$. We can show that the bilinear map $\langle\cdot,\cdot\rangle_{-j-1}^{j+1}$ satisfies the equation (3.52) from the equation (3.61) and that it also satisfies the equations (3.50) and (3.51) by the same argument to the argument of the case where $j=0$. □

By Proposition 3.20, we can obtain pairings $\langle\cdot,\cdot\rangle_{-j}^j$ for all $j\geq 0$ inductively. Then, we can define a pairing between $(\tilde{\pi}^+,\tilde{U}^+)$ and $(\tilde{\varpi}^-,\tilde{\mathcal{U}}^-)$.

DEFINITION 3.21. We define a bilinear map $\langle\cdot,\cdot\rangle:\tilde{U}^+\times\tilde{\mathcal{U}}^-\to F$ by:

$$(3.62)\qquad \langle u_n^+,w_{-m}^-\rangle:=\begin{cases}\langle u_n^+,w_{-n}^-\rangle_{-n}^n & (n=m)\\ 0 & (n\neq m)\end{cases}$$

for any $n,m\geq 0$, $u_n^+\in U_n^+\subset\tilde{U}^+$ and $w_{-m}^-\in\mathcal{U}_{-m}^-\subset\tilde{\mathcal{U}}^-$.

By Definition 3.19 and Proposition 3.20, we have that $\langle\cdot,\cdot\rangle$ satisfies

$$(3.63)\qquad \langle\tilde{\pi}^+(z_j\otimes\tilde{u}^+),\tilde{w}^-\rangle=-\langle\tilde{u}^+,\tilde{\pi}^-(z_j\otimes\tilde{w}^-)\rangle$$

for $j=0,\pm1$ and any $z_j\in V_j$, $\tilde{u}^+\in\tilde{U}^+$, $\tilde{w}^-\in\tilde{\mathcal{U}}^-$.

Proposition 3.22. *The bilinear form $\langle\cdot,\cdot\rangle:\tilde{U}^+\times\tilde{\mathcal{U}}^-\to F$ is non-degenerate and $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$-invariant (cf.* [9, Definition 1.4 and Remark 1.4]).

Proof. First, let us show that the bilinear form $\langle\cdot,\cdot\rangle$ is non-degenerate. For this, it is sufficient to show that the bilinear map $\langle\cdot,\cdot\rangle_{-j}^j:U_j^+\times\mathcal{U}_{-j}\to F$ is non-degenerate for each $j\geq 0$. We show it by induction on j. For $j=0$, it follows that $\langle\cdot,\cdot\rangle_0^0$ is non-degenerate from the assumption. For $j+1$, we take an element $u_{j+1}^+\in U_{j+1}^+$ which satisfies $\langle u_{j+1}^+,r_{-j}^-(y_{-1}\otimes w_{-j}^-)\rangle_{-j-1}^{j+1}=0$ for any $y_{-1}\in V_{-1}$ and $w_{-j}^-\in\mathcal{U}_{-j}^-$. Then, we have

$$0=\langle u_{j+1}^+,r_{-j}^-(y_{-1}\otimes w_{-j}^-)\rangle_{-j-1}^{j+1}=-\langle\tilde{\pi}^+(y_{-1}\otimes u_{j+1}^+),w_{-j}^-\rangle_{-j}^j=-\langle u_{j+1}^+(y_{-1}),w_{-j}^-\rangle_{-j}^j.$$

By the induction hypothesis that $\langle\cdot,\cdot\rangle_{-j}^j$ is non-degenerate, we can obtain that $u_{j+1}^+(y_{-1})=0$ for any $y_{-1}\in V_{-1}$, and, thus, we have $u_{j+1}^+=0\in U_{j+1}^+\subset\mathrm{Hom}(V_{-1},U_j^+)$. Similarly, we can show that an element $w_{-j-1}^-\in\mathcal{U}_{-j-1}^-$ which satisfies $\langle r_j^+(x_1\otimes u_j^+),w_{-j-1}^-\rangle_{-j-1}^{j+1}=0$ for any $x_1\in V_1$ and $u_i^+\in U_i^+$ is 0 by (3.63). Summarizing the above argument, we can obtain that the map $\langle\cdot,\cdot\rangle_{-j-1}^{j+1}$ is non-degenerate. Therefore, by induction, we can obtain that the bilinear map $\langle\cdot,\cdot\rangle:\tilde{U}^+\times\tilde{\mathcal{U}}^-\to F$ is non-degenerate.

Next, let us show that the bilinear map $\langle\cdot,\cdot\rangle$ is $L(\mathfrak{g},\rho,V,\mathcal{V},B_0)$-invariant. For this, it is sufficient to show that the following equation holds:

$$\langle \tilde{\pi}^+(x_j \otimes u_n^+), w^-_{-n-j}\rangle^{n+j}_{-n-j} + \langle u_n^+, \tilde{\pi}^-(x_j \otimes w^-_{-n-j})\rangle^n_{-n} = 0 \tag{3.64}$$

for any $j, n \in \mathbb{Z}$, $x_j \in V_j$, $u_n \in U_n^+$ and $w^-_{-n-j} \in \mathcal{U}^-_{-n-j}$. We shall show it by induction on j. Assume that $j \geq 0$. For $j = 0, 1$, the equation (3.64) follows from (3.63) immediately. For $j + 1$, by induction hypothesis, we have

$$\begin{aligned}(3.65)\quad &\langle \tilde{\pi}^+([v_1, x_j] \otimes u_n^+), w^-_{-n-j-1}\rangle^{n+j+1}_{-n-j-1}\\ &= \langle \tilde{\pi}^+(v_1 \otimes \tilde{\pi}^+(x_j \otimes u_n^+)), w^-_{-n-j-1}\rangle^{n+j+1}_{-n-j-1} - \langle \tilde{\pi}^+(x_j \otimes \tilde{\pi}^+(v_1 \otimes u_n^+)), w^-_{-n-j-1}\rangle^{n+j+1}_{-n-j-1}\\ &= -\langle \tilde{\pi}^+(x_j \otimes u_n^+), \tilde{\pi}^-(v_1 \otimes w^-_{-n-j-1})\rangle^{n+j}_{-n-j} + \langle \tilde{\pi}^+(v_1 \otimes u_n^+), \tilde{\pi}^-(x_j \otimes w^-_{-n-j-1})\rangle^{n+1}_{-n-1}\\ &= \langle u_n^+, \tilde{\pi}^-(x_j \otimes \tilde{\pi}^-(v_1 \otimes w^-_{-n-j-1}))\rangle^n_{-n} - \langle u_n^+, \tilde{\pi}^-(v_1 \otimes \tilde{\pi}^-(x_j \otimes w^-_{-n-j-1}))\rangle^n_{-n}\\ &= -\langle u_n^+, \tilde{\pi}^-([v_1, x_j] \otimes w^-_{-n-j-1})\rangle^n_{-n}\end{aligned}$$

for any $n \in \mathbb{Z}$, $x_1 \in V_1$, $v_j \in V_j$, $u_n^+ \in U_n^+$ and $w^-_{-n-j-1} \in \mathcal{U}^-_{-n-j-1}$. Thus, by induction, we can show the equation (3.64) for all $j \geq 0$. Similarly, we can obtain the equation (3.64) for all $j \leq 0$. Thus, we have the equation (3.64) for all $j \in \mathbb{Z}$. Therefore the bilinear map $\langle\cdot,\cdot\rangle : \tilde{U}^+ \times \tilde{\mathcal{U}}^- \to F$ is $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-invariant. □

By Proposition 3.22, we can regard $\tilde{\mathcal{U}}^-$ as an $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$-submodule of $\mathrm{Hom}(\tilde{U}^+, F)$.

3.4. The Φ-map between $(\tilde{\pi}^+, \tilde{U}^+)$ and $(\tilde{\varpi}^-, \tilde{\mathcal{U}}^-)$. We retain to assume that a pentad $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ is standard and the bilinear form B_0 is symmetric. As I proved in section 3.3, a pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ satisfies the condition (2.3). Let us construct the Φ-map of the pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ and show that it is standard.

DEFINITION 3.23. Assume that pentads $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ are standard and that B_0 is symmetric. We define a linear map $\tilde{\Phi}^0_0 : U_0^+ \otimes \mathcal{U}_0^- \to V_0$ as:

$$\tilde{\Phi}^0_0(u_0^+ \otimes w_0^-) := \Phi_\pi(u_0^+ \otimes w_0^-) \tag{3.66}$$

where $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $u_0^+ \in U_0^+$, $w_0^- \in \mathcal{U}_0^-$ and Φ_π is the Φ-map of $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$.

Moreover, for each $i \geq 0$, we inductively define a linear map $\tilde{\Phi}^{i+1}_0 : U^+_{i+1} \otimes \mathcal{U}_0^- \to V_{i+1}$ by:

$$\tilde{\Phi}^{i+1}_0(r_i^+(x_1 \otimes u_i^+) \otimes w_0^-) := [x_1, \tilde{\Phi}^i_0(u_i^+ \otimes w_0^-)], \tag{3.67}$$

where $x_1 \in V_1$, $y_{-1} \in V_{-1}$, $u_i^+ \in U_i^+$ and $w_0^- \in \mathcal{U}_0^-$.

Assume that an integer $j \geq 0$ satisfies a condition that we have linear maps $\tilde{\Phi}^k_{-j} : U_k^+ \otimes \mathcal{U}^-_{-j} \to V_{k-j}$ for all $k \geq 0$. Then, for any $k \geq 0$, we define a linear map $\tilde{\Phi}^k_{-j-1} : U_k^+ \otimes \mathcal{U}^-_{-j-1} \to V_{k-j-1}$ by:

$$\begin{aligned}(3.68)\quad &\tilde{\Phi}^k_{-j-1}(u_k^+ \otimes r^-_{-j}(y_{-1} \otimes w^-_{-j}))\\ &:= \begin{cases} [y_{-1}, \tilde{\Phi}^0_{-j}(u_0^+ \otimes w^-_{-j})] & (k = 0)\\ [y_{-1}, \tilde{\Phi}^k_{-j}(u_k^+ \otimes w^-_{-j})] - \tilde{\Phi}^{k-1}_{-j}(\tilde{\pi}^+(y_{-1} \otimes u_k^+) \otimes w^-_{-j}) & (k \geq 1)\end{cases}\end{aligned}$$

where $y_{-1} \in V_{-1}$, $u_k^+ \in U_k^+$ and $w^-_{-j} \in \mathcal{U}^-_{-j}$.

Consequently, we can define linear maps $\tilde{\Phi}^i_{-j} : U_i^+ \otimes \mathcal{U}^-_{-j} \to V_{i-j}$ for all $i, j \geq 0$.

Proposition 3.24. *The linear map $\tilde{\Phi}^i_{-j}$ is well-defined and satisfies the following equation:*

$$B_L(a_{-i+j}, \tilde{\Phi}^i_{-j}(u_i^+ \otimes w_{-j}^-)) = \langle \tilde{\pi}^+(a_{-i+j} \otimes u_i^+), w_{-j}^- \rangle \tag{3.69}$$

for any $i, j \geq 0$, $a_{-i+j} \in V_{-i+j}$, $u_i^+ \in U_i^+$ *and* $w_{-j}^- \in \mathcal{U}_{-j}^-$.

Proof. Let us show that the linear maps defined by the equations (3.66), (3.67) and (3.68) satisfy our claim by induction. First, let us show that the linear map $\tilde{\Phi}_0^{i+1}$ ($i \geq 0$) defined in (3.67) is well-defined by induction on i. For $i = 0$, under the above notation, we have

$$\begin{aligned} B_L(a_{-1}, [x_1, \tilde{\Phi}_0^0(u_0^+ \otimes w_0^-)]) =& B_L([a_{-1}, x_1], \tilde{\Phi}_0^0(u_0^+ \otimes w_0^-)) = \langle \tilde{\pi}^+([a_{-1}, x_1] \otimes u_0^+), w_0^- \rangle \\ =& \langle r_0^+(x_1 \otimes u_0^+)(a_{-1}), w_0^- \rangle = \langle \tilde{\pi}^+(a_{-1} \otimes r_0^+(x_1 \otimes u_0^+)), w_0^- \rangle \end{aligned} \tag{3.70}$$

for any $a_{-1} \in V_{-1}$. Thus, if $x_1^1, \ldots, x_1^l \in V_1$ and $u_0^{+,1}, \ldots, u_0^{+,l} \in U_0^+$ satisfy $\sum_{s=1}^l r_0^+(x_1^s \otimes u_0^{+,s}) = 0$, then we have

$$\sum_{s=1}^l B_L(a_{-1}, [x_1^s, \tilde{\Phi}_0^0(u_0^{+,s} \otimes w_0^-)]) = 0 \tag{3.71}$$

for any $a_{-1} \in V_{-1}$. Since the restriction of B_L to $V_{-1} \times V_1$ is non-degenerate, we have

$$\sum_{s=1}^l [x_1^s, \tilde{\Phi}_0^0(u_0^{+,s} \otimes w_0^-)] = 0, \tag{3.72}$$

and, thus, the map $\tilde{\Phi}_0^1$ is well-defined. The equation (3.69) follows from (3.70).

For $i \geq 1$, under the notation of (3.67), we have

$$\begin{aligned} B_L(a_{-i-1}, [x_1, \tilde{\Phi}_0^i(u_i^+ \otimes w_0^-)]) &= B_L([a_{-i-1}, x_1], \tilde{\Phi}_0^i(u_i^+ \otimes w_0^-)) \\ &= \langle \tilde{\pi}^+([a_{-i-1}, x_1] \otimes u_i^+), w_0^- \rangle \\ &= \langle \tilde{\pi}^+(a_{-i-1} \otimes \tilde{\pi}^+(x_1 \otimes u_i^+)), w_0^- \rangle - \langle \tilde{\pi}^+(x_1 \otimes \tilde{\pi}^+(a_{-i-1} \otimes u_i^+)), w_0^- \rangle \\ &= \langle \tilde{\pi}^+(a_{-i-1} \otimes r_i^+(x_1 \otimes u_i^+)), w_0^- \rangle \end{aligned} \tag{3.73}$$

by the induction hypothesis for any $a_{-i-1} \in V_{-i-1}$. Thus, by the same argument to the argument of the case where $i = 0$, we have the well-definedness of $\tilde{\Phi}_0^{i+1}$ and that $\tilde{\Phi}_0^{i+1}$ satisfies the equation (3.69). Therefore, by induction, we can obtain our claim on $\tilde{\Phi}_0^{i+1}$ for all $i \geq 0$.

Let us show that the linear maps defined in (3.68) are well-defined. We assume that an integer $i \geq 0$ satisfies the condition that we have linear maps $\tilde{\Phi}^k_{-i} : U_k^+ \otimes \mathcal{U}_{-i}^- \to V_{k-i}$ for all $k \geq 0$ which satisfy the equation (3.69). When $i = 0$, it has been shown that this assumption holds. Then, we can show the well-definedness of the linear maps $\tilde{\Phi}^k_{-1}$ ($k \geq 0$) by induction on k. When $k = 0$, we can show that $\tilde{\Phi}^0_{-1}$ is well-defined and satisfies (3.69) by a similar argument to the argument of (3.67). When $k \geq 1$, we have

$$\begin{aligned} B_L(a_{-k+1}, [y_{-1}, \tilde{\Phi}_0^k(u_k^+ \otimes w_0^-)] &- \tilde{\Phi}_0^{k-1}(\tilde{\pi}^+(y_{-1} \otimes u_k^+) \otimes w_0^-)) \\ &= B_L([a_{-k+1}, y_{-1}], \tilde{\Phi}_0^k(u_k^+ \otimes w_0^-)) - B_L(a_{-k+1}, \tilde{\Phi}_0^{k-1}(\tilde{\pi}^+(y_{-1} \otimes u_k^+) \otimes w_0^-)) \\ &= \langle \tilde{\pi}^+([a_{-k+1}, y_{-1}] \otimes u_k^+), w_0^- \rangle - \langle \tilde{\pi}^+(a_{-k+1} \otimes \tilde{\pi}^+(y_{-1} \otimes u_k^+)), w_0^- \rangle \\ &= -\langle \tilde{\pi}^+(y_{-1} \otimes \tilde{\pi}^+(a_{-k+1} \otimes u_k^+)), w_0^- \rangle \\ &= \langle \tilde{\pi}^+(a_{-k+1} \otimes u_k^+), \tilde{\pi}^-(y_{-1} \otimes w_0^-) \rangle = \langle \tilde{\pi}^+(a_{-k+1} \otimes u_k^+), r_0^-(y_{-1} \otimes w_0^-) \rangle \end{aligned} \tag{3.74}$$

for any $k \geq 1$ and $a_{-k+1} \in V_{-k+1}$ under the notation of (3.68). Thus, by a similar argument to the argument of (3.67), we have the well-definedness of $\tilde{\Phi}^k_{-1}$ for all $k \geq 1$ and that $\tilde{\Phi}^k_{-1}$ satisfies the equation (3.69). For $i \geq 1$, by the same argument to the argument of the case where $i = 0$, we have the well-definedness of $\tilde{\Phi}^k_{-i-1}$ for all $k \geq 0$ and that $\tilde{\Phi}^k_{-i-1}$ satisfies the equation (3.69). Thus, by induction, we have linear maps $\tilde{\Phi}^i_{-j}$ for all $i, j \geq 0$ which satisfies the equation (3.69). This completes the proof. □

As a corollary of Propositions 3.22 and 3.24, we have the following theorem.

Theorem 3.25. *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ be standard pentads and assume that B_0 is symmetric. Then a pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is also a standard pentad whose Φ-map, denoted by $\tilde{\Phi}^+_\pi$, is defined by:*

$$\tilde{\Phi}^+_\pi(u^+_i \otimes w^-_{-j}) := \tilde{\Phi}^i_{-j}(u^+_i \otimes w^-_{-j}) \tag{3.75}$$

for any $i, j \geq 0$, $u^+_i \in U^+_i$ and $w^-_{-j} \in \mathcal{U}^-_{-j}$, where $\tilde{\Phi}^i_{-j}$ is the linear map defined in Definition 3.23.

3.5. Chain rule. Under the assumptions of sections 3.3 and 3.4, let us construct the Lie algebra associated with a standard pentad of the form $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, U^+, \mathcal{U}^-, B_L)$. To find the structure of the Lie algebra $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, U^+, \mathcal{U}^-, B_L)$, we give the following theorem.

Theorem 3.26 (chain rule)**.** *Let $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ be standard pentads. Assume that B_0 is symmetric. Then a pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is also a standard pentad and the Lie algebra associated with it is isomorphic to $L(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$, i.e. we have*

$$L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) \simeq L(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0) \tag{3.76}$$

as Lie algebras up to grading.

Proof. Note that the pentad $(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$ is a standard pentad whose Φ-map $\Phi_{\rho\oplus\pi}$ is given by:

$$\Phi_{\rho\oplus\pi}((v, u) \otimes (\phi, \psi)) = \Phi_\rho(v \otimes \phi) + \Phi_\pi(u \otimes \psi)$$

where $v \in V$, $\phi \in \mathcal{V}$, $u \in U$, $\psi \in \mathcal{U}$ and Φ_ρ and Φ_π are the Φ-maps of the pentads $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\mathfrak{g}, \pi, U, \mathcal{U}, B_0)$ respectively. It has been already shown in Theorem 3.25 that the pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is standard. We denote the n-graduations of $(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ by V_n and $(\tilde{U}^+)_n$, i.e.

$$L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0) = \bigoplus_{n\in\mathbb{Z}} V_n, \quad L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) = \bigoplus_{m\in\mathbb{Z}} (\tilde{U}^+)_m. \tag{3.77}$$

Moreover, we denote $(\tilde{U}^+)_1$ and $(\tilde{U}^+)_{-1}$ by:

$$(\tilde{U}^+)_1 = \bigoplus_{i\geq 0} U^+_i, \quad (\tilde{U}^+)_{-1} = \bigoplus_{j\geq 0} \mathcal{U}^-_{-j}. \tag{3.78}$$

Denote a bilinear form on $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ defined in Definition 2.18 by $\overline{B}_L$. By Lemmas 2.37 and 2.38, we can define derivations α and β on $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+,$

$U^+, \mathcal{U}^-, B_L)$ which satisfy

$$\alpha(v_n) = nv_n, \quad \alpha(u_i^+) = iu_i^+, \quad \alpha(w_{-j}^-) = -jw_{-j}^-, \quad \beta(\tilde{u}_m^+) = m\tilde{u}_m^+ \tag{3.79}$$

and

$$\overline{B}_L(\alpha(\overline{z}), \overline{\omega}) + \overline{B}_L(\overline{z}, \alpha(\overline{\omega})) = \overline{B}_L(\beta(\overline{z}), \overline{\omega}) + \overline{B}_L(\overline{z}, \beta(\overline{\omega})) = 0 \tag{3.80}$$

for any $n, m \in \mathbb{Z}$, $i, j \geq 0$, $v_n \in V_n$, $u_i^+ \in U_i^+ \subset (\tilde{U}^+)_1$, $w_{-j}^- \in \mathcal{U}_{-j}^- \subset (\tilde{U}^+)_{-1}$, $\tilde{u}_m^+ \in (\tilde{U}^+)_m$ and $\overline{z}, \overline{\omega} \in L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$. Since $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is generated by $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\tilde{U}^+)_{\pm 1}$ and since $L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0)$ and $(\tilde{U}^+)_{\pm 1}$ are generated by V_0, $V_{\pm 1}$, $U = U_0^+$ and $\mathcal{U} = \mathcal{U}_0^-$, we have that $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is generated by V_0, $V_{\pm 1}$, U_0^+ and $\mathcal{U}_0^-$. Put

$$W_{(n,m)} := \left\{ \overline{X} \in L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) \,\middle|\, \alpha(\overline{X}) = n\overline{X},\ \beta(\overline{X}) = m\overline{X} \right\}$$

for any $n, m \in \mathbb{Z}$. Then we can easily show that all eigenvalues of α and β are integers by induction and that $[W_{(n,m)}, W_{(k,l)}] \subset W_{(n+k,m+l)}$. Thus, we can obtain the following $\mathbb{Z}$-grading of $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ induced by the eigenspace decomposition of $\gamma := \alpha + \beta$:

$$L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) = \bigoplus_{k \in \mathbb{Z}} \Big(\bigoplus_{n+m=k} W_{(n,m)} \Big).$$

If we put $W_k^\gamma := \left\{ \overline{X} \mid \gamma(\overline{X}) = k\overline{X} \right\}$, then we have $W_k^\gamma = \bigoplus_{n+m=k} W_{(n,m)}$ and, thus, we can obtain the following $\mathbb{Z}$-grading of $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$:

$$L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) = \bigoplus_{k \in \mathbb{Z}} W_k^\gamma.$$

In particular,

$$W_0^\gamma = V_0, \quad W_1^\gamma = V_1 \oplus U_0^+, \quad W_{-1}^\gamma = V_{-1} \oplus \mathcal{U}_0^-. \tag{3.81}$$

We can easily show that $W_{k+1}^\gamma = [W_1^\gamma, W_k^\gamma]$, $W_{-k-1}^\gamma = [W_{-1}^\gamma, W_{-k}^\gamma]$ for all $k \geq 1$ and that the restriction of $\overline{B}_L$ to $W_k^\gamma \times W_{-k}^\gamma$ is non-degenerate for any $k \in \mathbb{Z}$ from (3.80). Therefore, by Theorem 2.20, we have the isomorphism (3.76). □

Example 3.27. We retain to use the notations of Examples 2.35 and 3.16. Put $\mathcal{U} := \mathbb{C}$ and define a representation $\varpi : \mathfrak{g} \otimes \mathcal{U} \to \mathcal{U}$ and a bilinear map $\langle \cdot, \cdot \rangle_U : U \times \mathcal{U} \to \mathbb{C}$ by:

$$\varpi((a, b, A) \otimes w) := -aw, \quad \langle u, w \rangle_U := uw.$$

We can identify $\mathcal{U}$ with $\mathrm{Hom}(U, \mathbb{C})$ via $\langle \cdot, \cdot \rangle_U$. Then pentads $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ and $(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$ are standard. Let us show that the Lie algebra $L(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$ is isomorphic to $\mathfrak{sl}_4$. Put elements

$$H_0 := \begin{pmatrix} \frac{5}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{-3}{4} & \\ & & & \frac{-3}{4} \end{pmatrix}, \quad H_1 := \begin{pmatrix} \frac{3}{4} & & & \\ & \frac{-1}{4} & & \\ & & \frac{-1}{4} & \\ & & & \frac{-1}{4} \end{pmatrix}, \quad H_2 := \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & \frac{-1}{2} & \\ & & & \frac{-1}{2} \end{pmatrix} \in \mathfrak{sl}_4.$$

Then we can obtain a $\mathbb{Z}$-grading of $\mathfrak{sl}_4$ by the eigenspace decomposition of $\mathrm{ad}\, H_0$:

$$\mathfrak{sl}_4 = \bigoplus_{i=-2}^{2} \mathfrak{l}_i \quad (\mathfrak{l}_i := \{X \in \mathfrak{sl}_4 \mid [H_0, X] = iX\}). \tag{3.82}$$

In particular,

$$\mathfrak{l}_0 = \left\{ \left. \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{A}} \\ 0 & 0 & & \end{pmatrix} \right| a, b \in \mathbb{C}, A \in \mathfrak{gl}_2, a + b + \mathrm{Tr}(A) = 0 \right\} \simeq \mathfrak{gl}_1 \oplus \mathfrak{gl}_1 \oplus \mathfrak{sl}_2,$$

$$\mathfrak{l}_1 = \left\{ \left. \begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & v_1 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| u, v_1, v_2 \in \mathbb{C} \right\}, \quad \mathfrak{l}_{-1} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ \psi & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 & 0 \end{pmatrix} \right| \psi, \phi_1, \phi_2 \in \mathbb{C} \right\}.$$

Then, we have that $\mathfrak{l}_0 \simeq \mathbb{C}H_1 \oplus \mathbb{C}H_2 \oplus \mathfrak{sl}_2$ and that the restriction of a bilinear form T, defined by $T(X, X') := \mathrm{Tr}(XX')$ $(X, X' \in \mathfrak{sl}_4)$, to $\mathfrak{l}_0 \times \mathfrak{l}_0$ satisfies:

$$T\,|_{\mathfrak{l}_0 \times \mathfrak{l}_0} ((a, b, A), (a', b', A')) = \frac{3}{4} aa' + bb' + \frac{1}{2}(ab' + a'b) + \mathrm{Tr}(AA'),$$

where $a, a' \in \mathbb{C}H_1$, $b, b' \in \mathbb{C}H_2$, $A, A' \in \mathfrak{sl}_2$. Thus, we can easily show that the grading (3.82) and the Killing form of $\mathfrak{sl}_4$, denoted by $K_{\mathfrak{sl}_4}$, satisfy the assumptions of Theorem 2.20 and that a pentad $(\mathfrak{l}_0, \mathrm{ad}, \mathfrak{l}_1, \mathfrak{l}_{-1}, K_{\mathfrak{sl}_4}\,|_{\mathfrak{l}_0 \times \mathfrak{l}_0})$ is equivalent to $(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0)$ (cf. [4, 5, 6, the theory of prehomogeneous vector spaces of parabolic type]). Thus, by Theorems 2.20 and 3.26, we have

$$L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L) \simeq L(\mathfrak{g}, \rho \oplus \pi, V \oplus U, \mathcal{V} \oplus \mathcal{U}, B_0) \simeq \mathfrak{sl}_4.$$

In this case, we can directly check that the Lie algebra $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is isomorphic to $\mathfrak{sl}_4$ using Examples 2.34, 2.35 and 3.16. In fact, by the results of Examples 2.35 and 3.16, we have that the pentad $(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is equivalent to the pentad $(\mathfrak{gl}_1 \oplus \mathfrak{sl}_3, \Lambda_1, \mathbb{C}^3, \mathbb{C}^3, \kappa_3)$, which is defined in Example 2.34. Thus, we have that the Lie algebra $L(L(\mathfrak{g}, \rho, V, \mathcal{V}, B_0), \tilde{\pi}^+, \tilde{U}^+, \tilde{\mathcal{U}}^-, B_L)$ is isomorphic to $\mathfrak{sl}_4$.

References

[1] N. Bourbaki: Lie groups and Lie algebra. Springer, Berlin, 1989.

[2] V.G. Kac: *Simple irreducible graded Lie algebras of finite growth*, Math. USSR-Izvestija vol.2 (1968), 1271–1311.

[3] V.G. Kac: Infinite dimensional Lie algebras, third edition, Cambridge University Press, Cambridge, 1990.

[4] H. Rubenthaler: *Espaces préhomogènes de type parabolique*, Lect. Math. Kyoto Univ. **14** (1982), 189–221.

[5] H. Rubenthaler: *Espaces préhomogènes de type parabolique*, Thèse d'Etat, Université de Strasbourg, 1982.

[6] H. Rubenthaler: Algèbres de Lie et espaces préhomogènes (Travaux en Cours), Hermann, Paris, 1992.

[7] H. Rubenthaler: *Graded Lie algebras associated to a representation of a quadratic algebra*, arXiv: 1410.0031v2 (2014).

[8] N. Sasano: *Lie algebras generated by Lie modules*, Kyushu J. Math. **68** (2014), 377–403.

[9] G. Shen: *Graded modules of graded Lie algebras of Cartan type (II)-positive and negative graded modules*, Sci. Senia Ser. A **29** (1986), 1009–1019.

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