Takeda, M., Tawara, Y. and Tsuchida, K.
Osaka J. Math.
**54** (2017), 517–532

# COMPACTNESS OF MARKOV AND SCHRÖDINGER SEMI-GROUPS: A PROBABILISTIC APPROACH

Masayoshi TAKEDA, Yoshihiro TAWARA and Kaneharu TSUCHIDA

(Received June 10, 2016, revised July 20, 2016)

**Abstract**

It is proved if an irreducible, strong Feller symmetric Markov process possesses a tightness property, then its semi-group is an $L^2$-compact operator. In this paper, applying this fact, we prove probabilistically the compactness of Dirichlet-Laplacians and Schrödinger operators.

## 1. Introduction

Let $E$ be a locally compact separable metric space and $m$ a positive Radon measure on $E$ with full support. Let $X$ be an $m$-symmetric Markov process on $E$. We assume that $X$ is irreducible and has strong (resolvent) Feller property. Moreover, we assume that $X$ possesses the *tightness property*, i.e., for any $\epsilon > 0$ there exists a compact set $K$ such that $\sup_{x\in E} R_1 1_{K^c}(x) \leq \epsilon$. Here $R_1$ is the 1-resolvent of $X$ and $1_{K^c}$ is the indicator function of the complement of $K$. When $X$ has these properties, we say in this paper that $X$ belongs to Class (T). One of the authors proved in [14] that if $X$ belongs to Class (T), its semi-group turns out to be a compact operator on $L^2(E; m)$ (Theorem 2.1). In this paper, we apply this criterion to Dirichlet Laplacians $\Delta_D$ and Schrödinger operators $\Delta - V$ with positive potential and show probabilistically the compactness of these operators.

Let $X = (\mathbb{P}_x, B_t)$ be the Brownian motion on $\mathbb{R}^d$ and $X^D$ the absorbing Brownian motion on a domain $D$. We then prove that if $D \subset \mathbb{R}^d$ satisfies $\lim_{x\in D,|x|\to\infty} m(D \cap B(x, 1)) = 0$, then $X^D$ is in Class (T) and consequently its semi-group is compact. Here $m$ denotes the Lebesgue measure and $B(x, R)$ the ball centered at $x$ with radius $R$. If $x$ is the origin 0, we write $B(R)$ for $B(0, R)$.

We denote by $\mathcal{B}_0$ the set of Borel sets $B$ such that $\lim_{|x|\to\infty} m(B \cap B(x, 1)) = 0$. In [8], a Borel set in $\mathcal{B}_0$ is said to be *thin at infinity*. Let $V$ be a positive Borel function on $\mathbb{R}^d$ locally in the Kato class. Let $X^V = (\mathbb{P}^V_x, B_t)$ be the subprocess defined by $\mathbb{P}^V_x(d\omega) = \exp\left(-\int_0^t V(B_s(\omega))ds\right)\mathbb{P}_x(d\omega)$. We show that if the set $D_M := \{x \in \mathbb{R}^d \mid V(x) \leq M\}$ belongs to $\mathcal{B}_0$ for any $M > 0$, then $X^V$ is in Class (T) and its semi-group, Schrödinger semi-group of $-\Delta + V$, is compact. This fact is proved in [11], [8] analytically, while it is done in this paper probabilistically; the key to the proof of this fact is to show that the condition on $V$ implies the tightness property of $X^V$.

---

2010 Mathematics Subject Classification. Primary 47D08, 60J25.
M. Takeda was supported in part by Grant-in-Aid for Scientific Research (No.26247008(A)) and Grant-in-Aid for Challenging Exploratory Research (No.25610018), Japan Society for the Promotion of Science.

We apply Theorem 2.1 to time changed processes. Let $X$ be an irreducible symmetric Markov process with strong Feller property. We assume, in addition, that $X$ is transient. We then see that for a Green-tight measure $\mu$ with full fine support, the time-changed process $\check{X}$ by $A_t^\mu$, the positive continuous additive functional in the Revez correspondence to $\mu$, belongs to Class (T). As a results, the space $(\check{\mathcal{F}}, \check{\mathcal{E}}_1 = (\check{\mathcal{E}} + ( \ , \ )_\mu))$ is compactly embedded in $L^2(E;\mu)$, where $(\check{\mathcal{E}}, \check{\mathcal{F}})$ is the Dirichlet form generated by $\check{X}$. Moreover, let $(\mathcal{F}_e, \mathcal{E})$ be the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ associated with $X$, which turns out to be a Hilbert space under the condition for $X$ being transient. We then see that $\mathcal{F}_e$ is identical with $\check{\mathcal{F}}$, $\mathcal{E}$ is equivalent to $\check{\mathcal{E}}_1$ and thus $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$. Therefore, we can conclude that for any Green-tight measure $\mu$, the extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$. This fact says that if $\mu$ is Green-tight with respect to 1-resolvent, then $(\mathcal{F}, \mathcal{E}_1 = (\mathcal{E} + ( \ , \ )_m))$ is compactly embedded in $L^2(E;\mu)$.

Applying this result to the Brownian motion, we see that if $B \in \mathcal{B}(\mathbb{R}^d)$ satisfies that the measure $1_B dx$ is Green-tight, then 1-order Sobolev space $H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d; 1_B dx)$. We see from [4, Lemma 6.11] that for a domain $B$, this is also necessary.

## 2. Preliminaries

Let $E$ be a locally compact separable metric space, $E_\Delta = E \cup \{\Delta\}$ the one point compactification of $E$, and $m$ a positive Radon measure on $E$ with full support. Let $X = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, X_t, \mathbb{P}_x, \zeta)$ be an $m$-symmetric Borel right process having left limits on $(0,\zeta)$. Here $\zeta$ is the lifetime $\zeta(\omega) = \inf\{s \geq 0 \mid X_s(\omega) = \Delta\}$ and $\Omega$ is specifically taken to be the space of all right continuous functions from $[0,\infty]$ into $E_\Delta$ with $\omega(t) = \Delta$ for any $t \geq \zeta(\omega) = \inf\{s \geq 0 \mid w(s) = \Delta\}$ and $\omega(\infty) = \Delta$. The random variable $\zeta$ is called the lifetime which can be finite and $X_t$ is defined by $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$, $t \geq 0$. $\{\mathcal{F}_t\}_{t\geq 0}$ is the minimal (augmented) admissible filtration.

Let $\{p_t\}_{t\geq 0}$ be the semi-group of $X$, $p_t f(x) = \mathbb{E}_x(f(X_t))$. By Lemma 1.4.3 in [5], $\{p_t\}_{t\geq 0}$ uniquely determines a strongly continuous Markovian semi-group $\{T_t\}_{t\geq 0}$ on $L^2(E;m)$. We define the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(E;m)$ generated by $X$:

$$(2.1)\qquad \begin{cases} \mathcal{F} = \left\{ u \in L^2(E;m) \mid \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, u)_m < \infty \right\} \\ \mathcal{E}(u,v) = \lim_{t\to 0} \dfrac{1}{t}(u - T_t u, v)_m \ \text{ for } u, v \in \mathcal{F}. \end{cases}$$

We denote by $\mathcal{F}_e$ the family of $m$-measurable functions $u$ on $X$ such that $|u| < \infty$ $m$-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of functions in $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ $m$-a.e. We call $\mathcal{F}_e$ the *extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$. Every $u \in \mathcal{F}_e$ has a quasi-continuous version $\widetilde{u}$ ([5, Theorem 2.1.3]). In the sequel, we always assume that every function $u \in \mathcal{F}_e$ is represented by its quasi-continuous version.

Let us denote by $\{R_\alpha\}_{\alpha>0}$ the resolvent of $X$,

$$R_\alpha f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t} f(X_t) dt\right), \quad f \in \mathcal{B}_b(E),$$

where $\mathcal{B}_b(E)$ is the space of bounded Borel functions on $E$. We now make three assumptions on $X$:

I. (**Irreducibility**) If a Borel set $A$ is $p_t$-invariant, i.e., $\int_A p_t 1_{A^c} dm = 0$ for any $t > 0$, then $A$ satisfies either $m(A) = 0$ or $m(A^c) = 0$. Here $1_{A^c}$ is the indicator function of the complement of $A$.

II. (**Resolvent Strong Feller Property**) $R_\alpha(\mathcal{B}_b(E)) \subset C_b(E)$, $\alpha > 0$, where $C_b(E)$ is the space of bounded continuous functions.

III. (**Tightness Property**) For any $\epsilon > 0$, there exists a compact set $K$ such that $\sup_{x\in E} R_1 1_{K^c}(x) \leq \epsilon$. Here $1_{K^c}$ is the indicator function of the complement of $K$.

We here say that a Markov process belongs to **Class (T)** if it possess the properties I, II, III.

REMARK 2.1. (i) If $R_1 1 \in C_\infty(E)$, then X is explosive and satisfies the assumption III. In fact, it follows from the maximum property that

$$\sup_{x\in E} R_1 1_{K^c}(x) = \sup_{x\in K^c} R_1 1_{K^c}(x) \leq \sup_{x\in K^c} R_1 1(x).$$

Here $C_\infty(E)$ is the set of continuous functions vanishing at infinity.

(ii) If $C_\infty(E)$ is invariant under $R_1$, $R_1(C_\infty(E)) \subset C_\infty(E)$, then $R_1 1 \in C_\infty(E)$ is equivalent to III. In fact, assume III. For a compact set $K$, take a positive function $g \in C_\infty(E)$ such that $1_K \leq g$. We then see from the invariance of $C_\infty(E)$ that $0 \leq \lim_{x\to\infty} R_1 1_K(x) \leq \lim_{x\to\infty} R_1 g(x) = 0$. Hence for any $\epsilon > 0$ there exists a compact set $K$ such that

$$\limsup_{x\to\infty} R_1 1(x) \leq \limsup_{x\to\infty} R_1 1_K(x) + \limsup_{x\to\infty} R_1 1_{K^c}(x) \leq \sup_{x\in E} R_1 1_{K^c}(x) \leq \epsilon,$$

which implies $R_1 1 \in C_\infty(E)$. Hence, if $C_\infty(E)$ is invariant under $R_1$ and $X$ is conservative, $R_1 1 = 1$, then $X$ does not have the tightness property, in particular, the Ornstein-Uhlenbeck process does not.

(iii) Assume that $m$ is finite, $m(E) < \infty$ and that $\{p_t\}_{t\geq 0}$ is ultra-contractive, $\|p_t\|_{1,\infty} = c_t < \infty$ for any $t > 0$. Here $\|\cdot\|_{1,\infty}$ is the operator norm from $L^1(E;m)$ to $L^\infty(E;m)$. Note that $c_t$ is non-increasing because $\|p_t\|_{1,\infty} \leq \|p_s\|_{1,\infty} \cdot \|p_{t-s}\|_{\infty,\infty} \leq \|p_s\|_{1,\infty}$ for $0 < s < t$. We then see that $X$ has the tightness property III. Indeed, for any $\delta > 0$ and a compact set $K \subset \mathbb{R}^d$

$$R_1 1_{K^c}(x) \leq \int_0^\delta e^{-t} p_t 1_{K^c}(x)dt + \int_\delta^\infty e^{-t} p_t 1_{K^c}(x)dt \leq (1 - e^{-\delta}) + c_\delta \cdot m(K^c).$$

Hence for any $\varepsilon > 0$ $\|R_1 1_{K^c}\|_\infty < \varepsilon$, if $\delta > 0$ and a compact set $K$ satisfy $1-\exp(-\delta) < \varepsilon/2$ and $c_\delta \cdot m(K^c) < \varepsilon/2$.

It follows from the assumption II that the resolvent kernel is absolutely continuous with respect to $m$,

$$R_\beta(x, dy) = R_\beta(x, y)m(dy), \quad \text{for each } \alpha > 0,\ x \in E.$$

As a result, the transition probability $p_t(x, dy)$ is also absolutely continuous with respect to $m$,

$$p_t(x, dy) = p_t(x, y)m(dy) \quad \text{for each } t > 0,\ x \in E$$

([5, Theorem 4.2.4]). By [5, Lemma 4.2.4] the density $R_\beta(x, y)$ is assumed to be a non-negative Borel function such that $R_\beta(x, y)$ is symmetric and $\beta$-excessive in $x$ and in $y$. Under the absolute continuity condition, "quasi-everywhere" statements are strengthened to "everywhere" ones.

One of the authors proved the next theorem ([14, Theorem 4.1]).

**Theorem 2.1** ([14])**.** *If a Markov process $X$ is in Class $(T)$, then its semi-group $p_t$ is compact on $L^2(E; m)$.*

We denote by $S_{00}$ the set of positive Borel measures $\mu$ such that $\mu(E) < \infty$ and $R_1\mu(x)(= \int_E R_1(x, y)\mu(dy))$ is uniformly bounded in $x \in E$. A positive Borel measure $\mu$ on $E$ is said to be *smooth* if there exists a sequence $\{E_n\}_{n=1}^{\infty}$ of Borel sets increasing to $E$ such that $1_{E_n} \cdot \mu \in S_{00}$ for each $n$ and

$$\mathbb{P}_x\left(\lim_{n\to\infty} \sigma_{E\setminus E_n} \geq \zeta\right) = 1, \quad \forall x \in E,$$

where $\sigma_{E\setminus E_n}$ is the first hitting time of $E \setminus E_n$. The totality of smooth measures is denoted by $S_1$.

If an additive functional $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to $t$ for each $\omega \in \Omega$, it is said to be a *positive continuous additive functional* (PCAF in abbreviation). By [5, Theorem 5.1.7][1], there exists a one-to-one correspondence between positive smooth measures and PCAF's (**Revuz correspondence**): for each smooth measure $\mu$, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any positive Borel function $f$ on $E$ and $\gamma$-excessive function $h$ $(\gamma \geq 0)$, that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_{h\cdot m}\left(\int_0^t f(X_s)dA_s\right) = \int_E f(x)h(x)\mu(dx). \tag{2.2}$$

Here $\mathbb{E}_{h\cdot m}(\,\cdot\,) = \int_E \mathbb{E}_x(\,\cdot\,)h(x)m(dx)$. We denote by $A_t^{\mu}$ the PCAF corresponding to the smooth measure $\mu$.

We now introduce two classes of positive smooth measures which play a crucial role.

Definition 2.1. (i) A positive measure $\mu \in S_1$ is said to be in the *Kato class* (in notation, $\mu \in \mathcal{K}$), if

$$\lim_{\beta\to\infty} \sup_{x\in E} \int_E R_\beta(x, y)d\mu(y) = 0.$$

A positive measure $\mu \in S_1$

(ii) Suppose $X$ is transient. A measure $\mu \in \mathcal{K}$ is said to be *Green-tight* (in notation, $\mu \in \mathcal{K}_\infty(R)$), if for any $\epsilon > 0$ there exists a compact set $K$ such that

$$\sup_{x\in E} \int_{K^c} R(x, y)d\mu(y) \leq \epsilon.$$

If the measure $\mu(dx) = V(x)m(dx)$ is in $\mathcal{K}$ (resp. $\mathcal{K}_\infty$), we also denote $V \in \mathcal{K}$ (resp. $\mathcal{K}_\infty$).

Note that if $X$ is transient, then $(\mathcal{F}_e, \mathcal{E})$ is a Hilbert space. The next theorem is proved by Stollmann-Voigt [13].

[1] In [5], the measure $\mu$ (resp. PCAF $A_t$) is said to be a *smooth measure in the strict sense* (resp. a *PCAF in the strict sense*). We treat only smooth measures in the strict sense and PCAF's in the strict sense, and omit the term "in the strict sense".

**Theorem 2.2.** *For $\mu \in \mathcal{K}_\infty(R)$*

$$\int_E u^2(x)\mu(dx) \le \|R\mu\|_\infty \cdot \mathcal{E}(u,u), \quad u \in \mathcal{F}_e. \tag{2.3}$$

*Here,* $R\mu(x) = \int_E R(x,y)d\mu(y)$.

Note that $\|R\mu\|_\infty$ is finite by [2, Proposition 2.2]. Let $\check{X} = (\check{\mathbb{P}}, \check{X}_t)$ be the time-changed process, that is, $\check{\mathbb{P}}_x = \mathbb{P}_x$, $\check{X}_t = X_{\tau_t}$, $\tau_t = \inf\{s > 0 : A_s^\mu > t\}$. Define

$$F = \{x \in X : \mathbb{P}_x(\tau_0 = 0) = 1\}.$$

We call $F$ the *fine support* of $\mu$. Note that the 0-resolvent $\check{R}$ of $\check{X}$ is written as

$$\check{R}f(x) = \int_F R(x,y)f(x)d\mu(x), \quad f \in L^2(F;\mu),$$

We then see from (2.3) that for $\mu \in \mathcal{K}_\infty(R)$, $(\mathcal{F}_e, \mathcal{E})$ is continuously embedded in $L^2(E;\mu)$ and so $\check{R}$ is a bounded operator on $L^2(F;\mu)$.

**Theorem 2.3** ([14])**.** *Assume that a Markov process X satisfies I and II. If X is transient, then for $\mu \in \mathcal{K}_\infty(R)$, $(\mathcal{F}_e, \mathcal{E})$ is compactly embedded in $L^2(E;\mu)$.*

Theorem 2.3 is an extension of Theorem 2.1. Indeed, $(\mathcal{F}, \mathcal{E}_1)$ is a transient regular Dirichlet space and its extended Dirichlet space equals $(\mathcal{F}, \mathcal{E}_1)$. Notice that the 1-resolvent $R_1$ is identical with the 0-resolvent of $(\mathcal{F}, \mathcal{E}_1)$. We then see from Theorem 2.3 that if $\mu$ is Green-tight with respect to the 1-resolvent $R_1$ (in notation, $\mu \in \mathcal{K}_\infty(R_1)$), then $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;\mu)$. It is known in [2, Theorem 4.2] that if $X$ is in Class (T), then $m$ belongs to $\mathcal{K}_\infty(R_1)$. We then obtain Theorem 2.1 because the semi-group $p_t$ is compact if and only if $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;m)$.

**Corollary 2.1.** *Assume that a Markov process X satisfies I and II. If $\mu$ is a smooth measure in $\mathcal{K}_\infty(R_1)$, then $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;\mu)$. In particular, if X is in Class (T), $(\mathcal{F}, \mathcal{E}_1)$ is compactly embedded in $L^2(E;m)$.*

Theorem 2.3 and Corollary 2.1 tell us that the 0-resolvent and 1-resolvent of $\check{X}$ define compact operators on $L^2(F;\mu)$ respectively.

## 3. Dirichlet Laplacian

In this section, we deal with the Brownian motion $X = (\mathbb{P}_x, B_t)$ on $\mathbb{R}^d$ and give, as an application of Theorem 2.1, a sufficient condition for the compactness of semi-groups of Dirichlet-Laplacians.

**Lemma 3.1.** *Let $p_t$ be the semi-group of the Brownian motion. Then*

$$\|p_t\|_{p,\infty} \le \frac{C}{t^{d/(2p)}}, \quad p \ge 1,$$

*where $\|\cdot\|_{p,\infty}$ is the operator norm from $L^p(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$.*

Proof. Note that for $f \in L^p(\mathbb{R}^d)$,

$$|p_t f(x)| \le (p_t|f|^p(x))^{1/p}.$$

Hence we have, on account of $\|p_t\|_{1,\infty} \le C/t^{d/2}$

$$\|p_t f\|_\infty \le \|p_t|f|^p\|_\infty^{1/p} \le \|p_t\|_{1,\infty}^{1/p} \cdot \||f|^p\|_1^{1/p} = \|p_t\|_{1,\infty}^{1/p} \cdot \|f\|_p \le \frac{C}{t^{d/(2p)}} \cdot \|f\|_p.$$

□

Let $\mathcal{D}$ the set of domains in $\mathbb{R}^d$. We set

$$\mathcal{D}_0 = \left\{ D \in \mathcal{D} \mid \lim_{x \in D, |x| \to \infty} m(D \cap B(x,1)) = 0 \right\},$$

where $m$ denotes the Lebesgue measure on $\mathbb{R}^d$.

Denote by $\tau_B$ be the first exit time from a Borel set $B$, $\tau_B = \inf\{t > 0 \mid B_t \notin B\}$.

**Lemma 3.2.** *If a domain $D$ belongs to $\mathcal{D}_0$, then* $\lim_{x \in D, |x| \to \infty} p_t^D 1(x) = 0$ *for any* $t > 0$.

Proof. Note that for $t > 0$

$$\int_0^t 1_{D \cap B(x,1)^c}(B_s) ds \le \int_0^t 1_{B(x,1)^c}(B_s) ds \le (t - \tau_{B(x,1)})^+$$

($a^+ = a \vee 0$) and that

$$\{\tau_D > t\} \subset \left\{ \int_0^t 1_D(B_s) ds = t \right\}.$$

We then have for any $0 < \varepsilon < t$

$$\begin{aligned} \mathbb{P}_x(\tau_D > t) &\le \mathbb{P}_x\left( \int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) + \mathbb{P}_x\left( \int_0^t 1_{D \cap B(x,1)^c}(B_s) ds \ge t - \varepsilon \right) \\ &\le \mathbb{P}_x\left( \int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) + \mathbb{P}_x\left( (t - \tau_{B(x,1)})^+ \ge t - \varepsilon \right). \end{aligned}$$

By Lemma 3.1

$$\mathbb{E}_x\left( \int_0^t 1_{D \cap B(x,1)}(B_s) ds \right) \le \int_0^t \frac{C}{s^{d/(2p)}} ds \cdot m(D \cap B(x,1))^{1/p},$$

and $\int_0^t 1/s^{d/(2p)} ds < \infty$ for $p > d/2$. Hence the right-hand side above tends to 0 as $|x| \to \infty$ in $D$ by the assumption on $D$, and thus

$$\lim_{x \in D, |x| \to \infty} \mathbb{P}_x\left( \int_0^t 1_{D \cap B(x,1)}(B_s) ds \ge \varepsilon \right) = 0.$$

Noting that $\mathbb{P}_x\left( (t - \tau_{B(x,1)})^+ \ge t - \varepsilon \right) = \mathbb{P}_0(\tau_{B(1)} \le \varepsilon)$, we have

$$\limsup_{x \in D, |x| \to \infty} p_t^D 1(x) = \limsup_{x \in D, |x| \to \infty} \mathbb{P}_x(\tau_D > t) \le \mathbb{P}_0(\tau_{B(1)} \le \varepsilon) \to 0$$

as $\varepsilon \to 0$. □

From Lemma 3.2, we immediately obtain the next corollary.

**Corollary 3.1.** *If a domain $D$ belongs to $\mathcal{D}_0$, then* $\lim_{x \in D, |x| \to \infty} R_1^D 1(x) = 0$.

**Lemma 3.3.** *If a domain $D$ belongs to $\mathcal{D}_0$, then the absorbing BM on $D$ is in Class $(T)$.*

Proof. The irreducibility I and the resolvent strong Feller property II follow from [5, Exercise 4.6.3] and [3, Theorem 2.2] respectively.

Note that for a compact subset $K$ of $D$

$$R_1^D 1_{K^c} = R_1^D 1_{B(R)^c \cap K^c} + R_1^D 1_{B(R) \cap K^c} \leq R_1^D 1_{B(R)^c \cap K^c} + R_1 1_{D \cap B(R) \cap K^c}.$$

We see that by the maximum principle

$$\sup_{x \in D} R_1^D 1_{B(R)^c \cap K^c}(x) = \sup_{x \in D \cap B(R)^c} R_1^D 1_{B(R)^c \cap K^c}(x) \leq \sup_{x \in D \cap B(R)^c} R_1^D 1(x)$$

and that by Corollary 3.1 the right-hand side above tends to 0 as $R \to \infty$. Hence, for any $\epsilon > 0$ there exists $R > 0$ such that $\sup_{x \in D} R_1^D 1_{B(R)^c \cap K^c}(x) \leq \epsilon/2$ for any compact subset $K \subset D$.

Let $\{K_n\}_{n=1}^\infty$ be an increasing sequence of compact subsets of $D \cap B(R)$ such that $\lim_{n \to \infty} m(D \cap B(R) \cap K_n^c) = 0$. Then by using the maximum principle again

$$\lim_{n \to \infty} \sup_{x \in D} R_1 1_{D \cap B(R) \cap K_n^c}(x) = \lim_{n \to \infty} \sup_{x \in \mathbb{R}^d} R_1 1_{D \cap B(R) \cap K_n^c}(x) = 0.$$

Hence, $\sup_{x \in D} R_1 1_{D \cap B(R) \cap K_n^c}(x) \leq \epsilon/2$ for a large $n$. Therefore, the tightness property III of the absorbing BM on $D$ is proved. □

We now obtain the next corollary as an application of Theorem 2.1.

**Corollary 3.2.** *If a domain $D$ belongs to $\mathcal{D}_0$, then the semi-group of the Dirichlet Laplacian on $D$ is compact.*

## 4. Compact Embedding of the Sobolev Spaces

At the first part of this section, the 1-resolvent is associated with the $d$-dimensional Brownian motion.

We set

$$\mathcal{B}_0 = \left\{ B \in \mathcal{B}(\mathbb{R}^d) \mid \lim_{|x| \to \infty} m(B \cap B(x, 1)) = 0 \right\}.$$

Note that for $B \in \mathcal{B}_0$

$$\lim_{|x| \to \infty} m(B \cap B(x, R)) = 0, \quad \forall R > 0. \tag{4.1}$$

The 1-resolvent kernel of the $d$-dimensional Brownian motion ($d \geq 3$) has the following bound ([9, Theorem 6.23])[2]:

$$R_1(x, y) \simeq \begin{cases} \dfrac{1}{|x-y|^{d-2}}, & |x-y| \leq 1, \\ \dfrac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}}, & |x-y| \geq 1. \end{cases}$$

**Lemma 4.1.** *$B$ belongs to $\mathcal{B}_0$ if and only if $m^B(\bullet)(= m(B \cap \bullet))$ is in $\mathcal{K}_\infty(R_1)$.*

[2]For positive functions $f(z)$ and $g(z)$ on some set $Z$, we write $f \simeq g$ if there exists a positive constant $C$ such that $C^{-1} \leq f(z)/g(z) \leq C, \quad \forall z \in Z$.

Proof. Suppose $B \in \mathcal{B}_0$. For $R > l > 1$

$$B_1 = B(R)^c \cap B(x,l) \cap B, \quad B_2 = B(R)^c \cap B(x,l)^c \cap B.$$

Since $B(R)^c \cap B(x,l) = \emptyset$ for $x \in B(R-l)$, we have

$$\begin{aligned} R_1 1_{B(R)^c \cap B}(x) &\le C_1 \int_{B_1} \frac{1}{|x-y|^{d-2}} dy + C_2 \int_{B_2} \frac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}} dy \\ &\le C_1 \sup_{x \in B(R-l)^c} \int_{B(x,l) \cap B} \frac{1}{|x-y|^{d-2}} dy + C_2 \int_{B(x,l)^c} \frac{e^{-\sqrt{2}|x-y|}}{|x-y|^{(d-1)/2}} dy. \end{aligned}$$

For any $\varepsilon > 0$, the second term of the right-hand side is less than $\varepsilon/2$ for a large $l$, and the first term is less than $\varepsilon/2$ for a large $R$ because $B \in \mathcal{B}_0$. Hence $m^B$ belongs to $\mathcal{K}_\infty(R_1)$.

Suppose $m^B \in \mathcal{K}_\infty(R_1)$. Then for $x \in B(R+1)^c$

$$\begin{aligned} \int_{B(R)^c} R_1(x,y) m^B(dy) &\ge \int_{B(R)^c \cap B \cap B(x,1)} R_1(x,y) dy \ge c_1 \int_{B \cap B(x,1)} \frac{1}{|x-y|^{d-2}} dy \\ &\ge c_1 m(B \cap B(x,1)), \end{aligned}$$

and thus

$$\limsup_{R \to \infty} \sup_{|x| \ge R+1} m(B \cap B(x,1)) \le \lim_{R \to \infty} \sup_{x \in \mathbb{R}^d} \int_{B(R)^c} R_1(x,y) m^B(dy) = 0.$$

□

We obtain the next corollary from Corollary 2.1.

**Corollary 4.1.** *If $B \in \mathcal{B}_0$, then $H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(B)$.*

Corollary 4.1 is known (cf. [4, Chapter X, Lemma 6.11, Lemma 6.12]). Moreover, it is shown in [4, Lemma 6.11] that the condition for an open set $D$ being in $\mathcal{B}_0$ is a necessary and sufficient one for $H^1(\mathbb{R}^d)$ being compactly embedded in $L^2(D)$. Hence we can summarize as follows:

**Theorem 4.1.** *Let $D$ be a domain of $\mathbb{R}^d$. The following statements are equivalent.*

(i) $D \in \mathcal{B}_0$;
(ii) $m^D \in \mathcal{K}_\infty(R_1)$;
(iii) *$H^1(\mathbb{R}^d)$ is compactly embedded in $L^2(D)$.*

**4.1. Existence of Ground States.** In the sequel, let us consider the symmetric $\alpha$-stable process on $\mathbb{R}^d$, the Lévy process with generator $-(-\Delta)^{\alpha/2}$, $0 < \alpha \le 2$, and denote it by $X^\alpha = (\mathbb{P}_x, X_t)$. We suppose, in addition, the transience of $X^\alpha$, $d > \alpha$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $X^\alpha$ on $L^2(\mathbb{R}^d)$ is expressed by

$$\mathcal{E}^\alpha(u,v) = \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (u(y) - u(x))(v(y) - v(x)) \frac{A_{d,\alpha}}{|x-y|^{d+\alpha}} dx dy, \quad \mathcal{F} = H^{\alpha/2}(\mathbb{R}^d),$$

where $H^{\alpha/2}(\mathbb{R}^d)$ is the Sobolev space of order $\alpha/2$ and

$$A_{d,\alpha} = \frac{\alpha \cdot 2^{\alpha-1} \Gamma(\frac{\alpha+d}{2})}{\pi^{\frac{d}{2}} \Gamma(1 - \frac{\alpha}{2})}, \qquad \Gamma(s) := \int_0^\infty x^{s-1} e^{-x} dx.$$

The transition density of $X^\alpha$, $p(t,x,y)$, satisfies

(4.2)
$$p(t,x,y) \simeq t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}},$$

and the 1-resolvent density $R_1(x,y)$

$$R_1(x,y) \simeq \int_0^{|x-y|^\alpha} e^{-t}\frac{t}{|x-y|^{d+\alpha}}dt + \int_{|x-y|^\alpha}^\infty e^{-t}t^{-\frac{d}{\alpha}}dt.$$

The first trem of the right-hand side above equals

$$\frac{1-(1+|x-y|^\alpha)e^{-|x-y|^\alpha}}{|x-y|^{d+\alpha}} \simeq \begin{cases} \dfrac{1}{|x-y|^{d-\alpha}}, & |x-y| \le 1, \\ \dfrac{1}{|x-y|^{d+\alpha}}, & |x-y| \ge 1, \end{cases}$$

and the second term is less than

$$e^{-|x-y|^\alpha}\int_{|x-y|^\alpha}^\infty t^{-\frac{d}{\alpha}}dt = \frac{\alpha}{d-\alpha}\frac{e^{-|x-y|^\alpha}}{|x-y|^{d-\alpha}}.$$

We then see that

(4.3)
$$R_1(x,y) \simeq \begin{cases} \dfrac{1}{|x-y|^{d-\alpha}}, & |x-y| \le 1, \\ \dfrac{1}{|x-y|^{d+\alpha}}, & |x-y| \ge 1. \end{cases}$$

For $V \in \mathcal{B}_+(\mathbb{R}^d)$ let

$$M(r) = \operatorname*{ess\,sup}_{x\in B(r)^c} V(x)$$

and set

$$\mathcal{V} = \left\{ V \in \mathcal{B}_+(\mathbb{R}^d) \mid \lim_{|x|\to\infty} \|V1_{B(x,1)}\|_1 = 0,\ \exists r_0 > 0 \text{ s.t. } M(r_0) < \infty \right\}.$$

**Corollary 4.2.** *If $V$ is in $\mathcal{V} \cap \mathcal{K}$, then $V$ belongs to $\mathcal{K}_\infty(R_1)$. In particular, if $B \in \mathcal{B}(\mathbb{R}^d)$ is in $\mathcal{B}_0$, then $H^\beta(\mathbb{R}^d)$, $0 < \beta \le 1$ is compactly embedded in $L^2(B)$.*

Let

$$V_\gamma(x) = \frac{1}{|x|^\gamma} \wedge 1, \quad \gamma > 0.$$

**Lemma 4.2.** *Let $R$ be the Green function of the transient symmetric $\alpha$-stable process, $R(x,y) \simeq 1/|x-y|^{d-\alpha}$. Then $V_\gamma$ belongs to $\mathcal{K}_\infty(R)$ if and only if $\gamma > \alpha$.*

Proof. If $\gamma > \alpha$, then $V_\gamma \in \mathcal{K}_\infty(R)$. Indeed, take $p$ so that $d/\alpha > p > d/\gamma$ and let $q = p/(p-1)$. For $R > 1 > \varepsilon > 0$

$$\begin{aligned}\int_{\{|y|\ge R\}\cap\{|y-x|\ge\varepsilon\}} \frac{1}{|x-y|^{d-\alpha}}\frac{1}{|y|^\gamma}dy &\le \left(\int_{\{|y-x|\ge\varepsilon\}} \frac{1}{|x-y|^{(d-\alpha)q}}\right)^{1/q}\left(\int_{\{|y|\ge R\}} \frac{1}{|y|^{\gamma p}}dy\right)^{1/p} \\ &= \omega_1\left(\int_\varepsilon^\infty \frac{1}{r^{(d-\alpha)q-d+1}}dr\right)^{1/q}\left(\int_R^\infty \frac{1}{r^{\gamma p-d+1}}dr\right)^{1/p}.\end{aligned}$$

Since $(d-\alpha)q-d+1=(d-\alpha p)/(p-1)+1>1$ and $\gamma p-d+1>1$, the right hand side tends to 0 as $R\to\infty$. Hence

$$\limsup_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy\leq\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}\cap\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy.$$

Since

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}\cap\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy\leq\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^d}\int_{\{|y-x|\leq\varepsilon\}}\frac{1}{|x-y|^{d-\alpha}}dy=0,$$

we have

$$\lim_{R\to\infty}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy=0.$$

For $\gamma\leq\alpha$

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}\int_{\{|y|\geq R\}}\frac{1}{|x-y|^{d-\alpha}}V_\gamma(y)dy&\geq\int_{\{|y|\geq R\}}\frac{1}{|y|^{d-\alpha}}\frac{1}{|y|^\gamma}dy\\&=\omega_1\int_R^\infty\frac{1}{r^{\gamma-\alpha+1}}dr=\infty,\end{aligned}$$

and thus $V_\gamma\notin\mathcal{K}_\infty(R)$. □

For any $\gamma>0$, $V_\gamma$ belongs to $\mathcal{K}\cap\mathcal{V}$ and so to $\mathcal{K}_\infty(R_1)$ by Corollary 4.2. The lemma above tells us that $\mathcal{K}_\infty(R)$ is strictly included in $\mathcal{K}_\infty(R_1)$.

We see that for $\alpha<\gamma\leq d$, $V_\gamma$ is in $\mathcal{K}_\infty(R)$ with $\int_{\mathbb{R}^d}V_\gamma(x)dx=\infty$. Combining Theorem 2.3 with Lemma 4.2, we see that if $\gamma>\alpha$, then the extended Dirichlet space $H_e^{\alpha/2}(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d;V_\gamma dx)$. However, we see that the embedding is not compact if $\gamma=\alpha$. Indeed, we see from Hardy's inequality,

$$\int_{\mathbb{R}^d}u^2(x)\frac{1}{|x|^\alpha}dx\leq C\mathcal{E}^\alpha(u,u)$$

that $H_e^{\alpha/2}(\mathbb{R}^d)$ is continuously embedded in $L^2(\mathbb{R}^d;V_\alpha dx)$. In other words, the 0-order resolvent operator $\check{R}$ of the time-changed process by $\int_0^t V_\alpha(X_s)ds$,

$$\check{R}^\alpha f(x)=R(V_\alpha f)(x)$$

is a bounded operator on $L^2(\mathbb{R}^d;V_\alpha dx)$ and so is

$$T^\alpha f(x):=\int_{\mathbb{R}^d}K^\alpha(x,y)f(y)dy,\quad K^\alpha(x,y)=\frac{\sqrt{V_\alpha(x)V_\alpha(y)}}{|x-y|^{d-\alpha}}$$

on $L^2(\mathbb{R}^d)$ because of the unitary equivalence between $\check{R}^\alpha$ and $T^\alpha$. Moreover, the compact embedding of $H_e^{\alpha/2}(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d;V_\alpha dx)$ is equivalent to the compactness of the operator $T^\alpha$ on $L^2(\mathbb{R}^d)$. The kernel $K^\alpha$ is called the *Birman-Schwinger Kernel* (cf. [12, Section 7.9]). Note that the time changed operator $\check{R}$ can be defined for a smooth measure $\mu$ by $R^\alpha(f\mu)$; however, $T^\alpha$ cannot be defined because the root of measure $\mu$ has no meaning.

Let $\varphi_0=1_{B(2)\setminus B(1)}$ and define

$$\varphi_n(x)=2^{-\frac{d(d-\alpha)}{2}n}\varphi_0(2^{-(d-\alpha)n}x).$$

Then we can check that $||\varphi_n||_2 = ||\varphi_0||_2$, $\varphi_n$ converges $L^2$-weakly to 0, and

$$(\varphi_n, T^\alpha \varphi_n) = \iint_{1\le|x|\le 2, 1\le|y|\le 2} \frac{1}{|x|^{\alpha/2}|x-y|^{d-\alpha}|y|^{\alpha/2}} dxdy.$$

If $T^\alpha$ is compact, then $T^\alpha\varphi_n$ converges $L^2$-strongly to 0 and $(\varphi_n, T^\alpha\varphi_n)$ converges to 0 as $n \to \infty$, which is contradictory. Hence, we have the next proposition.

**Proposition 4.1.** *Suppose $d > \alpha$. The extended Dirichlet space $H_e^{\alpha/2}(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d; V_\gamma dx)$ if and only if $\gamma > \alpha$.*

Using Corollary 2.1, we show existence of ground states of Schrödinger operators. There exists a decreasing function $g$ on $[0, \infty)$ and $R_1(x, y)$ is written as

$$R_1(x, y) = g(|x - y|).$$

and for $V \in \mathcal{K} \cap L^1(\mathbb{R}^d)$

$$\begin{aligned} \int_{\mathbb{R}^d} R_1(x, y)V(y)dy &= \int_{|x-y|\le\varepsilon} g(|x - y|)V(y)dy + \int_{|x-y|>\varepsilon} g(|x - y|)V(y)dy \\ &\le k(\varepsilon) + g(\varepsilon)||V||_1, \end{aligned} \tag{4.4}$$

where

$$k(\varepsilon) = \sup_{x\in\mathbb{R}^d} \int_{|x-y|\le\varepsilon} g(|x - y|)V(y)dy.$$

It is known in [1] that

$$V \in \mathcal{K} \iff \lim_{\varepsilon\downarrow 0} k(\varepsilon) = 0. \tag{4.5}$$

Lemma 4.1 can be extended as follows:

**Proposition 4.2.** *If $V$ is in $\mathcal{V} \cap \mathcal{K}$, then $V$ belongs to $\mathcal{K}_\infty(R_1)$.*

Proof. For $R > l > r_0$,

$$\begin{aligned} \int_{B(R)^c} R_1(x, y)V(y)dy &= \int_{B(R)^c\cap B(x,l)^c} g(|x - y|)V(y)dy + \int_{B(R)^c\cap B(x,l)} g(|x - y|)V(y)dy \\ &\le M(r_0)\omega_1 \int_l^\infty g(r)r^{d-1}dr + \int_{B(R)^c} g(|x - y|)(V1_{B(x,l)})(y)dy, \end{aligned}$$

where $\omega_1$ is the surface area of the unit sphere. By (4.4) the second term of the right-hand side is less than

$$\sup_{x\in B(R-l)^c} \int_{\mathbb{R}^d} g(|x - y|)(V1_{B(x,l)})(y)dy \le \sup_{x\in B(R-l)^c} (k(\varepsilon) + g(\varepsilon)||V1_{B(x,l)}||_1).$$

By the assumption $V \in \mathcal{V}$,

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(R)^c} R_1(x, y)V(y)dy \le M(r_0)\omega_1 \int_l^\infty g(r)r^{d-1}dr + k(\varepsilon)$$

and by (4.5) the second term of the right-hand side tends to 0 as $\varepsilon \downarrow 0$. Letting $l \uparrow \infty$ leads us to $V \in \mathcal{K}_\infty(R_1)$. □

Note that the equivalence (4.5) is valid for $X^\alpha$(cf. [7]). Then the estimate (4.3) of $R_1$ leads us to Proposition 4.2 for $X^\alpha$ by the same argument.

**Proposition 4.3.** *If $V \in \mathcal{K}$ satisfies $V1_{\{V\geq\varepsilon\}} \in L^1(\mathbb{R}^d)$ for any $\varepsilon > 0$, then $V \in \mathcal{K}_\infty(R_1)$.*

Proof. Since the 1-resolvent kernel $R_1(x,y)$ can be written as $g(|x-y|)$,

$$\int_{B(R)^c} R_1(x,y)V(y)dy = \int_{B(R)^c\cap\{V\geq\varepsilon\}} g(|x-y|)V(y)dy + \int_{B(R)^c\cap\{V<\varepsilon\}} g(|x-y|)V(y)dy$$
$$\leq \int_{B(R)^c\cap\{V\geq\varepsilon\}} g(|x-y|)V(y)dy + \varepsilon\omega_1 \int_0^\infty g(r)r^{d-1}dr.$$

Noting $\mathcal{K}\cap L^1(\mathbb{R}^d) \subset \mathcal{K}_\infty(R)$ by [16, Proposition 1], we have

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \int_{B(R)^c} R_1(x,y)V(y)dy \leq \varepsilon\omega_1 \int_0^\infty g(r)r^{d-1}dr \longrightarrow 0, \quad \varepsilon \downarrow 0.$$

□

For $V = V^+ - V^- \in \mathcal{K}_{loc} - \mathcal{K}$ we define

$$\mathcal{E}^V(u,u) = \frac{1}{2}\mathbb{D}(u,u) + \int_{\mathbb{R}^d} u^2 V dx, \quad u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx),$$

where $\mathbb{D}$ denotes the Dirichlet integral.

**Corollary 4.3.** *Let $V = V^+ - V^- \in \mathcal{K}_{loc} - \mathcal{K}\cap\mathcal{V}$. If*

$$\lambda_0 := \inf\left\{\mathcal{E}^V(u,u) \mid u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d : V^+dx), \int_{\mathbb{R}^d} u^2dx = 1\right\} < 0, \tag{4.6}$$

*then a minimizer for $\lambda_0$ exists.*

Proof. Let $\gamma_0$ be the positive constant such that

$$\inf\left\{\mathcal{E}^{V^+}(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2V^-dx = 1,\ u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx)\right\} = 1. \tag{4.7}$$

$V^-$ belongs to $\mathcal{K}_\infty(R_1) \subset \mathcal{K}_\infty(R_1^{V^+})$ by Proposition 4.2 and a minimizer, $\varphi_0$, in (4.7) exists by Corollary 2.1. Put $\phi_0 = \varphi_0/\|\varphi_0\|_2$. Then $\|\phi_0\|_2 = 1$, $\mathcal{E}^V(\phi_0,\phi_0) + \gamma_0(\phi_0,\phi_0)_m = 0$ and thus

$$\inf\left\{\mathcal{E}^V(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2dx = 1,\ u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx)\right\} \leq 0. \tag{4.8}$$

We see from the same argument as in [15, Lemma 2.2] that

$$\inf\left\{\mathcal{E}^{V^+}(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2V^-dx = 1,\ u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx)\right\} \geq 1$$

if and only if

$$\inf\left\{\mathcal{E}^V(u,u) + \gamma_0(u,u)_m \mid \int_{\mathbb{R}^d} u^2dx = 1,\ u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx)\right\} \geq 0.$$

Hence by combing (4.7) with (4.8) we conclude that

$$\gamma_0 + \inf\left\{\mathcal{E}^V(u,u) \mid \int_{\mathbb{R}^d} u^2dx = 1,\ u \in H^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d; V^+dx)\right\} = 0,$$

$\lambda_0$ equals $-\gamma_0$ and $\varphi_0/\|\varphi_0\|_2$ is a minimizer for $\lambda_0$. □

Suppose that $V \in L^{d/2}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ on $\mathbb{R}^d$ *vanishes at infinity*, that is, it satisfies

$$m(\{x \mid |V(x)| > \varepsilon\}) < \infty \quad \text{for all } \varepsilon > 0. \tag{4.9}$$

Then it is known in [9, Theorem 11.5] that if, in addition, $V$ satisfies (4.6), then a minimizer exists. Note that $V \in \mathcal{V}$ does not satisfy (4.9) in general. Indeed, for $B \in \mathcal{B}_0$ with $m(B) = \infty$, $V := 1_B$ does not satisfy (4.9).

## 5. Schrödinger Semi-groups

Recall that $E$ is a locally compact separable metric space and $m$ is a positive Radon measure on $E$ with full support. Let $X$ be an $m$-symmetric Borel right process having left limits on $(0, \zeta)$, where $\zeta$ is the life time (see section 2). In this section, we assume that $X$ has the properties I and II. We define the Schrödinger semi-group $\{p_t^\mu\}_{t\geq 0}$ by

$$p_t^\mu f(x) = \mathbb{E}_x\left(e^{-A_t^\mu} f(X_t)\right), \quad f \in \mathcal{B}_b(E),$$

and consider the compactness of the operator $p_t^\mu$ on $L^2(E; m)$.

**Lemma 5.1.** $\lim_{x\to\infty} R_1^\mu 1(x) = 0$ *if and only if* $\lim_{x\to\infty} p_t^\mu 1(x) = 0$ *for any* $t > 0$.

Proof. The "if" part is clear. Noting

$$R_1^\mu 1(x) = \int_0^\infty e^{-s} p_s^\mu 1(x) ds \geq \int_0^t e^{-s} p_s^\mu 1(x) ds \geq te^{-t} p_t^\mu 1(x),$$

we have this lemma. □

A measure $\mu$ is said to be in $\mathcal{K}_{loc}$ if $1_G\mu$ is of Kato class for any relatively compact open set $G \subset E$.

**Theorem 5.1.** *Let* $\mu \in \mathcal{K}_{loc}$. *Assume that for any* $M > 0$ *there exists a Borel set* $D_M$ *such that*

(i) $\mu \geq M \cdot m$ *on* $D_M^c$,

(ii) *for any* $t > 0$ *and any* $\epsilon > 0$

$$\lim_{|x|\to\infty} \mathbb{P}_x\left(\int_0^t 1_{D_M}(X_s) ds > \epsilon,\ t < \zeta\right) = 0.$$

*Then* $p_t^\mu$ *is compact.*

Proof. Owing to Remark 2.1 (i) and Lemma 5.1, it is sufficient to show that $\lim_{x\to\infty} p_t^\mu 1(x) = 0$ for any $t > 0$.

Since

$$\left\{\omega \in \Omega \mid \int_0^t 1_{D_M^c}(X_s) ds \geq t - \epsilon,\ t < \zeta\right\} = \left\{\omega \in \Omega \mid \int_0^t 1_{D_M}(X_s) ds \leq \epsilon,\ t < \zeta\right\},$$

we have

$$
\begin{aligned}
p_t^\mu 1(x) &= \mathbb{E}_x\left(e^{-A_t^\mu};\int_0^t 1_{D_M}(X_s)ds > \epsilon,\ t<\zeta\right) \\
&\quad + \mathbb{E}_x\left(e^{-A_t^\mu};\int_0^t 1_{D_M^c}(X_s)ds \geq t-\epsilon,\ t<\zeta\right).
\end{aligned} \tag{5.1}
$$

It follows from the assumption (i) that if $\int_0^t 1_{D_M^c}(X_s)ds \geq t-\epsilon$, then $\int_0^t 1_{D_M^c}(X_s)dA_s^\mu \geq M(t-\epsilon)$. Hence the second term of (5.1) is less than $\exp(-M(t-\epsilon))$ and thus

$$\limsup_{|x|\to\infty} p_t^\mu 1(x) \leq e^{-M(t-\epsilon)}$$

by the assumption (ii). We have the desired claim by letting $M$ to $\infty$. □

In the sequel, let us consider the symmetric $\alpha$-stable process on $\mathbb{R}^d$, the Lévy process with generator $-(-\Delta)^{\alpha/2}$, $0<\alpha\leq 2$, and denote it by $X^\alpha = (\mathbb{P}_x, X_t)$. Let $V$ be a positive function on $\mathbb{R}^d$ in the local Kato class. Set

$$V_M = \{x\in\mathbb{R}^d \mid V(x)\leq M\}.$$

**Lemma 5.2.** *If $V_M \in \mathcal{B}_0$, then*

$$\lim_{|x|\to\infty}\mathbb{P}_x\left(\int_0^t 1_{V_M}(X_s)ds > \epsilon\right) = 0.$$

Proof. We have

$$
\begin{aligned}
\mathbb{P}_x\left(\int_0^t 1_{V_M}(X_s)ds > \epsilon\right) &= \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds + \int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds > \epsilon\right) \\
&\leq \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds > \frac{\epsilon}{2}\right) + \mathbb{P}_x\left(\int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds > \frac{\epsilon}{2}\right).
\end{aligned} \tag{5.2}
$$

Note that by the same argument as in Lemma 3.1, the semi-group $p_t$ of $X^\alpha$ satisfies $\|p_t\|_{p,\infty} \leq C/t^{d/(\alpha p)}$. We then see that for $p > d/\alpha$ the first term of the right-hand side is dominated by

$$\frac{2}{\epsilon}\mathbb{E}_x\left(\int_0^t 1_{V_M\cap B(x,R)}(X_s)ds\right) \leq C(\varepsilon,t)\cdot m(V_M\cap B(x,R))^{1/p}$$

and tends to 0 as $|x|\to\infty$ on account of (4.1), where $m$ means the Lebesgue measure on $\mathbb{R}^d$.

Since $\int_0^t 1_{V_M\cap B(x,R)^c}(X_s)ds \leq (t-\tau_{B(x,R)})^+$, the second term of the right-hand side of (5.2) is dominated by

$$\mathbb{P}_x(t-\tau_{B(x,R)} > \epsilon/2) = \mathbb{P}_0(\tau_{B(R)} < t-\epsilon/2) \longrightarrow 0$$

as $R\to\infty$. Here $\tau_{B(x,R)}$ is the first exist time from $B(x,R)$. Therefore, we have this lemma. □

Lemma 5.2 is valid for any $B \in \mathcal{B}_0$. Combining Theorem 5.1 with Lemma 5.2, we have the next theorem.

**Theorem 5.2.** *Let $V\in\mathcal{K}_{loc}$. If $V_M\in\mathcal{B}_0$ for any $M>0$, then the semi-group of $(-\Delta)^{\alpha/2}+V$ is compact.*

For the symmetric $\alpha$-stable process, the compactness of $p_t^V$ is equivalent to $\lim_{|x|\to\infty} p_t^V 1(x) = 0$ ([6, Lemma 9]). On account of Remark 2.1 (ii) and Lemma 5.1 we have the next corollary.

**Corollary 5.1.** *For $V \in \mathcal{K}_{loc}$, let $X^V$ be the subprocess of the symmetric $\alpha$-stable process by the multiplicative functional* $\exp(-\int_0^t V(X_s)ds)$. *Then the following statements are equivalent.*

(i) *$X^V$ is in Class (T);*
(ii) $\lim_{|x|\to\infty} p_t^V 1(x) = 0$*;*
(iii) *$p_t^V$ is compact on* $L^2(\mathbb{R}^d)$.

## References


[1] M. Aizenman and B. Simon: *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math. **35** (1982), 209–273.
[2] Z.-Q. Chen: *Gaugeability and Conditional Gaugeability*, Trans. Amer. Math. Soc. **354** (2002), 4639–4679.
[3] K.L. Chung and Z.X. Zhao: From Brownian motion to Schrödinger's equation, Springer-Verlag, Berlin, 1995.
[4] D.E. Edmunds and W.D. Evans: Spectral theory and differential operators, Oxford University Press, 1987.
[5] M. Fukushima, Y. Oshima and M. Takeda: Dirichlet Forms and Symmetric Markov Processes, de Gruyter, 2nd rev. and ext. ed., 2011.
[6] K. Kaleta and T. Kulczycki: *Intrinsic ultracontractivity for Schrödinger operators based on fractional Laplacians*, Potential Anal. **33** (2010), 313–339.
[7] K. Kuwae and M. Takahashi: *Kato class measures of symmetric Markov processes under heat kernel estimates*, J. Funct. Anal. **250** (2007), 86–113.
[8] D. Lenz, P. Stollmann and D. Wingert: *Compactness of Schrödinger semigroups*, Math. Nachr. **283** (2010), 94–103.
[9] E.H. Lieb and M. Loss: Analysis, Graduate Studies in Mathematics **14**, American Mathematical Society, 2001.
[10] M. Reed and B. Simon: Method of Modern Mathematical Physics I: Functional Analysis, Academic Press, 1980.
[11] B. Simon: *Schrödinger operators with purely discrete spectra*, Methods Funct. Anal. Topology **15** (2009), 61–66.
[12] B. Simon: Operator theory, A Comprehensive Course in Analysis, Part 4. American Mathematical Society, 2015.
[13] P. Stollmann and J. Voigt:*Perturbation of Dirichlet forms by measures*, Potential Anal. **5** (1996), 109–138.
[14] M. Takeda: *Compactness of symmetric Markov semi-groups and boundedness of eigenfunctions*, preprint.
[15] M. Takeda and K. Tsuchida: *Differentiability of spectral functions for symmetric $\alpha$-stable processes*, Trans. Amer. Math. Soc. **359** (2007), 4031–4054.
[16] Z. Zhao: *Subcriticality and gaugeability of the Schrödinger operator*, Trans. Amer. Math. Soc. **334** (1992), 75–96.

Masayoshi Takeda
Mathematical Institute
Tohoku University
Aoba, Sendai 980-8578
Japan
e-mail: takeda@math.tohoku.ac.jp

Yoshihiro Tawara
Division of General Education
National Institute of Technology, Nagaoka College
Nagaoka, Niigata 940-8532
Japan
e-mail: tawara@nagaoka-ct.ac.jp

Kaneharu Tsuchida
Department of Mathematics
National Defense Academy
Yokosuka, Kanagawa 239-8686
Japan
e-mail: tsuchida@nda.ac.jp