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MORI DREAM SPACES EXTREMAL CONTRACTIONS OF K3 SURFACES

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Abstract

We will give a criterion to assure that an extremal contraction of a K3 surface which is not a Mori Dream Space produces a singular surface which is a Mori Dream Space. We list the possible Néron–Severi groups of K3 surfaces with this property and an extra geometric condition such that the Picard number is greater than or equal to 10. We give a detailed description of two geometric examples for which the Picard number of the K3 surface is 3, i.e. the minimal possible in order to have the required property. Moreover we observe that there are infinitely many examples of K3 surfaces with the required property and Picard number equal to 3.

1. Introduction

The Mori Dream Spaces are projective varieties for which the Minimal Model Program can be applied successfully, i.e. the necessary flops and contractions exist and the program terminates. They were introduced in [4] where it is also proved that the property of being a Mori Dream Space is essentially equivalent to the finite generation of the Cox ring of the variety.

In this paper we discuss the following situation: let us consider a variety which is not a Mori Dream Space and let us contract exactly one extremal ray of its effective cone. Is it possible to obtain in such a way a Mori Dream Space? The answer to this question is known to be yes, so the property "to be a Mori Dream Space" is yet not preserved by resolution of simple singularities, even if the resolution is a crepant resolution. The examples which allow one to give a positive answer to this question are provided in [7, Theorem 5.2]. These examples are constructed as follows: let S be a K3 surface which is a Mori Dream Space. Then the singular variety $V := (S \times S)/\mathfrak{S}_2$, where $\mathfrak{S}_2$ is the symmetric group, is known to be a Mori Dream Space. The variety V is singular along the diagonal and there exists an extremal crepant resolution of V which is the Hyperkähler variety $H := Hilb^2(S)$. The variety V is obtained from H by contracting the exceptional divisor over the diagonal, so V is an extremal contraction of H. In [7] Oguiso gives examples of K3 surfaces S such that S and V are Mori Dream Spaces but H is not. The proof of the fact that S and V are a Mori Dream Space and H is not is based on the computation of the automorphism groups of these varieties. Indeed, it is possible to construct K3 surfaces S such that S and V have a finite automorphism groups, but the automorphism group of H is not finite.

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The relation among the finiteness of the automorphism group and the finite generation of the Cox ring (so the property of being a Mori Dream Space), is deep as shown for example by the following result, by Artebani, Hausen and Laface, [1], which will be be fundamental in this paper: a K3 surface is a Mori Dream Space if and only if its automorphism group is finite.

In view of the result by Oguiso on the varieties H and V, it is quite natural to ask if there exist similar results in lower dimension, in particular for K3 surfaces. So the aim of this note is to positively answer to the following question:

Question 1.1. Are there K3 surfaces X which are not Mori Dream Spaces and such that the surface X' obtained by contracting exactly one extremal ray of X is a Mori Dream Space?

We will answer to the question 1.1 both by providing an abstract algorithm which produces the Néron Severi groups of admissible examples and describing the geometry of some of them.

Since X is a K3 surface, it is by definition smooth and minimal and so the extremal contraction of X produces a singular surface X'. From this point of view, X is a crepant extremal resolution of the singular surface X', and we are constructing a crepant extremal resolution of a Mori Dream Space which is not a Mori Dream Space. The same is true for the Oguiso's examples, indeed the variety H is crepant extremal resolution on V.

In Section 2 we prove that under a geometrical condition on the smooth rational curves on X, the singular surface X' is a Mori Dream Space if and only if the K3 surface Y whose Néron–Severi group is isometric to the one of X' is a Mori Dream Space. In Section 3 we give a criterion to find the Néron–Severi group of the K3 surfaces X such that there exists an extremal contraction producing a singular surface X' which is a Mori Dream Space. Here we use the results of [1] on K3 surfaces which are Mori Dream Spaces and the results of [5] on K3 surfaces with finite automorphism group. The main result of the Section 3 is Theorem 3.4 where we prove that, under a geometric condition on the rational curves on X, if X is a K3 surface which is not a Mori Dream Space, but which admits an extremal contraction producing a singular surface which is a Mori Dream Space, then $\rho(X) \geq 3$ and $\rho(X) \neq 19$. Moreover, we classify all the admissible Néron–Severi groups of X if $\rho(X) \geq 10$ (the reason of this bound is only computational, and is essentially due to the fact that in [5] the same bound is considered). Section 4 is the geometric heart of this paper. Here we give two examples of K3 surfaces X which are not Mori Dream Spaces, but which admit an extremal contraction producing a singular surface which is a Mori Dream Space. In both the cases $\rho(X) = 3$ and the Néron–Severi groups of the surfaces obtained after the contraction are the same. We will show that the automorphism group of the K3 surface X is infinite and we show that some automorphisms do not descend to automorphisms of the singular model. In Section 5 we provide other examples, giving some geometric details. In particular we will show that there exists an infinite number of K3 surfaces of Picard number 3 which are not Mori Dream Spaces but such that the singular surface obtained by one extremal contraction is a Mori Dream Space, see Proposition 5.2 for a more precise statement.

2. A criterion to conclude that the contraction of an extremal ray of a K3 surface produces a Mori Dream Space

2.1. The K3 surfaces X and Y and the singular surface X'.

Definition 2.1. A K3 surface is a complex compact surface with trivial canonical bundle and trivial irregularity.

We denote by Λ_{K3} the unique even unimodular lattice with signature $(3, 19)$ and we observe that if S is a K3 surface, then $H^2(S, \mathbb{Z}) \simeq \Lambda_{K3}$.

In the following for each lattice L and each element $l \in L$ we denote by $l^{\perp_L}$ the sublattice of L whose elements are orthogonal to l, i.e. $l^{\perp_L} := \{k \in L$ such that $kl = 0\}$.

Let X be an algebraic K3 surface (i.e. a K3 surface with an ample line bundle). The Néron–Severi group of X is a sublattice of Λ_{K3} and it is hyperbolic (i.e. its signature is $(1, \rho(X) - 1)$). We assume that $\rho(X) := \mathrm{rank}(NS(X)) \geq 2$. We denote by L the abstract lattice such that $L \simeq NS(X)$. Let us assume that there exists a smooth rational curve N on X. We denote by N also the class of the curve N in $NS(X) \simeq L$ and we recall that $N^2 = -2$. We denote by M the sublattice of $NS(X) \simeq L$ which is orthogonal to N, i.e. $M := N^{\perp_L}$. Since L is a hyperbolic lattice and $\langle N \rangle$ is a negative definite lattice, the lattice M is a hyperbolic lattice. Moreover, the lattice M is primitively embedded in L, by definition. Since there exists a primitive embedding of L in Λ_{K3}, there exists a primitive embedding of M in Λ_{K3}. As a consequence of the surjectivity of the period map for K3 surfaces, there exists at least a K3 surface (and in fact infinitely many) whose Néron–Severi group is isometric to M. Let Y be a K3 surface such that $NS(Y) \simeq M$. We observe that $\rho(Y) = \rho(X) - 1$.

The curve N is a smooth irreducible curves on X, so it determines a wall of the chamber of the positive cone which coincides with the ample cone of X. Since the set of $\mathbb{Q}$-divisors is dense in the positive cone, there exists at least one pseudoample divisor which is on the wall determined by N, i.e. there exists a pseudoample divisor orthogonal to N. Since the set of wall is locally finite, there exists at least one pseudoample divisor which is on the wall determined by N, but not on other walls, i.e. there exists a pseudoample divisor, say H_X, which is orthogonal to N, but not to any other smooth rational curve on X. Since the divisor H_X is orthogonal to N, it is represented by a vector in $M \subset L \simeq NS(X)$. The same vector represent a divisor $H_Y \in M \simeq NS(Y)$ on Y, which is a divisor with a positive self intersection and with a non zero-intersection with all the irreducible classes on M with self intersection -2. Hence we can chose the ample cone of Y to be the chamber of the positive cone of Y which contains H_Y. To recap we have a K3 surface X on which we fixed a smooth rational curve N and a pseudoample divisor H_X (which is orthogonal to N and not to other irreducible smooth rational curves on X). We associate to (X, N, H_X) a K3 surface Y such that $NS(Y) \simeq M \simeq N^{\perp_{NS(X)}}$ and H_Y (the divisor represented in M by the same vector which represents H_X) is ample.

Let us denote by $\mathcal{B}_M = \{B_1, \ldots B_{\mathrm{rank}(M)}\}$ a $\mathbb{Z}$-basis of the abstract lattice M. The set $\mathcal{B}_L = \{B_1, \ldots B_{\mathrm{rank}(M)}, N\}$ is a $\mathbb{Q}$-basis of the lattice L, i.e. every element in L is a linear combination with rational coefficients of $\{B_1, \ldots B_{\mathrm{rank}(M)}, N\}$. For certain specific lattice L the set $\mathcal{B}_L$ is also a $\mathbb{Z}$-basis for L, but this can not be guarantee in a general context. However, every element in L which has a trivial intersection with N is a linear combination with integer coefficients of $\{B_1, \ldots B_{\mathrm{rank}(M)}\}$.

Let $\phi : X \to X'$ be the map which contracts the curve N to a point. Then X' is a singular variety with exactly one singular point, of type A_1, which is the point $\phi(N)$.

The surface X' is a normal surface and its Néron–Severi group is isometric to M. So, the Néron–Severi group $NS(X')$ and the Néron–Severi group $NS(Y)$ are isometric, and both isometric to M.

By construction, we fixed a specific primitive embedding of M in $NS(X)$, which is in fact a marking. Let us denote this embedding by $f_X : M \to NS(X)$. So $NS(X')$ and $NS(Y)$ are identified with $N^{\perp_L}$, and the marking f_X induces the isometries $f_{X'} : M \to NS(X')$ by $f_Y : M \to NS(Y)$. For each vector $v \in M$ we denote by $D_X := f_X(v)$, $D_{X'} := f_{X'}(v)$ and $D_Y := f_Y(v)$. We say that the divisors D_X, $D_{X'}$ and D_Y are associated if they are the images (for the map f_X, $f_{X'}$ and f_Y respectively) of the same vector $v \in M$. Let D_X be a divisor on X such that $D_X = f_X(v)$ for a certain $v \in M$. Then $D_XN = 0$ and so the associated vectors are $D_{X'} = f_{X'}(f_X^{-1}(D_X))$ and $D_Y = f_Y(f_X^{-1}(D_X))$.

2.2. The Nef cones of Y and X'.

Lemma 2.2. *Let D_X be a divisor on X such that D_X is nef and $D_XN = 0$. The divisor $D_{X'} \in NS(X')$ associated to D_X is nef.*

Proof. The Lemma follows directly by [3, Example 1.4.4,ii)], but here we give a direct proof. We denote by v_D the vector in M such that $f_X(v_D) = D_X$, and we recall that $D_{X'} = f_{X'}(v_D)$. Since D_X is nef, for every curve $C_X \subset X$, $D_XC_X \geq 0$. Let us now consider a curve $C_{X'} \subset X'$. The strict transform of $C_{X'}$ on X is a curve, C_X, whose class is $\alpha v_C + \eta N$, where $v_C \in f_X(M)$ and $\alpha, \eta \in \mathbb{Q}$. Hence the class of $C_{X'}$ is the $\mathbb{Q}$-divisor $\alpha f_{X'}(v_C)$. So we have

$$D_{X'}C_{X'} = v_D\,(\alpha v_C) = v_D\,(\alpha v_C + \eta N) = D_XC_X \geq 0,$$

where we used that f_X and $f_{X'}$ are isometries and the fact that $v_DN = 0$. Hence $D_{X'}$ has a non negative intersection with all the curves in X'. We conclude that $D_{X'}$ is nef. □

Lemma 2.3. *Let D_X be a divisor on X such that D_X is nef and $D_XN = 0$. The divisor $D_Y \in NS(Y)$ associated to D_X is nef.*

Proof. We consider a divisor D_X which is nef. Let us consider the associated divisor $D_Y \in NS(Y)$. It suffices to show that for every effective (-2)-class $R_Y \in NS(Y)$, $R_YD_Y \geq 0$. Let R_Y be an effective (-2)-class in $NS(Y)$. It is associated to a divisor $R_X \in NS(X)$. Clearly $R_X^2 = R_Y^2 = -2$. Since X is a K3 surface, by the Riemann-Roch theorem there are only the following two possibilities: either R_X is effective or $-R_X$ is effective. If R_X is effective, then $R_XD_X \geq 0$. By definition, $R_XN = 0$ and $D_XN = 0$ so $R_XD_X = R_YD_Y$. Hence, $R_YD_Y \geq 0$ and D_Y is nef. So it now suffices to exclude that $-R_X$ is effective. Let us consider the intersection R_XH_X, since both R_X and H_X are contained in $M \subset NS(X)$, we have $R_XH_X = R_YH_Y$, which is non negative, since H_Y is ample and R_Y is effective. So $R_XH_X > 0$ and H_X is pseudoample, this implies that $-R_X$ can not be effective. □

Lemma 2.4. *Let $D_{X'} \in NS(X')$ be a nef divisor. Let D_X be the divisor associated to $D_{X'}$. Then $D_X \in NS(X)$ is nef (and $D_XN = 0$).*

Proof. The Lemma follows directly by [3, Example 1.4.4,i)], but here we give a direct proof.

In order to show that D_X is nef we show that for every curve $C_X \in NS(X)$, $D_X C_X \geq 0$. If $C_X = N$, then $D_X C_X = 0$. We now assume that $C_X \neq N$. So $\phi(C_X) \subset X'$ is a curve in X' and C_X is the strict transform of $\phi(C_X)$ with respect to the blow up ϕ. The class of C_X is represented in $NS(X)$ by the class $\alpha f_X(v_C) + \eta N$ for a certain $v_C \in M$ and $\alpha, \eta \in \mathbb{Q}$. Then $\phi(C_X)$ is represented by $\alpha f_{X'}(v_C)$. Let us denote by $v_D \in M$ the vector such that $D_X = f_X(v_D)$. Thus,

$$D_X C_X = f_X(v_D)(\alpha f_X(v_C) + \eta N) = \alpha v_D v_C = f_{X'}(v_D)\,(\alpha f_{X'}(v_C)) = D_{X'}\phi(C_X)$$

where we used that $f_X(v_D)N = 0$ and that f_X and $f_{X'}$ are isometries. Since $D_{X'}$ is nef, $D_{X'}\phi(C_X) \geq 0$, so $D_X C_X \geq 0$ for every curve C_X in X. □

Lemma 2.5. *Let $D_Y \in NS(Y)$ be a nef divisor on Y. Let $D_X \in NS(X)$ be the associated divisor on X. If there exists no a curve $B_X \subset X$ with self intersection -2 such that $B_X N = 1$, then D_X is nef (and $D_X N = 0$).*

Proof. It suffices to prove that $D_X C_X \geq 0$ for every irreducible C_X curve on X with self intersection -2. If $C_X = N$, then $D_X C_X = D_X N = 0$. So we assume that $C_X \neq N$ and thus $C_X N \geq 0$. Let us denote by $c := C_X N$. By the hypothesis we know that $c \neq 1$, hence either $c = 0$, or $c \geq 2$. The divisor C_X can be written in $NS(X) \otimes \mathbb{Q}$ as $C_X = A_X - \frac{c}{2}N$, where $A_X \in NS(X) \otimes \mathbb{Q}$ is a $\mathbb{Q}$-divisor orthogonal to N. Since $D_X N = 0$, $D_X C_X = D_X A_X$. If $c \geq 2$, then $A_X^2 = \left(C_X + \frac{c}{2}N\right)^2 = -2 - \frac{c^2}{2} + c^2 \geq 0$. So any positive multiple of A_X has a positive self intersection, in particular there exists a multiple mA_X, $m \in \mathbb{N}$ of A_X which is a divisor in $NS(X)$. By Riemann–Roch theorem either mA_X or $-mA_X$ is effective. Since C_X is a curve, $C_X H_X > 0$ for the pseudoample divisor H_X, and thus $mA_X H_X > 0$, so mA_X is an effective divisor in X. Thus the $\mathbb{Q}$-divisor $A_Y(= f_Y(f_X^{-1}(A_X))$ is an effective $\mathbb{Q}$-divisor on Y (since $A_Y^2 \geq 0$ and $A_Y H_Y > 0$). The divisor D_Y is nef on Y, so $D_Y A_Y \geq 0$. By $D_X C_X = D_X A_X = D_Y A_Y \geq 0$ we conclude the proof in case $c \geq 2$.

If $c = 0$, i.e. $CX_N = 0$, C_X is by definition an effective divisor on X and the associated divisor C_Y is an effective divisor on Y. So $D_Y C_Y \geq 0$ and we conclude by $D_X C_X = D_Y C_Y \geq 0$. □

Remark 2.6. In Lemma 2.5, the assumption that there exists no a (-2)-curve B_X such that $B_X N = 1$ is essential. Indeed if the curve B_X exists, then the divisor D_X in the statement can not be nef. Because of the duality between the effective cone and the nef cone of a surface, to prove that D_X can not be nef, it suffices to produce an effective class E_X on X such that $E_X N = 0$, but $E_Y(= f_Y^{-1}(f_X(E_X)))$ is not effective on Y.

So we consider the curve $E_X = 2B_X + N$, which is clearly an effective class in X such that $E_X N = 0$.

Since E_X is an effective divisor such that $E_X B_X < 0$ and $E_X N = 0$, we conclude that any multiple of E_X is supported on $B_X \cup N$. Thus no positive multiples of E_X can be linearly equivalent to a positive sum of (-2) curves orthogonal to N and hence no multiples of E_Y can be in the effective cone of Y.

Proposition 2.7. *Let us denote by* $\mathrm{Nef}(X')$ *and* $\mathrm{Nef}(Y)$ *the nef cones of X' and Y respec-*

tively. Then

$$\mathrm{Nef}(X') \subseteq \mathrm{Nef}(Y)$$

and the equality holds if and only if there exists no a (-2)*-curve* B_X *on* X *such that* $B_X N = 1$.

Proof. We first recall that the lattices $NS(X')$, $NS(Y)$ and M are all isometric.

Let $D_{X'}$ be a nef divisor on X'. Then the associated divisor D_X is a nef divisor on X and $D_X N = 0$, by Lemma 2.4. This implies, by Lemma 2.3, that the corresponding divisor D_Y on Y is nef. So $\mathrm{Nef}(X') \subseteq \mathrm{Nef}(Y)$.

Let D_Y be a nef divisor on Y. If there exists no a (-2)-curve B_X on X such that $B_X N = 1$, then the associated divisor D_X is nef on X and $D_X N = 0$ by Lemma 2.5. This implies, by Lemma 2.2 that the corresponding divisor $D_{X'}$ on X' is nef.

If there exists a (-2)-curve B_X on X such that $B_X N = 1$, then $\mathrm{Nef}(X') \neq \mathrm{Nef}(Y)$, by Remark 2.6. □

Lemma 2.8. *Let* $D_X \in NS(X)$ *be a semi-ample divisor* (*i.e. there exists a positive integer* $m > 0$ *such that* $|mD_X|$ *is without base points*) *such that* $D_X N = 0$*. Then the associated divisor* $D_{X'} \in NS(X')$ *is semi-ample.*

Proof. Since D_X is semi-ample, there exists an integer $m \in \mathbb{N}$ such that $|mD_X|$ is base points free. In particular $|mD_X|$ does not have fixed component. Since $D_X N = 0$, $mD_X N = 0$ and thus $H^0(X, mD_X)$ contains sections which do not pass through any point of N. The images of these sections under the map ϕ do not pass through the point $\phi(N) := P$.

The map $\phi : X \to X'$ is an isomorphism outside N. Let us consider the sections of $H^0(X, mD_X)$, we call them s_j, $j = 1, \ldots, 1 + (mD_X)^2/2$. The curves $\phi(s_j)$ are sections of $H^0(X', mD_{X'})$ and form a basis of this space. Viceversa, let us consider a section s' in $H^0(X, mD_{X'})$. It is the image of a curve in X which is in fact the strict transform of s' and which is clearly a section of mD_X. Since $(mD_X)^2 = (mD_{X'})^2$, $h^0(X, mD_X) = h^0(X, mD_{X'})$ and so a basis of one of these two spaces induces a basis of the other. Let us assume that $|mD_{X'}|$ has a fixed point Q, i.e. all the sections in $H^0(X', mD_{X'})$ passes through Q. We already observed that $Q \neq P = \phi(N)$, since P is not a base point for $|mD_{X'}|$. This would imply that $\phi^{-1}(Q)$ is a base point for $|mD_X|$, which is impossible since $|mD_X|$ is base points free. So $|mD_{X'}|$ is base points free and thus $D_{X'}$ is semi-ample. □

2.3. Mori Dream Spaces: the surfaces Y **and** X'**.** We recall the definition of Mori Dream Space, in the context of the surfaces (and we recall that any small $\mathbb{Q}$-factorial modification is an isomorphism if we are considering surfaces).

Definition 2.9. ([4]) We will call a normal projective surface X a Mori Dream Space provided the following hold:

(1) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X) \otimes \mathbb{Q} \equiv NS(X) \otimes \mathbb{Q}$;
(2) $\mathrm{Nef}(X)$ is the convex hull of finitely many semi–ample line bundles.

By [4, Proposition 2.9] a variety X such that $\mathrm{Pic}(X) \otimes \mathbb{Q} = NS(X) \otimes \mathbb{Q}$ is a Mori Dream Space if and only if its Cox ring $\mathcal{R}(X)$ is finitely generated.

Proposition 2.10. *Let us assume that there exists no a* (-2)*-curve* B_X *on* X *such that*

$B_X N = 1$. *Then the surface Y is a Mori Dream Space if and only if the surface X' is a Mori Dream Space.*

Proof. We recall that every semi-ample divisor is nef. On a K3 surface also the viceversa holds, i.e. a divisor on a K3 surface is semi-ample if and only if it is nef.

We recall that under the assumptions, $\mathrm{Nef}(X') = \mathrm{Nef}(Y)$, by Proposition 2.7.

Let Y be a Mori Dream Space. By definition $\mathrm{Nef}(Y)$ is the convex hull of a finite set of semi-ample divisors, $D_Y^{(1)}, \dots D_Y^{(r)}$. Then $\mathrm{Nef}(X')$ is the convex hull of a finite set of nef divisors, which are the divisors $D_{X'}^{(1)}, \dots D_{X'}^{(r)}$ associated to $D_Y^{(1)}, \dots D_Y^{(r)}$. It remains to show that $D_{X'}^{(1)}, \dots D_{X'}^{(r)}$ are semi-ample.

The associated divisors on X, $D_X^{(1)}, \dots D_X^{(r)}$ are nef and such that $D_X^i N = 0$ for each $i = 1, \dots r$, by Lemma 2.5. So they are semi-ample divisors on X (which is a K3 surface). By Lemma 2.8, the divisors $D_{X'}^{(1)}, \dots D_{X'}^{(r)}$ are semi-ample on X'. Thus X' is a Mori Dream Space.

Let us now assume that Y is not a Mori Dream Space, then X' is not a Mori Dream Space. Indeed, since Y is a K3 surface, it is $\mathbb{Q}$-factorial, $\mathrm{Pic}(Y) \otimes \mathbb{Q} \equiv NS(Y) \otimes \mathbb{Q}$, and the nef divisors are semi-ample. So if Y is not a Mori Dream Space, then $\mathrm{Nef}(Y)$ is not the convex hull of finitely many nef divisors, which implies that also $\mathrm{Nef}(X')$ is not the convex hull of finitely many nef divisors, and thus it can not be a Mori Dream Space. □

3. Admissible pairs (X, X')

Definition 3.1. An admissible pair is a pair of surfaces (X, X') such that X is a K3 surface that is not a Mori Dream Space and X' is obtained by contracting exactly one extremal ray of X and is a Mori Dream Space.

Let us consider an admissible pair (X, X'). By assumption the Picard number of X is greater than or equal to 2. Under this condition, by [6, Theorem 2] the extremal rays of the effective cone of X are curves with self intersection either 0 or -2. Since we are assuming that X' is a surface, the contraction associated to the extremal ray can not be a fibration and so we are contracting a (-2)-curve.

In order to answer positively to the question 1.1 it suffices to find an admissible pair. In this section we describe how to find the Néron–Severi groups of the K3 surfaces X in admissible pairs (see Condition 3.3) and we give a list of admissible pairs such that $\rho(X) \geq 10$ (cf. Theorem 3.4).

Let us assume that X and Y are K3 surfaces as in Section 2 (i.e. X admits a (-2) curve N such that the lattice orthogonal to N in $NS(X)$ is the Néron–Severi group of Y). We recall that $NS(X') \simeq NS(Y)$, where X' is obtained by X contracting N. By Proposition 2.10, if there are no (-2)-curves B_X on X such that $B_X N = 1$, then the pair of surfaces (X, X') is an admissible pair if and only if X is not a Mori Dream Space and Y is a Mori Dream Space. Both X and Y are K3 surfaces and since the K3 surfaces which are Mori Dream Spaces are classified, this provides a way to construct pairs (X, X') as required.

Here we summarize some results on K3 surfaces, which will be used in the following.

Theorem 3.2. ([1, Theorems 2.7,2.11, 2.12]) *An algebraic K3 surface S is a Mori Dream Space if and only if the automorphism group* $\mathrm{Aut}(S)$ *is finite.*

In particular, if $\rho(S) = 1$, then S is a Mori Dream Space;

if $\rho(S) = 2$, then S is a Mori Dream Space if and only if $NS(S)$ contains at least an element with self intersection either 0 or -2;
if $\rho(S) \geq 3$, then S is a Mori Dream Space if and only if $NS(S)$ belongs to a finite known list of hyperbolic lattices.

The previous Theorem gives a constructive way to produce admissible pairs (X, X'), indeed it suffices to have a K3 surface X with the following properties:

Condition 3.3.
- $|\mathrm{Aut}(X)| = \infty$;
- *there exists a rational curve N on X;*
- *there are no (-2)-curves B_X on X such that $B_X N = 1$;*
- *if $M := N^{\perp_{NS(X)}}$ and Y is a K3 surface such that $NS(Y) \simeq M$, then $|\mathrm{Aut}(Y)| < \infty$.*

Theorem 3.4. *Let (X, X') be an admissible pair of surfaces, then $\rho(X) \geq 3$.*
Let us assume that there exists no a (-2)-curve $B_X \subset X$ such that $B_X N = 1$. Then $\rho(X) \neq 19$ and if $\rho(X) \geq 4$, then there are finitely many possible choices for the lattice $NS(X)$.

The complete list of the Néron–Severi groups $NS(X)$ of admissible pairs (X, X') such that there exists no a (-2)-curve $B_X \subset X$ with $B_X N = 1$ and $\rho(X) \geq 10$ is given in Table (3.1).

If L is as in Table (3.1) and X is such that $NS(X) \simeq L$, then there exists a smooth irreducible rational curve $N \subset X$ such that, denoted by X' the surface obtained contracting N, $NS(X') \simeq M$ and thus (X, X') is an admissible pair.

(3.1)

$\rho(X)$	$L \simeq NS(X)$	$M \simeq NS(X')$
20	$U \oplus E_8^2 \oplus A_1^2$	$U \oplus E_8^2 \oplus A_1$
18	$U \oplus E_8 \oplus E_7 \oplus A_1$	$U \oplus E_8 \oplus E_7$
17	$U \oplus E_8 \oplus D_6 \oplus A_1$	$U \oplus E_8 \oplus D_6$
16	$U \oplus E_8 \oplus D_4 \oplus A_1^2$	$U \oplus E_8 \oplus D_4 \oplus A_1$
15	$U \oplus E_8 \oplus A_1^5$	$U \oplus D_8 \oplus D_4$
15	$U \oplus E_8 \oplus A_1^5$	$U \oplus E_8 \oplus A_1^4$
14	$U \oplus E_7 \oplus A_1^5$	$U \oplus E_7 \oplus A_1^4$
14	$U \oplus E_8 \oplus A_3 \oplus A_1$	$U \oplus E_8 \oplus A_3$
13	$U \oplus D_6 \oplus A_1^5$	$U \oplus D_6 \oplus A_1^4$
13	$U \oplus E_8 \oplus A_2 \oplus A_1$	$U \oplus E_8 \oplus A_2$
12	$U \oplus D_4 \oplus A_1^6$	$U \oplus D_4 \oplus A_1^5$
11	$U \oplus E_6 \oplus A_2 \oplus A_1$	$U \oplus E_6 \oplus A_2$
11	$U \oplus A_1^9$	$U \oplus A_1^8$
10	$U(2) \oplus A_1^8$	$U(2) \oplus A_1^7$
10	$U \oplus E_8(2)$	$U(2) \oplus A_1^7$
10	$U \oplus A_7 \oplus A_1$	$U \oplus A_7$
10	$U \oplus D_4 \oplus A_3 \oplus A_1$	$U \oplus D_4 \oplus A_3$
10	$U \oplus D_5 \oplus A_2 \oplus A_1$	$U \oplus D_5 \oplus A_2$
10	$U \oplus D_7 \oplus A_1$	$U \oplus D_7$
10	$U \oplus E_6 \oplus A_1 \oplus A_1$	$U \oplus E_6 \oplus A_1$

Proof. If X is a K3 surface and it is not a Mori Dream Space then $\mathrm{Aut}(X)$ is not finite.

In particular this implies that $\rho(X) \geq 2$ and that if $\rho(X) = 2$, then X does not admit curves with self intersection equal either to 0 or -2. Since we require that X admits a (-2)-curve, $\rho(X) \geq 3$.

We now assume that (X, X') is an admissible pair such that there exists no a (-2)-curve B_X with $B_X N = 1$. Under the latter condition, the fact that (X, X') is an admissible pair is equivalent to the conditions $|\mathrm{Aut}(X)| = \infty$ and $|\mathrm{Aut}(Y)| < \infty$, and hence to a condition on the lattices L and M. We first investigate the lattice condition on L and M and after that we discuss of the existence of the curves N and B_X.

By hypothesis, $NS(X)$ is an overlattice of finite index of $M \oplus N$ where N is the class of the (-2)-curve contracted and M is primitively embedded in $NS(X)$. We denote by r the index of the inclusions $M \oplus N \hookrightarrow NS(X)$. If $r \neq 1$, then there exists a class $(m+N)/r \in NS(X)$ such that $m \in M$, $(m+N)/r \notin M \oplus N$. Clearly this implies that $N((m+N)/r) = N^2/r = -2/r \in \mathbb{Z}$ and thus r is either 1 or 2. So $NS(X)$ is either $M \oplus N$ or an overlattice of index 2 of $M \oplus N$.

The number of hyperbolic lattices M with $\mathrm{rank}(M) \geq 3$ such that if Y is a K3 surface with $NS(Y) \simeq M$, then $|\mathrm{Aut}(Y)| < \infty$ is finite. If (X, X') is an admissible pair, then $NS(Y) \simeq NS(X') \simeq M$, and the admissible choices for M are finite. The lattice $NS(X)$ is an overlattice of index 1 or 2 of $NS(X') \oplus N \simeq M \oplus A_1$. Since the number of overlattices of index two of $M \oplus A_1$ is finite (up to isometries), it follows that the possible choices for $NS(X)$ are finite if $\mathrm{rank}(M) \geq 3$, i.e. if $\rho(NS(X)) \geq 4$.

In order to construct the list of the Néron–Severi groups of the admissible pairs (X, X') such that there exists no a (-2)-curve $B_X \subset X$ with $B_X N = 1$ (and to exclude the case $\rho(X) = 19$), we check the list of the Néron–Severi groups of the K3 surfaces with a finite group of automorphisms, given in [5]. We denote by M a lattice in this list. If $NS(X') \simeq M$, then X' is a Mori Dream Space: Indeed, if there exists no a (-2)-curve $B_X \subset X$ with $B_X N = 1$, X' is a Mori Dream Space if and only if Y is a Mori Dream Space by Proposition 2.10 and Y is a Mori Dream Space if and only if Y is a K3 surface with a finite automorphism group, by Theorem 3.2. But $NS(Y) \simeq M$, so Y is a Mori Dream Space.

So for each M in the list given in [5] we have to construct the lattice $M \oplus N$ and the overlattices of index 2 of $M \oplus N$. Each of these lattices is a good candidate to be the Néron–Severi group of X. Let us denote by L one of these lattices and by X a K3 surface such that $NS(X) \simeq L$. We now have to check that X is not a Mori Dream Space, so we have to check that $|\mathrm{Aut}(X)| = \infty$, i.e that L is not in the list given in [5]. In this way one produces the list of the possible Néron–Severi groups of X and X'. Then a geometric argument can be used in order to show that there exists a model of X such that the class N represents a smooth irreducible (-2)-curve and so an extremal ray and the analysis of the lattice L allows to conclude that there exists no a (-2)-curve $B_X \subset X$ such that $B_X N = 1$.

Our first step is to construct the list of the Nèron–Severi groups given in Table (3.1). We give all the details for the first lines of the Table, the other cases are very similar. Let us check the list given in [5] of the lattices M such that if Y is a K3 surface with $NS(Y) \simeq M$, then $|\mathrm{Aut}(Y)| < \infty$. If $\mathrm{rank}(M) = 19$, then $M \simeq U \oplus E_8^2 \oplus A_1$. So the lattice L is either $M \oplus N \simeq U \oplus E_8^2 \oplus A_1^2$ or an overlattice of index 2 of $U \oplus E_8^2 \oplus A_1^2$. The discriminant group of the lattice $U \oplus E_8^2 \oplus A_1^2$ is $(\mathbb{Z}/2\mathbb{Z})^2$, so an overlattice of index two of this lattice is unimodular. But there exits no a hyperbolic even unimodular lattice of rank 20. So $L \simeq U \oplus E_8^2 \oplus A_1^2$. We now assume $NS(X) \simeq L$. Since L is not contained in the list given in [5] we conclude that $|\mathrm{Aut}(X)| = \infty$ and thus X is not a Mori Dream Space. Moreover, since $L \simeq M \oplus \mathbb{Z}N$, all the vectors in $v \in L$ are of type $v := m + \eta N$, $\eta \in \mathbb{Z}$, $m \in M$, hence $vN \in 2\mathbb{Z}$ and thus there are no divisors $D \in NS(X)$ such that $DN = 1$. In particular there is no a (-2)-curve $B_X \subset X$ with $B_X N = 1$. This guarantee that if the pair (X, X') is admissible and $\rho(X) = 20$, then $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$ and $NS(X') \simeq U \oplus E_8^2 \oplus A_1$.

Let us consider the lattice M of the list in [5] with $\mathrm{rank}(M) = 18$. In this case $M \simeq U \oplus E_8^2$, which is a unimodular lattice. A priori L could be either $L \simeq M \oplus N \simeq U \oplus E_8^2 \oplus A_1$ or an overlattice of index 2 of $M \oplus N \simeq U \oplus E_8^2 \oplus A_1$. The discriminant group of $U \oplus E_8^2 \oplus A_1$ is $\mathbb{Z}/2\mathbb{Z}$, so there are no overlattices of index two of $U \oplus E_8^2 \oplus A_1$, and thus $L \simeq M \oplus N \simeq U \oplus E_8^2 \oplus A_1$. Hence, if there exists an admissible pair (X, X') such that $\rho(X) = 19$, then $NS(X) \simeq U \oplus E_8^2 \oplus A_1$. But if $NS(X) \simeq U \oplus E_8^2 \oplus A_1$, then $|\mathrm{Aut}(X)| < \infty$ by [5], and so X is a Mori Dream Space, by Theorem 3.2. We conclude that there are no admissible pairs (X, X') such that $\rho(X) = 19$. The other cases in the Table are similar.

Let L be a lattice given in the second column of Table (3.1) and M the corresponding lattice given in the third column of the same Table. Now we prove that the generic K3 surface X such that $NS(X) \simeq L$ admits an extremal contraction (of the curve N) such that the corresponding surface X' has the property $NS(X') \simeq M$ and that there is no a (-2)-curve $B_X \subset X$ with $B_X N = 1$. This implies that (X, X') is an admissible pair.

First we observe that for all the pairs (L, M) in Table 3.1, except $(U \oplus E_8(-2), U(2) \oplus A_1^7)$, $L \simeq M \oplus A_1 \simeq M \oplus \mathbb{Z}N$ and this implies that for any divisor $D \in NS(X) \simeq M \oplus \mathbb{Z}$, $DN \in 2\mathbb{Z}$. Thus there exists no a (-2)-class $B_X \subset X$ with $B_X N = 1$. For the pair $(L, M) \simeq (U \oplus E_8(2), U(2) \oplus A_1^7)$, one has to deeply analyze the lattice L, which is an overlattice of index 2 of $U(2) \oplus A_1^8$. Denoted by $(u_1, u_2, N_1, \dots, N_8)$ the basis of $U(2) \oplus A_1^8$, L is obtained adding to this set of divisors the divisor $(\sum_{i=1}^8 N_i)/2$. Each $l \in L$ is of the form $l := a_1 u_1 + a_2 u_2 + \sum_{i=1}^8 b_i N_1 - k(\sum_{i=1}^8 N_1)/2$, where $a_j, b_i \in \mathbb{Z}$ and k is either 0 or 1. Choosing the curve N to be N_8, the condition $lN = lN_8 = 1$ implies that $k = 1$ and $b_8 = 0$. The condition $l^2 = -2$ is now equivalent to $4a_1a_2 - 2\sum_{i=1}^8 b_i^2 + 2\sum_{i=1}^8 b_i - 4 = -2$, which is impossible modulo 4. Thus also the pair $(L, M) \simeq (U \oplus E_8(-2), U(2) \oplus A_1^7)$ is such that there exists no a (-2)-curve $B_X \subset X$ with $B_X N = 1$.

Now it suffices to prove that there exists a smooth irreducible rational curve which is represented in $NS(X)$ by the class N such that $N^{\perp_{NS(X)}} \simeq M \simeq NS(X')$. In all the cases but $(L, M) \simeq (U \oplus E_8 \oplus A_1^5, U \oplus D_6 \oplus D_4), (U(2) \oplus A_1^8, U(2) \oplus A_1^7), (U \oplus E_8(2), U(2) \oplus A_1^7)$, the lattice $L \simeq NS(X)$ is $L \simeq U \oplus R \oplus A_1$ for a certain root lattice R and the lattice $M \simeq NS(X')$ is $M \simeq U \oplus R$.

Since $L \simeq U \oplus R \oplus A_1$, the surface X admits an elliptic fibration such that the irreducible components of the reducible fibers which do not intersect the zero section are represented by the lattice $R \oplus A_1$. In particular, the lattice A_1 which appears as direct summand in the decomposition $L \simeq U \oplus R \oplus A_1$, represents an irreducible component of a fiber of type I_2 (or III). Thus A_1 is generated by the class of a smooth irreducible rational curve. The contraction of this curve gives the singular surface X', whose Nèron–Severi group is naturally identified with $U \oplus R \simeq M$.

In Section 5.1, an explicit equation of the elliptic fibration on X associated to the decomposition $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$ is provided, as example.

The existence of the smooth irreducible rational curve N in the remaining cases (i.e.

$(L, M) \simeq (U \oplus E_8 \oplus A_1^5, U \oplus D_6 \oplus D_4), (U(2) \oplus A_1^8, U(2) \oplus A_1^7), (U \oplus E_8(2), U(2) \oplus A_1^7))$ is proved in Section 5, (more precisely in Example 5.1, in Section 5.2 and in Section 5.3 respectively). □

REMARK 3.5. In Theorem 3.4 we proved that if (X, X') is an admissible pair, there exists no a (-2)-curve $B_X \subset X$ such that $B_X N = 1$ and $\rho(X) \geq 4$, then the lattice $NS(X)$ is isometric to a lattice in a finite list of hyperbolic lattice. On the other hand, if $\rho(X) = 3$ then it is possible to construct infinitely many admissible pairs (X_n, X_n') such that $n \in \mathbb{N}$ and $NS(X_n) \not\simeq NS(X_{n'})$ if $n \neq n'$. Examples are provided in Proposition 5.2.

REMARK 3.6. There exist examples of both the following cases:

- (X_1, X_1') and (X_2, X_2') are two admissible pairs such that $NS(X_1) \simeq NS(X_2)$, but $NS(X_1') \not\simeq NS(X_2')$
- (X_1, X_1') and (X_2, X_2') are two admissible pairs such that $NS(X_1') \simeq NS(X_2')$, but $NS(X_1) \not\simeq NS(X_2)$.

An example of the first case is given in the Table 3.1, $\rho(X) = 15$ and an example of the second case is given in Table 3.1, $M \simeq U(2) \oplus A_1^7$. We briefly describe the geometry of these case in Example 5.1, Section 5.2 and Section 5.3.

An more exhaustive example of the second case is provided in Section 4, where the geometric details are given. Moreover, an infinite series of examples is presented in Proposition 5.2.

REMARK 3.7. In the proof of Theorem 3.4 we proved and used the following fact: if $L \simeq M \oplus \mathbb{Z}N$, then there exists no a (-2)-curve $B_X \subset X$ such that $B_X N = 1$. To be more precise, the existence of such a curve implies that L is an overlattice of index 2 of $M \oplus \mathbb{Z}N$ which contains the class $(m + N)/2$, where $m \in M$, $m^2 = -6$ and $m/2 \in M^\vee/M$. Hence sufficient conditions to conclude that there exists no a (-2)-curve $B_X \subset X$ with $B_X N = 1$ are:

- $M \oplus \mathbb{Z}N$ has index 1 in L;
- M does not contain vectors with self intersection -6;
- M does not contain vectors m of self intersection -6 such that $m/2 \in M^\vee/M$ (for example this is the case if the discriminant quadratic form of M takes value in $\mathbb{Z}$).

Since we described the deep relation between the automorphism group and the property of being a Mori Dream Space, we now consider the relation between the automorphism groups of the surfaces involved in our construction. Since X' is obtained from X by contracting a curves, $\mathrm{Aut}(X') \subset \mathrm{Aut}(X)$. More precisely, every automorphism α which does not preserve the curve N does not descend to an automorphism of X' and every automorphism α which preserves N descend to an automorphism α' of X'. Every automorphism α' of X' lifts to an automorphism α of X which leaves invariant the rational curve N. Moreover there is also a strong relation among the automorphism group of X' and Y:

Corollary 3.8. *Let us assume that there is no a (-2)-curve $B_X \subset X$ such that $B_X N = 1$. Then group of automorphisms of X' is contained in the group of automorphisms of Y.*

Proof. Let α' be an automorphism of X'. It lifts to an automorphism α of X which preserves N. So α induces an effective Hodge isometry of $H^2(X, \mathbb{Z})$. We denote by T_X the transcendental lattice of X and we observe that the isometry induced by α preserves the splitting $M \oplus N \oplus T_X$. Hence it is a Hodge isometry of $H^2(Y, \mathbb{Z})$, and the Néron–Severi group $NS(Y)$ is M. Moreover, the Nef cone of Y can be identified with the one of X' by the Proposition 2.7. Since α' is an automorphism of X', it preserves the nef cone of X' and so α^* preserves the ample cone of Y. So α^* is an Hodge effective isometry for Y and thus it is induced by an automorphism α_Y of Y. Let α_Y be an automorphism of Y induced by $\alpha' \in \mathrm{Aut}(X')$ as above. If α_Y is the identity, then α_Y^* is the identity on $H^2(Y, \mathbb{Z})$ and thus α^* is the identity on $H^2(X, \mathbb{Z})$. So $\alpha \in \mathrm{Aut}(X)$ is the identity, by Torelli theorem, hence $\alpha' \in \mathrm{Aut}(X')$ is the identity. So the map $\alpha' \mapsto \alpha_Y$ is injective. □

4. Two examples

This section is devoted to the geometric description of two examples. First we consider a lattice M of rank 2 such that if Y is a K3 surface with $NS(Y) \simeq M$, then Y is a Mori Dream Space. We describe the geometry and the automorphism group of Y in Section 4.2.1: Y admits a model as quartic hypersurface with a node and a model as double cover of $\mathbb{P}^2$. For a generic choice of Y the automorphism group is generated by the cover involution.

In Section 4.1 we construct two different lattices L of rank 3: one of them is isometric to $M \oplus A_1$, the other one is the unique even overlattice of index 2 of $M \oplus A_1$. If the Néron–Severi group of a K3 surface is isometric to one of these two lattices, then the K3 surface is not a Mori Dream Space, but the contraction of a (-2)-curve on it produces a Mori Dream Space whose Néron–Severi group is isometric to M, so we construct the Nèron–Severi groups of two admissible pairs.

We describe the geometry of the K3 surfaces of these two admissible pairs in Sections 4.3.1 and 4.3.3: one of them admits a model as quartic in $\mathbb{P}^3$ with two nodes (which clearly specializes the model of Y) and as double cover of $\mathbb{P}^2$, the other is an elliptic fibration and admits a model as double cover of the Hirzebruch surface $\mathbb{F}_4$. We show that the automorphism group of both these K3 surfaces are infinite, but that several automorphisms do not descend to automorphism of the contracted model, which is in fact a Mori Dream Space.

In the following proposition we summarize the results obtained in this section.

Proposition 4.1. *Let M be the lattice $\langle 4 \rangle \oplus \langle -2 \rangle$. Let L be an overlattice of M such that:*

- *There exists a vector $n \in L$, such that $n^2 = -2$ and $n^{\perp_L} \simeq M$*
- *L admits a primitive embedding in Λ_{K3}.*

Then L is a hyperbolic even lattice of rank 3 and it is isometric either to $\langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$ or to $U \oplus \langle -4 \rangle$ (see Section 4.1*). Moreover,*

(1) *Let Y be a generic K3 surface such that $NS(Y) \simeq M$. Then Y admits a map $\phi : Y \to \mathbb{P}^2$ which is a $2:1$ cover branched along a smooth sextic $C_6 \subset \mathbb{P}^2$. There exists a conic c_2 which is tangent to C_6 in their six intersection points. The automorphism group of Y is generated by the cover involution ι, in particular it is finite (see* Section 4.2.1).

(2) *Let X be a generic K3 surface such that $NS(X) \simeq \langle 4 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle$. Then X admits two different maps $\phi_i : X \to (\mathbb{P}^2)_i$, $i = 1, 2$. Each of them is a $2:1$ cover branched along a sextic $(C_6)_i \subset (\mathbb{P}^2)_i$ with one node in the point $(P)_i \in (C_6)_i$. There exists a conic $(c_2)_i$ such that $(P)_i \notin (c_2)_i$ and $(c_2)_i$ is tangent to $(C_6)_i$ in their six intersection points. The automorphism group of X is infinite and contains the two*

(*non commutative!*) *cover involutions* ι_i.

The map ϕ_1 (*resp.* ϕ_2) *contracts exactly one rational curve, which is* $(c_2)_2$ (*resp.* $(c_2)_1$). *Let* X' *be the double cover of* $\mathbb{P}^2$ *branched along* $(C_6)_1$. *Then* $((c_2)_2)^{\perp_{NS(X)}} \simeq NS(X')$ *is isometric to* M *and* (X, X') *is an admissible pair. Moreover* Aut(X') *is generated by the cover involution* ι_1 (*in particular it is finite*) *and the induced effective Hodge isometry coincides with the one induced by the involution* ι *of* Y (*see* Section 4.3.1).

(3) *Let* S *be a generic K3 surface such that* $N(S) \simeq U \oplus \langle -4 \rangle$. *Then* S *admits an elliptic fibration* $\mathcal{E} : S \to \mathbb{P}^1$ *such that* $MW(\mathcal{E}) = \mathbb{Z}$ *is generated by a section* s_1 *which has a trivial intersection with the zero section. The automorphism group of* S *is infinite and contains the translation by* s_1 (*which is an automorphism of infinite order*).

There exists a $2:1$ *map* $\varphi : S \to \mathbb{P}^5$ *whose image is the cone, C, over the twisted rational quartic in* $\mathbb{P}^4$ *and let* $B \subset C$ *be the branch locus. The map* φ *contracts exactly one rational curve* s*, which is a section of the fibration* $\mathcal{E}$ *and whose image is the vertex of the cone. Let* S' *be the double cover of* C *branched along* B. *Then* $s^{\perp_{NS(S)}} \simeq NS(S')$ *is isometric to* M *and* (S, S') *is an admissible pair. Moreover* Aut(S') *is generated by the cover involution* (*in particular it is finite*) *and the induced effective Hodge isometry coincides with the one induced by the involution* ι *of* Y (*see* Section 4.3.3).

The following lemma will be essential, since it implies that Proposition 2.10 can be applied for both the pairs (X, X') and (S, S').

Lemma 4.2. *The lattice* $M \simeq \langle 4 \rangle \oplus \langle -2 \rangle$ *does not contain a vector of length* -6. *Hence* X (*respectively* S) *does not contain a* (-2) *curve which intersects* $(c_2)_2$ (*respectively* s) *in 1 point.*

Proof. The quadratic form of M computed on (x, y) is $4x^2 - 2y^2$. So a vector has length -6 if and only if $2x^2 - y^2 = -3$. This condition implies that $x \equiv 0 \mod 3$ and $y \equiv 0 \mod 3$. So we write $x = 3h$, $y = 3k$. The requirement $(3h, 3k)$ has length -6, is equivalent to $6h^2 - 3k^2 = -1$, which is clearly impossible modulo 3. By Remark 3.7, this implies that, denoted by N a vector with self intersection -2, neither $M \oplus \mathbb{Z}N$ or an overlattice of index 2 of $M \oplus \mathbb{Z}N$ contains a vector B with length -2 such that $BN = 1$. In particular this applies to the lattices $NS(X) \simeq M \oplus \mathbb{Z}(c_2)_2$ and $NS(S)$, which is an overlattice of index 2 of $M \oplus \mathbb{Z}s$, and thus X and S are as in the statement. □

4.1. Lattice enhancements. Let M be the lattice $\langle 4 \rangle \oplus \langle -2 \rangle$ and let us denote by $\{n_1, n_2\}$ its basis. The generators of the discriminant group are $M^\vee/M \simeq \langle n_1/4, n_2/2 \rangle$.

Let L be a lattice such that L admits a primitive embedding in Λ_{K3}. Since Λ_{K3} is an even lattice, L is an even lattice.

We now require that there exists a vector $n \in L$, such that $n^2 = -2$ and $n^{\perp_L} \simeq M$. In particular this implies that L is an overlattice of finite index $r \in \mathbb{N}$ of $M \oplus \mathbb{Z}n$. Since $M \oplus \mathbb{Z}n$ is a hyperbolic lattice of rank 3, L is a hyperbolic lattice of rank 3.

We recall that every even hyperbolic lattice of rank less than 11 admits a primitive embedding in Λ_{K3}. So its enough to construct all the non isometric even overlattices of $M \oplus \mathbb{Z}n$ of index $r \in \mathbb{N}$ in order to classify the admissible lattices L. We already proved that r is either 1 or 2 in proof of Theorem 3.4.

The first admissible choice for L is $M\oplus\mathbb{Z}n \simeq \langle 4\rangle\oplus\langle -2\rangle\oplus\langle -2\rangle$ (which clearly corresponds to $r = 1$).

Now we look for an overlattice of index $r = 2$. A $\mathbb{Q}$-basis of L is given by n_1, n_2 and n. If this is not a $\mathbb{Z}$ basis, then there exists a vector $w := (a_1n_1+a_2n_2+a_3n)/2$, $a_i \in \mathbb{Z}/2\mathbb{Z}$, such that $w \notin (M\oplus\mathbb{Z}n)$, $wn_1 \in \mathbb{Z}$, $wn_2 \in \mathbb{Z}$, $wn \in \mathbb{Z}$ and $ww \in 2\mathbb{Z}$. The condition $ww \in 2\mathbb{Z}$ immediately implies $a_2 \equiv a_3 \mod 2$. If a_2 and a_3 are both even, then $w \equiv n_1/2 \mod (M \oplus \mathbb{Z}n)$ and $(n_1/2)^2 = 1 \notin 2\mathbb{Z}$. So $a_2 \equiv a_3 \equiv 1 \mod 2$. Again the condition $ww \in 2\mathbb{Z}$ implies that $a_1 \equiv 1 \mod 2$. So there exists a unique overlattice of index $r \neq 1$ of $M \oplus \mathbb{Z}n$ and it is the lattice obtained by adding to the $\mathbb{Q}$ basis $\{n_1, n_2, n\}$ the vector $(n_1 + n_2 + n)/2$. We can now find a change of bases in order to give a better description of this overlattice: let us consider the $\mathbb{Z}$-basis $\{(n_1 - n_2 - n)/2, (n_1 + n_2 - n)/2, -n_1 + 2n\}$. Computing the bilinear form on this basis we find $U \oplus \langle -4\rangle$, so the unique even hyperbolic overlattice of index 2 of $M \oplus \mathbb{Z}n$ is isometric to $U \oplus \langle -4\rangle$.

4.1.1. Remarks on the orthogonal to M. There exists a unique embedding (up to isometries) of M in $\Lambda_{K3} \simeq U \oplus U \oplus U \oplus E_8 \oplus E_8$ which is given by

$$n_1 := \begin{pmatrix} 1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \underline{0} \oplus \underline{0},\quad n_2 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ -1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \underline{0} \oplus \underline{0}.$$

Let us denote by T the lattice $M^{\perp_{\Lambda_{K3}}}$. It is generated by the generators of the third copy of U, by the generators of $E_8 \oplus E_8$ and by the two vectors:

$$t_1 := \begin{pmatrix} -1 \\ 2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \underline{0} \oplus \underline{0},\quad t_2 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \underline{0} \oplus \underline{0}.$$

Since Λ_{K3} is unimodular, it is an overlattice of index 8 of $M\oplus T$ and indeed $\Lambda_{K3}/(L\oplus M)$ is generated by $(t_1 + n_1)/4$ and $(t_2 + n_2)/2$.

In order to construct a lattice isometric to $M\oplus\mathbb{Z}n$ we have to identify a vector $n \in T$ with $n^2 = -2$. Let us consider the following two possibilities:

- $n := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ -1 \end{pmatrix} \oplus \underline{0} \oplus \underline{0}$. In this case there is no overlattice of $L \oplus \mathbb{Z}n$ contained in Λ_{K3} and so L is $\langle 4\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$. The orthogonal of such a lattice in Λ_{K3} is generated by the generators of $E_8 \oplus E_8$, by t_1, by t_2 and by the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \underline{0} \oplus \underline{0}$. It is isometric to $E_8 \oplus E_8 \oplus \langle -4\rangle \oplus \langle 2\rangle \oplus \langle 2\rangle$.
- $n := t_1 + t_2$. In this case the lattice generated by $n_1, n_2, (n_1 + n_2 + n)/2$ is primitively embedded in Λ_{K3} and is clearly an overlattice of index 2 of $M\oplus\mathbb{Z}n$. The orthogonal of such a lattice in Λ_{K3} is generated by the generators of the third copy of U, the generators of $E_8 \oplus E_8$ and by $t_1 + 2t_2$. This lattice is isometric to $U \oplus E_8 \oplus E_8 \oplus \langle 4\rangle$.

4.2. Some K3 surfaces with Picard number 2.

4.2.1. Quartic with a node. Let us consider a K3 surface Y which admits a model as quartic with exactly one ordinary double point, which is a singularity of type A_1. Then there exists a pseudo-ample divisor H in $NS(Y)$ with self intersection 4 and which is orthogonal to a (-2)-vector. Indeed $\varphi_{|H|} : Y \to \mathbb{P}^3$ contracts exactly one rational curve to the ordinary double point. The class of this curve, called N_1, has self intersection -2 (since the curve is a rational curve on a K3 surface) and is orthogonal to H (since the curve is contracted by $\varphi_{|H|}$). Hence the lattice $\langle H, N_1\rangle \simeq \langle 4\rangle \oplus \langle -2\rangle$ is primitively embedded in the Néron–Severi group of a K3 surface which admits a model as quartic with a singularity of type A_1. Moreover the computation of the moduli of the family of the nodal quartics, implies that generically $NS(Y) \simeq \langle 4\rangle \oplus \langle -2\rangle$.

Up to projective transformations, we can assume that the node of the quartic is in the point $(1 : 0 : 0 : 0) \in \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$ and so an equation for Y is of the form

$$x_0^2 F_2(x_1 : x_2 : x_3) + x_0 F_3(x_1 : x_2 : x_3) + F_4(x_1 : x_2 : x_3) = 0, \tag{4.1}$$

where F_i are generic homogenous polynomials of degree i.

Let us consider the projection of the quartic (4.1) from $(1 : 0 : 0 : 0)$ to $\mathbb{P}^2_{(x_1:x_2:x_3)}$. It exhibits Y as double cover of $\mathbb{P}^2_{(x_1:x_2:x_3)}$ branched along the sextic curve

$$C_6 := V\left(F_3(x_1 : x_2 : x_3)^2 - 4F_2(x_1 : x_2 : x_3)F_4(x_1 : x_2 : x_3)\right) \subset \mathbb{P}^2_{(x_1:x_2:x_3)} \tag{4.2}$$

The conic $c_2 := V(F_2(x_1 : x_2 : x_3))$ intersects the branch locus in the six points $F_2(x_1 : x_2 : x_3) = F_3(x_1 : x_2 : x_3) = 0$ and each of them has multiplicity 2.

The projection from the singular point of the quartic is associated to the divisor $H - N_1$.

The cover involution of the $2 : 1$ map $\varphi_{|H-N_1|} : Y \to \mathbb{P}^2$ is an involution, called ι, of Y and it does not preserve the symplectic structure of Y (indeed Y/ι is birational to $\mathbb{P}^2$). The class of $H - N_1$ is clearly preserved by the isometry ι^*, but the class N_1 is not. Indeed, since c_2 is everywhere tangent to the branch locus, its inverse image consists of two disjoint curves. One of them is N_1 and the other is $2H - 3N_1$. So ι^* acts as -1 on the transcendental lattice and as

$$\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} \tag{4.3}$$

on the basis $\{H, N_1\}$ of the Néron–Severi group.

Remark 4.3. The K3 surface Y admits two different (equivalent up to automorphism of the surface, but not up to projectivity of $\mathbb{P}^3$) models as singular quartic in $\mathbb{P}^3$. One of them is associated to the divisor H, the other one to $\iota^*(H) = 3H - 4N_1$.

Since Y has a smooth rational curve and its Picard number is 2, the automorphism group of Y is finite. To be more precise it is known that the automorphism group of Y generically coincides with $\mathbb{Z}/2\mathbb{Z} \simeq \langle\iota\rangle$ (see e.g. [2]).

4.2.2. K3 surfaces with an elliptic fibration. Let V be a K3 surface and $\mathcal{E} : V \to \mathbb{P}^1$ be an elliptic fibration (i.e. a fibration in curves of genus 1 which admits at least one section). We will denote by F the class of the fiber of $\mathcal{E}$ and by s_0 the class of a given section (called zero section) of $\mathcal{E}$.

Hence $NS(V) \supset \langle F, s_0\rangle$. We obseve that $F^2 = 0$, $Fs_0 = 1$ and $s_0^2 = -2$ so the bilinear form computed on the basis $\{F, F + s_0\}$ is given by the matrix U.

If V is generic among the K3 surfaces admitting an elliptic fibration, then $NS(V) \simeq U$.

On each elliptic curve (so on each smooth fiber of $\mathcal{E}$) the hyperelliptic involution is well defined. The hyperelliptic involutions on the fibers glue together giving an involution, called h, of V. The quotient by this involution is the Hirzebruch surface $\mathbb{F}_4$ and the ramification locus consists of the zero section and of the trisection t_{10} passing through the 2-torsion of each fiber. The trisection t_{10} is a curve of genus 10 and is represented by the class $6F + 3s_0$.

Since both the class of the section s_0 and the class of the fiber F are preserved by h, h^* is the identity on $NS(V)$ and -1 on the transcendental lattice.

The divisor $4F + 2s_0$ defines a $2:1$ map $\varphi_{|4F+2s_0|} : V \to \mathcal{C} \subset \mathbb{P}^5$ where $\mathcal{C}$ is a cone over the twisted quartic curve in $\mathbb{P}^4$ (see [8]). We observe that $\mathcal{C}$ is a model of $\mathbb{F}_4$ obtained contracting the exceptional curve.

The zero section s_0 is contracted by $\varphi_{|4F+2s_0|}$ and its image is the vertex of the cone. So the double cover $V \to \mathcal{C}$ is branched along the image of the trisection, i.e. on $\varphi_{|4F+2s_0|}(t_{10})$. The involution h is exactly the cover involution of this double cover.

Since V contains a rational curve and $\rho(V) = 2$, $|\mathrm{Aut}(V)| < \infty$. To be more precise, for a generic choice of V, $\mathrm{Aut}(V) = \langle h\rangle$ (see e.g. [2]).

4.3. K3 surfaces with Picard number 3.

4.3.1. Quartic with two nodes (lattice $L \simeq \langle 4\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$). Let us now consider a K3 surface X such that $NS(X) \simeq \langle 4\rangle \oplus \langle -2\rangle \oplus \langle -2\rangle$. We assume that X is generic among the ones with this property and we denote by $\{H, N_1, N_2\}$ the basis of $NS(X)$. We can assume that H is a pseudo ample divisor. The map $\varphi_{|H|} : X \to \mathbb{P}^3$ exhibits X as quartic surface with two double points (the contraction of N_1 and N_2). Up to projective transformations we can assume that the singular point $\varphi_{|H|}(N_1)$ is $(1:0:0:0) \in \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$ (as in Section 4.2.1) and the singular point $\varphi_{|H|}(N_2)$ is $(0:0:0:1) \in \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$. So any quartic in this family has the following equation:

$$\begin{aligned} x_0^2F_2(x_1:x_2:x_3) + x_0\Big(G_3(x_1:x_2) + x_3G_2(x_1:x_2) + x_3^2G_1(x_1:x_2)\Big) + \\ + x_3^2H_2(x_1:x_2) + x_3H_3(x_1:x_2) + H_4(x_1:x_2) = 0 \end{aligned} \tag{4.4}$$

where F_i, G_i, H_i are homogenous polynomial of degree i.

As in Section 4.2.1, we consider the projection from $(1:0:0:0)$ which corresponds to the divisor $H - N_1$. This gives a $2:1$ map $\varphi_{|H-N_1|} : X \to \mathbb{P}^2_{(x_1:x_2:x_3)}$ which is a double cover branched along the sextic:

$$\begin{aligned} C_6 := V\Big(\Big(G_3(x_1:x_2) + x_3G_2(x_1:x_2) + x_3^2G_1(x_1:x_2)\Big)^2 + \\ -4F_2(x_1:x_2:x_3)\Big(x_3^2H_2(x_1:x_2) + x_3H_3(x_1:x_2) + H_4(x_1:x_2)\Big)\Big). \end{aligned}$$

The sextic C_6 has a unique singular point, which is an ordinary node, in $(0:0:1)$, i.e. in the point $\varphi_{|H-N_1|}(N_2)$. Moreover (as in section 4.2.1), the conic $V(F_2(x_1:x_2:x_3))$ intersect C_6 in six smooth points and is tangent to C_6 in each of these points. This conic is the image for $\varphi_{|H-N_1|}$ of the rational curve N_1.

The cover involution ι_1 preserves both the class $H - N_1$ (which is the pull back of the hyperplane section of $\mathbb{P}^2_{(x_1:x_2:x_3)}$) and the class N_2 (which is sent to the node of the sextic C_6 and thus is preserved by the cover involution). Viceversa, the curve corresponding to N_1 is not preserved by ι_1 and $\iota_1^*(N_1) = 2H - 3N_1$ (see Section 4.2.1). So the involution ι_1^* on $NS(X)$ is represented, with respect to the basis $\{H, N_1, N_2\}$, by the matrix:

$$\iota_1^* := \begin{bmatrix} 3 & 2 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the quartic surface (4.4) has two nodes, we can consider two different projections to $\mathbb{P}^2$: the projection (already considered) from $(1:0:0:0)$ to $\mathbb{P}^2_{(x_1:x_2:x_3)}$, associated to the linear system $|H-N_1|$, which exhibits X as double cover of $\mathbb{P}^2_{(x_1:x_2:x_3)}$ whose cover involution is ι_1; and the projection from $(0:0:0:1)$ to $\mathbb{P}^2_{(x_0:x_1:x_2)}$, which is associated to the linear system $|H-N_2|$. It exhibits X as double cover of $\mathbb{P}^2_{(x_0:x_1:x_2)}$ and we call the cover involution ι_2. The involution ι_2^* of $NS(X)$ is represented with respect to the basis $\{H, N_1, N_2\}$ by the matrix

$$\iota_2^* := \begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ -4 & 0 & -3 \end{bmatrix}.$$

We observe that $\iota_1\iota_2$ is an automorphism of infinite order of X, indeed the associated isometry of $NS(X)$ has infinite order and is represented on the basis $\{H, N_1, N_2\}$ by the matrix

$$(\iota_1\iota_2)^* := \begin{bmatrix} 9 & 2 & 6 \\ -12 & -3 & -8 \\ -4 & 0 & -3 \end{bmatrix}.$$

So $\mathrm{Aut}(X)$ is infinite (indeed $\iota_1\iota_2 \in \mathrm{Aut}(X)$).

The map $\varphi_{|H-N_1|}$ gives a model of X which contracts exactly one rational curve, the curve corresponding to the class N_2. Indeed $(H-N_1)^{\perp_{NS(X)}} \simeq \langle -4\rangle \oplus \langle -2\rangle$ which clearly contains exactly two vectors with self intersection -2 and only one of them is effective. We observe that ι_2^* does not preserves the class $H-N_1$, so it does not descend to an automorphism of $\varphi_{|H-N_1|}(X)$. Viceversa ι_1 is (by definition) an automorphism of the model associated to $\varphi_{|H-N_1|}$. The restriction of ι_1^* to $N_2^{\perp_{NS(X)}} \simeq NS(Y)$ coincides with the action of ι^* (where ι is the unique non trivial automorphism of Y) on $NS(Y)$, given in (4.3).

Denoted by X' the (singular) double cover of $\mathbb{P}^2$ branched along C_6, (X, X') is an admissible pair and $\mathrm{Aut}(X')$ is generated by the cover involution. Indeed, ι_1 induces the cover involution on X', so $\mathrm{Aut}(X') \supset \mathbb{Z}/2\mathbb{Z}$. By Corollary 3.8, $\mathrm{Aut}(X') \subset \mathrm{Aut}(Y)$ and since $\mathrm{Aut}(Y) = \mathbb{Z}/2\mathbb{Z}$, we deduce that $\mathrm{Aut}(X') = \mathbb{Z}/2\mathbb{Z}$.

4.3.2. Elliptic fibrations with non trivial Mordell Weil group (lattices $U \oplus \langle -2d\rangle$). Here we consider K3 surfaces with Picard number 3 and an elliptic fibration. Since U is primitively embedded in the Néron–Severi group of any K3 surface admitting an elliptic fibration and it is a unimodular lattice, the Néron–Severi group of a K3 surface with an elliptic fibration and with Picard number 3 is isometric to $U \oplus \langle -2d\rangle$, $d \in \mathbb{N}_{>0}$.

We will denote by S_d a K3 surface such that $NS(S_d) \simeq U \oplus \langle -2d\rangle$ and we will assume that S_d is generic among the K3 surfaces with this property. First we discuss the geometric properties of these surfaces S_d and then we focus on the case $d = 2$.

Let us denote by $\{b_1, b_2, b_3\}$ the basis of $NS(S_d)$ on which the bilinear form is represented by $U \oplus \langle -2d\rangle$. Then we put $F := b_1$ and $s_0 := b_2 - b_1$. The lattice $F^{\perp_{NS(S_d)}}$ consists of the vector $w := (w_1, w_2, w_3)$ such that $w_2 = 0$. The bilinear form computed on w is $2w_1w_2 - 2dw_3^2$, so $F^{\perp_{NS(S_d)}}$ contains a (-2)-class if and only if $d = 1$.

So if $d = 1$, then the elliptic fibration $\varphi_{|F|} : S_d \to \mathbb{P}^1$ admits exactly one reducible fiber, which is necessarily of type I_2, since S_1 is generic.

If $d>1$, then the elliptic fibration $\varphi_{|F|}: S_d \to \mathbb{P}^1$ has no reducible fibers and thus, by Shioda–Tate formula, the rank of the Mordell Weil group is 1. From now on we assume $d>1$ and we denote by s_1 a section of $\varphi_{|F|}: S_d \to \mathbb{P}^1$ which generates the Mordell–Weil group. Hence $\{F, s_0, s_1\}$ is a basis of $NS(S_d)$. It is related to the basis $\{b_1, b_2, b_3\}$ by the following: $F=b_1$, $s_0=b_2-b_1$, $s_1=(d-1)b_1+b_2+b_3$. It is immediate to check that the bilinear form computed on this basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & d-2 \\ 1 & d-2 & -2 \end{bmatrix}.$$

So the K3 surfaces S_d, $d>1$, can be geometrically described as the K3 surfaces admitting an elliptic fibration such that the rank of the Mordell-Weil group is 1 and the intersection among the zero section and a generator of the Mordell–Weil group is $d-2$.

We observe that the classes $s_n := (dn^2-1)b_1+b_2+nb_3$, $n\in\mathbb{Z}$ are the sections of the fibration $\varphi_{|F|}: S_d \to \mathbb{P}^1$ and there is an isomorphism of groups between the Mordell–Weil group of $\varphi_{|F|}$ and $\mathbb{Z}$ given by $s_n \mapsto n$. Fixed a value d, we have $s_0s_n = dn^2-2$. Moreover $s_is_{i+n} = s_0s_n$, since s_i is the translation (by s_i) of s_0 and s_{i+n} is the translation (by s_i) of s_n. It is immediate to check that $s_0s_k > s_0s_h$ if $|k|>|h|$, so s_0s_1 is the minimal possible intersection number among two sections of $\varphi_{|F|}: S_d \to \mathbb{P}^1$.

4.3.3. The K3 surface $S := S_2$ ($NS(S) \simeq L \simeq U\oplus\langle -4\rangle$). In case $d=2$, $NS(S_2)\simeq U\oplus\langle -4\rangle$. In the following we will denote by S the surface S_2, in order to simplify the notation. The K3 surface S is the generic K3 surface with an elliptic fibration, such that there is a section of infinite order which has a trivial intersection with the zero section.

Our purpose is to describe a map (a geometric model) of S which contracts exactly one rational curve (indeed we already know that there exists a rational curve on S such that the orthogonal to the class of this curve in $NS(S)$ is isometric to the lattice $M\simeq\langle 4\rangle\oplus\langle -2\rangle$, by Section 4.1).

We consider the map $4F+2s_0$. It exhibits S as double cover of a cone $C\subset\mathbb{P}^5$ over the twisted rational curve of degree 4 in $\mathbb{P}^4$, as in Section 4.2.2. The curve s_0 is contracted and it is the unique rational curve which is contracted by this map. Indeed the orthogonal to $4F+2s_0 = 2b_1+2b_2$ in $NS(S_2)$ is generated by $b_2-b_1=s_0$ and b_3. So, if r is the class of a rational curve in S contracted by $4F+2s_0$, then $r=z_1(b_2-b_1)+z_2b_3$, $z_1, z_2\in\mathbb{Z}$; $r^2=-2$; $rF=rb_1\geq 0$. This implies that $z_1=1$ and $z_2=0$, so $r=s_0$.

The family of K3 surfaces which are double covers of $C\subset\mathbb{P}^5$ is 18-dimensional (the K3 surface V described in Section 4.2.2 is a general member of such a family), but S is the general member of a 17-dimensional subfamily. Indeed there is a peculiarity in the branch locus, $\varphi_{|4F+2s_0|}(t_{10})$, of the double cover $\varphi_{|4F+2s_0|}: S\to C$: there exists a curve which is tangent to $\varphi_{|4F+2s_0|}(t_{10})\subset C$ in each of their intersection points. Indeed the class $4F+2s_0$ is equivalent to s_1+s_{-1} in $NS(S)$. This means that $\varphi_{|4F+2s_0|}(s_1)=\varphi_{|4F+2s_0|}(s_{-1})$ and thus the inverse image of $\varphi_{|4F+2s_0|}(s_1)$ splits in the double cover. So $\varphi_{|4F+2s_0|}(s_1)$ is tangent to $\varphi_{|4F+2s_0|}(t_{10})$ in all their intersection points, which are $6=s_1t_{10}$. We observe that since $s_1s_0=0$, the curve $\varphi_{|4F+2s_0|}(s_1)$ does not pass through the vertex of C.

The automorphism group of S is surely infinite and contains at least the following automorphisms: h which is the cover involution of the double cover $S \to C$ (i.e. it is the hyperelliptic involution of the elliptic fibration $\varphi_{|F|} : S \to \mathbb{P}^1$) and the automorphism T_{s_1} which is the translation by the section s_1. We observe that: $h(F) = F$, $h(s_0) = s_0$, $h(s_n) = s_{-n}$ (indeed h switches for example the sections s_1 and s_{-1}) and that $T_{s_1}(F) = F$ and $T_{s_1}(s_n) = s_{n+1}$, $n \in \mathbb{Z}$. With respect to the basis $\{b_1, b_2, b_3\}$ these automorphisms are represented by

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad T_{s_1} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is immediate to check that the class $4F + 2s_0 = 2b_1 + 2b_2$ is preserved by h but not by T_{s_1}, so T_{s_1} does not descend to an automorphism of the singular model $\varphi_{|4F+2s_0|} : S \to C$. Indeed the class s_0, which is the class of the unique curve contracted by $\varphi_{|4F+2s_0|}$, is not preserved by T_{s_1}.

The map $\varphi_{|4F+2s_0|}$ contracts s_0 and we observe that the lattice $s_0^{\perp_{NS(Y_2)}} \simeq M = \langle 4\rangle \oplus \langle -2\rangle$ is generated by $2b_1 + 2b_2 + b_3$ and $b_1 + b_2 + b_3$. The involution h^* restricted to the lattice $\langle (2b_1 + 2b_2 + b_3), (b_1 + b_2 + b_3)\rangle$ is represented by the matrix:

$$\begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix},$$

which in fact coincides with the involution ι^* defined on M by the automorphism ι of Y.

Denoted by S' the (singular) double cover of C branched along $\varphi_{|4F+2s_0|}(t_{10})$, (S, S') is an admissible pair and $\mathrm{Aut}(S')$ is generated by the cover involution. Indeed, h induces the cover involution on S', so $\mathrm{Aut}(S') \supset \mathbb{Z}/2\mathbb{Z}$. By Corollary 3.8, $\mathrm{Aut}(S') \subset \mathrm{Aut}(Y)$ and since $\mathrm{Aut}(Y) = \mathbb{Z}/2\mathbb{Z}$, we deduce that $\mathrm{Aut}(S') = \mathbb{Z}/2\mathbb{Z}$.

5. Other examples

In this section we briefly describe other geometric examples: We conclude the proof of the Theorem 3.4 in cases $(L, M) \simeq (U \oplus E_8 \oplus A_1^5, U \oplus D_6 \oplus D_4), (U(2) \oplus A_1^8, U(2) \oplus A_1^7), (U \oplus E_8(2), U(2) \oplus A_1^7)$, showing the existence of the smooth irreducible rational curve N which has to be contracted in order to obtain the Mori Dream Space X'. We also give an explicit equation of an elliptic fibration mentioned in the proof of the same theorem.

The principal results of this section are geometric descriptions of the following situations:
• there exists two admissible pairs (X_1, X_1') and (X_2, X_2') such that $X_1 \simeq X_2$ but $X_1' \not\simeq X_2'$ (see Example 5.1);
• there exists two infinite series of admissible pairs (S_d, S_d') and (Q_d, Q_d') such that $\rho(Q_d) = \rho(S_d) = 3$, (the minimal possible), and moreover $NS(S_d) \not\simeq NS(Q_d)$ but $NS(S_d') \simeq NS(Q_d')$, see Proposition 5.2.

5.1. The K3 surface with $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$. Let X be the K3 surface such that $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$. In the proof of Theorem 3.4 we showed that X admits a smooth irreducible rational curve N which can be contracted in order to obtain a Mori Dream Space X' and that this curve is one of the two components of one of the two fibers of type I_2 of a certain elliptic fibration $\mathcal{E} : X \to \mathbb{P}^1$. The existence of this fibration immediately follows from the decomposition of $NS(X)$ in the direct sum $U \oplus E_8^2 \oplus A_1^2$. Here we give also an explicit equation of this fibration.

Let us consider the pencil of plane cubics

$$V((x_1^3 + x_0^2x_2 + x_1^2x_2) + tx_2^3) \subset \mathbb{P}^2_{(x_0:x_1:x_2)}$$

generated by a triple line l and a smooth cubic C_3 which admits the line l as inflectional tangent. It is well known (and easy to check) that this pencil induces an elliptic fibration on the rational surface which is the blow up of $\mathbb{P}^2$ nine times in the intersection point between l and C_3. This elliptic fibration has one fiber of type II^* (which is the pull back of l) and two fibers of type I_1. With the coordinates $x_2 = 1$, $y = x_0$ and $x = x_1 - 1/3$ we immediately obtain the Weierstrass form

$$y^2 = x^3 - \frac{1}{3}x + t$$

which has a fiber of type II^* at infinity and two fibers of type I_1 in $t = \pm 2/27$. So the double cover $\mathbb{P}^1_{(T:S)} \to \mathbb{P}^1_{(t:s)}$ which is branched in $t = \pm\frac{2}{27}$, induces the base change $s := \frac{27}{4}(S^2 - T^2)$ and $t := \frac{1}{2}(S^2 + T^2)$ which gives the elliptic fibration

$$y^2 = x^3 - \frac{1}{3}x\left(\frac{27}{4}S^2 - \frac{27}{4}T^2\right)^4 + \left(\frac{1}{2}S^2 + \frac{1}{2}T^2\right)\left(\frac{27}{4}S^2 - \frac{27}{4}T^2\right)^5.$$

This is in fact the equation of the unique (up to projective transformation) elliptic fibration over $\mathbb{P}^1_{(T:S)}$ with reducible fibers $2II^* + 2I_2$ and so it is an equation of (a singular model of) the unique K3 surface X with $NS(X) \simeq U \oplus E_8^2 \oplus A_1^2$. The fibers over $(1:0)$ and $(0:1)$ are of type I_2, so each of them consists of two smooth irreducible rational curves meeting in 2 points. One of these rational curves meets the zero section of the fibration. The other can be chosen to be the curve N.

5.2. K3 surfaces with $NS(X) \simeq U(2) \oplus A_1^8$. We give a geometric description of the K3 surface X, which is generic among the K3 surfaces such that $NS(X) \simeq U(2) \oplus A_1^8$. We will show that it surely contains a smooth irreducible rational curve which can be contracted in order to obtain a singular surface X' whose Néron–Severi group is isometric to $U(2) \oplus A_1^7$. This concludes the proof of Theorem 3.4 in case $NS(X) \simeq U(2) \oplus A_1^8$.

Since there exists a unique even hyperbolic lattice L such that: $\mathrm{rank}(L) = 10$, the discriminant group is $(\mathbb{Z}/2\mathbb{Z})^{10}$ and the discriminant form takes values in $\frac{1}{2}\mathbb{Z}$ (and not in $\mathbb{Z}$), we find $U(2) \oplus A_1^8 \simeq \langle 2\rangle \oplus \langle -2\rangle^9$. So $NS(X) \simeq \langle 2\rangle \oplus \langle -2\rangle^9$. We can assume that one of the primitive generators of the sublattice $\langle 2\rangle \hookrightarrow NS(X)$ is pseudo ample. We denote this divisor by A and we observe that $\phi_{|A|} : X \to \mathbb{P}^2$ exhibits X as double cover of $\mathbb{P}^2$ branched along an irreducible sextic C_6 with 9 nodes $P_1, \dots, P_9$. The double cover of the blow up of $\mathbb{P}^2$ in these 9 points is a smooth minimal model of X and it contains 9 smooth rational curves which are the double cover of the 9 exceptional divisors. The classes of these rational curves are represented by (-2)-classes orthogonal to A and mutually orthogonal. So X admits at least one smooth rational curve (indeed at least 9) such that the contraction of this curve produces a singular surface X' whose Néron–Severi group is isometric to $\langle 2\rangle \oplus \langle -2\rangle^8 \simeq U(2) \oplus A_1^7$. A geometric construction of X' is the following: let us consider the irreducible sextic curve C_6 with 9 nodes $P_1, \dots P_9$ such that $\phi_{|A|} : X \to \mathbb{P}^2$ is branched along C_6. Let us blow up $\mathbb{P}^2$ in the eight points $P_1, \dots, P_8$ and let us denote by $\widetilde{\mathbb{P}^2}$ the surface obtained by this blow up. Let us denote by $\widetilde{C_6}$ the strict transform of C_6 for this blow up. We observe that $\widetilde{C_6}$ has exactly one singular point. The double cover of $\widetilde{\mathbb{P}^2}$ branched along $\widetilde{C_6}$ is X', indeed the following diagram commute:

$$\begin{tikzcd}
X' \arrow[r, "2:1"] & \widetilde{\mathbb{P}^2} \arrow[dr] & \\
X \arrow[u, "\phi"] \arrow[r, "2:1"] & \widetilde{\widetilde{\mathbb{P}^2}} \arrow[u] \arrow[r] & \mathbb{P}^2
\end{tikzcd}$$

where $\widetilde{\widetilde{\mathbb{P}^2}}$ is the blow up of $\widetilde{\mathbb{P}^2}$ in the unique singular point of $\widetilde{C_6}$ and it coincides with the blow up of $\mathbb{P}^2$ in the nine points $P_1, \dots, P_9$ and $\phi : X \to X'$ is the contraction of the smooth rational curve of X which is the double cover of the exceptional divisor of the blow up $\widetilde{\widetilde{\mathbb{P}^2}} \to \widetilde{\mathbb{P}^2}$.

5.3. K3 surfaces with $NS(X) \simeq U \oplus E_8(2)$. We give a geometric description of the K3 surface X, which is generic among the K3 surfaces such that $NS(X) \simeq U \oplus E_8(2)$. We will show that it surely contains a smooth irreducible rational curve which can be contracted in order to obtain a singular surface X' whose Néron–Severi group is isometric to $U(2) \oplus A_1^7$. This concludes the proof of Theorem 3.4 in case $NS(X) \simeq U \oplus E_8(2)$.

Since there exists a unique even hyperbolic lattice L such that: $\mathrm{rank}(L) = 10$, the discriminant group is $(\mathbb{Z}/2\mathbb{Z})^8$ and the discriminant form takes values in $\mathbb{Z}$, we find that $U \oplus E_8(2)$ is the unique overlattice of index 2 of $U(2) \oplus A_1^8$, and it is generated by the generators of $U(2) \oplus A_1^8$, $\{u_1, u_2, N_1, \dots N_8\}$, and by the class $(\sum_{i=1}^8 N_i)/2$.

We can assume that a choice of the primitive generators of the sublattice $U(2) \hookrightarrow NS(X)$ consists of nef divisors, and we denote them by u_1 and u_2. We observe that $\phi_{|u_1+u_2|} : X \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ exhibits X as double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a (possibly reducible) curve $C_{4,4}$ of bidegree $(4, 4)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ with 8 nodes $P_1 = \phi_{|u_1+u_2|}(N_1), \dots, P_8 = \phi_{|u_1+u_2|}(N_8)$. The class of the reduced curve $\phi^{-1}_{u_1+u_2}(C_{4,4})$ in $NS(X)$ is given by $2u_1 + 2u_2 - \sum_{i=1}^8 N_i$. The curve $C_{(4,4)}$ is in fact reducible and it is the union of two curves of bidegree $(2, 2)$, which corresponds on X to the class $\left(2u_1 + 2u_2 - \left(\sum_{i=1}^8 N_i\right)\right)/2 \in NS(X)$. These two components meet exactly in the 8 points P_i. The double cover of the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in these 8 points is a smooth minimal model of X and it contains 8 smooth rational curves, N_i, which are the double cover of the 8 exceptional divisors. The classes of these rational curves are represented by (-2)-classes orthogonal to $u_1 + u_2$ and mutually orthogonal. So X admits at least one smooth rational curve (indeed at least 8) such that the contraction of this curve produces a singular surface X'. The Néron–Severi group of X' is the orthogonal to an N_i, say to N_8, in the lattice spanned by $\{u_1, u_2, N_1, \dots N_8, (\sum_{i=1}^8 N_i)/2\}$ and it is generated by $\{u_1, u_2, N_1, \dots N_7\}$. So $NS(X') \simeq U(2) \oplus A_1^7$. A geometric construction of X' is the following: let us consider the reducible curve $C_{4,4}$ with 8 nodes $P_1, \dots P_8$ such that $\phi_{|u_1+u_2|} : X \to \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ is branched along $C_{4,4}$. Let $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ be the blow up $\mathbb{P}^1 \times \mathbb{P}^1$ in the seven points $P_1, \dots, P_7$. Let us denote by $\widetilde{C_{4,4}}$ the strict transform of $C_{4,4}$ for this blow up. We observe that $\widetilde{C_{4,4}}$ has exactly one singular point. The double cover of $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ branched along $\widetilde{C_{4,4}}$ is X', indeed the following diagram commute:

$$\begin{tikzcd}
X' \arrow[r, "2:1"] & \widetilde{\mathbb{P}^1\times\mathbb{P}^1} \arrow[dr] & \\
X \arrow[u, "\phi"] \arrow[r, "2:1"] & \widetilde{\widetilde{\mathbb{P}^1\times\mathbb{P}^1}} \arrow[u] \arrow[r] & \mathbb{P}^1\times\mathbb{P}^1
\end{tikzcd}$$

where $\widetilde{\widetilde{\mathbb{P}^1\times\mathbb{P}^1}}$ is the blow up of $\widetilde{\mathbb{P}^1\times\mathbb{P}^1}^2$ in the unique singular point of $\widetilde{C_{4,4}}$ and it coincides with the blow up of $\mathbb{P}^1\times\mathbb{P}^1$ in the eight points $P_1,\ldots,P_8$ and $\phi : X \to X'$ is the contraction of the smooth rational curve of X which is the double cover of the exceptional divisor of the blow up $\widetilde{\widetilde{\mathbb{P}^1\times\mathbb{P}^1}} \to \widetilde{\mathbb{P}^1\times\mathbb{P}^1}$.

5.4. Two admissible pairs (X, X'_1) and (X, X'_2) with $X'_1 \not\simeq X'_2$ and $\rho(X) = 15$. We consider a K3 surface X whose Néron–Severi group is isometric to $U \oplus D_8 \oplus D_4 \oplus A_1$, so it is not a Mori Dream Space (cf. Proposition 3.4 and [5]). We show that it admits two different rational curves N_1 and N_2 such that, denoted by X'_i the singular surface contraction of the curve N_i, the two pairs (X, X'_1) and (X, X'_2) are both admissible and $X'_1 \not\simeq X'_2$. This gives geometric examples of the cases with $\rho(X) = 15$ of the Table 3.1 and concludes the proof of Theorem 3.4.

EXAMPLE 5.1. Let X be a K3 surface admitting an elliptic fibration $\mathcal{E} : X \to \mathbb{P}^1$ whose singular fibers are $I_4^* + I_0^* + I_2 + 6I_1$ and whose Mordell–Weil group is trivial. The Néron–Severi group of the surface is isomorphic to $U\oplus D_8\oplus D_4\oplus A_1$ and is generated by the classes $F, s_0, \Theta_i^{(1)}, i = 1,\ldots 8, \Theta_j^{(2)}, j = 1,2,3,4, \Theta_1^{(3)}$, where F is the class of the fiber of the fibration $\mathcal{E}$, s_0 is the class of the unique section of the fibrations, $\Theta_i^{(1)}$ are the eight components of the fibers of type I_4^* which do not intersect the zero section, $\Theta_j^{(2)}$ are the 4 components of the fiber of type I_0^* which do not intersect the zero section, $\Theta_1^{(3)}$ is the component of the fiber of type I_2 which does not intersect the zero section. The map $\phi : X \to X'_1$ which contracts the curve $N_1 := \Theta_1^{(3)}$ produces the singular surface X'_1 such that $NS(X'_1) \simeq U \oplus D_8 \oplus D_4$.

In the following we will assume that the intersection properties of the components of the fibers of type I_n^* are numbered as follow:

$$\begin{tikzcd}[column sep=small]
\Theta_0 \arrow[dr, dash] & & & & & & \Theta_{n+3} \\
& \Theta_2 \arrow[r, dash] & \Theta_3 \arrow[r, dash] & \cdots & \Theta_{n+1} \arrow[r, dash] & \Theta_{n+2} \arrow[ur, dash] \arrow[dr, dash] & \\
\Theta_1 \arrow[ur, dash] & & & & & & \Theta_{n+4}
\end{tikzcd}$$

Let us consider the divisor

$$D := \Theta_5^{(1)} + 2\Theta_4^{(1)} + 3\Theta_3^{(1)} + 4\Theta_2^{(1)} + 5\Theta_0^{(1)} + 6s_0 + 4\Theta_0^{(2)} + 2\Theta_2^{(2)} + 3\Theta_0^{(3)}$$

on X.

We observe that $D^2 = 0$ and D is an effective divisor. Moreover the curve $\Theta_6^{(1)}$ is a rational curve such that $D\Theta_6^{(1)} = 1$. Thus, the linear system $|D|$ defines an elliptic fibration $\phi_{|D|} : X \to \mathbb{P}^1$. The class of D is the class of the fiber of the fibration and the divisor D exhibits a reducible fiber of type II^* of this fibration. Let us denote by R the lattice

$\langle\Theta_5^{(1)},\Theta_4^{(1)},\Theta_3^{(1)},\Theta_2^{(1)},\Theta_0^{(1)},s_0,\Theta_0^{(2)},\Theta_2^{(2)},\Theta_3^{(3)},\Theta_6^{(1)}\rangle$. Then $R \simeq U\oplus E_8$ and the orthogonal complement of R in $NS(X)$ is isometric to A_1^5 (since the discriminant group of $NS(X)$ is $(\mathbb{Z}/2\mathbb{Z})^5$). So the reducible fibers of the elliptic fibration induced by $|D|$ are II^*+5I_2. In particular the class

$$\Theta_0^{(1)}-\Theta_1^{(1)}+2s_0+2\Theta_0^{(2)}+\Theta_1^{(2)}+2\Theta_2^{(2)}+\Theta_3^{(2)}+\Theta_0^{(3)}$$

is the class of an effective (-2)-curve orthogonal to R (and in fact a bisection of the fibration $\mathcal{E}$) and so it is the class of one of the components of one of the fibers of type I_2 of the fibration $\phi_{|D|}:X\to\mathbb{P}^1$. The map which contracts exactly this curve produces a surface X_2' which is singular in a point and whose Néron–Severi group is $U\oplus E_8\oplus A_1^4$.

5.5. Infinite admissible pairs (S_d,S_d') with $\rho(S_d)=3$. Let S_d be a generic K3 surface admitting an elliptic fibration $\mathcal{E}_d:S_d\to\mathbb{P}^1$ such that $MW(\mathcal{E}_d)=\langle s_1\rangle$ and $s_0s_1=d-2$ (described in Section 4.3.2).

Proposition 5.2. *Let $\phi:S_d\to S_d'$ be the contraction of the curve s_1. Then $NS(S_d')\simeq\langle 2\rangle\oplus\langle -2d\rangle$. If d is even, then there is no a (-2)-curve $B_{S_d}\subset S_d$ with $B_{S_d}s_1=1$.*

If d is even and a square, then (S_d,S_d') is an admissible pair. In particular this gives an infinite number of admissible pairs such that the Picard number of the K3 surface is 3 (the minimal possible).

If d is not a square and $d\equiv 0 \mod 4$, then S_d' is not a Mori Dream Space, so (S_d,S_d') is not an admissible pair for infinitely many values of d.

Moreover, for almost all the d such that d is a square, the pair (Q_d,Q_d') is also an admissible if $NS(Q_d')\simeq NS(S_d')$ and $NS(Q_d)\simeq\langle 2\rangle\oplus\langle -2d\rangle\oplus\langle -2\rangle$. So we have an infinite number of admissible pairs (S_d,S_d') and (Q_d,Q_d') such that $NS(S_d)\not\simeq NS(Q_d)$ and $NS(S_d')\simeq NS(Q_d')$.

Proof. We already observed that S_d is not a Mori Dream Space, since the translation by the section s_1 is an automorphism of infinite order of S_d.

Let us assume that there exists a (-2)-curve $B_{S_d}\subset S_d$ such that $B_{S_d}s_1=1$. Then there exists a vector $b\in NS(S_d)$ such that $b^2=-2$ and $bs_1=1$. The vector b is of the form $xF+ys_0+zs_1$. So $bs_1=1$ implies $x+(d-2)y-2z=1$, i.e. $x=1+2z-(d-2)y$ and $b^2=-2$ implies $-2y^2-2z^2+2xy+2xz+2(d-2)yz=-2$. These two conditions together give $y^2+z^2+y+2zy-y^2d+z+1=0$, which is impossible modulo 2 if d is even. We conclude that if d is even there exists no a (-2)-curve $B_{S_d}\subset S_d$ such that $B_{S_d}s_1=1$ and thus Proposition 2.10 applies.

The lattice $(s_1)^{\perp_{NS(S_d)}}$ is generated by $\langle 2F+s_1,-dF+s_0-s_1\rangle\simeq\langle 2\rangle\oplus\langle -2d\rangle$. If d is even, the surface S_d' is a Mori Dream Space if and only if the K3 surface Y_d with Néron–Severi group isometric to $\langle 2\rangle\oplus\langle -2d\rangle$ is a Mori Dream Space. Since the rank of the lattice is 2, we know that the K3 surface Y_d is a Mori Dream Space if and only if the lattice $\langle 2\rangle\oplus\langle -2d\rangle$ represents 0 or -2.

The quadratic form associated to $\langle 2\rangle\oplus\langle -2d\rangle$ is $2x^2-2dy^2$. The form represents the zero if and only if there exists $(x,y)\in\mathbb{Z}^2$ such that $2x^2-2dy^2=0$ and represents -2 if and only if $2x^2-2dy^2=-2$.

If d is a square, then there exists $b\in\mathbb{Z}$ such that $d=b^2$ and it suffices to chose $(x,y)=(b,1)$. So if d is a square, then the quadratic form represents zero which implies that Y_d is a Mori Dream Space. So if d is an even square, then S_d' is a Mori Dream Space.

Viceversa if d is not a square, then the quadratic form does not represent 0. Let us assume that the quadratic form represents -2. So there exists $(x, y) \in \mathbb{Z}^2$, $x^2 - dy^2 = -1$. Let us consider this equation modulo 4 (where the square are either 0 or 1). We have the following possible values for (x^2, y^2) modulo 4, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$. The choices $(0, 0)$ and $(1, 0)$ give a contradiction. So either $(x^2, y^2) \equiv (0, 1) \mod 4$ and in this case $d \equiv 1 \mod 4$ or $(x^2, y^2) \equiv (1, 1) \mod 4$ and in this case $d \equiv 2 \mod 4$. Therefore, if $d \equiv 0 \mod 4$ or $d \equiv 3 \mod 4$, then the quadratic form does not represent -2. If $d \equiv 0 \mod 4$, then d is even and thus there is no a (-2)-curve $B_{S_d} \subset S_d$ such that $B_{S_d} s_1 = 1$. So if $d \equiv 0 \mod 4$, then S'_d is a Mori Dream space if and only if $\langle 2 \rangle \oplus \langle -2d \rangle$ represents either 0 or -2. But if $d \equiv 0 \mod 4$ and d is not a square, then $\langle 2 \rangle \oplus \langle -2d \rangle$ does not represent neither 0 or -2 and thus S'_d in not a Mori Dream Space.

We observe that $NS(Q_d) \simeq \langle 2 \rangle \oplus \langle -2d \rangle \oplus \langle -2 \rangle \simeq NS(Y_d) \oplus \langle -2 \rangle$ does not contain a vector of length -2 which meets the last generator with multiplicity 1, by Remark 3.7. So by Proposition 2.10 Q'_d is a Mori Dream Space if and only if Y_d is a Mori Dream Space.

If d is a square, then the K3 surface Y_d is a Mori Dream Space. As a consequence Q'_d are Mori Dream Space. Hence the pair (Q_d, Q'_d) is an admissible pair if and only if Q_d is not a Mori Dream Space. Since there are exactly 27 hyperbolic lattices L of rank 3 such that if the Néron–Severi group of a K3 surface is isometric to one of these lattices, the K3 surface has a finite automorphism group and so is a Mori Dream Space, we conclude that for almost all the even squares d, Q_d is not a Mori Dream Space and (Q_d, Q'_d) is an admissible pair.

In particular for almost all the $d \in \mathbb{N}$ such that d is an even square, both (Q_d, Q'_d) and (S_d, S'_d) are admissible pair such that $NS(S_d) \not\simeq NS(Q_d)$ and $NS(S'_d) \simeq NS(Q'_d)$. □

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