

FREE PRODUCT OF TWO ELLIPTIC QUATERNIONIC MÖBIUS TRANSFORMATIONS

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(Received April 14, 2015, revised May 6, 2016)

Abstract

Suppose that f and g are two elliptic quaternionic Möbius transformations of orders m and n respectively. If the hyperbolic distance $\delta(f, g)$ between $\text{fix}(f)$ and $\text{fix}(g)$ satisfies

$$\cosh \delta(f, g) \geq \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

then the group $\langle f, g \rangle$ is discrete non-elementary and isomorphic to the free product $\langle f \rangle * \langle g \rangle$.

1. Introduction

Elements of the quaternions \mathbb{H} have the form $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathbb{H}$, where $q_i \in \mathbb{R}$ and $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. Let $\bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$ and $|q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ be the conjugate and modulus of q , respectively. We define $\Re(q) = (q + \bar{q})/2$ and $\Im(q) = (q - \bar{q})/2$.

Let $\mathbf{H}^3 = \{x + y\mathbf{i} + t\mathbf{j} : x, y, t \in \mathbb{R}, t > 0\}$ and $\partial\mathbf{H}^3 = \bar{\mathbb{C}}$. We associate with each Möbius transformation

$$f(z) = \frac{az + b}{cz + d}, \quad ad - bc = 1,$$

the matrix

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}).$$

The Poincaré extension of f is given by

$$f(x + y\mathbf{i} + t\mathbf{j}) = (a(x + y\mathbf{i} + t\mathbf{j}) + b)(c(x + y\mathbf{i} + t\mathbf{j}) + d)^{-1}.$$

For $f \in \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I_2\}$, we define

$$\text{fix}(f) = \{z \in \bar{\mathbf{H}}^3 : f(z) = z\}.$$

It is well known that

$$\text{PSL}(2, \mathbb{C}) = \text{Isom}(\mathbf{H}^3).$$

2010 Mathematics Subject Classification. Primary 30F40, 20H10; Secondary 57S30.

The project supported by NSF of China (10801107) and NSFS of Guangdong Province (2015A030313644, 2016A030313002).

A Kleinian group is a discrete nonelementary subgroup of isometries of $\text{Isom}(\mathbf{H}^3)$. Equivalently such groups are identified with discrete groups of Möbius transformations of the Riemann sphere $\overline{\mathbb{C}} = \partial\mathbf{H}^3$.

We recall the following normalization of two elliptic elements of $\text{PSL}(2, \mathbb{C})$ [6].

Proposition 1.1. *Suppose that f and g are elliptic elements of $\text{PSL}(2, \mathbb{C})$ of order m and n respectively. Let $\delta = \delta(f, g)$ be the hyperbolic distance between $\text{fix}(f)$ and $\text{fix}(g)$ and $\phi = \phi(f, g)$ the angle between the spheres or hyperplanes which contain $\text{fix}(f)$ or $\text{fix}(g)$ and the common perpendicular of $\text{fix}(f)$ and $\text{fix}(g)$. Then there is an $h \in \text{PSL}(2, \mathbb{C})$ such that*

$$hfh^{-1} = \begin{pmatrix} \cos \frac{\pi}{m} & \mathbf{i}\omega \sin \frac{\pi}{m} \\ \mathbf{i} \sin \frac{\pi}{m} / \omega & \cos \frac{\pi}{m} \end{pmatrix}$$

and

$$hgh^{-1} = \begin{pmatrix} \cos \frac{\pi}{n} & \mathbf{i} \sin \frac{\pi}{n} / \omega \\ \mathbf{i}\omega \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix},$$

where $\omega^2 = e^{\delta + \mathbf{i}\phi}$.

Based on the above observation, Gehring, Maclachlan and Martin [6] proved the following theorem.

Theorem GMM *Suppose that f and g are elliptic elements of $\text{PSL}(2, \mathbb{C})$ of order m and n respectively. If*

$$\cosh \delta(f, g) \geq \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

then $\langle f, g \rangle$ is discrete non-elementary and isomorphic to the free product $\langle f \rangle * \langle g \rangle$.

In this paper we regard $\mathfrak{J}(\mathbb{H}) \times \mathbb{R}^+$ as the model of \mathbf{H}^4 and identify $\text{Isom}(\mathbf{H}^4)$ with the group $\text{PSp}(1, 1)$. The main aim of this paper is to establish an analog of Theorem GMM in $\text{PSp}(1, 1)$. Our main result is the following theorem.

Theorem 1.1. *Suppose that f and g are elliptic elements of $\text{PSp}(1, 1)$ of orders m and n respectively. If the hyperbolic distance $\delta(f, g)$ between $\text{fix}(f)$ and $\text{fix}(g)$ satisfies*

$$\cosh \delta(f, g) \geq \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

then $\langle f, g \rangle$ is discrete non-elementary and isomorphic to the free product $\langle f \rangle * \langle g \rangle$.

The paper is organized as follows. In Section 2 we shall collect some basic facts in quaternionic hyperbolic geometry. Section 3 is devoted to obtaining some properties of elliptic elements. Section 4 contains the proof of Theorem 1.1.

2. Background

We briefly recall some necessary material on quaternionic hyperbolic geometry here and we refer to [1, 3, 5, 7] for further details.

2.1. Quaternionic Möbius transformations. Let $\mathbb{H}^{1,1}$ be the vector space with the Hermitian form of signature (1, 1) given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J_1 \mathbf{z} = \bar{w}_1 z_2 + \bar{w}_2 z_1$$

with the matrix

$$J_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We define

$$\mathrm{Sp}(1, 1) = \{f \in \mathrm{GL}(2, \mathbb{H}) : f^* J_1 f = J_1\}.$$

From this we find $f^{-1} = J_1^{-1} f^* J_1$. That is:

$$(1) \quad f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad f^{-1} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.$$

Use of the identities $ff^{-1} = f^{-1}f = I_2$ proves the following lemma.

Lemma 2.1. $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(1, 1)$ if and only if

$$\begin{aligned} a\bar{d} + b\bar{c} &= 1, \quad \bar{d}a + \bar{b}c = 1, \\ \Re(a\bar{b}) &= \Re(c\bar{d}) = \Re(\bar{d}b) = \Re(\bar{c}a) = 0. \end{aligned}$$

Following the terminology in [5], we let

$$V_0 = \{\mathbf{z} \in \mathbb{H}^{1,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}, \quad V_- = \{\mathbf{z} \in \mathbb{H}^{1,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}.$$

It is obvious that V_0 and V_- are invariant under $\mathrm{Sp}(1, 1)$. Let $\mathbb{P} : \mathbb{H}^{1,1} - \{0\} \rightarrow \mathbb{H}\mathbb{P}^1$ be the right projection map given by

$$\mathbb{P}(z_1, z_2)^T = z_1 z_2^{-1} \text{ provided } z_2 \neq 0$$

and

$$(2) \quad \mathbb{P}(z_1, 0)^T = \infty, \quad \mathbb{P}(0, z_2)^T = o.$$

The Siegel domain model of quaternionic hyperbolic 1-space is defined to be

$$\mathbf{H}_{\mathbb{H}}^1 = \mathbb{P}(V_-), \quad \partial\mathbf{H}_{\mathbb{H}}^1 = \mathbb{P}(V_0).$$

The Bergman metric on $\mathbf{H}_{\mathbb{H}}^1$ is given by the distance formula

$$(3) \quad \cosh^2 \frac{\rho(z, w)}{2} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle},$$

where $z, w \in \mathbf{H}_{\mathbb{H}}^1$, $\mathbf{z} \in \mathbb{P}^{-1}(z)$, $\mathbf{w} \in \mathbb{P}^{-1}(w)$.

The elements f in $\mathrm{Sp}(1, 1)$ act on $\mathbf{H}_{\mathbb{H}}^1 \cup \partial\mathbf{H}_{\mathbb{H}}^1$ according to the following formula:

$$f(z) = \mathbb{P}f\mathbb{P}^{-1}(z) = (az + b)(cz + d)^{-1}.$$

We define $\mathrm{PSp}(1, 1) = \mathrm{Sp}(1, 1)/\{\pm I_2\}$, which is the group of holomorphic isometries of $\mathbf{H}_{\mathbb{H}}^1$.

Let $z = v - u = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} - u$, where $u, X, Y, Z \in \mathbb{R}$. We denote

$$z = (v, u) \in \mathfrak{I}(\mathbb{H}) \times \mathbb{R}.$$

Then $z \in \mathbf{H}_{\mathbb{H}}^1$ if and only if $u > 0$, and $z \in \partial\mathbf{H}_{\mathbb{H}}^1$ if and only if $u = 0$. Thus we have the following identification:

$$\mathbf{H}_{\mathbb{H}}^1 = \mathfrak{I}(\mathbb{H}) \times \mathbb{R}^+,$$

where $\mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$. Hence we can regard $\mathbf{H}_{\mathbb{H}}^1$ as the upper half-space model of \mathbf{H}^4 with the isometric group $\text{PSp}(1, 1)$.

In [1], Cao, Parker and Wang considered the classification of quaternionic Möbius transformations in the ball model. The Hermitian form of the ball model is given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J_2 \mathbf{z} = \overline{w_1} z_1 - \overline{w_2} z_2, \quad \text{where } J_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Cayley transformation C between the ball model and the Siegel domain model is

$$(4) \quad C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

That is, $C^* J_2 C = C^{-1} J_2 C = J_1$.

For $f \in \text{PSp}(1, 1)$, we define

$$\text{fix}(f) = \{z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : f(z) = z\}.$$

Following Chen and Greenberg [5], we say that a non-trivial element $f \in \text{PSp}(1, 1)$ is:

- (i) *elliptic* if it has a fixed point in $\mathbf{H}_{\mathbb{H}}^1$;
- (ii) *parabolic* if it has exactly one fixed point which lies in $\partial\mathbf{H}_{\mathbb{H}}^1$;
- (iii) *loxodromic* if it has exactly two fixed points which lie in $\partial\mathbf{H}_{\mathbb{H}}^1$.

2.2. Isometric sphere. Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSp}(1, 1)$ with $f(\infty) \neq \infty$. We know that $c \neq 0$ and set

$$r_f = \frac{1}{|c|}.$$

As in [3, 7], we define the *isometric sphere* $I(f)$ of $f \in \text{PSp}(1, 1)$ not fixing ∞ as

$$I(f) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |cz + d| = 1 \right\} = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |z + c^{-1}d| = r_f \right\}.$$

By (1) we have

$$I(f^{-1}) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |\bar{c}z + \bar{d}| = 1 \right\} = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |z + \bar{c}^{-1}\bar{d}| = r_f \right\}.$$

We define

$$\text{Ext } I(f) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |cz + d| > 1 \right\}, \quad \text{Int } I(f) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : |cz + d| < 1 \right\}.$$

The following proposition will be used later on.

Proposition 2.1. *Let $f \in \text{PSp}(1, 1)$ of the form (1) with $f(\infty) \neq \infty$. Then*

- (i) $f(I(f)) = I(f^{-1})$;
- (ii) $f(\text{Ext } I(f)) \subset \text{Int } I(f^{-1})$;

(iii) $f(\text{Int } I(f)) \subset \text{Ext } I(f^{-1})$.

Proof. Let $z \in I(f)$. Then $|cz + d| = 1$. By Lemma 2.1, we have

$$|\bar{c}f(z) + \bar{a}| = \frac{|\bar{c}(az + b) + \bar{a}(cz + d)|}{|cz + d|} = 1.$$

This implies that $f(z) \in I(f^{-1})$, which proves (i). The proofs of (ii) and (iii) follow from similar reasoning. □

3. Some properties of elliptic elements

Let f be an elliptic element in $\text{PSp}(1, 1)$ with Λ_0 its negative class of eigenvalues and Λ_1 its positive class. According to [5], there are two types of elliptic elements in $\text{PSp}(1, 1)$. If $\Lambda_0 = \Lambda_1$, then f is a boundary elliptic element; if $\Lambda_0 \neq \Lambda_1$, then f is a regular elliptic element. As in [2], we have the following conjugacy classification of elliptic elements in $\text{PSp}(1, 1)$.

Lemma 3.1. *Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSp}(1, 1)$ and $\alpha, \beta, \theta \in [0, \pi]$.*

(i) *f is a boundary elliptic element if f is conjugate to*

$$(5) \quad E(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}, \text{ where } \theta \neq 0, \pi;$$

(ii) *f is a regular elliptic element if f is conjugate to*

$$(6) \quad E(\alpha, \beta) = \begin{pmatrix} \frac{e^{i\alpha} + e^{i\beta}}{2} & \frac{e^{i\alpha} - e^{i\beta}}{2} \\ \frac{e^{i\alpha} - e^{i\beta}}{2} & \frac{e^{i\alpha} + e^{i\beta}}{2} \end{pmatrix}, \text{ where } \alpha \neq \beta.$$

Proof. By conjugation, if necessary, we may assume that f fixes o and ∞ . That is $f = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. Then there is a $\mu \in \mathbb{H}$ of unit modulus such that $\mu a \mu^{-1} = e^{i\theta}$, where $0 < \theta < \pi$.

This implies that $hfh^{-1} = E(\theta)$, where $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \text{PSp}(1, 1)$. This proves (i).

In the ball model, if f is a regular elliptic element, then f is conjugate to the form

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \text{ where } \alpha \neq \beta.$$

By the connection of the ball model and the Siegel domain model, we have $E(\alpha, \beta) = C \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} C^{-1}$. This proves (ii). □

By direct computation, we have

$$\text{fix}(E(\theta)) = \{z = X\mathbf{i} - u : u \geq 0\} = \{(v, u) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^{\geq}\}$$

and

$$\text{fix}(E(\alpha, \beta)) = \{1\} = \{(0, 1) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^+\},$$

where $\mathbb{R}^{\geq} = \{u \in \mathbb{R} : u \geq 0\}$.

Lemma 3.2. *Let g be a regular elliptic element of the form*

$$(7) \quad g = \begin{pmatrix} \frac{e^{i\alpha} + e^{i\beta}}{2} & \frac{e^{i\alpha} - e^{i\beta}}{2} \\ \frac{e^{i\alpha} - e^{i\beta}}{2} & \frac{e^{i\alpha} + e^{i\beta}}{2} \end{pmatrix} \in \text{PSp}(1, 1),$$

where $\alpha = \frac{\pi}{p}, \beta = \frac{\pi}{q}, p \neq q$ and $\text{ord}(g) = n$. Then

$$\frac{|p - q|}{2pq} \geq \frac{1}{n}.$$

Proof. Let C be the cayley transformation given by (4). Then

$$h = C^{-1}gC = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}$$

is a regular elliptic element in the ball model. Let the least common multiple of p and q be

$$\text{lcm}(p, q) = l.$$

Without loss of generality, we assume that $p > q$. We only need to consider the following three cases.

Case (1): Both $\frac{l}{p}$ and $\frac{l}{q}$ are odd. In this case, $\text{ord}(g) = l = n$. Let $t = \frac{n}{p}$ and $s = \frac{n}{q}$. Then t and s are odd and $s > t$. This implies that

$$n(p - q) = pq(s - t) \geq 2pq.$$

Case (2): Both $\frac{l}{p}$ and $\frac{l}{q}$ are even. In this case, $\text{ord}(g) = l = n$. Let $t = \frac{n}{p}$ and $s = \frac{n}{q}$. Then t and s are even and $s > t$. This implies that

$$n(p - q) = pq(s - t) \geq 2pq.$$

Case (3): Only one of $\frac{l}{p}$ and $\frac{l}{q}$ is even. In this case, $\text{ord}(g) = 2l = n$. Let $t = \frac{l}{p}$ and $s = \frac{l}{q}$. Then $s > t$. This implies that

$$n(p - q) = 2pq(s - t) \geq 2pq.$$

□

The span of $M \subset \overline{\mathbf{H}^4}$, denoted by $\text{span}(M)$, is the smallest hyperbolic surface containing M . For two elliptic elements of $\text{PSp}(1, 1)$, we define $\delta = \delta(f, g)$ to be the hyperbolic distance between $\text{fix}(f)$ and $\text{fix}(g)$.

DEFINITION 3.1. Let f and g be two boundary elliptic elements of $\text{PSp}(1, 1)$ with $\text{fix}(f) \cap \text{fix}(g) = \emptyset$. Then there is a unique geodesic γ such that

$$\text{fix}(f) \cap \gamma = \gamma_f, \text{fix}(g) \cap \gamma = \gamma_g, \rho(\gamma_f, \gamma_g) = \delta(f, g).$$

We define $\phi = \phi(f, g)$ to be the angle between $\text{span}(\text{fix}(f) \cup \gamma)$ and $\text{span}(\text{fix}(g) \cup \gamma)$.

For each pair of boundary elliptic elements f and g of $\text{PSp}(1, 1)$ such that $\text{fix}(f) \cap \text{fix}(g) = \emptyset$, we call the four numbers

$$\text{ord}(f), \text{ord}(g), \phi(f, g), \delta(f, g)$$

the parameters of the 2-generator group $\langle f, g \rangle$ and write

$$\text{par}(\langle f, g \rangle) = (\text{ord}(f), \text{ord}(g), \phi(f, g), \delta(f, g)).$$

Lemma 3.3. *If $\text{fix}(f) \cap \text{fix}(g) = \emptyset$, then the group $\langle f, g \rangle$ generated by two boundary elliptic elements f and g is uniquely determined (up to conjugacy) by $\text{par}(\langle f, g \rangle) = (\text{ord}(f), \text{ord}(g), \phi(f, g), \delta(f, g))$.*

Proof. Suppose that f and g are two boundary elliptic elements of $\text{PSp}(1, 1)$ of order m and n respectively, with $\text{fix}(f) \cap \text{fix}(g) = \emptyset$. Let $\phi = \phi(f, g)$ and $\delta = \delta(f, g)$. Then there is a unique geodesic γ such that

$$\text{fix}(f) \cap \gamma = \gamma_f, \text{fix}(g) \cap \gamma = \gamma_g, \rho(\gamma_f, \gamma_g) = \delta.$$

Note that $\text{par}(\langle f, g \rangle) = (m, n, \phi, \delta)$ is invariant under conjugation. We may assume that γ is the u -axis. Then $\text{fix}(f)$ and $\text{fix}(g)$ are two semi-2-spheres. Let

$$F_f = \text{fix}(f) \cap \mathfrak{I}(\mathbb{H}), F_g = \text{fix}(g) \cap \mathfrak{I}(\mathbb{H}).$$

Then F_f and F_g are two circles on $\mathfrak{I}(\mathbb{H})$. Let $A_f X + B_f Y + C_f Z = 0$ and $A_g X + B_g Y + C_g Z = 0$ be the planes in $\mathfrak{I}(\mathbb{H})$ containing F_f and F_g , respectively. Note that the angle between the two vectors (A_f, B_f, C_f) and (A_g, B_g, C_g) is just the angle $\phi(f, g)$. Due to the action of $\text{Sp}(1)$ by conjugation in \mathbb{H} coincides with the action of $\text{SO}(3)$, therefore we can find an $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \text{PSp}(1, 1)$ which maps any two vectors with angle ϕ to two fixed vectors with angle ϕ . This implies that the group $\langle f, g \rangle$ generated by two boundary elliptic elements f and g is uniquely determined (up to conjugacy) by $\text{par}(\langle f, g \rangle) = (m, n, \phi, \delta)$. \square

Proposition 3.1. *We have that*

$$\text{par}(\langle f, g \rangle) = (m, n, \phi, \delta)$$

for the following two elliptic elements:

$$f = \begin{pmatrix} \cos \frac{\pi}{m} & \mathbf{k}e^\delta \sin \frac{\pi}{m} \\ \mathbf{k}e^{-\delta} \sin \frac{\pi}{m} & \cos \frac{\pi}{m} \end{pmatrix},$$

$$g = \begin{pmatrix} \cos \frac{\pi}{n} & (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \sin \frac{\pi}{n} \\ (\mathbf{k} \cos \phi + \mathbf{i} \sin \phi) \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}.$$

Proof. Let

$$F = \begin{pmatrix} e^{i\frac{\pi}{m}} & 0 \\ 0 & e^{i\frac{\pi}{m}} \end{pmatrix}$$

and

$$G = \begin{pmatrix} \cos \frac{\pi}{n} + (\mathbf{i} \cos \phi + \mathbf{k} \sin \phi) \sin \frac{\pi}{n} & 0 \\ 0 & \cos \frac{\pi}{n} + (\mathbf{i} \cos \phi + \mathbf{k} \sin \phi) \sin \frac{\pi}{n} \end{pmatrix}.$$

Then $F, G \in \text{PSp}(1, 1)$ with $\text{ord}(F) = m$ and $\text{ord}(G) = n$. By direct computation, we get

$$\text{fix}(F) = \{(v, u) \in \mathbb{R}\mathbf{i} \times \mathbb{R}^{\geq}\}$$

and

$$\text{fix}(G) = \{z \in \mathbb{R}(\mathbf{i} \cos \phi + \mathbf{k} \sin \phi) : \Re(z) \leq 0\} = \{(v, u) \in (\mathbf{i} \cos \phi + \mathbf{k} \sin \phi)\mathbb{R} \times \mathbb{R}^{\geq}\},$$

where $\mathbb{R}(\mathbf{i} \cos \phi + \mathbf{k} \sin \phi)$ is the smallest subfield containing \mathbb{R} and $\mathbf{i} \cos \phi + \mathbf{k} \sin \phi$ [5]. Note that $\text{fix}(F) \cap \mathfrak{I}(\mathbb{H})$ is the X -axis and $\text{fix}(G) \cap \mathfrak{I}(\mathbb{H})$ is the line $Z = \tan \phi X$. The angle between these two lines is ϕ .

Let

$$K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{j} & 1 \\ 1 & \mathbf{j} \end{pmatrix}, D_\delta = \begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2} \end{pmatrix} \in \text{PSp}(1, 1).$$

Set

$$f = D_\delta K_1^{-1} F K_1 D_\delta^{-1} = \begin{pmatrix} \cos \frac{\pi}{m} & \mathbf{k} e^\delta \sin \frac{\pi}{m} \\ \mathbf{k} e^{-\delta} \sin \frac{\pi}{m} & \cos \frac{\pi}{m} \end{pmatrix},$$

$$g = K_1^{-1} G K_1 = \begin{pmatrix} \cos \frac{\pi}{n} & (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \sin \frac{\pi}{n} \\ (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}.$$

We note that $\text{fix}(f)$ is a semi-2-sphere in the XYu space and $\text{fix}(f) \cap \mathfrak{I}(\mathbb{H}) = \{(X, Y) : X^2 + Y^2 = e^\delta\}$. We also note that $\text{fix}(g) \cap \mathfrak{I}(\mathbb{H})$ is a circle represented by the parameter equations:

$$X = \frac{-2t \cos \phi}{1+t^2}, Y = \frac{1-t^2}{1+t^2}, Z = \frac{-2t \sin \phi}{1+t^2}, -\infty < t < \infty.$$

Hence $\text{fix}(g) \cap \mathfrak{I}(\mathbb{H})$ lies in the plane given by

$$X \sin \phi - Z \cos \phi = 0.$$

It is obvious that the u -axis is orthogonal to $\text{fix}(f)$ and $\text{fix}(g)$ and intersects with them in the points e^δ and 1, respectively. By (3) and Lemma 3.3, we have $\delta(f, g) = \delta$ and $\phi(f, g) = \phi$. \square

Lemma 3.4. *Suppose that F is a boundary elliptic element of order m and G is a regular elliptic element of order n with $\text{fix}(f) \cap \text{fix}(g) = \emptyset$. Then there is an $h \in \text{PSp}(1, 1)$ such that*

$$(8) \quad f = h F h^{-1} = \begin{pmatrix} \cos \frac{\pi}{m} & \mathbf{k} e^\delta \sin \frac{\pi}{m} \\ \mathbf{k} e^{-\delta} \sin \frac{\pi}{m} & \cos \frac{\pi}{m} \end{pmatrix},$$

$$(9) \quad g = h G h^{-1} = \begin{pmatrix} \mu \frac{e^{i\alpha} + e^{i\beta}}{2} \mu^{-1} & \mu \frac{e^{i\alpha} - e^{i\beta}}{2} \mu^{-1} \\ \mu \frac{e^{i\alpha} - e^{i\beta}}{2} \mu^{-1} & \mu \frac{e^{i\alpha} + e^{i\beta}}{2} \mu^{-1} \end{pmatrix},$$

where $\alpha = \frac{\pi}{p}, \beta = \frac{\pi}{q}, p \neq q$ and $\text{ord}(g) = n$.

Proof. In the ball model, we assume that g' is of the form

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}, \text{ where } \alpha \neq \beta$$

and f' is a boundary elliptic element of order n . Without loss of generality, we may assume that $G = E(\alpha, \beta)$ and the u -axis is orthogonal to $\text{fix}(F)$. Similarly as in the proof of Lemma

3.3, we can find an $h = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} \in \text{PSp}(1, 1)$ such that hFh^{-1} and hGh^{-1} are of the forms (8) and (9). □

Similarly we can prove the following lemma.

Lemma 3.5. *Suppose that F and G are two regular elliptic elements of order m and n respectively, with $\text{fix}(f) \cap \text{fix}(g) = \emptyset$. Then there is an $h \in \text{PSp}(1, 1)$ such that*

$$f = \begin{pmatrix} \frac{e^{i\theta} + e^{i\eta}}{2} & e^\delta \frac{e^{i\theta} - e^{i\eta}}{2} \\ e^{-\delta} \frac{e^{i\theta} - e^{i\eta}}{2} & \frac{e^{i\theta} + e^{i\eta}}{2} \end{pmatrix},$$

where $\theta = \frac{\pi}{p'}$, $\eta = \frac{\pi}{q'}$, $p' \neq q'$ and $\text{ord}(f) = m$;

$$g = hGh^{-1} = \begin{pmatrix} \mu \frac{e^{i\alpha} + e^{i\beta}}{2} \mu^{-1} & \mu \frac{e^{i\alpha} - e^{i\beta}}{2} \mu^{-1} \\ \mu \frac{e^{i\alpha} - e^{i\beta}}{2} \mu^{-1} & \mu \frac{e^{i\alpha} + e^{i\beta}}{2} \mu^{-1} \end{pmatrix},$$

where $\alpha = \frac{\pi}{p}$, $\beta = \frac{\pi}{q}$, $p \neq q$ and $\text{ord}(g) = n$.

4. The proof of the main theorem

In this section we will prove our main theorem. We divide the proof into three cases.

Case (1): Both f and g are boundary elliptic elements. By Lemma 3.3 and Proposition 3.1, we may assume that

$$f = \begin{pmatrix} \cos \frac{\pi}{m} & \mathbf{k}e^\delta \sin \frac{\pi}{m} \\ \mathbf{k}e^{-\delta} \sin \frac{\pi}{m} & \cos \frac{\pi}{m} \end{pmatrix},$$

$$g = \begin{pmatrix} \cos \frac{\pi}{n} & (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \sin \frac{\pi}{n} \\ (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \sin \frac{\pi}{n} & \cos \frac{\pi}{n} \end{pmatrix}.$$

Obviously, the isometric spheres of f and f^{-1} are

$$I(f) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z - \mathbf{k}e^\delta \cot \frac{\pi}{m} \right| = \frac{e^\delta}{\sin \frac{\pi}{m}} \right\},$$

$$I(f^{-1}) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z + \mathbf{k}e^\delta \cot \frac{\pi}{m} \right| = \frac{e^\delta}{\sin \frac{\pi}{m}} \right\},$$

and the isometric spheres of g and g^{-1} are

$$I(g) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z - (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \cot \frac{\pi}{n} \right| = \frac{1}{\sin \frac{\pi}{n}} \right\},$$

$$I(g^{-1}) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z + (\mathbf{k} \cos \phi - \mathbf{i} \sin \phi) \cot \frac{\pi}{n} \right| = \frac{1}{\sin \frac{\pi}{n}} \right\}.$$

The fundamental domain for the action of f on $\overline{\mathbf{H}}_{\mathbb{H}}^1$ is the union set of

$$\text{Ext } I(f) \cap \text{Ext } I(f^{-1})$$

and

$$\text{Int } I(f) \cap \text{Int } I(f^{-1}).$$

It is obvious that $I(g)$ and $I(g^{-1})$ lie in the semi-Euclidean ball $B(0, r) \cap \overline{\mathbf{H}_{\mathbb{H}}^1}$, where

$$r = \frac{1 + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}}$$

and $B(0, r) = \{x \in \mathbb{H} : |x| < r\}$. Additionally, the two isometric sphere $I(f)$ and $I(f^{-1})$ also contain the semi-ball $B(0, s) \cap \overline{\mathbf{H}_{\mathbb{H}}^1}$, where

$$s = \frac{e^\delta(1 - \cos \frac{\pi}{m})}{\sin \frac{\pi}{m}}.$$

Since

$$\cosh \delta(f, g) = \cosh \delta \geq \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

we have

$$e^\delta \geq \frac{(1 + \cos \frac{\pi}{n}) \sin \frac{\pi}{m}}{(1 - \cos \frac{\pi}{m}) \sin \frac{\pi}{n}},$$

which implies $r \leq s$. We have therefore seen that the exterior of a fundamental domain for $\langle g \rangle$ lies inside a fundamental domain for $\langle f \rangle$. It follows from Proposition 2.1 and the simplest version of the Klein-Maskit combination theorem that the group $\langle f, g \rangle$ is discrete and isomorphic to the free product of cyclic groups:

$$\langle f, g \rangle \cong \langle f \rangle * \langle g \rangle \cong \mathbb{Z}_m * \mathbb{Z}_n.$$

Case (2): f is a boundary elliptic element and g is a regular elliptic element. By Lemma 3.4, we may assume that f has the same representation as in Case (1) and g is of the form given by (9). Then the isometric spheres of g and g^{-1} are

$$I(g) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : \left| z + \mu i \mu^{-1} \cot \frac{\beta - \alpha}{2} \right| = \frac{1}{|\sin \frac{\beta - \alpha}{2}|} \right\}$$

and

$$I(g^{-1}) = \left\{ z \in \overline{\mathbf{H}_{\mathbb{H}}^1} : \left| z - \mu i \mu^{-1} \cot \frac{\beta - \alpha}{2} \right| = \frac{1}{|\sin \frac{\beta - \alpha}{2}|} \right\}.$$

It is obvious that the isometric spheres $I(g)$ and $I(g^{-1})$ lie in the semi-ball $B(0, r) \cap \overline{\mathbf{H}_{\mathbb{H}}^1}$, where

$$r = \frac{1 + \cos \frac{|p-q|\pi}{2pq}}{\sin \frac{|p-q|\pi}{2pq}}.$$

It follows from Lemma 3.2 that

$$\frac{|p - q|\pi}{2pq} \geq \frac{\pi}{n}.$$

Hence

$$r \leq \frac{1 + \cos \frac{\pi}{n}}{\sin \frac{\pi}{n}}.$$

As in Case (1), the result follows from the above inequality.

Case (3): Both f and g are regular elliptic elements. By Lemma 3.5, we may assume that g has the same representation as in Case (2) and f is of the form

$$f = \begin{pmatrix} \frac{e^{i\theta} + e^{i\eta}}{2} & e^\delta \frac{e^{i\theta} - e^{i\eta}}{2} \\ e^{-\delta} \frac{e^{i\theta} - e^{i\eta}}{2} & \frac{e^{i\theta} + e^{i\eta}}{2} \end{pmatrix} \in \text{PSp}(1, 1),$$

where $\theta = \frac{\pi}{p'}$, $\eta = \frac{\pi}{q'}$, $p' \neq q'$ and $\text{ord}(f) = m$. Note that $\text{fix}(f) = \{e^\delta\}$. The isometric spheres of f and f^{-1} are

$$I(f) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z + \mathbf{i}e^\delta \cot \frac{\eta - \theta}{2} \right| = \frac{e^\delta}{|\sin \frac{\eta - \theta}{2}|} \right\}$$

and

$$I(f^{-1}) = \left\{ z \in \overline{\mathbf{H}}_{\mathbb{H}}^1 : \left| z - \mathbf{i}e^\delta \cot \frac{\eta - \theta}{2} \right| = \frac{e^\delta}{|\sin \frac{\eta - \theta}{2}|} \right\}.$$

The two isometric spheres $I(f)$ and $I(f^{-1})$ contain the semi-ball $B(0, s) \cap \overline{\mathbf{H}}_{\mathbb{H}}^1$, where

$$s = \frac{e^\delta (1 - \cos \frac{|p' - q'|\pi}{2p'q'})}{\sin \frac{|p' - q'|\pi}{2p'q'}}.$$

It follows from Lemma 3.2 that

$$\frac{|p' - q'|\pi}{2p'q'} \geq \frac{\pi}{m}.$$

Hence

$$s \geq \frac{e^\delta (1 - \cos \frac{\pi}{m})}{\sin \frac{\pi}{m}}.$$

As in Cases (1) and (2), the result easily follows from the above inequality. □

REMARK 4.1. As in [4], $\text{SL}(2, \mathbb{C})$ can be embedded as a subgroup of $\text{Sp}(1, 1)$ as follows:

$$f \in \text{SL}(2, \mathbb{C}) \leftrightarrow T f T^{-1} \in \text{Sp}(1, 1),$$

where

$$T = \begin{pmatrix} \frac{1-j}{2} & \frac{1-j}{2} \\ \frac{1+j}{2} & -\frac{1-j}{2} \end{pmatrix}.$$

Hence Theorem 1.1 is a natural generalization of the result for real hyperbolic spaces of dimension 2 and 3 [6, 8].

ACKNOWLEDGEMENTS. The author would like to express much gratitude to the referee for a careful reading of this paper as well as many useful suggestions.

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