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A GENERALIZATION OF NAKAI'S THEOREM ON LOCALLY FINITE ITERATIVE HIGHER DERIVATIONS

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Abstract

Let k be a field of arbitrary characteristic. In 1978, Nakai proved a structure theorem for k-domains admitting a nontrivial locally finite iterative higher derivation when k is algebraically closed. In this paper, we generalize Nakai's theorem to cover the case where k is not algebraically closed. As a consequence, we obtain a cancellation theorem of the following form: Let A and A' be finitely generated k-domains with $A[x] \simeq_k A'[x]$. If A and $\bar{k} \otimes_k A$ are UFDs and $\operatorname{trans.deg}_k A = 2$, then we have $A \simeq_k A'$. This generalizes the cancellation theorem of Crachiola.

1. Introduction

Let A be a commutative ring with identity, and $A[x]$ the polynomial ring in one variable over A. A homomorphism $\sigma : A \to A[x]$ of rings is called an *exponential map* on A if the following condition holds for each $a \in A$, where $a_0, \dots, a_m \in A$ are such that $\sigma(a) = \sum_{i=0}^{m} a_i x^i$, and y is a new variable:

(E1) $a_0 = a$. (E2) $\sum_{i=0}^{m} \sigma(a_i) y^i = \sum_{i=0}^{m} a_i (x+y)^i$ in $A[x, y]$.

The condition above is equivalent to the condition that σ is a coaction of the group scheme $\mathbf{G}_a = \operatorname{Spec}(\mathbf{Z}[x])$ on $\operatorname{Spec}(A)$. For each exponential map σ on A, a collection $(\delta_i)_{i=0}^{\infty}$ of endomorphisms of the additive group A is defined by $\sigma(a) = \sum_{i \geq 0} \delta_i(a) x^i$ for each $a \in A$. The thus-obtained $(\delta_i)_{i=0}^{\infty}$ is called a *locally finite iterative higher derivation* on A. The notions of exponential maps and locally finite iterative higher derivations are equivalent, so we consider exponential maps.

For each $a \in A$, we define $\deg_\sigma(a)$ and $\operatorname{lc}_\sigma(a)$ to be the degree and leading coefficient of $\sigma(a)$ as a polynomial in x over A, respectively. By (E1), the *ring* $A^\sigma := \{a \in A \mid \sigma(a) = a\}$ *of σ-invariants* is equal to $\sigma^{-1}(A)$. Assume that σ is *nontrivial*, i.e., $A^\sigma \neq A$. Then, we have $\deg_\sigma(a) \geq 1$ for each $a \in A \setminus A^\sigma$. We call $s \in A$ a *local slice* of σ if $\deg_\sigma(s)$ is equal to the minimum among $\deg_\sigma(a)$ for $a \in A \setminus A^\sigma$. A local slice s of σ is called a *slice* of σ if $\operatorname{lc}_\sigma(s) = 1$. There always exists a local slice, but a slice does not exist in general. It is known that, if s is a slice of σ, then A is the polynomial ring in s over A^σ (Theorem 2.1). Even if σ has no slice, A can be a polynomial ring in one variable over A^σ in some special cases. For example, let k be a field of arbitrary characteristic. Then, this is the case if A is the polynomial ring in two variables over k (cf. [15] when $\operatorname{char} k = 0$, [11] when k is algebraically closed, and [7] for the general case).

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Nakai [14, Thm. 1] proved the following theorem. Here, for a subring R of A, we say that σ is an exponential map *over* R if A^σ contains R.

Theorem 1.1 (Nakai). *Let A be a k-domain, and σ a nontrivial exponential map on A over k. Assume that k is algebraically closed, A^σ is a finitely generated PID over k, and every prime element of A^σ is a prime element of A. Then, A is the polynomial ring in one variable over A^σ.*

The purpose of this paper is to generalize Theorem 1.1, and derive some useful consequences. Our main theorem (Theorem 3.2) generalizes Theorem 1.1 in the following points. First of all, the ground field k is not necessary algebraically closed. Second, the k-algebra A^σ is not necessary finitely generated. Third, A is not necessarily an integral domain. Furthermore, locally finite iterative higher derivation is replaced by exponential map, which simplifies the proof of Nakai.

One of the consequences of our main theorem implies the following result.

Theorem 1.2. *Let k be any field, and A and A' finitely generated k-domains with $A[x] \simeq_k A'[x]$. If A and $\bar{k} \otimes_k A$ are UFDs and* $\operatorname{trans.deg}_k A = 2$, *then we have $A \simeq_k A'$. Here, $\bar{k}$ is an algebraic closure of k.*

This theorem is a generalization of Crachiola [2, Cor. 3.2] which says that $A[x] \simeq_k A'[x]$ implies $A \simeq_k A'$ if A and A' are finitely generated UFDs over an algebraically closed field k with $\operatorname{trans.deg}_k A = \operatorname{trans.deg}_k A' = 2$. One benefit of this generalization is that Theorem 1.2 covers the case where A is the polynomial ring in two variables over an arbitrary field k.

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2. Exponential map with a slice

Let A be any commutative ring, and σ a nontrivial exponential map on A. In this section, we prove the following theorem for the lack of a suitable reference (cf. e.g. [3, Lem. 2.2], [12, §1.4], [13, Appendix] when A is a domain).

Theorem 2.1. *If σ has a slice s, then A is the polynomial ring in s over A^σ.*

First, note that (E1) and (E2) imply the following statements. By the statement (i), we know that $\operatorname{lc}_\sigma(a)$ belongs to A^σ for each $a \in A$.

Lemma 2.2. (i) *For each $0 \le i \le m$, we have $\deg_\sigma(a_i) \le m - i$. Hence, a_m belongs to A^σ.*
(ii) *Assume that $p := \operatorname{char}(A)$ is a prime number, and write $m = lp^e$, where $e \ge 0$ and $l \ge 1$ with $p \nmid l$. If $a_m \neq 0$, then we have $\deg_\sigma(a_{(l-1)p^e}) = p^e$.*
(iii) *If $\sigma(a_i) = a_i$ for all $1 \le i \le m$, then $f(x) := \sigma(a) - a = \sum_{i=1}^m a_i x^i$ is* additive, *i.e., satisfies $f(x+y) = f(x) + f(y)$.*

Proof. Considering total degrees, (i) is clear from (E2). In the case (ii), we have $l \in A^*$ and

$$(x+y)^m = (x+y)^{lp^e} = (x^{p^e} + y^{p^e})^l = y^{lp^e} + lx^{p^e} y^{(l-1)p^e} + \cdots + x^{lp^e}.$$

By (E2), this implies $\sigma(a_{(l-1)p^e}) = la_m x^{p^e} +$ (terms of lower degree in x) with $la_m \neq 0$. In the case (iii), we have

$$\sum_{i=0}^{m} \sigma(a_i)y^i = \sigma(a) + \sum_{i=1}^{m} a_i y^i = \sum_{i=0}^{m} a_i x^i + \sum_{i=1}^{m} a_i y^i = a_0 + f(x) + f(y),$$

since $a_0 = a$ by (E1). Hence, we know by (E2) that $f(x)$ is additive. □

For each integer $n \geq 2$, let $d(n)$ be the greatest common divisor of the binomial coefficients $\binom{n}{i}$ for $1 \leq i < n$. If $n = p^d$ for some prime number p and $d \geq 1$, then we have $d(n) = p$, since $p^2 \nmid \binom{p^d}{p^{d-1}}$. Otherwise, we have $d(n) = 1$. Hence, the following lemma holds.

Lemma 2.3. *For an integer $n \geq 2$ and $a \in A \setminus \{0\}$, we have $a(x+y)^n = a(x^n + y^n)$ if and only if there exist a prime number p and $d \geq 1$ such that $n = p^d$ and $\{l \in \mathbf{Z} \mid la = 0\} = p\mathbf{Z}$.*

Let s be a local slice of σ. Then, by the minimality of $n := \deg_\sigma(s)$, we see from Lemma 2.2 (i) that the coefficient of x^i in $\sigma(s)$ belongs to A^σ for $i = 1, \ldots, n$. Hence, $\sigma(s) - s$ is additive by Lemma 2.2 (iii). If furthermore s is a slice of σ, then this implies either $n = 1$, or $p := \operatorname{char}(A)$ is a prime number and $n = p^d$ for some $d \geq 1$ by Lemma 2.3 with $a = 1$.

Now, let us prove Theorem 2.1. First, we show that each $a \in A \setminus \{0\}$ belongs to $A^\sigma[s]$ by induction on $m := \deg_\sigma(a)$. If $m = 0$, then a belongs to A^σ. Assume that $m > 0$. We show that $n := \deg_\sigma(s)$ divides m. We may assume that $n \geq 2$. Then, $p := \operatorname{char}(A)$ is a prime number, and $n = p^d$ for some $d \geq 1$ as mentioned. Write $m = lp^e$ with $p \nmid l$ and $e \geq 0$. Then, $\deg_\sigma(b) = p^e$ holds for some $b \in A$ by Lemma 2.2 (ii). By the minimality of n, it follows that $n = p^d$ divides p^e, and hence divides m. Now, set $a' := \operatorname{lc}_\sigma(a)$ and $c := a - a' s^{m/n}$. Then, the degree $\deg_\sigma(c)$ of $\sigma(c) = \sigma(a) - a'\sigma(s)^{m/n}$ is less than m. Hence, c belongs to $A^\sigma[s]$ by induction assumption. Thus, a belongs to $A^\sigma[s]$, since so does $a' s^{m/n}$. Therefore, we have $A = A^\sigma[s]$. Next, suppose that $A^\sigma[s]$ is not the polynomial ring in s over A^σ. Then, there exist $m \geq 1$ and $a_0, \ldots, a_m \in A^\sigma$ with $a_m \neq 0$ such that $\sum_{i=0}^m a_i s^i = 0$. Since $\sum_{i=0}^m a_i \sigma(s)^i = 0$, and $\sigma(s)$ is a monic polynomial of positive degree, we are led to a contradiction. This completes the proof of Theorem 2.1.

The following corollary is a consequence of Theorem 2.1, since σ extends to a nontrivial exponential map $\tilde{\sigma}$ on $A[a^{-1}]$ with $A[a^{-1}]^{\tilde{\sigma}} = A^\sigma[a^{-1}]$ which has a slice $a^{-1}s$.

Corollary 2.4. *Let s be a local slice of σ such that $a := \operatorname{lc}_\sigma(s)$ is a nonzero divisor of A. Then, $A[a^{-1}]$ is the polynomial ring in s over $A^\sigma[a^{-1}]$. Hence, we have* $\operatorname{trans.deg}_{A^\sigma} A = 1$ *if A is a domain.*

3. Main results

Recall that, if σ is a nontrivial exponential map on a commutative ring A, then the *plinth ideal*

$$\operatorname{pl}(\sigma) := \{\operatorname{lc}_\sigma(s) \mid s \in A \text{ is a local slice of } \sigma\} \cup \{0\}$$

is an ideal of A^σ. Actually, if $s, s' \in A$ are local slices with $\operatorname{lc}_\sigma(s) + \operatorname{lc}_\sigma(s') \neq 0$, and $a \in A^\sigma$ is such that $a \operatorname{lc}_\sigma(s) \neq 0$, then $s + s'$ and as are local slices with $\operatorname{lc}_\sigma(s + s') = \operatorname{lc}_\sigma(s) + \operatorname{lc}_\sigma(s')$ and $\operatorname{lc}_\sigma(as) = a \operatorname{lc}_\sigma(s)$. The notion of plinth ideal already appeared in Nakai [14], although not called by this name. A local slice s of σ is said to be *minimal* if there does not exist

$a \in \mathrm{pl}(\sigma)$ such that $\mathrm{lc}_\sigma(s)A^\sigma \subsetneq aA^\sigma$. If $s \in A$ satisfies $\mathrm{pl}(\sigma) = \mathrm{lc}_\sigma(s)A^\sigma$, then s is a minimal local slice of σ.

Lemma 3.1. *If s is a minimal local slice with $\mathrm{lc}_\sigma(s)$ a nonzero divisor of A^σ, and if $q \in A^\sigma$ is not a unit of A, then the image of s in A/qA is not contained in the image of A^σ in A/qA.*

Proof. If $s - b = qs'$ holds for some $b \in A^\sigma$ and $s' \in A$, then we have $\sigma(s) - b = q\sigma(s')$, and so $\mathrm{lc}_\sigma(s) = qc$ for some $c \in A$. Since $\mathrm{lc}_\sigma(s)$ is a nonzero divisor of A^σ, it follows that so is q. Since $\mathrm{lc}_\sigma(s')$ belongs to A^σ, we get $\deg_\sigma(s) = \deg_\sigma(s')$ and $\mathrm{lc}_\sigma(s) = q\,\mathrm{lc}_\sigma(s')$, contradicting the minimality of s. □

Now, let k be any field, A a commutative k-algebra, and σ a nontrivial exponential map on A over k. Set $\bar{A} := \bar{k} \otimes_k A$ and $\bar{\sigma} := \mathrm{id}_{\bar{k}} \otimes \sigma$, where $\bar{k}$ is an algebraic closure of k. Then, $\bar{\sigma}$ is an exponential map on $\bar{A}$ over $\bar{k}$ with $\bar{A}^{\bar{\sigma}} = \bar{k} \otimes_k A^\sigma$. In this notation, we have the following theorem.

Theorem 3.2. *Let k, A and σ be as above, and let $s \in A$ be a minimal local slice of $\bar{\sigma}$. Assume that $\mathrm{lc}_\sigma(s)$ is a nonzero divisor of $\bar{A}$, and is written as $p_1 \cdots p_l$ with $l \geq 0$, where $p_1, \ldots, p_l \in \bar{A}^{\bar{\sigma}}$ are such that $\bar{A}^{\bar{\sigma}}/p_i\bar{A}^{\bar{\sigma}} = \bar{k}$ and $\bar{A}/p_i\bar{A}$ is a domain for $i = 1, \ldots, l$. Then, A is the polynomial ring in s over A^σ.*

Proof. By assumption, s is a minimal local slice of $\bar{\sigma}$, and $a := \mathrm{lc}_\sigma(s) = p_1 \cdots p_l$ is a nonzero divisor of $\bar{A}$. Hence, $\bar{A}[a^{-1}]$ is the polynomial ring in s over $\bar{A}^{\bar{\sigma}}[a^{-1}]$ by Corollary 2.4. Since A^σ is contained in $\bar{A}^{\bar{\sigma}}[a^{-1}]$, it suffices to verify $A = A^\sigma[s]$. Since $A \supset A^\sigma[s]$, we prove that $\bar{k} \otimes_k A = \bar{k} \otimes_k A^\sigma[s]$, that is, $\bar{A} = \bar{A}^{\bar{\sigma}}[s]$. This is clear if $a = 1$. So assume that $l \geq 1$. We remark that, if $c \in \bar{A}$ satisfies $p_1 \cdots p_i c \in \bar{A}^{\bar{\sigma}}$ for some $1 \leq i \leq l$, then c belongs to $\bar{A}^{\bar{\sigma}}$. In fact, since $p_1, \ldots, p_i$ are elements of $\bar{A}^{\bar{\sigma}}$ by assumption, we have $p_1 \cdots p_i \bar{\sigma}(c) = p_1 \cdots p_i c$, and so $\bar{\sigma}(c) = c$. Similarly, $p_i\bar{A} \cap \bar{A}^{\bar{\sigma}} = p_i\bar{A}^{\bar{\sigma}}$ holds for each i. Now, take any $b \in \bar{A}$, and write $b = a^{-n} \sum_{i \geq 0} b_i s^i$, where $n \geq 0$ and $b_i \in \bar{A}^{\bar{\sigma}}$ for each i. We may assume that n is minimal among such expressions. To conclude $b \in \bar{A}^{\bar{\sigma}}[s]$, we show that $n = 0$ by contradiction. Suppose that $n \geq 1$. Then, $\sum_{i \geq 0} b_i s^i = a^n b$ belongs to $a\bar{A}$. Let $1 \leq u \leq l + 1$ be the maximal number satisfying $\{b_i \mid i \geq 0\} \subset p_1 \cdots p_{u-1}\bar{A}$. Then, by the minimality of n, we have $1 \leq u \leq l$. Set $c_i := (p_1 \cdots p_{u-1})^{-1} b_i \in \bar{A}^{\bar{\sigma}}$ for each i. Then, $\sum_{i \geq 0} c_i s^i$ belongs to $p_u\bar{A}$, but c_{i_0} does not belong to $p_u\bar{A}$ for some i_0 by the maximality of u. By assumption, $\bar{A}/p_u\bar{A}$ is a domain. Moreover, the image of $\bar{A}^{\bar{\sigma}}$ in $\bar{A}/p_u\bar{A}$ is equal to $\bar{k}$, since $p_u\bar{A} \cap \bar{A}^{\bar{\sigma}} = p_u\bar{A}^{\bar{\sigma}}$, and $\bar{A}^{\bar{\sigma}}/p_u\bar{A}^{\bar{\sigma}} = \bar{k}$ by assumption. Hence, the image of s in $\bar{A}/p_u\bar{A}$ is algebraic over $\bar{k}$, and thus belongs to the image of $\bar{A}^{\bar{\sigma}}$ in $\bar{A}/p_u\bar{A}$. Since $\mathrm{lc}_\sigma(s)$ is a nonzero divisor of $\bar{A}$, this contradicts the minimality of s by Lemma 3.1. Therefore, b belongs to $\bar{A}^{\bar{\sigma}}[s]$, proving $\bar{A} = \bar{A}^{\bar{\sigma}}[s]$. □

Observe that every local slice of $\bar{\sigma}$ is written as a $\bar{k}$-linear combination of local slices of σ. Hence, we get $\mathrm{pl}(\bar{\sigma}) = \bar{k} \otimes_k \mathrm{pl}(\sigma)$. Thus, if $s \in A$ satisfies $\mathrm{pl}(\sigma) = \mathrm{lc}_\sigma(s)A^\sigma$, then we have $\mathrm{pl}(\bar{\sigma}) = \bar{k} \otimes_k \mathrm{pl}(\sigma) = \mathrm{lc}_\sigma(s)\bar{A}^{\bar{\sigma}}$, and so s is a minimal local slice of $\bar{\sigma}$. Therefore, if A admits a nontrivial exponential map σ over k with the following three conditions, then A is a polynomial ring in one variable over A^σ by Theorem 3.2:

(N1) $\mathrm{pl}(\sigma)$ is a principal ideal of A^σ generated by a nonzero divisor of $\bar{A}$.
(N2) $\bar{A}^{\bar{\sigma}}$ is a PID with $\mathrm{trans.deg}_k \bar{A}^{\bar{\sigma}} \leq 1$.

(N3) $\bar{A}/p\bar{A}$ is a domain for every prime element p of $\bar{A}^{\bar{\sigma}}$.

Since a finitely generated PID over a field k has transcendence degree at most one over k, we see that the assumption of Theorem 1.1 implies (N1), (N2) and (N3). Therefore, we obtain Theorem 1.1 from Theorem 3.2.

Next, assume that A is a domain. It is well known that $A^\sigma = \sigma^{-1}(A)$ is *factorially closed* in A, i.e., $ab \in A^\sigma$ implies $a, b \in A^\sigma$ for each $a, b \in A \setminus \{0\}$, since A is factorially closed in $A[x]$, and σ is injective by (E1). This implies that $(A^\sigma)^* = A^*$, and every irreducible element of A^σ is an irreducible element of A. Note that, if $p \in A^\sigma$ is a prime element of A, then p is a prime element of A^σ, since $pA^\sigma = pA \cap A^\sigma$ is a prime ideal. Hence, if A is a UFD, then A^σ is also a UFD, and every prime element of A^σ is a prime element of A. We also note that, if A contains a field k, then every exponential map σ on A is an exponential map over k, since $k \setminus \{0\} \subset A^* \subset A^\sigma$.

The following corollary is also a consequence of Theorem 3.2.

Corollary 3.3. *Let A be a UFD over a field k, and σ a nontrivial exponential map on A. If* $\operatorname{trans.deg}_k A = 2$ *and $\bar{k} \otimes_k A$ is a UFD, then A is a polynomial ring in one variable over A^σ.*

Proof. It suffices to check (N1), (N2) and (N3). Since A and $\bar{A}$ are UFDs, we know from the previous observation that A^σ and $\bar{A}^{\bar{\sigma}}$ are UFDs, and (N3) is fulfilled. Recall that a UFD is a PID if every nonzero principal prime ideal is maximal. Since $\operatorname{trans.deg}_k A = 2$ by assumption, we have $\operatorname{trans.deg}_k A^\sigma = \operatorname{trans.deg}_k \bar{A}^{\bar{\sigma}} = 1$ by Corollary 2.4, and so A^σ and $\bar{A}^{\bar{\sigma}}$ are PIDs. Therefore, we get (N1) and (N2). □

In Corollary 3.3, the assumption that $\bar{k} \otimes_k A$ is a UFD is necessary. In fact, there exists a UFD A over k such that $\operatorname{trans.deg}_k A = 2$, $\bar{k} \otimes_k A$ is not a UFD, and A admits an exponential map σ for which A is not a polynomial ring in one variable over A^σ. For instance, consider the k-domain

$$A := k[S, T, U]/(S^n U - f(T)),$$

where S, T and U are variables, $n \geq 1$ and $f(T) \in k[T]$ is an irreducible polynomial with $\deg f(T) \geq 2$. We note that A is the coordinate ring of a *Danielewski surface* (cf. e.g. [10]; see also [4, Ex. 4] for factoriality). Let s, t and u be the images of S, T and U in A, respectively. Then, $A/sA \simeq k[T, U]/(f(T))$ is a domain by the irreducibility of $f(T)$. Hence, s is a prime element of A. Since $A[s^{-1}] = k[s, s^{-1}, t, s^{-n} f(t)] = k[s, s^{-1}, t]$ is a UFD, we know that A is a UFD. We define an exponential map τ on $A[s^{-1}]$ over $k[s, s^{-1}]$ by $\tau(t) = t + s^n x$. Then, τ restricts to an exponential map σ on A over k, since $\tau(s^{-n} f(t)) = s^{-n} f(t + s^n x)$ belongs to $A[x] = k[s, t, s^{-n} f(t)][x]$. Moreover, we have

$$A^\sigma = A[s^{-1}]^\tau \cap A = k[s, s^{-1}] \cap A = k[s].$$

We show that A is not a polynomial ring in one variable over A^σ. If the assertion is false, then $\bar{A}$ is a polynomial ring in two variables over $\bar{k}$, and hence is a UFD. We prove that this is not the case. Since $\bar{A}/u\bar{A} \simeq \bar{k}[S, T]/(f(T))$ and $\deg f(T) \geq 2$, we see that u is neither a prime element nor a unit of $\bar{A}$. Thus, if $\bar{A}$ is a UFD, then there exist $u_1, u_2 \in \bar{A} \setminus \bar{A}^*$ such that $u_1 u_2 = u = s^{-n} f(t)$. In $\bar{A}[s^{-1}] = \bar{k}[s, s^{-1}, t]$, we may write $u_i = s^{n_i} w_i$ for $i = 1, 2$, where $n_i \in \mathbf{Z}$ and $w_i \in \bar{k}[s, t] \setminus s\bar{k}[s, t]$. Since $s^{n_1+n_2+n} w_1 w_2 = f(t)$, we get $w_1 w_2 = f(t)$ and

$w_1, w_2 \in \bar{k}[t]$. Then, we know from $s^{n_i} w_i = u_i \in A = \bar{k}[s, t, s^{-n} f(t)]$ that $n_i \geq 0$ or $w_i = cf(t)$ for some $c \in k^*$. It follows that u_1 or u_2 belongs to k^*, a contradiction.

Next, we explain how to prove Theorem 1.2. Crachiola [2, Thm. 3.1] showed Corollary 3.3 when k is algebraically closed. He derived from this result the cancellation theorem [2, Cor. 3.2] mentioned in Section 1 by making use of Crachiola–Makar-Limanov [3, Thm. 3.1] and Abhyankar-Eakin-Heinzer [1, Thm. 3.3]. His argument in fact proved the following statement for an arbitrary field k (see the proof of [2, Cor. 3.2]).

Lemma 3.4. *For $i = 1, 2$, let A_i be a finitely generated k-domain with* $\operatorname{trans.deg}_k A_i = 2$ *having the following property*: If A_i admits a nontrivial exponential map σ over k, then A_i is a polynomial ring in one variable over A_i^{σ}. *Then, it holds that $A_1[x] \simeq_k A_2[x]$ implies $A_1 \simeq_k A_2$.*

Under the assumption of Theorem 1.2, $A'[x]$ and $\bar{k} \otimes_k A'[x]$ are UFDs, and $\operatorname{trans.deg}_k A' = 2$. Since A' and $\bar{k} \otimes_k A'$ are factorially closed in $A'[x]$ and $\bar{k} \otimes_k A'[x]$, respectively, it follows that A' and $\bar{k} \otimes_k A'$ are also UFDs. Hence, using Lemma 3.4, we can derive Theorem 1.2 from Corollary 3.3.

Finally, let $A = R[x, y]$ be the polynomial ring in two variables over a domain R. Corollary 3.3 also implies the result that, if R is a field, and σ is a nontrivial exponential map on A, then A is a polynomial ring in one variable over A^{σ} (cf. §1). The following theorem is a consequence of this result. We call $f \in R[x, y]$ a *coordinate* of $R[x, y]$ if there exists $g \in R[x, y]$ such that $R[x, y] = R[f, g]$. A domain R is called an *HCF-ring* if, for any $a, b \in R$, there exists $c \in R$ such that $aR \cap bR = cR$. For example, UFDs are HCF-rings.

Theorem 3.5. *Let R be an HCF-ring, K the field of fractions of R, and σ a nontrivial exponential map on $R[x, y]$ over R. Then, there exists $f \in R[x, y]$ such that f is a coordinate of $K[x, y]$ and $R[x, y]^{\sigma} = R[f]$.*

Proof. $R[x, y]^{\sigma}$ is factorially closed in $R[x, y]$, and is of transcendence degree one over R by Corollary 2.4. Since R is an HCF-ring by assumption, this implies that $R[x, y]^{\sigma} = R[f]$ for some $f \in R[x, y]$ by Abhyankar-Eakin-Heinzer [1, Prop. 4.8]. Let $\tilde{\sigma}$ be the extension of σ to $K[x, y]$. Then, we have $K[x, y]^{\tilde{\sigma}} = K[f]$, and $K[x, y] = K[x, y]^{\tilde{\sigma}}[g] = K[f, g]$ for some $g \in K[x, y]$ by the result mentioned above. □

When R contains $\mathbf{Q}$, this result is found in Freudenburg [5, Thms. 4.11 and 4.13]. The proof above is similar to Freudenburg's. When R is a field, Theorem 3.5 can be derived from the famous result of Jung [6] and van der Kulk [8] very easily (cf. [9]).

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