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ON TWO MODULI SPACES OF SHEAVES SUPPORTED ON QUADRIC SURFACES

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Abstract

We show that the moduli space of semi-stable sheaves on a smooth quadric surface, having dimension 1, multiplicity 4, Euler characteristic 2, and first Chern class $(2,2)$, is the blow-up at two points of a certain hypersurface in a weighted projective space.

Let $\mathbf{M}$ be the moduli space of Gieseker semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ having Hilbert polynomial $P_{\mathcal{F}}(m) = 4m+2$, relative to the polarization $\mathcal{O}(1,1)$, and first Chern class $c_1(\mathcal{F}) = (2,2)$. Let $\mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ be the moduli space of Gieseker semi-stable sheaves $\mathcal{F}$ on $\mathbb{P}^3$ having Hilbert polynomial $P_{\mathcal{F}}(m) = m^2+3m+2$. Such sheaves are supported on quadric surfaces. The purpose of this note is to show that $\mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ is isomorphic to a certain hypersurface in a weighted projective space (see Proposition 6) and to give an elementary proof of a result of Chung and Moon [3] stating that $\mathbf{M}$ is the blow-up of $\mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ at two regular points.

Let l, m, n be positive integers. Let V be a vector space over $\mathbb{C}$ of dimension l. The reductive group $G = (\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}))/\mathbb{C}^*$ acts by conjugation on the vector space $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)$ of $m \times n$-matrices with entries in V. The resulting good quotient

$$\mathrm{N}(V;m,n) = \mathrm{N}(l;m,n) = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^{\mathrm{ss}} /\!\!/ G$$

is called a *Kronecker moduli space*. Kronecker moduli spaces arise from the study of moduli spaces of torsion-free sheaves, as in [4]. According to [10, Corollary 3.7] and [3, Lemma 5.2], the map

$$\mathrm{Hom}(2\mathcal{O}_{\mathbb{P}^3}(-1), 2\mathcal{O}_{\mathbb{P}^3})^{\mathrm{ss}} \longrightarrow \mathrm{M}_{\mathbb{P}^3}(m^2+3m+2), \qquad \langle\varphi\rangle \longmapsto \langle Coker(\varphi)\rangle,$$

is a good quotient modulo $(\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}))/\mathbb{C}^*$. Thus, the above moduli space is isomorphic to $\mathrm{N}(4;2,2)$. According to [10, Remark 3.9], $\mathrm{M}_{\mathbb{P}^3}(m^2+3m+2)$ is rational; this result was reproved in [3] using the wall-crossing method.

Lemma 1. *Assume that* $\mathrm{N}(l;m,n)$ *contains stable points. Then the same is true of* $\mathrm{N}(k;m,n)$ *for all integers* $k > l$*, and, moreover,* $\mathrm{N}(k;m,n)$ *is birational to* $\mathbb{A}^{(k-l)mn} \times \mathrm{N}(l;m,n)$.

Proof. Let U, V be vector spaces over $\mathbb{C}$ of dimension $k-l$, respectively, l, and put $W = U \oplus V$. The projection of W onto the second factor induces a G-equivariant projection

$$\pi\colon\ \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes W) \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V).$$

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From King's criterion of semi-stability [8] we see that

$$\pi^{-1}(\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^{\mathrm{s}}) \subset \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes W)^{\mathrm{s}}.$$

The left term, denoted by E, is a trivial G-linearized vector bundle over $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^{\mathrm{s}}$ with fiber $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes U)$. The geometric quotient map

$$\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m \otimes V)^{\mathrm{s}} \longrightarrow \mathrm{N}(V; m, n)^{\mathrm{s}}$$

is a principal G-bundle, so we can apply [7, Theorem 4.2.14] to deduce that E descends to a vector bundle F over $\mathrm{N}(V; m, n)^{\mathrm{s}}$. Clearly, F is the geometric quotient of E by G, hence F is isomorphic to an open subset of $\mathrm{N}(W; m, n)^{\mathrm{s}}$. We conclude that $\mathrm{N}(W; m, n)$ is birational to $\mathbb{A}^{(k-l)mn} \times \mathrm{N}(V; m, n)$. □

Proposition 2.
(i) *For* $l \geq 3$, $\mathrm{N}(l; 2, 2)$ *is rational.*
(ii) *For* $l \geq 3$ *and* $n \geq 1$, $\mathrm{N}(l; n, n+1)$ *is rational.*

Proof. According to [4, Lemma 25], $\mathrm{N}(3; 2, 2)$ is isomorphic to $\mathbb{P}^5$. Identifying $\mathbb{P}^5$ with the space of conic curves in $\mathbb{P}^2$, the stable points correspond to irreducible conics. Applying Lemma 1, yields (i).

According to [5, Propositions 4.5 and 4.6], the subset of $\mathrm{N}(3; n, n+1)$ of matrices whose maximal minors have no common factor is isomorphic to the subset of $\mathrm{Hilb}_{\mathbb{P}^2}(n(n+1)/2)$ of schemes that are not contained in any curve of degree $n-1$. Thus, $\mathrm{N}(3; n, n+1)$ is birational to $\mathrm{Hilb}_{\mathbb{P}^2}(n(n+1)/2)$, so it is rational. Moreover, $\mathrm{N}(3; n, n+1)$ consists only of stable points. Applying Lemma 1, yields (ii). □

Proposition 3. *For* $l \geq 3$ *and* $n \geq 1$, $\mathrm{N}(l; n, n)$ *is a rational variety.*

Proof. The argument is inspired by [10, Remark 3.9]. In view of [4, Section 3], $\mathrm{N}(3; n, n)$ contains stable points. This is due to the fact that we have the inequality $x < n/n < 1/x$, where x is the smaller solution to the equation $x^2 - 3x + 1 = 0$. Thus, we are in the context of Lemma 1, which asserts that $\mathrm{N}(l; n, n)$ is rational for $l \geq 3$ if $\mathrm{N}(3; n, n)$ is rational. We may, therefore, restrict to the case when $l = 3$. Let V be a vector space over $\mathbb{C}$ with basis $\{x, y, z\}$. An element $\varphi \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)$ can be written uniquely in the form $\varphi = \varphi_1 x + \varphi_2 y + \varphi_3 z$, where $\varphi_1, \varphi_2, \varphi_3 \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. Let

$$\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 \subset \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)^{\mathrm{s}}$$

be the open invariant subset given by the condition that φ_1 be invertible. Let $X \subset \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0$ be the closed subset given by the condition $\varphi_1 = I$. The group $\mathrm{PGL}(n, \mathbb{C})$ acts on X by conjugation. The composite map

$$X \hookrightarrow \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 \longrightarrow \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 / G$$

is surjective and its fibers are precisely the $\mathrm{PGL}(n, \mathbb{C})$-orbits. Thus, it factors through a bijective morphism

$$X / \mathrm{PGL}(n, \mathbb{C}) \longrightarrow \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes V)_0 / G.$$

In characteristic zero, bijective morphisms of irreducible varieties are birational. We have reduced to the following problem. Let U be a complex vector space of dimension 2 and let $\mathrm{PGL}(n, \mathbb{C})$ act on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)$ by conjugation. Then the resulting good quotient is rational.

Choose a basis $\{y, z\}$ of U. An element $\psi \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)$ can be uniquely written in the form $\psi = y\psi_1 + z\psi_2$, where $\psi_1, \psi_2 \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. Let

$$\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)$$

be the open invariant subset given by the conditions that ψ have trivial stabilizer and that ψ_1 be invertible and have distinct eigenvalues. Let $Y \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0$ be the closed subset given by the condition that ψ_1 be a diagonal matrix. Let $S, T \subset \mathrm{PGL}(n, \mathbb{C})$ be the image of the canonical embedding of the group of permutations of n elements, respectively, the subgroup of diagonal matrices. Then $H = ST$ is a closed subgroup of $\mathrm{PGL}(n, \mathbb{C})$ leaving Y invariant. The composite map

$$Y \hookrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 / \mathrm{PGL}(n, \mathbb{C})$$

is surjective and its fibers are precisely the H-orbits. Thus, it factors through a bijective morphism

$$Y/H \longrightarrow \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n \otimes U)_0 / \mathrm{PGL}(n, \mathbb{C})$$

that must be birational. We have reduced the problem to showing that Y/H is rational.

Let $Y_0 \subset Y$ be the open H-invariant subset given by the condition that all entries of ψ_2 be non-zero. Concretely, $Y_0 = D \times E$, where $D, E \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ are the subset of invertible diagonal matrices with distinct entries on the diagonal, respectively, the subset of matrices without zero entries. The normal subgroup $T \leq H$ acts trivially on D, hence $(D \times E)/T$ is a trivial bundle over D with fiber E/T. The induced action of $S = H/T$ is compatible with the bundle structure. The stabilizer in S of any $\psi_1 \in D$ acts trivially on the fiber over ψ_1, because it is trivial. It follows that $(D \times E)/T$ descends to a fiber bundle F over D/S. Clearly, F is isomorphic to $(D \times E)/H$, hence $(D \times E)/H$ is birational to $D/S \times E/T$. Both D/S and E/T are rational, namely D/S is isomorphic to an open subset of $\mathrm{S}^n(\mathbb{A}^1) \simeq \mathbb{A}^n$, while $E/T \simeq (\mathbb{A}^1 \setminus \{0\})^{n^2-n+1}$. In conclusion, Y/H is rational. □

Let $r > 0$ and χ be integers. Let $\mathrm{M}_{\mathbb{P}^2}(r, \chi)$ denote the moduli space of Gieseker semi-stable sheaves on $\mathbb{P}^2$ having Hilbert polynomial $P(m) = rm + \chi$. It is well known that $\mathrm{M}_{\mathbb{P}^2}(r, 0)$ is birational to $\mathrm{N}(3; r, r)$ and, if r is even, $\mathrm{M}_{\mathbb{P}^2}(r, r/2)$ is birational to $\mathrm{N}(6; r/2, r/2)$. We obtain the following.

Corollary 4. *The moduli spaces* $\mathrm{M}_{\mathbb{P}^2}(r, 0)$ *and, if r is even,* $\mathrm{M}_{\mathbb{P}^2}(r, r/2)$*, are rational.*

The rationality of $\mathrm{M}_{\mathbb{P}^2}(3, 0)$ and $\mathrm{M}_{\mathbb{P}^2}(4, 2)$ is already known from [9].

The maps

$$\det\colon \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \longrightarrow \mathrm{S}^2\, V, \qquad \det(\varphi) = \varphi_{11}\varphi_{22} - \varphi_{12}\varphi_{21},$$

and

$$\mathrm{e}\colon \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \longrightarrow \Lambda^4\, V, \qquad \mathrm{e}(\varphi) = \varphi_{11} \wedge \varphi_{22} \wedge \varphi_{12} \wedge \varphi_{21}$$

are semi-invariant in the sense that for any $(g, h) \in G$ and $\varphi \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V)$,

$$\det((g,h)\varphi) = \det(g)^{-1}\det(h)\det(\varphi), \qquad \mathrm{e}((g,h)\varphi) = \det(g)^{-2}\det(h)^2\,\mathrm{e}(\varphi).$$

Using King's criterion of semi-stability [8], it is easy to see that φ is semi-stable if and only if $\det(\varphi) \neq 0$ and is stable if and only if $\det(\varphi)$ is irreducible in $\mathrm{S}^* V$. In the case when $\dim(V) = 3$, the isomorphism $\mathrm{N}(V;2,2) \to \mathbb{P}(\mathrm{S}^2 V)$ of [4] is given by $\langle\varphi\rangle \mapsto \langle\det(\varphi)\rangle$.

In the sequel we will assume that $\dim(V) = 4$ and that $m = 2$, $n = 2$. Choose bases $\{x, y, z, w\}$ of V and $\{v_1, v_2, v_3, v_4\}$ of V^*. Consider the semi-invariant functions

$$\epsilon, \rho\colon\ \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \longrightarrow \mathbb{C}, \qquad \epsilon(\varphi) = \mathrm{i}_{v_1\wedge v_2\wedge v_3\wedge v_4}\,\mathrm{e}(\varphi),$$

$$\rho(\varphi) = \mathrm{i}_{v_1\wedge v_2\wedge v_3\wedge v_4}(\mathrm{i}_{v_1}\det(\varphi) \wedge \mathrm{i}_{v_2}\det(\varphi) \wedge \mathrm{i}_{v_3}\det(\varphi) \wedge \mathrm{i}_{v_4}\det(\varphi)).$$

Here i_v denotes the internal product with a vector $v \in V^*$.

Proposition 5. *We have the relation $\epsilon^2 = \rho$.*

Proof. Let $\{v_1', v_2', v_3', v_4'\}$ be another basis of V^* and let $\upsilon \in \mathrm{GL}(4,\mathbb{C})$ be the change-of-basis matrix. With respect to this basis we define the functions ρ' and ϵ' as above. Then $\epsilon'(\varphi) = \det(\upsilon)\epsilon(\varphi)$ and $\rho'(\varphi) = \det(\upsilon)^2\rho(\varphi)$, hence $\epsilon(\varphi)^2 = \rho(\varphi)$ if and only if $\epsilon'(\varphi)^2 = \rho'(\varphi)$. Put $U = \mathrm{span}\{x, y, z\}$ and let

$$\pi\colon\ \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V) \longrightarrow \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes U)$$

be the morphism induced by the projection of $V = U \oplus \mathbb{C}w$ onto the first factor. It is enough to verify the relation on the Zariski open subset given by the condition that $\det(\pi(\varphi))$ be irreducible. Changing, possibly, the basis of U, we may assume that $\det(\pi(\varphi)) = x^2 - yz$. Since $\pi(\varphi)$ is stable, and since $\mathrm{N}(U;2,2)$ is isomorphic to $\mathbb{P}(\mathrm{S}^2 U)$, we have

$$\pi(\varphi) \sim \begin{bmatrix} x & y \\ z & x \end{bmatrix}, \quad \text{so we may write} \quad \varphi = \begin{bmatrix} x + aw & y + bw \\ z + cw & x + dw \end{bmatrix}.$$

We have

$$\det(\varphi) = x^2 - yz + (a + d)xw - cyw - bzw + (ad - bc)w^2,$$

$$\mathrm{e}(\varphi) = (d - a)x \wedge y \wedge z \wedge w.$$

Since we are free to choose the basis of V^*, we choose $\{v_1, v_2, v_3, v_4\}$ to be the dual of $\{x, y, z, w\}$. We have

$$\begin{aligned}
\mathrm{i}_{v_1}\det(\varphi) &= \frac{\partial}{\partial x}\det(\varphi) = 2x + (a + d)w,\\
\mathrm{i}_{v_2}\det(\varphi) &= \frac{\partial}{\partial y}\det(\varphi) = -z - cw,\\
\mathrm{i}_{v_3}\det(\varphi) &= \frac{\partial}{\partial z}\det(\varphi) = -y - bw,\\
\mathrm{i}_{v_4}\det(\varphi) &= \frac{\partial}{\partial w}\det(\varphi) = (a + d)x - cy - bz + 2(ad - bc)w,
\end{aligned}$$

$$\epsilon(\varphi) = d - a, \qquad \rho(\varphi) = \begin{vmatrix} 2 & 0 & 0 & a+d \\ 0 & 0 & -1 & -c \\ 0 & -1 & 0 & -b \\ a+d & -c & -b & 2(ad-bc) \end{vmatrix} = (a-d)^2.$$

In conclusion, $\epsilon(\varphi)^2 = (d-a)^2 = \rho(\varphi)$. □

Consider the action of $\mathbb{C}^*$ on $\mathrm{S}^2 V \oplus \Lambda^4 V$ given by $t(q, p) = (tq, t^2 p)$ and let $\mathbb{P}$ denote the weighted projective space $((\mathrm{S}^2 V \oplus \Lambda^4 V) \setminus \{0\})/\mathbb{C}^*$. Consider the map

$$\eta\colon\ \mathrm{N}(V; 2, 2) \longrightarrow \mathbb{P}, \qquad \eta(\langle\varphi\rangle) = \langle \det(\varphi), \mathrm{e}(\varphi)\rangle.$$

Choose coordinates on $\mathbb{P}$ given by the choice of basis $\{x, y, z, w\}$ of V. In view of Proposition 5, the image of η is contained in the hypersurface $H \subset \mathbb{P}$ given by the equation $\mathrm{dis}(q) = p^2$, where $\mathrm{dis}(q)$ denotes the discriminant of the quadratic form q.

Proposition 6. *Assume that* $\dim(V) = 4$. *Then the map* $\eta\colon\ \mathrm{N}(V; 2, 2) \to H$ *is an isomorphism.*

Proof. The singular points of the cone $\hat{H} \subset \mathrm{S}^2 V \oplus \Lambda^4 V$ over H are of the form $(q, 0)$, where $q \in \mathrm{S}^2 V$ is a singular point of the vanishing locus of the discriminant. It follows that $\hat{H}$ is regular in codimension 1. From Serre's criterion of normality we deduce that $\hat{H}$ is normal (condition S2 is satisfied because $\hat{H}$ is a hypersurface in a smooth variety). Normality is inherited by a good quotient, hence $H = (\hat{H} \setminus \{0\})/\mathbb{C}^*$ is normal, too. In view of the Main Theorem of Zariski, it is enough to show that η is bijective. Since $\mathrm{N}(V; 2, 2)$ is complete, and since $\mathrm{N}(V; 2, 2)$ and H are irreducible of the same dimension, it is enough to show that η is injective.

Assume that $\eta(\langle\varphi_1\rangle) = \eta(\langle\varphi_2\rangle)$. Varying φ_1 and φ_2 in their respective orbits, we may assume that $\det(\varphi_1) = \det(\varphi_2)$ and $\mathrm{e}(\varphi_1) = \mathrm{e}(\varphi_2)$. If $\det(\varphi_1)$ is reducible, say $\det(\varphi_1) = uu'$ for some $u, u' \in V$, then it is easy to see that

$$\varphi_1 \sim \begin{bmatrix} u & u_1 \\ 0 & u' \end{bmatrix}, \qquad \varphi_2 \sim \begin{bmatrix} u & u_2 \\ 0 & u' \end{bmatrix}$$

for some $u_1, u_2 \in V$. But then $\langle\varphi_1\rangle = \langle\varphi_2\rangle = \langle \mathrm{diag}(u, u')\rangle$. Assume now that $\det(\varphi_1)$ is irreducible. There exists a vector $w \in V$ and a subspace $U \subset V$ such that $V = U \oplus \mathbb{C}w$ and $\det(\pi(\varphi_1))$ is irreducible (notations as at Proposition 5). As mentioned at Proposition 5, we may choose a basis $\{x, y, z\}$ of U such that $\det(\pi(\varphi_1)) = x^2 - yz$, forcing

$$\pi(\varphi_1) \sim \pi(\varphi_2) \sim \begin{bmatrix} x & y \\ z & x \end{bmatrix}.$$

Thus, we may write

$$\varphi_1 = \begin{bmatrix} x + a_1 w & y + b_1 w \\ z + c_1 w & x + d_1 w \end{bmatrix}, \qquad \varphi_2 = \begin{bmatrix} x + a_2 w & y + b_2 w \\ z + c_2 w & x + d_2 w \end{bmatrix}.$$

The relation $\det(\varphi_1) = \det(\varphi_2)$ yields the relations $b_1 = b_2$, $c_1 = c_2$, $a_1 + d_1 = a_2 + d_2$. The relation $\mathrm{e}(\varphi_1) = \mathrm{e}(\varphi_2)$ yields the relation $a_1 - d_1 = a_2 - d_2$. We conclude that $\varphi_1 = \varphi_2$, hence $\langle\varphi_1\rangle = \langle\varphi_2\rangle$. □

REMARK 7. It was already known to Le Potier [10, Remark 3.8] that the map

$$\det\colon \mathrm{N}(V;2,2) \longrightarrow \mathbb{P}(\mathrm{S}^2 V)$$

is a double cover branched over the locus of singular quadratic surfaces.

In the sequel, we will use the abbreviations $\mathcal{O}(r,s) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(r,s)$, $\omega = \omega_{\mathbb{P}^1\times\mathbb{P}^1}$, and $\mathcal{F}^{\mathrm{D}} = \mathcal{E}xt^1_{\mathcal{O}}(\mathcal{F},\omega)$ for a sheaf $\mathcal{F}$ on $\mathbb{P}^1\times\mathbb{P}^1$ of dimension 1. We quote below [3, Proposition 3.8].

Proposition 8. *The sheaves $\mathcal{F}$ giving points in $\mathbf{M}$ are precisely the sheaves having one of the following three types of resolution:*

$$0 \longrightarrow 2\mathcal{O}(-1,-1) \xrightarrow{\varphi} 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathcal{O}(-2,-1) \longrightarrow \mathcal{O}(0,1) \longrightarrow \mathcal{F} \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow \mathcal{O}(-1,-2) \longrightarrow \mathcal{O}(1,0) \longrightarrow \mathcal{F} \longrightarrow 0. \tag{3}$$

This proposition was proved in [3] by the wall-crossing method, however, it was also nearly obtained in [1]. At [1, Lemma 20] it is mistakenly claimed that all sheaves in $\mathbf{M}$ have resolution (1). At a closer inspection, the argument of [1, Lemma 20] shows that the sheaves in $\mathbf{M}$ satisfying the conditions $\mathrm{H}^0(\mathcal{F}^{\mathrm{D}}(1,0)) = 0$ and $\mathrm{H}^0(\mathcal{F}^{\mathrm{D}}(0,1)) = 0$ are precisely the sheaves given by resolution (1). Indeed, the exact sequence (50) in [1] reads

$$0 \longrightarrow \mathcal{H} \longrightarrow 2\mathcal{O} \longrightarrow \mathcal{F} \longrightarrow 0, \tag{4}$$

where $\mathcal{H}$ is a locally free sheaf of rank 2 and determinant ω. Dualizing this sequence, we get the exact sequence

$$0 \longrightarrow 2\mathcal{O}(-2,-2) \longrightarrow \mathcal{H}^{\mathrm{D}} \simeq \mathcal{H}^* \otimes \omega \simeq \mathcal{H} \otimes \det(\mathcal{H})^* \otimes \omega \simeq \mathcal{H} \longrightarrow \mathcal{F}^{\mathrm{D}} \longrightarrow 0. \tag{5}$$

From this we get the relations

$$\mathrm{h}^1(\mathcal{H}(1,0)) = \mathrm{h}^0(\mathcal{F}^{\mathrm{D}}(1,0)) \quad \text{and} \quad \mathrm{h}^1(\mathcal{H}(0,1)) = \mathrm{h}^0(\mathcal{F}^{\mathrm{D}}(0,1)).$$

The vanishing of $\mathrm{H}^1(\mathcal{H}(1,0))$ and $\mathrm{H}^1(\mathcal{H}(0,1))$ implies that $\mathcal{H} \simeq 2\mathcal{O}(-1,-1)$, in which case (4) yields resolution (1).

According to [11, Theorem 13], if $\mathcal{F}$ gives a point in $\mathbf{M}$, then $\mathcal{F}^{\mathrm{D}}(0,1)$ and $\mathcal{F}^{\mathrm{D}}(1,0)$ give points in the moduli space $\mathbf{M}'$ of semi-stable sheaves on $\mathbb{P}^1\times\mathbb{P}^1$ having Hilbert polynomial $P(m) = 4m$ and first Chern class $c_1 = (2,2)$. We claim that the sheaves $\mathcal{E}$ giving points in $\mathbf{M}'$ and satisfying the condition $\mathrm{H}^0(\mathcal{E}) \neq 0$ are precisely the structure sheaves of curves $E \subset \mathbb{P}^1\times\mathbb{P}^1$ of type $(2,2)$. By the argument of [1, Lemma 9], $\mathcal{O}_E$ gives a stable point in $\mathbf{M}'$. Conversely, if $\mathcal{E}$ gives a point in $\mathbf{M}'$ and $\mathrm{H}^0(\mathcal{E}) \neq 0$, then, by the argument of [6, Proposition 2.1.3], there is an injective morphism $\mathcal{O}_C \to \mathcal{E}$ for a curve $C \subset \mathbb{P}^1\times\mathbb{P}^1$. If C did not have type $(2,2)$, then the semi-stability of $\mathcal{E}$ would get contradicted. Thus, C has type $(2,2)$ and, comparing Hilbert polynomials, we see that $\mathcal{O}_C \simeq \mathcal{E}$. In conclusion, if $\mathrm{H}^0(\mathcal{F}^{\mathrm{D}}(0,1)) \neq 0$, then $\mathcal{F} \simeq \mathcal{O}_E(0,-1)^{\mathrm{D}} \simeq \mathcal{O}_E(0,1)$, hence $\mathcal{F}$ has resolution (2). If $\mathrm{H}^0(\mathcal{F}^{\mathrm{D}}(1,0)) \neq 0$, then $\mathcal{F} \simeq \mathcal{O}_E(-1,0)^{\mathrm{D}} \simeq \mathcal{O}_E(1,0)$, hence $\mathcal{F}$ has resolution (3).

We denote by $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2 \subset \mathbf{M}$ the subsets of sheaves given by resolution (1), (2), respectively, (3). Clearly, $\mathbf{M}_0$ is open and $\mathbf{M}_1$, $\mathbf{M}_2$ are divisors isomorphic to $\mathbb{P}^8$. Let

$\mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0$ denote the subset of injective morphisms.

Corollary 9. *The canonical map from below is a good quotient modulo G:*

$$\gamma\colon\ \mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 \longrightarrow \mathbf{M}_0, \qquad \gamma(\varphi) = \langle \mathit{Coker}(\varphi)\rangle.$$

Proof. According to [2, Lemma 1], for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1\times\mathbb{P}^1$ there is a spectral sequence converging to $\mathcal{F}$ in degree zero and to 0 in degrees different from zero, similar to the Beilinson spectral sequence. Its first level E_1^{ij} is given by

$$\mathrm{E}_1^{ij} = 0 \quad \text{if } i > 0 \text{ or } i < -2,$$

$$\mathrm{E}_1^{0j} = \mathrm{H}^j(\mathcal{F})\otimes\mathcal{O}, \qquad \mathrm{E}_1^{-2,j} = \mathrm{H}^j(\mathcal{F}(-1,-1))\otimes\mathcal{O}(-1,-1),$$

and by the exact sequences

$$\mathrm{H}^j(\mathcal{F}(0,-1))\otimes\mathcal{O}(0,-1) \longrightarrow \mathrm{E}_1^{-1,j} \longrightarrow \mathrm{H}^j(\mathcal{F}(-1,0))\otimes\mathcal{O}(-1,0).$$

If $\mathcal{F}$ gives a point in $\mathbf{M}_0$, then

$$\mathrm{H}^0(\mathcal{F})\simeq\mathbb{C}^2, \quad \mathrm{H}^1(\mathcal{F})=0, \quad \mathrm{H}^0(\mathcal{F}(-1,-1))=0, \quad \mathrm{H}^1(\mathcal{F}(-1,-1))\simeq\mathbb{C}^2,$$

$$\mathrm{H}^0(\mathcal{F}(0,-1))=0, \quad \mathrm{H}^1(\mathcal{F}(0,-1))=0, \quad \mathrm{H}^0(\mathcal{F}(-1,0))=0, \quad \mathrm{H}^1(\mathcal{F}(-1,0))=0.$$

Thus, E_1 has only two non-zero terms: $\mathrm{E}_1^{-2,1} = 2\mathcal{O}(-1,-1)$ and $\mathrm{E}_1^{0,0} = 2\mathcal{O}$. The relevant part of E_2 is represented in the following table:

$$\begin{array}{ccc} 2\mathcal{O}(-1,-1) & 0 & 0 \\ & \searrow^{\varphi} & \\ 0 & 0 & 2\mathcal{O} \end{array}$$

The sequence degenerates at E_3, hence φ is injective and $\mathit{Coker}(\varphi)\simeq\mathcal{F}$. This shows that resolution (1) can be obtained from the Beilinson spectral sequence of $\mathcal{F}$. Arguing as at [6, Theorem 3.1.6], we can see that resolution (1) can be obtained for local flat families of sheaves in $\mathbf{M}_0$, hence γ is a categorical quotient. By the uniqueness of the categorical quotient, we deduce that γ is a good quotient map. □

We fix vector spaces V_1 and V_2 over $\mathbb{C}$ of dimension 2 and we make the identifications

$$\mathbb{P}^1\times\mathbb{P}^1 = \mathbb{P}(V_1)\times\mathbb{P}(V_2), \quad \mathrm{H}^0(\mathcal{O}(r,s)) = \mathrm{S}^r\,V_1^*\otimes\mathrm{S}^s\,V_2^*, \quad V = V_1^*\otimes V_2^*.$$

Let

$$\mathbf{W}\subset\mathrm{Hom}\,(2\mathcal{O}(-1,-1)\oplus\mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1),\ \mathcal{O}(-1,0)\oplus\mathcal{O}(0,-1)\oplus 2\mathcal{O})$$

be the open subset of injective morphisms ψ for which $\mathit{Coker}(\psi)$ is Gieseker semi-stable. We represent ψ by a matrix

$$\psi = \left[\begin{array}{c|c} \psi_{11} & \psi_{12} \\ \hline \psi_{21} & \psi_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1\otimes u_{12} & 1\otimes v_{12} & a_1 & 0 \\ u_{11}\otimes 1 & v_{11}\otimes 1 & 0 & a_2 \\ \hline f_{11} & f_{12} & u_{21}\otimes 1 & 1\otimes u_{22} \\ f_{21} & f_{22} & v_{21}\otimes 1 & 1\otimes v_{22} \end{array}\right],$$

where $a_1, a_2 \in \mathbb{C}$, $u_{ij}, v_{ij} \in V_j^*$, $f_{ij} \in V$. The algebraic group

$$\mathbf{G} = (\,\mathrm{Aut}(2\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1)) \times \mathrm{Aut}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \oplus 2\mathcal{O}))/\mathbb{C}^*$$

acts on $\mathbf{W}$ by conjugation. We represent elements of $\mathbf{G}$ by pairs (g, h), where

$$g = \begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix}, \qquad h = \begin{bmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{bmatrix},$$

$g_{11} \in \mathrm{Aut}(2\mathcal{O}(-1,-1))$, $h_{22} \in \mathrm{Aut}(2\mathcal{O})$, etc.

Proposition 10. *The canonical map* $\theta\colon \mathbf{W} \to \mathbf{M}$, $\theta(\psi) = \langle \mathit{Coker}(\psi)\rangle$ *is a good quotient modulo* $\mathbf{G}$.

Proof. Let $\mathbf{W}_0 \subset \mathbf{W}$ be the open subset given by the condition that ψ_{12} be invertible. Concretely, $\mathbf{W}_0$ is the set of morphisms ψ such that ψ_{12} is invertible and $\alpha(\psi) = \psi_{21} - \psi_{22}\psi_{12}^{-1}\psi_{11}$ is injective. In view of Proposition 8, $\mathbf{M}_0 = \theta(\mathbf{W}_0)$. The restriction $\theta_0\colon \mathbf{W}_0 \to \mathbf{M}_0$ is the composition

$$\mathbf{W}_0 \xrightarrow{\alpha} \mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 \xrightarrow{\gamma} \mathbf{M}_0,$$

where γ is the good quotient map from Corollary 9. Let $\mathbf{G}_0 \trianglelefteq \mathbf{G}$ be the closed normal subgroup given by the conditions $g_{11} = cI$, $h_{22} = cI$, $c \in \mathbb{C}^*$. We have the relation $\alpha(h\psi g^{-1}) = h_{22}\alpha(\psi)g_{11}^{-1}$, hence α is constant on the orbits of $\mathbf{G}_0$. Since any $\psi \in \mathbf{W}_0$ is equivalent to

$$\begin{bmatrix} 0 & I \\ \alpha(\psi) & 0 \end{bmatrix},$$

it follows that the fibers of α are precisely the $\mathbf{G}_0$-orbits, and that α has a section. We deduce that α is a geometric quotient modulo $\mathbf{G}_0$. Since γ is a good quotient modulo $\mathbf{G}/\mathbf{G}_0$, we conclude that θ_0 is a good quotient modulo $\mathbf{G}$. Let $\mathbf{M}_0^{\mathrm{s}} \subset \mathbf{M}_0$ be the subset of stable points. Since $\gamma^{-1}(\mathbf{M}_0^{\mathrm{s}}) \to \mathbf{M}_0^{\mathrm{s}}$ is a geometric quotient modulo $\mathbf{G}/\mathbf{G}_0$, we deduce that $\theta^{-1}(\mathbf{M}_0^{\mathrm{s}}) \to \mathbf{M}_0^{\mathrm{s}}$ is a geometric quotient modulo $\mathbf{G}$.

Assume now that $\psi \in \mathbf{W} \setminus \mathbf{W}_0$. Denote $\mathcal{F} = \mathit{Coker}(\psi)$. Then $\psi_{12} \neq 0$, otherwise $\mathit{Coker}(\psi_{22})$ would be a destabilizing subsheaf of $\mathcal{F}$. Thus, $\mathbf{W} \setminus \mathbf{W}_0$ is the disjoint union of two subsets $\mathbf{W}_1$ and $\mathbf{W}_2$. The former is given by the relations $a_1 \neq 0$, $a_2 = 0$; the latter is given by the relations $a_1 = 0$, $a_2 \neq 0$. Assume that $\psi \in \mathbf{W}_1$. Then u_{11}, v_{11} are linearly independent, otherwise $\mathcal{F}$ would have a destabilizing quotient sheaf of slope zero. Likewise, u_{22}, v_{22} are linearly independent, otherwise $\mathcal{F}$ would have a destabilizing subsheaf of slope 1. Consider the morphism

$$\xi \in \mathrm{Hom}(2\mathcal{O}(-1,-1) \oplus \mathcal{O}(0,-1),\ \mathcal{O}(0,-1) \oplus 2\mathcal{O}),$$

$$\xi = \begin{bmatrix} u_{11} \otimes 1 & v_{11} \otimes 1 & 0 \\ f_{11} - a_1^{-1}u_{21} \otimes u_{12} & f_{12} - a_1^{-1}u_{21} \otimes v_{12} & 1 \otimes u_{22} \\ f_{21} - a_1^{-1}v_{21} \otimes u_{12} & f_{22} - a_1^{-1}v_{21} \otimes v_{12} & 1 \otimes v_{22} \end{bmatrix}.$$

Clearly, $\mathcal{F} \simeq \mathit{Coker}(\xi)$. Applying the snake lemma to the exact diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(0,-1) & \xrightarrow{\begin{bmatrix} 1\otimes u_{22} \\ 1\otimes v_{22} \end{bmatrix}} & 2\mathcal{O} & \xrightarrow{\begin{bmatrix} -1\otimes v_{22} & 1\otimes u_{22} \end{bmatrix}} & \mathcal{O}(0,1) \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2\mathcal{O}(-1,-1)\oplus\mathcal{O}(0,-1) & \xrightarrow{\xi} & \mathcal{O}(0,-1)\oplus 2\mathcal{O} & \longrightarrow & \mathcal{F} \\
 & & \downarrow & & \downarrow & & \\
\mathcal{O}(-2,-1) & \xrightarrow{\begin{bmatrix} -v_{11}\otimes 1 \\ u_{11}\otimes 1 \end{bmatrix}} & 2\mathcal{O}(-1,-1) & \xrightarrow{\begin{bmatrix} u_{11}\otimes 1 & v_{11}\otimes 1 \end{bmatrix}} & \mathcal{O}(0,-1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

we obtain resolution (2). This shows that $\theta(\mathbf{W}_1) \subset \mathbf{M}_1$. It is now easy to see that the restricted map $\mathbf{W}_1 \to \mathbf{M}_1$ is surjective and that its fibers are precisely the $\mathbf{G}$-orbits. By symmetry, the same is true of the restricted map $\mathbf{W}_2 \to \mathbf{M}_2$.

Let $\mathbf{M}^{\mathrm{s}} \subset \mathbf{M}$ be the open subset of stable points and $\mathbf{W}^{\mathrm{s}} = \theta^{-1}(\mathbf{M}^{\mathrm{s}})$. We have proved above that the fibers of the restricted map $\theta^{\mathrm{s}} \colon \mathbf{W}^{\mathrm{s}} \to \mathbf{M}^{\mathrm{s}}$ are precisely the $\mathbf{G}$-orbits. Since $\mathbf{M}^{\mathrm{s}}$ is normal (being smooth), we can apply [12, Theorem 4.2] to deduce that θ^{s} is a geometric quotient modulo $\mathbf{G}$. Since $\mathbf{M} = \mathbf{M}_0 \cup \mathbf{M}^{\mathrm{s}}$, we deduce that θ is a good quotient map. □

Choose bases $\{u_1, v_1\}$ of V_1^* and $\{u_2, v_2\}$ of V_2^*. Then $x = u_1 \otimes u_2$, $y = v_1 \otimes u_2$, $z = u_1 \otimes v_2$, $w = v_1 \otimes v_2$ form a basis of V. An easy calculation shows that the set of injective morphisms

$$\mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 \subset \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes V)$$

is the subset of matrices whose determinant is not a multiple of $xw - yz$. Thus,

$$\mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 /\!\!/ G \simeq \mathrm{N}(V; 2, 2) \setminus \det^{-1}\{\langle xw - yz \rangle\}.$$

According to Remark 7, $\det^{-1}\{\langle xw - yz\rangle\}$ consists of two points ν_1 and ν_2, where $\epsilon(\nu_1) = 1$, $\epsilon(\nu_2) = -1$. We saw at Corollary 9 that γ induces an isomorphism

$$\mathrm{Hom}(2\mathcal{O}(-1,-1), 2\mathcal{O})_0 /\!\!/ G \longrightarrow \mathbf{M}_0.$$

The inverse of this isomorphism is denoted by

$$\beta_0 \colon \mathbf{M}_0 \longrightarrow \mathrm{N}(V; 2, 2) \setminus \{\nu_1, \nu_2\}.$$

It is natural to ask whether $\mathbf{M}$ is the blow-up of $\mathrm{N}(V; 2, 2)$ at ν_1 and ν_2. This is, indeed, one of the main results in [3], where a blowing-down map $\beta \colon \mathbf{M} \to \mathrm{N}(V; 2, 2)$ is constructed via Fourier-Mukai transforms of sheaves, in view of the identification of $\mathrm{N}(V; 2, 2)$ with $\mathrm{M}_{\mathbb{P}^3}(m^2 + 3m + 2)$. We give below an alternate construction.

Proposition 11. *The map β_0 extends to a blowing-down map $\beta \colon \mathbf{M} \to \mathrm{N}(V; 2, 2)$ with exceptional divisor $\mathbf{M}_1 \cup \mathbf{M}_2$ and blowing-up locus $\{\nu_1, \nu_2\}$.*

Proof. Recall that on $\mathbf{M}_0 = \mathbf{W}_0 /\!\!/ \mathbf{G}$, β_0 is induced by the map sending ψ to

$$\psi_{21} - a_1^{-1}\begin{bmatrix} u_{21}\otimes 1 \\ v_{21}\otimes 1 \end{bmatrix}\begin{bmatrix} 1\otimes u_{12} & 1\otimes v_{12} \end{bmatrix} - a_2^{-1}\begin{bmatrix} 1\otimes u_{22} \\ 1\otimes v_{22} \end{bmatrix}\begin{bmatrix} u_{11}\otimes 1 & v_{11}\otimes 1 \end{bmatrix}.$$

Equivalently, β_0 is induced by the map sending ψ to

$$a_2\psi_{21} - a_1^{-1}a_2\begin{bmatrix} u_{21}\otimes 1 \\ v_{21}\otimes 1 \end{bmatrix}\begin{bmatrix} 1\otimes u_{12} & 1\otimes v_{12} \end{bmatrix} - \begin{bmatrix} 1\otimes u_{22} \\ 1\otimes v_{22} \end{bmatrix}\begin{bmatrix} u_{11}\otimes 1 & v_{11}\otimes 1 \end{bmatrix}$$

which is defined on $\mathbf{W}_0 \cup \mathbf{W}_1$. This map factors through a morphism $\mathbf{M}_0 \cup \mathbf{M}_1 \to \mathrm{N}(V;2,2)$, which maps $\mathbf{M}_1$ to the class of the matrix

$$\begin{bmatrix} 1\otimes u_2 \\ 1\otimes v_2 \end{bmatrix}\begin{bmatrix} u_1\otimes 1 & v_1\otimes 1 \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

that is, to ν_1. Analogously, β_0 extends to a morphism defined on $\mathbf{M}_0 \cup \mathbf{M}_2$, which maps $\mathbf{M}_2$ to the class of the matrix

$$\begin{bmatrix} u_1\otimes 1 \\ v_1\otimes 1 \end{bmatrix}\begin{bmatrix} 1\otimes u_2 & 1\otimes v_2 \end{bmatrix} = \begin{bmatrix} x & z \\ y & w \end{bmatrix},$$

that is, to ν_2. Finally, the two morphisms we have constructed thus far glue to a morphism $\beta\colon \mathbf{M} \to \mathrm{N}(V;2,2)$. Since ν_1 and ν_2 are smooth points, β is a blow-down. □

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