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SCATTERING FOR QUASILINEAR HYPERBOLIC EQUATIONS OF KIRCHHOFF TYPE WITH PERTURBATION

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Abstract

This paper is concerned with the abstract quasilinear hyperbolic equations of Kirchhoff type with perturbation. We show the existence of the wave operators and the scattering operator for small data, and that these operators are homeomorphic with respect to a suitable metric in a neighborhood of the origin.

Introduction

Let H be a separable complex Hilbert space H with the inner product $(\cdot,\cdot)_H$ and the norm $\|\cdot\|$. Let A be a non-negative injective self-adjoint operator with domain $D(A)$, and let m be a function satisfying

$$m \in C^2([0,\infty);[m_0,\infty)),$$

with a positive constant m_0. Let $b(t)$ be a C^1 function on $\mathbb{R}$. We consider the initial value problem of the abstract quasilinear hyperbolic equations of Kirchhoff type with perturbation

$$u''(t) + b(t)u'(t) + m(\|A^{1/2}u(t)\|^2)^2 Au(t) = 0, \tag{0.1}$$

$$u(0) = \phi_0, \quad u'(0) = \psi_0. \tag{0.2}$$

The asymptotic behavior of the solution of the equation above depends on the integrability of b with respect to t.

If $b(t) = (1+t)^{-p}$ with $0 \le p \le 1$, it is known that the global solution of (0.1)-(0.2) exists uniquely and behaves like solutions of a corresponding parabolic equation, for small initial data $(\phi_0,\psi_0) \in D(A) \times D(A^{1/2})$ (see Yamazaki [14] for $0 \le p < 1$ and Ghisi and Gobbino [7] for $p = 1$, and see Ghisi [6] for a mildly degenerate case $m(\lambda) = \lambda^\gamma$ with $\gamma \ge 1$ and $0 \le p \le 1$). There are no result about the global solvability for large initial data in Sobolev spaces, even for the constant dissipation term.

On the other hand, if b satisfies the assumptions

$$\lim_{t\to\pm\infty} b(t) = 0, \tag{0.3}$$

$$\langle t\rangle^p t b'(t) \in L^1(\mathbb{R}), \tag{0.4}$$

where $p \ge 0$ is a constant and $\langle t\rangle := (1+|t|^2)^{1/2}$, the author [15] showed the global existence of a solution for small data in some class and showed the following (see Theorems B and C):

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(i) The solution of the Cauchy problem for the perturbed Kirchhoff equation has the same asymptotic behavior of a function obtained by a transformation of time variable from a solution of the free wave equation with an appropriate wave speed.

(ii) Conversely, there exists a solution of the Kirchhoff equation which has the same asymptotic behavior of a function obtained by a transformation of the time variable from the solution of the Cauchy problem of the free wave equation with an appropriate wave speed.

However, these facts do not imply that solutions of the perturbed Kirchhoff equation are not asymptotically free, in the sense that it has the same asymptotic behavior as that of solutions of free wave equations, since the transformation of time variable is used. The author [16] showed the following: If b satisfies (0.3) and (0.4) with $p = 0$ and does not change sign for sufficient large $|t|$, and there exists a solution with small data which is asymptotically free, then $tb(t)$ must be integrable (see Theorem D). The integrability of $tb(t)$ is equivalent to the condition (0.3) and (0.4) with $p = 1$ if b is monotone for sufficiently large $|t|$ (see remark 3). Hence, in order to show that the solutions are asymptotically free, it is necessary to assume that $p \geq 1$ in the assumption (0.4), if b is monotone for sufficiently large $|t|$ (see remark 3).

The purpose of this paper is to show the following assertions under the assumption that b satisfies (0.3) and (0.4) with $p \geq 1$.

(iii) The solutions of the perturbed Kirchhoff equation with small data in some class are asymptotically free.

(iv) For small initial data in some class and the solution of the free wave equation with an appropriate wave speed, there exists a solution of perturbed Kirchhoff equation which approaches to the solution.

(v) The wave operators exist and they are homeomorphic in a neighborhood of the origin with respect to a metric given in Notation 5 in Section 1. It follows that the scattering operator is also homeomorphic in a neighborhood of the origin.

Last we list previous results on the asymptotic behavior or scattering, in the case $b \equiv 0$ and $A = -\Delta$ on $L^2(\mathbb{R}^n)$. Ghisi [5] first showed (iii) above. The author [11] showed (iii) and (iv). However [11] did not prove (v), but proved the existence of the wave and its inverse operators for initial data in some class, and the continuity of the scattering operator only at the origin with respect to a metric somewhat different from the one in this paper, together with the continuity from a neighborhood of the origin in this topology to a Sobolev space with weaker norm. Kajitani [8, 9] considered the Kirchhoff equation of the type $m(\lambda) = 1 + \varepsilon\lambda$, where A is an elliptic differential operator with coefficients depending on space variables and he showed that, for every initial data in a class, there exists a positive constant $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ the property (i) and (ii) above hold. Hence, even in the case $b \equiv 0$ and $A = -\Delta$ on $L^2(\mathbb{R}^n)$, our result (v) is new.

The outline of this paper is as follows: In section 1, we state notations and results. In section 2, we state some examples of Sobolev spaces included in the spaces we consider. In section 3, we transform original problem. In section 4, we prove propositions. In section 5, we prove Theorem 1. In section 6, we prove Theorems 2 and 3.

1. Results

For a closed operator B in a Hilbert space H, $D(B)$ and $R(B)$ denote the domain of B and the range of B, respectively.

For a non-negative number J, the domain of $A^{J/2}$ becomes a Hilbert space $D(A^{J/2})$ equipped with the inner product

$$(f, g)_J := (A^{J/2} f, A^{J/2} g)_H + (f, g)_H.$$

and the norm $\|f\|_J^2 = (f, f)_J$. We note that $\|f\|_0 = \|f\|$.

Notation 1. Let $\mathcal{L}(H)$ denote the space of all bounded linear operator on Hilbert space H equipped with the operator norm $\|\cdot\|_{\mathcal{L}(H)}$.

We state notations and some classes of functions as in [15].

Notation 2. Let p be a non-negative number. For a continuous function f on $\mathbb{R}$, put

$$|f|_p = \int_{\mathbb{R}} \langle r \rangle^p |f(r)| dr.$$

Notation 3. For every $\alpha > 0$, let $\mathcal{H}_\alpha$ denote the completion of $D(A^\alpha)$ by the norm $\|A^\alpha \cdot\|$. Let $\mathcal{A}^\alpha$ be the extension of A^α on $\mathcal{H}_\alpha$. The fact that A^α is an injective self-adjoint operator implies that the range $R(A^\alpha)$ is dense in H, and thus $\mathcal{A}^\alpha : \mathcal{H}_\alpha \to H$ is surjective. From this fact and the definition, it follows that $\mathcal{A}^\alpha : \mathcal{H}_\alpha \to H$ is an isometric isomorphism.

For example, if $H = L^2(\mathbb{R}^n)$ and $A = -\Delta$ with $D(A) = H^2(\mathbb{R}^n)$, then $\mathcal{H}_\alpha$ is equal to the homogeneous Sobolev space $\dot{H}^{2\alpha}$.

Notation 4. Let k and p be non-negative numbers and let ε be a positive number. We put

$$\begin{aligned} X_{k,p} :=\{(f, g) \in \mathcal{H}_{1/2} \times H; (\mathcal{A}^{1/2} f, g) \in D(A^{k/4}) \times D(A^{k/4}),\\ |(f, g)|^2_{X_{k,p}} := |(A^{k/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f, A^{k/4} \mathcal{A}^{1/2} f)_H|_p\\ + |(A^{k/4} e^{2irA^{1/2}} g, A^{k/4} g)_H|_p\\ + 2|(A^{k/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f, A^{k/4} g)_H|_p < \infty\}, \end{aligned}$$

and

$$Y_{k,p} := X_{k,p} \cap (H \times H).$$

We abbreviate k if $k = 1$ as

$$X_p := X_{1,p}, \quad Y_p := Y_{1,p}.$$

We put

$$\tilde{Y}_p :=\{(f, g) \in Y_p; (f, g) \text{ satisfies the following (1.1)}\},$$

$$\lim_{t\to\pm\infty} (e^{itA^{1/2}} A^{1/2} f, A^{1/2} f)_H = \lim_{t\to\pm\infty} (e^{itA^{1/2}} g, g)_H = \lim_{t\to\pm\infty} (e^{itA^{1/2}} A^{1/2} f, g)_H = 0. \tag{1.1}$$

Remark 1. Kajitani [9] introduced notation $Y_{k,p}$ and noted

$$Y_p \cap Y_{0,p} = Y_{1,p} \cap Y_{0,p} \subset \tilde{Y}_p,$$

since the functions in (1.1) and their derivatives with respect to t are integrable.

REMARK 2. In the case $H = L^2(\mathbb{R}^n)$ and $A = -\Delta$ with $D(A) = H^2(\mathbb{R}^n)$, we can prove that (1.1) holds for every $(f, g) \in \mathcal{H}_{1/2} \times H$, in the same way as in the proof of Riemann-Lebesgue Theorem.

NOTATION 5. For $J \geq 0$, we put

$$\begin{aligned} X_p^J &= \{(f,g) \in X_p; (\mathcal{A}^{1/2} f, g) \in D(A^{J/2}) \times D(A^{J/2})\}, \\ Y_p^J &= Y_p \cap (D(A^{(J+1)/2}) \times D(A^{J/2})), \\ \tilde{Y}_p^J &= \tilde{Y}_p \cap (D(A^{(J+1)/2}) \times D(A^{J/2})), \end{aligned}$$

and put

$$\begin{aligned} |(f,g)|_{X_p^J} &:= |(f,g)|_{X_p} + \left\|\mathcal{A}^{1/2} f\right\|_J + \|g\|_J \quad \text{for } (f,g) \in X_p^J \\ |(f,g)|_{Y_p^J} &:= |(f,g)|_{X_p^J} + \|f\| = |(f,g)|_{X_p} + \|f\|_{J+1} + \|g\|_J \quad \text{for } (f,g) \in Y_p^J. \end{aligned}$$

We define a metric on X_p^J as

$$d_p^J((f_1,g_1),(f_2,g_2)) := d_p((f_1,g_1),(f_2,g_2)) + \left\|\mathcal{A}^{1/2}(f_1 - f_2)\right\|_J + \|g_1 - g_2\|_J$$

for $(f_1, g_1), (f_2, g_2) \in X_p^J$, where

$$\begin{aligned} & d_p((f_1,g_1),(f_2,g_2))^2 \\ &:= |(A^{1/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f_1, \mathcal{A}^{3/4} f_1)_H - (A^{1/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f_2, \mathcal{A}^{3/4} f_2)_H|_p \\ &\quad + |(A^{1/4} e^{2irA^{1/2}} g_1, A^{1/4} g_1)_H - (A^{1/4} e^{2irA^{1/2}} g_2, A^{1/4} g_2)_H|_p \\ &\quad + 2|(A^{1/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f_1, A^{1/4} g_1)_H - (A^{1/4} e^{2irA^{1/2}} \mathcal{A}^{1/2} f_2, A^{1/4} g_2)_H|_p. \end{aligned}$$

We define a metric on Y_p^J as

$$d_{Y_p}^J((f_1,g_1),(f_2,g_2)) := d_p^J((f_1,g_1),(f_2,g_2)) + \|f_1 - f_2\|$$

for $(f_1, g_1), (f_2, g_2) \in Y_p^J$.

It is easy to see that d_p^J and $d_{Y_p}^J$ become metrics on X_p^J and Y_p^J respectively, and thus, X_p^J and Y_p^J become metric spaces with the metrics d_p^J and $d_{Y_p}^J$ respectively.

NOTATION 6. For $J \geq 0$ and $\varepsilon > 0$, we define a subset $X_p^J(\varepsilon)$ of X_p^J by

$$X_p^J(\varepsilon) := \{(f,g) \in X_p^J;\ |(f,g)|_{X_p} + \|\mathcal{A}^{1/2} f\| + \|g\| \leq \varepsilon\}.$$

We define subsets $Y_p^J(\varepsilon)$ and $\tilde{Y}_p^J(\varepsilon)$ of Y_p^J by

$$\begin{aligned} Y_p^J(\varepsilon) &:= X_p^J(\varepsilon) \cap Y_p = X_p^J(\varepsilon) \cap (H \times H), \\ \tilde{Y}_p^J(\varepsilon) &:= X_p^J(\varepsilon) \cap \tilde{Y}_p, \end{aligned}$$

In [15], we extended the space considered in problem (0.1)–(0.2) to a larger one, namely, we consider the problem

$$u''(t) + b(t)u'(t) + m(\|\mathcal{A}^{1/2} u(t)\|^2)^2 A^{1/2} \mathcal{A}^{1/2} u(t) = 0, \qquad t > 0, \tag{1.2}$$

(1.3) $$u(0) = \phi_0, \quad u'(0) = \psi_0,$$

for $(\phi_0, \psi_0) \in \mathcal{H}_{1/2} \times D(A^{1/2})$ satisfying $\mathcal{A}^{1/2}\phi_0 \in D(A^{1/2})$.

Notation 7. Let I be one of the interval $\mathbb{R}$, $(0, \infty)$ or $(-\infty, 0)$ and let $b(t) \in C(I)$. We say that u is a solution of

(1.4) $$u''(t) + b(t)u'(t) + m(\|\mathcal{A}^{1/2}u(t)\|^2)^2 A^{1/2}\mathcal{A}^{1/2}u(t) = 0, \qquad t \in I,$$

if

$$u \in C^1(I; \mathcal{H}_{1/2}), \ (\mathcal{A}^{1/2}u, u') \in \bigcap_{j=0,1} C^j\left(I; D(A^{(1-j)/2}) \times D(A^{(1-j)/2})\right),$$

and (1.4) holds in H.

The author showed the following global existence of solution.

Theorem A (Theorem 1 of [15]). *Let $J \geq 1$ and $p \geq 0$. Assume that b satisfies* (0.3) *and* (0.4). *Then there exist a positive number ε_1 such that the following holds. For every $(\phi_0, \psi_0) \in X_p^J(\varepsilon_1)$, perturbed Kirchhoff equation* (1.2)–(1.3) *has a unique global solution $u \in C(\mathbb{R}; \mathcal{H}_{1/2})$ satisfying*

(1.5) $$(\mathcal{A}^{1/2}u, u') \in \bigcap_{j=0,1} C^j\left(\mathbb{R}; D(A^{(1-j)/2}) \times D(A^{(1-j)/2})\right).$$

Furthermore, the following holds.

$$(u, u') \in C\left(\mathbb{R}; X_p^J\right),$$
$$(\mathcal{A}^{1/2}u, u') \in C^1\left(\mathbb{R}; D(A^{(J-1)/2}) \times D(A^{(J-1)/2})\right).$$

If $(\phi_0, \psi_0) \in Y_p^J(\varepsilon_1)$, then $u \in \bigcap_{j=0,1,2} C^j\left(\mathbb{R}; D(A^{(J+1-j)/2})\right)$.

The global solutions of Kirchhoff equation given by Theorem A has the same asymptotic behavior as time-transformed function of the solution of a free equation:

Theorem B (Theorem 2 of [15]). *Let $J \geq 1$ and $p \geq 0$. Assume that b satisfies* (0.3) *and* (0.4). *Let ε_1 be the positive constant in Theorem A.*

(i) There exists a positive constant $\varepsilon_2(\leq \varepsilon_1)$ such that the following holds. Suppose that $(\phi_0, \psi_0) \in X_p^J(\varepsilon_2)$, and that $u(t) \in C^1(\mathbb{R}; \mathcal{H}_{1/2})$ is the solution of (1.2)–(1.3) *satisfying $(\mathcal{A}^{1/2}u, u') \in \bigcap_{j=0,1} C^j\left(\mathbb{R}; D(A^{(1-j)/2}) \times D(A^{(1-j)/2})\right)$. Then the limits*

$$C_{\pm\infty} = \lim_{t\to\pm\infty} m(\|A^{1/2}u(t)\|^2)$$

exist and satisfy

$$\lim_{t\to\pm\infty} \langle t\rangle^p |m(\|A^{1/2}u(t)\|^2) - C_{\pm\infty}| = 0,$$

and there uniquely exist global solutions $v_+, v_- \in C^1(\mathbb{R}; \mathcal{H}_{1/2})$ of

$$v_+''(t) + C_{+\infty}^2 A^{1/2}\mathcal{A}^{1/2}v_+ = 0, \quad t \in \mathbb{R},$$
$$v_-''(t) + C_{-\infty}^2 A^{1/2}\mathcal{A}^{1/2}v_- = 0, \quad t \in \mathbb{R},$$

satisfying

$$(\mathcal{A}^{1/2}v_\pm, v'_\pm) \in \bigcap_{j=0,1} C^j\left(\mathbb{R}; D(A^{(1-j)/2} \times D(A^{(1-j)/2})\right)$$

and

$$\lim_{t\to\pm\infty}\left(\|\mathcal{A}^{1/2}u(t) - \mathcal{A}^{1/2}v_\pm(C_{\pm\infty}^{-1}\tau(t))\| + \|u'(t) - v'_\pm(C_{\pm\infty}^{-1}\tau(t))\|\right) = 0,$$

where

$$\tau(t) := \int_0^t m(\|A^{1/2}u(s)\|^2)ds \quad \text{for} \quad t \in \mathbb{R}.$$

Furthermore, it holds that

$$(v_\pm, v'_\pm) \in C\left(\mathbb{R}; X_p^J\right), \tag{1.6}$$

$$(\mathcal{A}^{1/2}v_\pm, v'_\pm) \in C^1\left(\mathbb{R}; D(A^{(J-1)/2}) \times D(A^{(J-1)/2})\right), \tag{1.7}$$

$$\lim_{t\to\pm\infty}\langle t\rangle^p\left(\|\mathcal{A}^{1/2}u(t) - \mathcal{A}^{1/2}v_\pm(C_{\pm\infty}^{-1}\tau(t))\|_J + \|u'(t) - v'_\pm(C_{\pm\infty}^{-1}\tau(t))\|_J\right) = 0.$$

(ii) Assume moreover that (ϕ_0, ψ_0) *satisfies* (1.1). *Then*

$$C_{\pm\infty} = m\left(\frac{1}{2}\|\mathcal{A}^{1/2}\phi_\pm\|^2 + \frac{1}{2C_{\pm\infty}^2}\|\psi_\pm\|^2\right), \tag{1.8}$$

and the limits $\lim_{t\to\pm\infty}\|\mathcal{A}^{1/2}u(t)\|^2$ *and* $\lim_{t\to\pm\infty}\|u'(t)\|^2$ *exist and satisfy the equality*

$$\lim_{t\to\pm\infty}\|u'(t)\|^2 = m(\lim_{t\to\pm\infty}\|\mathcal{A}^{1/2}u(t)\|^2)^2\lim_{t\to\pm\infty}\|\mathcal{A}^{1/2}u(t)\|^2.$$

Conversely to Theorem B, [15] showed the existence of the solution of perturbed Kirchhoff equation which has similar asymptotic behavior as that of the solution of the free equation. By (1.8), it is necessary to assume the relation (1.9) below.

Theorem C (Theorem 3 of [15])**.** *Let* $J \geq 1$ *and* $p \geq 0$. *Assume that* b *satisfies* (0.3) *and* (0.4). *Then there exist a positive constant* ε_3 *such that the following holds. Suppose that* $(\phi, \psi) \in X_p^J(\varepsilon_3)$ *satisfies* (1.1). *Then there uniquely exists a positive number* c_∞ *satisfying the equality*

$$c_\infty = m\left(\frac{1}{2}\|\mathcal{A}^{1/2}\phi\|^2 + \frac{1}{2c_\infty^2}\|\psi\|^2\right). \tag{1.9}$$

Let v *be the solution of*

$$v''(t) + c_\infty^2 A^{1/2}\mathcal{A}^{1/2}v = 0 \quad t \in \mathbb{R}$$

with $v(0) = \phi, v'(0) = \psi$. *Then there uniquely exist global solutions* $u_\pm(t) \in C^1(\mathbb{R}; \mathcal{H}_{1/2})$ *of* (1.2) *satisfying*

$$(\mathcal{A}^{1/2}u_\pm, u'_\pm) \in \bigcap_{j=0,1} C^j\left(\mathbb{R}; D(A^{(1-j)/2}) \times D(A^{(1-j)/2})\right),$$

$$\lim_{t\to\pm\infty}\left(\|\mathcal{A}^{1/2}u_\pm(t) - \mathcal{A}^{1/2}v(C_{\pm\infty}^{-1}\tau(t))\| + \|u'_\pm(t) - v'(C_{\pm\infty}^{-1}\tau(t))\|\right) = 0,$$

and that $\frac{d}{dt}\|\mathcal{A}^{1/2}u_\pm(t)\|^2$ *is integrable on* $\mathbb{R}$.

Furthermore, $u_\pm$ *satisfy the following.*

$$
\begin{aligned}
&(u_\pm, u_\pm') \in C\left(\mathbb{R}; X_p^J\right),\\
&(\mathcal{A}^{1/2}u_\pm, u_\pm') \in C^1\left(\mathbb{R}; D(A^{(J-1)/2}) \times D(A^{(J-1)/2})\right).\\
&\lim_{t\to\pm\infty} \langle t\rangle^p \Big(\|\mathcal{A}^{1/2}u_\pm(t) - \mathcal{A}^{1/2}v(C_{\pm\infty}^{-1}\tau(t))\|_J + \|u_\pm'(t) - v'(C_{\pm\infty}^{-1}\tau(t))\|_J\Big) = 0,\\
&\langle t\rangle^p \frac{d}{dt}\|\mathcal{A}^{1/2}u_\pm(t)\|^2 \in L^1(\mathbb{R}).
\end{aligned}
$$

Theorem C does not mean that the non-trivial solutions of Kirchhoff equation which is given by Theorem A are asymptotically free, since the transformation $\tau(t)$ is used. Different from the linear wave equation, we need more decay condition on b for the asymptotically free property, in view of the next theorem.

Theorem D (Theorem, its proof and Remark 2 of [16])**.** *Assume that* m *satisfies*

$$m'(x) \neq 0 \quad \text{for every} \quad x > 0.$$

Assume that b *satisfies* (0.3) *and* (0.4) *with* $p = 0$ *and*

$$
\begin{aligned}
&b(t) \quad \text{does not change sign} \quad \text{for every} \ \ t \le -T_0,\\
&\qquad \text{and}\\
&b(t) \quad \text{does not change sign} \quad \text{for every} \ \ t \ge T_0,
\end{aligned}
$$

for a positive constant T_0*. Let* ε_2 *be a positive constant given by Theorem B. If there exists a solution* $u \in C(\mathbb{R}; \mathcal{H}_{1/2})$ *of perturbed Kirchhoff equation* (1.2)–(1.3) *with initial value* $(\phi_0, \psi_0)(\neq (0,0)) \in X_0^1(\varepsilon_2) \cap X_{0,0}$ *which is asymptotically free as* $t \to \pm\infty$*, in the sense that there exist positive constants* $c_{+\infty}, c_{-\infty}$ *and global solutions* v_+, v_- *of*

$$
\begin{aligned}
v_+''(t) + c_{+\infty}^2 A^{1/2}\mathcal{A}^{1/2}v_+ = 0, \quad t \in \mathbb{R},\\
v_-''(t) + c_{-\infty}^2 A^{1/2}\mathcal{A}^{1/2}v_- = 0, \quad t \in \mathbb{R},
\end{aligned}
$$

such that

$$\lim_{t\to\pm\infty} \Big(\|\mathcal{A}^{1/2}u(t) - \mathcal{A}^{1/2}v_\pm(t)\| + \|u'(t) - v_\pm'(t)\|\Big) = 0,$$

then

$$tb(t) \in L^1(\mathbb{R}).$$

Remark 3. If b satisfies (0.3) and (0.4), then Lemma 3 in section 4 with $f = b$ and $\alpha_{\pm\infty} = 0$ implies

$$\langle t\rangle^p b(t) \in L^1(\mathbb{R}). \tag{1.10}$$

This relation is also seen in the proof of Lemma 5 in [15].

If b is monotone for sufficiently large $|t|$, then the converse holds, that is, (1.10) implies (0.3) and (0.4).

In fact, if $b(t)$ is monotone-decreasing on $[T_0, \infty)$ for sufficiently large T_0 and satisfies (1.10), then it follows that b satisfies (0.3), $b(t) \ge 0$ on $[T_0, \infty)$ and

$$\begin{aligned}\int_{T_0}^{R} \langle t\rangle^{p+1}|b'(t)|\,dt &= -\int_{T_0}^{R} \langle t\rangle^{p+1} b'(t)\,dt \\ &= -\Big[\langle t\rangle^{p+1} b(t)\Big]_{t=T_0}^{R} + (p+1)\int_{T_0}^{R} \langle t\rangle^{p-1} t b(t)\,dt \\ &\leq \langle T_0\rangle^{p+1} b(T_0) + (p+1)\int_{T_0}^{R} \langle t\rangle^{p-1} t b(t)\,dt < \infty,\end{aligned}$$

for every $R \geq T_0$. Hence, $\langle t\rangle^p tb' \in L^1((0,\infty))$. In the same way, we can treat the case $b(t)$ is monotone-increasing on $[T_0,\infty)$ for sufficiently large T_0. In the same way, we can prove that the integrability of $\langle t\rangle^p b(t)$ on $(-\infty,0)$ implies $\langle t\rangle^p tb'(t) \in L^1((-\infty,0))$, under the assumption that b is monotone on $(-\infty,-T_0)$ for sufficiently large T_0, which completes the proof.

In view of Theorem D and Remark 3, it is natural to assume that b satisfies the decay condition (0.4) with $p \geq 1$, for solutions of perturbed Kirchhoff equation to be asymptotic free.

On the other hand, let us consider the case $H = L^2(\mathbb{R}^n)$, $A = -\Delta$ with $D(A) = H^2(\mathbb{R}^n)$. For every $p \in [0,1)$, we see from Matsuyama [10] that there exists a small initial data $(\phi, 0)$ in Y_p such that Kirchhoff equation (0.1)–(0.2) with $b \equiv 0$ has a solution whose asymptotic behavior is different from that of solutions of the corresponding free equations. In view of this fact, it is natural to assume that initial data belong to Y_p with $p \geq 1$, for solutions of Kirchhoff equation to be asymptotic free.

Hence assuming $p \geq 1$, we show that the global solutions of perturbed Kirchhoff equation given by Theorem A has the same asymptotic behavior of that of the solutions of free equations.

Theorem 1. *Let $J \geq 1$ and $p \geq 1$. Assume that b satisfies* (0.3) *and* (0.4). *Then there exist positive constants ε_4 and $K_2^{\pm}(\geq 1)$ such that the following holds for every $\varepsilon \in (0,\varepsilon_4]$. Suppose that $(\phi_0,\psi_0) \in Y_p^J(\varepsilon)$, and that $u(t)$ be the solution of* (0.1)–(0.2). *Then the limit*

$$C_{\pm\infty} = \lim_{t\to\pm\infty} m(\|A^{1/2}u(t)\|^2),$$

whose existence is guaranteed by Theorem B, satisfies decay estimates

$$\int_{-\infty}^{0} \langle t\rangle^{p-1}|m(\|A^{1/2}u(t)\|^2) - C_{-\infty}|dt < \infty, \tag{1.11}$$

$$\int_{0}^{\infty} \langle t\rangle^{p-1}|m(\|A^{1/2}u(t)\|^2) - C_{+\infty}|dt < \infty, \tag{1.12}$$

and there uniquely exist global solutions

$$v_+, v_- \in \bigcap_{j=0,1,2} C^j\left(\mathbb{R}; D(A^{(J+1-i)/2})\right)$$

of

$$v_+''(t) + C_{+\infty}^2 A v_+ = 0, \quad t \in \mathbb{R}, \tag{LE:$C_{+\infty}$}$$

$$v_-''(t) + C_{-\infty}^2 A v_- = 0, \quad t \in \mathbb{R}, \tag{LE:$C_{-\infty}$}$$

such that

$$\lim_{t\to\pm\infty}\left(\|A^{1/2}u(t)-A^{1/2}v_\pm(t)\|+\|u'(t)-v'_\pm(t)\|\right)=0. \tag{1.13}$$

Furthermore it holds that

$$\lim_{t\to\pm\infty}(\|u(t)-v_\pm(t)\|_{J+1}+\|u'(t)-v'_\pm(t)\|_J)=0, \tag{1.14}$$

$$\lim_{t\to\pm\infty}\langle t\rangle^{p-1}(\|u(t)-v_\pm(t)\|_J+\|u'(t)-v'_\pm(t)\|_{J-1})=0. \tag{1.15}$$

The operators

$$\Upsilon_\pm:(u(0),u'(0))=(\phi_0,\psi_0)\mapsto(\phi_\pm,\psi_\pm)=(v_\pm(0),v'_\pm(0)) \tag{1.16}$$

are continuous from $Y_p^J(\varepsilon)$ *to* $Y_p^J(K_2^\pm\varepsilon)$ *for every* $\varepsilon\in(0,\varepsilon_4]$.

If $p\geq 2$*, then*

$$\int_{-\infty}^0\langle t\rangle^{p-2}(\|u(t)-v_\pm(t)\|_J+\|u'(t)-v'_\pm(t)\|_{J-1})\,dt<\infty, \tag{1.17}$$

$$\int_0^{\infty}\langle t\rangle^{p-2}(\|u(t)-v_\pm(t)\|_J+\|u'(t)-v'_\pm(t)\|_{J-1})\,dt<\infty. \tag{1.18}$$

Remark 4. Assume that b is monotone for sufficiently large $|t|$. Then in view of Theorems D and 1 together with Remark 3, the assumption (0.3) and (0.4) with $p=1$ is necessary and sufficient condition for solutions of perturbed Kirchhoff equation with small initial data in Y_1^1 to be asymptotic free.

Remark 5. The operator $\Upsilon_\pm$ is the inverse of the wave operator $W_\pm$, as is seen in the next theorem.

Conversely, we show the existence of the solution of perturbed Kirchhoff equation which converges to the solution of the free equation. For this purpose, it is necessary to assume the relation (1.9) in view of (1.8) as well as Theorem C. Here we note that $\mathcal{A}^{1/2}\phi=A^{1/2}\phi$ in (1.8) and (1.9) if $(\phi,\psi)\in Y_p^1$.

Theorem 2. *Let* $J\geq 1$ *and* $p\geq 1$. *Assume that* b *satisfies* (0.3) *and* (0.4). *Then there exist positive constants* ε_5 *and* $K_3^\pm(\geq 1)$ *such that the following holds for every* $\varepsilon\in(0,\varepsilon_5]$: *Suppose that* $(\phi,\psi)\in\tilde{Y}_p^J(\varepsilon)$. *Let* c_∞ *be a uniquely determined positive number satisfying the equality* (1.9) *in Theorem C, and let* v *be the solution of*

$$\text{(LE:}c_\infty\text{)}\qquad \begin{aligned}&v''(t)+c_\infty^2Av=0\quad(t\in\mathbb{R}),\\ &v(0)=\phi,\quad v'(0)=\psi.\end{aligned}$$

Then there uniquely exist global solutions

$$u_+,u_-\in\bigcap_{j=0}^{2}C^i(\mathbb{R};D(A^{(J+1-j)/2}))$$

of (0.1) *such that*

$$\lim_{t\to\pm\infty}\left(\|A^{1/2}u_\pm(t)-A^{1/2}v(t)\|+\|u'_\pm(t)-v'(t)\|\right)=0, \tag{1.19}$$

and that

$$\frac{d}{dt}\|A^{1/2}u_\pm(t)\|^2 \in L^1(\mathbb{R}^n). \tag{1.20}$$

Furthermore, the following holds:

$$\lim_{t\to\pm\infty} m(\|A^{1/2}u_\pm(t)\|^2) = c_\infty, \tag{1.21}$$

$$\lim_{t\to\pm\infty} (\|u_\pm(t) - v(t)\|_{J+1} + \|u'_\pm(t) - v'(t)\|_J) = 0, \tag{1.22}$$

$$\lim_{t\to\pm\infty} \langle t\rangle^{p-1} (\|u_\pm(t) - v(t)\|_J + \|u'_\pm(t) - v'(t)\|_{J-1}) = 0, \tag{1.23}$$

and the wave operators

$$W_\pm : (\phi,\psi) = (v(0), v'(0)) \mapsto (\phi_{0,\pm}, \psi_{0,\pm}) = (u_\pm(0), u'_\pm(0))$$

are continuous from $\tilde{Y}^J_p(\varepsilon)$ *to* $\tilde{Y}^J_p(K_3^\pm\varepsilon)$.

If $p \ge 2$, *then*

$$\int_{-\infty}^0 \langle t\rangle^{p-2} \left(\|A^{1/2}u_\pm(t) - A^{1/2}v(t)\|_{J-1} + \|u'_\pm(t) - v'(t)\|_{J-1}\right) dt < \infty, \tag{1.24}$$

$$\int_0^{\infty} \langle t\rangle^{p-2} \left(\|A^{1/2}u_\pm(t) - A^{1/2}v(t)\|_{J-1} + \|u'_\pm(t) - v'(t)\|_{J-1}\right) dt < \infty. \tag{1.25}$$

Assume furthermore that $\varepsilon \le \min\{\varepsilon_4/K_3^\pm, \varepsilon_5/K_2^\mp\}$. *Then* $W_\pm$ *is a homeomorphism from* $\tilde{Y}^J_p(\varepsilon)$ *to* $W_\pm(\tilde{Y}^J_p(\varepsilon))(\subset \tilde{Y}^J_p(K_3^\pm\varepsilon) \subset \tilde{Y}^J_p(\varepsilon_4))$ *with respect to the metric* $d^J_{Y_p}$. *Let* $\Upsilon_\pm$ *be the mapping defined by* (1.16). *Then* $W_\pm\Upsilon_\pm$ *and* $\Upsilon_\pm W_\pm$ *are the identity mappings on* $\tilde{Y}^J_p(\varepsilon)$.

The existence and the continuity of the scattering operator immediately follows from Theorems 1 and 2.

Theorem 3. *Let* $J \ge 1/2$ *and* $p \ge 1$. *Let* ε_4 *and* K_2^+ *be the constants given by Theorem 1. Let* ε_5 *and* K_3^- *be the constants given by Theorem 2. Then for every* $\varepsilon \le \min\{\varepsilon_4/K_3^-, \varepsilon_5\}$, *the scattering operator* $S = W_+^{-1}W_-$ *is a homeomorphism from* $\tilde{Y}^J_p(\varepsilon)$ *to* $S(\tilde{Y}^J_p(\varepsilon))(\subset \tilde{Y}^J_p(K_2^+K_3^-\varepsilon))$ *with respect to the metric* $d^J_{Y_p}$.

2. Examples of Sobolev spaces included in the set $\tilde{Y}_p$

Let Ω be the whole space $\mathbb{R}^n$ or a non-trapping exterior domain with smooth boundary. In this section, we give some examples of Sobolev spaces which are included in $\tilde{Y}_p$, in the case $H = L^2(\mathbb{R}^n)$, $A = -\Delta$ with domain $D(A) = \{u \in H^2(\Omega);\ u(t,x) = 0 \ \text{ on } \partial\Omega\}$. Here $D(A) = H^2(\mathbb{R}^n)$ if $\Omega = \mathbb{R}^n$. As is noted in Remark 1, these follows from examples included in $Y_{k,p}$ for $k = 0, 1$, which were shown in the previous papers [11, 12, 13]. Before describing sufficient conditions, we prepare some notations.

Notation 8. Let $1 < p < \infty$ and $s \ge 0$. Let

$$W^s_p(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{f}) \in L^p(\mathbb{R}^n) \right\},$$

with the norm $\|f\|_{W^s_p} = \left\|\mathcal{F}^{-1}(\langle\xi\rangle^s \hat{f})\right\|_{L^p}$. Let

$$W^s_p(\Omega) = \left\{ f \;\middle|\; \exists g \in W^s_p(\mathbb{R}^n) \text{ such that } g|_\Omega = f \right\},$$

with the norm $\|f\|_{W_p^s(\Omega)} = \inf\left\{ \|g\|_{W_p^s(\mathbb{R}^n)} \,\middle|\, g|_\Omega = f \right\}$. Let $W_{p,0}^s(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_p^s(\Omega)}$. When $p = 2$, $W_2^s(\mathbb{R}^n)$ and $W_2^s(\Omega)$ are denoted by $H^s(\mathbb{R}^n)$ and $H^s(\Omega)$, respectively.

Notation 9. For non-negative numbers s and p,

$$H^{s,p} = \{f; \langle x\rangle^p f \in H^s(\mathbb{R}^n)\}(= \{f; \langle D\rangle^s \langle x\rangle^p f \in L^2(\mathbb{R}^n)\}).$$

Notation 10. For non-negative numbers k and q, we put

$$\begin{aligned}&|(\phi,\psi)|^2_{Z_{k,q}}\\ &:= \sup_{r\in\mathbb{R}} \langle r\rangle^q (|(A^{k/4}\mathcal{A}^{1/2}e^{2irA^{1/2}}\phi, A^{k/4}\mathcal{A}^{1/2}\phi)_H| + |(A^{k/4}e^{2irA^{1/2}}\psi, A^{k/4}\psi)_H| \\ &\qquad + 2|(A^{k/4}e^{2irA^{1/2}}\mathcal{A}^{1/2}\phi, \mathcal{A}^{k/4}\psi)_H|).\end{aligned}$$

Here we note that $|(\phi,\psi)|_{X_{k,p}} \le C|(\phi,\psi)|_{Z_{k,q}}$, if $1 \le p+1 < q$.

We first give examples of weighted Sololev spaces included in $Y_{k,p}$. Although [11] and [12] consider about the sufficient conditions for belonging to $Z_{1,q}$, it follows from the proof that these sufficient conditions are also sufficient conditions for belonging to $Z_{1,q} \cap Z_{0,q}$:

Example 1 (see [3, Lemma A] for integer p, and [11, Lemma 1.1] for general p). Let $H = L^2(\mathbb{R}^n)$, $A = -\Delta$ with domain $D(A) = H^2(\mathbb{R}^n)$. Let $p \ge 0$ in the case $n = 1$ and $0 \le p < n$ in the case $n \ge 2$. Let $q > p + 1$, and assume furthermore that $q \le n + 1$ in the case $n \ge 2$. Then the following inclusion holds.

$$H^{3/2,q}(\mathbb{R}^n) \times H^{1/2,q}(\mathbb{R}^n) \subset Z_{1,q} \cap Z_{0,q} \cap (H \times H) \subset Y_{1,p} \cap Y_{0,p} \subset \tilde{Y}_1.$$

We give examples of non-weighted Sobolev spaces included in $Y_{k,p}$:

Example 2 (see [12, Theorem 2]). Assume that $n \ge 4$. Let $2(n-1)/(n-3) < q < \infty$, and let q' be the dual exponent of q, that is, $1/q + 1/q' = 1$, and let $0 \le p < (n-1)(1/2 - 1/q) - 1$. Then

$$\begin{aligned}&\left(H^{\max\{n(1/2-1/q)+1,3\}}(\mathbb{R}^n) \cap W_{q'}^{(n+1)(1/2-1/q)+1}(\mathbb{R}^n)\right)\\ &\times \left(H^{\max\{n(1/2-1/q),2\}}(\mathbb{R}^n) \cap W_{q'}^{(n+1)(1/2-1/q)}(\mathbb{R}^n)\right)\\ &\subset Y_{1,p} \cap Y_{0,p} \subset \tilde{Y}_p.\end{aligned}$$

Example 3 (see [12, Theorem 4]). Assume that $n \ge 4$ and Ω is non-trapping exterior domain with smooth boundary. Let $2(n-1)/(n-3) < q < \infty$, and let q' be the dual exponent of q, that is, $1/q + 1/q' = 1$, and let $0 \le p < (n-1)(1/2 - 1/q) - 1$. Let M be the smallest integer such that $M \ge (n+1)(1/2 - 1/q)$. Then

$$W_{q,0}^{2M+1}(\Omega) \times W_{q,0}^{2M}(\Omega) \subset Y_{1,p} \cap Y_{0,p} \subset \tilde{Y}_p.$$

In the above examples, the sufficient condition for belonging to $Y_{p,k}$ follows from the condition for belonging to $Z_{q,k}$. In the space dimension 3, we [13] showed that the following unweighted Sobolev spaces are included in $Y_0 = Y_{1,0}$ directly, although we did not use the notation $Y_{1,0}$. It follows from the proof that these sufficient conditions are also sufficient conditions for belonging to $Y_0 \cap Y_{0,0}$:

EXAMPLE 4 (see [13, Theorem 3]).

$$\left(W_1^2(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)\right)^2 \subset Y_{1,0} \cap Y_{0,0} \subset \tilde{Y}_0.$$

EXAMPLE 5 (see [13, Theorem 5]). Let Ω be a non-trapping exterior domain in $\mathbb{R}^3$ with smooth boundary. Let p, $\tilde{p}$, q and $\tilde{q}$ be real numbers such that $1 < \tilde{q} < 3/2 < q \le 2, 3 < \tilde{p}$. Then there exists a positive constant C such that

$$\left(W^8_{\tilde{p},0}(\Omega) \cap W^9_{q,0}(\Omega) \cap W^9_{\tilde{q},0}(\Omega) \cap W_1^2(\Omega)\right)^2 \subset Y_{1,0} \cap Y_{0,0} \subset \tilde{Y}_0.$$

3. Transformed equation

Greenberg-Hu [4] introduced a transformation from a solution u of Kirchhoff equation into a pair of unknown functions expressed by using Fourier decomposition. D'Ancona-Spagnolo [1, 2, 3] improved their transformation in $\mathbb{R}^n$. [15] used a similar transformation by using spectral decomposition of the operator A instead of the Fourier transform.

We first state some classes and metrics which are transformed from X_p, Y_p, d_p and d_{Y_p} in Section 1.

NOTATION 11. Let $J \ge 0$, p be a non-negative number and let ε be a positive number. We define sets $\mathcal{X}_p$ and $\mathcal{Y}_p$ as

$$\mathcal{X}_p := \{(V, W) \in D(A^{1/4}) \times D(A^{1/4});\ |(V, W)|_{\mathcal{X}_p} < \infty\},$$
$$\mathcal{Y}_p := \{(V, W) \in \mathcal{X}_p;\ V - W \in R(A^{1/2})\},$$

where

$$|(V, W)|^2_{\mathcal{X}_p} := |(A^{1/4} e^{2irA^{1/2}} V, A^{1/4} V)_H|_p + |(A^{1/4} e^{2irA^{1/2}} W, A^{1/4} W)_H|_p + 2|(A^{1/4} e^{2irA^{1/2}} V, A^{1/4} W)_H|_p,$$

and subsets $\mathcal{X}_p(\varepsilon)$ of $\mathcal{X}_p$ and $\mathcal{Y}_p(\varepsilon)$ of $\mathcal{Y}_p$ as

$$\mathcal{X}_p(\varepsilon) := \{(V, W) \in \mathcal{X}_p; |(V, W)|_{\mathcal{X}_p} + \|V\| + \|W\| \le \varepsilon\},$$
$$\mathcal{Y}_p(\varepsilon) := \mathcal{X}_p(\varepsilon) \cap \mathcal{Y}_p.$$

We put

$$\mathcal{X}_p^J := \mathcal{X}_p \cap (D(A^{J/2}) \times D(A^{J/2})),\ \mathcal{X}_p^J(\varepsilon) := \mathcal{X}_p(\varepsilon) \cap (D(A^{J/2}) \times D(A^{J/2})),$$
$$\mathcal{Y}_p^J := \mathcal{Y}_p \cap (D(A^{J/2}) \times D(A^{J/2})),\ \mathcal{Y}_p^J(\varepsilon) := \mathcal{Y}_p(\varepsilon) \cap (D(A^{J/2}) \times D(A^{J/2})),$$

and

$$|(V, W)|_{\mathcal{X}_p^J} = |(V, W)|_{\mathcal{X}_p} + \|V\|_J + \|W\|_J,$$
$$|(V, W)|_{\mathcal{Y}_p^J} = |(V, W)|_{\mathcal{X}_p^J} + \|A^{-1/2}(V - W)\|.$$

For $(V_1, W_1), (V_2, W_2) \in \mathcal{X}_p$, we define

$$
\begin{aligned}
&\tilde{d}_p((V_1, W_1), (V_2, W_2))^2 \\
&:= |(A^{1/4} e^{2irA^{1/2}} V_1, A^{1/4} V_1)_H - (A^{1/4} e^{2irA^{1/2}} V_2, A^{1/4} V_2)_H|_p \\
&\quad + |(A^{1/4} e^{2irA^{1/2}} W_1, A^{1/4} W_1)_H - (A^{1/4} e^{2irA^{1/2}} W_2, A^{1/4} W_2)_H|_p \\
&\quad + 2|(A^{1/4} e^{2irA^{1/2}} V_1, A^{1/4} W_1)_H - (A^{1/4} e^{2irA^{1/2}} V_2, A^{1/4} W_2)_H|_p,
\end{aligned}
$$

and put

$$
\begin{aligned}
\tilde{d}_p^J((V_1, W_1), (V_2, W_2)) &:= \tilde{d}_p((V_1, W_1), (V_2, W_2)) + \|V_1 - V_2\|_J + \|W_1 - W_2\|_J, \\
\tilde{d}_{Y_p}^J((V_1, W_1), (V_2, W_2)) &:= \tilde{d}_p^J((V_1, W_1), (V_2, W_2)) \\
&\qquad + \|A^{-1/2}(V_1 - W_1) - A^{-1/2}(V_2 - W_2)\|.
\end{aligned}
$$

We see that $\tilde{d}_p^J$ and $\tilde{d}_{Y_p}^J$ become metrics on $\mathcal{X}_p^J$ and $\mathcal{Y}_p^J$, respectively. We equip $\mathcal{X}_p^J$ and a subset $\mathcal{X}_p^J(\varepsilon)$ of $\mathcal{X}_p^J$ with the metric $\tilde{d}_p^J$, and $\mathcal{Y}_p^J$ and a subset $\mathcal{Y}_p^J(\varepsilon)$ of $\mathcal{Y}_p^J$ with the metric $\tilde{d}_{Y_p}^J$

Notation 12. Let

$$
G_0 : (0, \infty) \times (D(A^{3/4}) \times D(A^{1/4})) \to D(A^{1/4}) \times D(A^{1/4})
$$

be the operator defined by

$$
G_0(\lambda, (\phi, \psi)) := \left(\lambda^{-1/2}\left(\psi - i\lambda A^{1/2}\phi\right), \lambda^{-1/2}\left(\psi + i\lambda A^{1/2}\phi\right)\right).
$$

Notation 13. Let

$$
\begin{aligned}
\mathcal{G}_0 : (0, \infty) &\times \{(V, W) \in D(A^{1/4}) \times D(A^{1/4}); V - W \in R(A^{1/2})\} \\
&\to D(A^{3/4}) \times D(A^{1/4})
\end{aligned}
$$

be the operator defined by

$$
\mathcal{G}_0(\lambda, (V, W)) := \left(\frac{i}{2}\lambda^{-1/2} A^{-1/2}(V - W), \frac{1}{2}\lambda^{1/2}(V + W)\right).
$$

Lemma A (Lemma 2 of [15])**.** *(i) For every positive numbers λ, p, ε, the following holds.*

$$
\begin{aligned}
G_0(\lambda, \cdot, \cdot) X_p^J(\varepsilon) &\subset \mathcal{X}_p^J(2\varepsilon'), \\
\mathcal{G}_0(\lambda, \cdot, \cdot) \mathcal{X}_p^J(\varepsilon) &\subset X_p^J(\varepsilon'),
\end{aligned}
$$

where $\varepsilon' = \sqrt{\max\{\lambda, 1/\lambda\}}\varepsilon$

(ii)

$$
\begin{aligned}
&\tilde{d}_p^J(G_0(\lambda_1, (\phi_1, \psi_1)), G_0(\lambda_2, (\phi_2, \psi_2)) \\
&\le 2 \max_{j=1,2}\left(\sqrt{\max\{\lambda_j, 1/\lambda_j\}} + |(\phi_j, \psi_j)|_{\mathcal{X}_p^J}\right) \\
&\qquad \times (|\lambda_1 - \lambda_2| + d_p^J((\phi_1, \psi_1)), (\phi_2, \psi_2)))
\end{aligned}
$$

for every and $(\lambda_j, (\phi_j, \psi_j)) \in (0, \infty) \times X_p^J (j = 1, 2)$, and

$$d_p^J(\mathcal{G}_0(\lambda_1,(V_1,W_1)),\mathcal{G}_0(\lambda_2,(V_2,W_2)))$$
$$\leq \max_{j=1,2}\Big(\sqrt{\max\{\lambda_j,1/\lambda_j\}}+|(V_J,W_j)|_{\mathcal{X}_p^J}\Big)$$
$$\times(|\lambda_1-\lambda_2|+\tilde{d}_p^J((V_1,W_1)),(V_2,W_2)))$$

for every $(\lambda_j,(V_j,W_j))\in(0,\infty)\times\mathcal{X}_p^J(j=1,2)$.

Lemma 1. *(i) For every positive numbers* λ,p,ε, *the following holds.*

$$G_0(\lambda,\cdot,\cdot)Y_p^J(\varepsilon)\subset\mathcal{Y}_p^J(2\varepsilon'),$$
$$\mathcal{G}_0(\lambda,\cdot,\cdot)\mathcal{Y}_p^J(\varepsilon)\subset Y_p^J(\varepsilon'),$$

where $\varepsilon'=\sqrt{\max\{\lambda,1/\lambda\}}\varepsilon$.

(ii)

$$\tilde{d}_{Y_p}^J(G_0(\lambda_1,(\phi_1,\psi_1)),G_0(\lambda_2,(\phi_2,\psi_2))$$
$$\leq 2\max_{j=1,2}\left(\sqrt{\max\{\lambda_j,1/\lambda_j\}}+|(\phi_j,\psi_j)|_{\mathcal{Y}_p^J}\right)$$
$$\times(|\lambda_1-\lambda_2|+d_{Y_p}^J((\phi_1,\psi_1)),(\phi_2,\psi_2)))$$

for every and $(\lambda_j,(\phi_j,\psi_j))\in(0,\infty)\times Y_p^J(j=1,2)$, *and*

$$d_{Y_p}^J(\mathcal{G}_0(\lambda_1,(V_1,W_1)),\mathcal{G}_0(\lambda_2,(V_2,W_2)))$$
$$\leq \max_{j=1,2}\Big(\sqrt{\max\{\lambda_j,1/\lambda_j\}}+|(V_j,W_j)|_{\mathcal{Y}_p^J}\Big)$$
$$\times(|\lambda_1-\lambda_2|+\tilde{d}_{Y_p}^J((V_1,W_1)),(V_2,W_2)))$$

for every $(\lambda_j,(V_j,W_j))\in(0,\infty)\times\mathcal{Y}_p^J(j=1,2)$.

Proof. Let $(V,W)=G_0(\lambda,(\phi,\psi))$. Then by definition, we have

$$A^{-1/2}(V-W)=2\lambda^{1/2}\phi,$$

with which Lemma A implies the conclusion. □

Lemma B (Lemma 3 of [15])**.** *Let* $p\geq 0$, $\varepsilon>0$, $J\geq 1/2$ *and* $a,b\in[m_0,\infty)$.

(i) *If* $(V,W)\in\mathcal{X}_p^J(\varepsilon)$, *then* $(e^{iaA^{1/2}}V,e^{-iaA^{1/2}}W)\in\mathcal{X}_p^J(2^{p/4}\langle a\rangle^{p/2}\varepsilon)$.

(ii) *The mapping*

$$\Phi_0:(a,(V,W))\to(e^{iaA^{1/2}}V,e^{-iaA^{1/2}}W)$$

is continuous from $\mathbb{R}\times\mathcal{X}_p^J$ *to* $\mathcal{X}_p^J$.

Lemma 2. *Let* $p\geq 0$, $\varepsilon>0$, $J\geq 1/2$ *and* $a,b\in[m_0,\infty)$.

(i) *If* $(V,W)\in\mathcal{Y}_p^J(\varepsilon)$, *then* $(e^{iaA^{1/2}}V,e^{-iaA^{1/2}}W)\in\mathcal{Y}_p^J(2^{p/4}\langle a\rangle^{p/2}\varepsilon)$.

(ii) *The mapping*

$$\Phi_0:(a,(V,W))\to(e^{iaA^{1/2}}V,e^{-iaA^{1/2}}W)$$

is continuous from $\mathbb{R}\times\mathcal{Y}_p^J$ *to* $\mathcal{Y}_p^J$.

Proof. (i) Since

$$e^{iaA^{1/2}}V - e^{-iaA^{1/2}}W = e^{iaA^{1/2}}(V-W) + 2iA^{1/2}A^{-1/2}\sin(aA^{1/2})W \in R(A^{1/2}), \tag{3.1}$$

where $A^{-1/2}\sin(aA^{1/2}) = \int_0^\infty \frac{\sin(a\lambda^{1/2})}{\lambda^{1/2}} dE_\lambda \in \mathcal{L}(H)$ is meaningful by the spectral decomposition of A with

$$\left\|A^{-1/2}\sin(aA^{1/2})\right\|_{\mathcal{L}(H)} = \left\|a\int_0^\infty \frac{\sin(a\lambda^{1/2})}{a\lambda^{1/2}} dE_\lambda\right\|_{\mathcal{L}(H)} \leq |a|. \tag{3.2}$$

Then Lemma B (i) and (3.1) imply the conclusion.

(ii) Let $(a_1,(V_1,W_1))$ be an arbitrary point of $\mathbb{R}\times\mathcal{Y}_p^J$, and put $U_1 = A^{-1/2}(V_1 - W_1)$. By Lemma B, we only have to prove the continuity of the mapping $(a,V,W) \mapsto A^{-1/2}\left(e^{iaA^{1/2}}V - e^{-iaA^{1/2}}W\right)$ from $\mathcal{Y}_p^0$ to H at the point $(a_1,(V_1,W_1))$. Let $(a,(V,W)) \in \mathbb{R}\times\mathcal{Y}_p^0$ and put $U = A^{-1/2}(V-W)$, $U_1 = A^{-1/2}(V_1-W_1)$. Since $(d/ds)e^{\pm isA^{1/2}}f = \pm iA^{1/2}e^{\pm isA^{1/2}}f$ for $f = V_1, W_1$, we have

$$\begin{aligned}
&(e^{iaA^{1/2}}V - e^{-iaA^{1/2}}W) - (e^{ia_1A^{1/2}}V_1 - e^{-ia_1A^{1/2}}W_1)\\
&= e^{iaA^{1/2}}((V-V_1)-(W-W_1)) + (e^{iaA^{1/2}} - e^{-iaA^{1/2}})(W-W_1)\\
&\quad + (e^{iaA^{1/2}} - e^{ia_1A^{1/2}})V_1 - (e^{-iaA^{1/2}} - e^{-ia_1A^{1/2}})W_1\\
&= e^{-ia_1A^{1/2}}A^{1/2}(U-U_1) + 2i\sin(aA^{1/2})(W-W_1)\\
&\quad + iA^{1/2}\int_{a_1}^{a}(e^{isA^{1/2}}V_1 + e^{-isA^{1/2}}W_1)ds.
\end{aligned}$$

Hence by using (3.2) and $\|e^{iaA^{1/2}}\|_{\mathcal{L}(H)} \leq 1$, we obtain

$$\begin{aligned}
&\|A^{-1/2}\left((e^{iaA^{1/2}}V - e^{-iaA^{1/2}}W) - (e^{ia_1A^{1/2}}V_1 - e^{-ia_1A^{1/2}}W_1)\right)\|\\
&\leq \|U-U_1\| + 2|a|\|W-W_1\| + 2|a-a_1|(\|V_1\|+\|W_1\|),
\end{aligned}$$

which implies the mapping $(a,V,W) \mapsto A^{-1/2}\left(e^{iaA^{1/2}}V - e^{-iaA^{1/2}}W\right)$ is continuous from $\mathcal{Y}_p^0$ to H. □

Now we transform the equation (0.1).

Assume that $u(t)$ is a solution of (0.1). Put

$$c(t) := m(\|A^{1/2}u(t)\|^2), \quad \tau(t) := \int_0^t c(s)ds, \quad \text{for} \quad t\in\mathbb{R}. \tag{3.3}$$

Then, $\tau(t)$ is a strictly increasing function on $\mathbb{R}$. Let $t(\tau)$ be the inverse of $\tau(t)$, and put

$$C(\tau) := c(t(\tau)), \tag{3.4}$$

$$\begin{cases} V(\tau) := C(\tau)^{-1/2}e^{i\tau A^{1/2}+B(t(\tau))/2}\left(u'(t(\tau)) - iC(\tau)A^{1/2}u(t(\tau))\right),\\ W(\tau) := C(\tau)^{-1/2}e^{-i\tau A^{1/2}+B(t(\tau))/2}\left(u'(t(\tau)) + iC(\tau)A^{1/2}u(t(\tau))\right), \end{cases} \tag{3.5}$$

for every $\tau\in\mathbb{R}$, where

$$B(t) := \int_0^t b(s)ds \quad \text{for} \quad t \in \mathbb{R}.$$

By this transformation, we easily see that the equation (0.1)–(0.2) is transformed into

$$\begin{cases} V'(\tau) = -q(\tau)e^{2i\tau A^{1/2}}W(\tau), \quad W'(\tau) = -q(\tau)e^{-2i\tau A^{1/2}}V(\tau), \\ C(\tau) = m\left(\dfrac{e^{-B(t(\tau))}}{4C(\tau)}\|e^{-i\tau A^{1/2}}V(\tau) - e^{i\tau A^{1/2}}W(\tau)\|^2\right), \\ (V(0), W(0)) = G_0(m(\|A^{1/2}\phi_0\|^2), \phi_0, \psi_0), \end{cases}$$

where

$$q(\tau) := \frac{C'(\tau) + b(t(\tau))}{2C(\tau)} \quad ('= \frac{d}{d\tau}).$$

The Hamiltonian $H(u, t)$ is defined as

$$H(u, t) := M(\|A^{1/2}u(t)\|^2) + \|u'(t)\|^2,$$

where

$$M(\rho) = \int_0^\rho m(s)^2 ds \qquad (\rho \geq 0).$$

Then we have

$$H(u, t) = H(u, T) - 2\int_T^t b(s)\|u'(s)\|^2 ds,$$

and thus, we have by Gronwall's inequality that

$$H(u, t) \leq \exp(2\|b\|_{L^1})H(u, T) \quad \text{for every} \ \ t \in \mathbb{R}. \tag{3.6}$$

Since $M(\rho) \geq m_0^2\rho$, inequality (3.6) implies

$$\|A^{1/2}u(t)\|^2 \leq \exp(2\|b\|_{L^1})H(u, T)/m_0^2 \quad \text{for every } t \in \mathbb{R}. \tag{3.7}$$

Put

$$L := \exp(2\|b\|_{L^1})(M(1) + 1)/m_0^2.$$

Then (3.7) implies

$$\|A^{1/2}u(t)\|^2 \leq L \quad \text{for every } t \in \mathbb{R},$$

if $\|A^{1/2}u(T)\| + \|u(T)\| \leq 1$.

Observing the fact above, we put

$$m_1 = \sup\{m(x); 0 \leq x \leq L\}, \quad m_2 = \sup\{|m'(x)|; 0 \leq x \leq L\}$$
$$m_3 := \sup\{|m''(x)|; 0 \leq x \leq L\}.$$

NOTATION 14. Let

$$\mathcal{C}_p = \mathcal{C}_p(\mathbb{R}) := \{C \in BC^1(\mathbb{R}); |C'|_p < \infty\}.$$

Here, $BC^1(\mathbb{R})$ is the set of all bounded C^1 functions with bounded derivatives on $\mathbb{R}$. The space $\mathcal{C}_p(\mathbb{R})$ becomes a Banach space with the norm

$$\|C\|_{\mathcal{C}_p} = \|C\|_{\mathcal{C}_p(\mathbb{R})} := \|C\|_{L^\infty(\mathbb{R})} + \left\|C'\right\|_{L^\infty(\mathbb{R})} + |C'|_p \ .$$

For a positive number δ, let

$$\mathcal{C}_{p,\delta}(\mathbb{R}) := \{C \in \mathcal{C}_p(\mathbb{R});\ m_0 \leq C(\tau) \leq m_1 \text{ for every } \tau \in \mathbb{R},\ |C'|_p \leq \delta\},$$

which is a closed subset of $\mathcal{C}_p(\mathbb{R})$.

Let $T \in [-\infty, \infty]$. Taking the aforementioned fact into account, we consider the following equation:

$$V'(\tau) = -q(\tau)e^{2i\tau A^{1/2}}W(\tau), \quad W'(\tau) = -q(\tau)e^{-2i\tau A^{1/2}}V(\tau), \tag{3.8}$$

$$C(\tau) = m\left(\frac{e^{-B(t(\tau))}}{4C(\tau)}\|e^{-i\tau A^{1/2}}V(\tau) - e^{i\tau A^{1/2}}W(\tau)\|^2\right), \tag{3.9}$$

$$V(T) = \tilde{V}, \quad W(T) = \tilde{W}, \tag{3.10}$$

where

$$t(\tau) = \int_0^\tau \frac{1}{C(\sigma)}d\sigma \quad \text{for} \quad \tau \in \mathbb{R}, \tag{3.11}$$

$$q(\tau) := \frac{C'(\tau) + b(t(\tau))}{2C(\tau)} \ ({}' = \frac{d}{d\tau}), \quad B(t) := \int_0^t b(s)ds. \tag{3.12}$$

Here, $V(\pm\infty) = \tilde{V}$ and $W(\pm\infty) = \tilde{W}$ mean that

$$\lim_{t\to\pm\infty} V(t) = \tilde{V} \quad \text{and} \quad \lim_{t\to\pm\infty} W(t) = \tilde{W} \quad \text{in } D(A^{J/2}).$$

Lemma C (Lemma 4 of [15])**.** *Let $C, C_j \in \mathcal{C}_p(\mathbb{R})$ $(j = 1, 2)$. Then the functions t defined by* (3.11) *and t_j defined by* (3.11) *with $C = C_j$ for $j = 1, 2$ satisfy the following.*

$$\frac{|\tau|}{m_1} \leq |t(\tau)| \leq \frac{|\tau|}{m_0} \quad \textit{for} \quad \tau \in \mathbb{R}, \tag{3.13}$$

$$|(t_1 - t_2)(\tau)| \leq \frac{|\tau|}{m_0^2}\|C_1 - C_2\|_{L^\infty} \quad \textit{for every} \quad \tau \in \mathbb{R}.$$

Lemma D (Lemma 5 of [15])**.** *Let $T \in [-\infty, \infty]$ and fix it. Let $J \geq 0$ and $p \geq 0$. Let $C_j \in \mathcal{C}_p$, and let t_j and q_j be the functions defined by* (3.11) *and* (3.12) *with $C = C_j$ for $j = 1, 2$.*

$$\langle t\rangle^p q_j \in L^1(\mathbb{R}) \quad \textit{and} \quad |q_j|_p \leq \frac{1}{2m_0}(|C_j'|_p + \frac{\langle m_1\rangle^p}{m_0}|b|_p)$$

for $j = 1, 2$ and

$$|q_1 - q_2|_p \leq \left(\frac{1}{2m_0} + \frac{m_1\langle m_1\rangle^p}{2m_0^2}(\frac{m_1|tb'|_p}{m_0} + |b|_p) + \frac{|C_2'|_p}{2m_0^2}\right)\|C_1 - C_2\|_{\mathcal{C}_p}.$$

Proposition A (Propositions 3 and 5, Proof of Proposition 5 and Lemma 3 of [15])**.** *Let $T \in [-\infty, \infty]$. Let $J \geq 1$ and $p \geq 0$. Then there exist positive constants ε_6 and K_5 such that for every $(\tilde{V}, \tilde{W}) \in \mathcal{X}_p^J(\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_6$, a solution $(C, (V, W)) \in C^1(\mathbb{R}) \times \cap_{j=0,1} C^j(\mathbb{R}; D(A^{(J-j)/2}) \times D(A^{(J-j)/2}))$ of* (3.8)–(3.10) *exists and satisfies*

$$C \in \mathcal{C}_{p,K_5\varepsilon}(\mathbb{R}),$$

(3.14) $$(V, W) \in C(\mathbb{R}; \mathcal{X}_p^J(K_5\varepsilon)),$$

(3.15) $$\|V(\tau)\|_J + \|W(\tau)\|_J \le \exp(\|q\|_{L^1(\mathbb{R})})\left(\|\tilde{V}\|_J + \|\tilde{W}\|_J\right).$$

Furthermore, the limits

(3.16) $$V_{\pm\infty} = \lim_{\tau\to\pm\infty} V(\tau) \text{ and } W_{\pm\infty} = \lim_{\tau\to\pm\infty} W(\tau) \text{ in } D(A^{J/2}),$$

(3.17) $$C_{\pm\infty} = \lim_{\tau\to\pm\infty} C(\tau) \in \mathbb{R},$$

exist and satisfy the following:

(3.18) $$\lim_{\tau\to\pm\infty} \langle\tau\rangle^p \|V(\tau) - V_{\pm\infty}\|_J = \lim_{\tau\to\pm\infty} \langle\tau\rangle^p \|W(\tau) - W_{\pm\infty}\|_J = 0,$$

(3.19) $$\lim_{\tau\to\pm\infty} \langle\tau\rangle^p |C(\tau) - C_{\pm\infty}| = 0.$$

Assume furthermore that

(3.20) $$\lim_{r\to\pm\infty} (e^{2irA^{1/2}}\tilde{V}, \tilde{V})_H = \lim_{r\to\pm\infty} (e^{2irA^{1/2}}\tilde{W}, \tilde{W})_H = \lim_{r\to\pm\infty} (e^{2irA^{1/2}}\tilde{V}, \tilde{W})_H = 0,$$

which condition we denote as $(\tilde{V}, \tilde{W}) \in \tilde{\mathcal{X}}$. *Then*

(3.21) $$C_{\pm\infty} = m\left(\frac{e^{-B_{\pm\infty}}}{4C_{\pm\infty}}(\|V_{\pm\infty}\|^2 + \|W_{\pm\infty}\|^2)\right),$$

where $B_{\pm\infty} = \int_0^{\pm\infty} b(s)ds$, *and* $(V_{\pm\infty}, W_{\pm\infty}) \in \tilde{\mathcal{X}}$.

Proposition B (Proposition 4 of [15]). *Let* $J \ge 1$, $p \ge 0$. *Let* ε_6 *be a constant given by Proposition A. There exist positive constants* $\varepsilon_7(\le \varepsilon_6)$ *and* K_6 *such that the following assertion holds. Let* $(\tilde{V}_j, \tilde{W}_j) \in \mathcal{X}^J_{p,\varepsilon_7}$, *and assume that* $(C_j, (V_j, W_j)) \in C^1(\mathbb{R}) \times \cap_{j=0,1} C^j(\mathbb{R}; D(A^{(J-j)/2}) \times D(A^{(J-j)/2}))$ *are solutions of* (3.8)–(3.10) *satisfying* (3.33) *and* $(\tilde{V}, \tilde{W}) = (\tilde{V}_j, \tilde{W}_j)$ *for* $j = 1, 2$. *Assume moreover that* $\langle t\rangle^p C \in L^1(\mathbb{R})$ *if* $T = \pm\infty$. *Then*

(3.22) $$\|C_1 - C_2\|_{\mathcal{C}_p} \le K_6 \tilde{d}_p((\tilde{V}_1, \tilde{W}_1), (\tilde{V}_2, \tilde{W}_2)),$$

(3.23) $$\tilde{d}^J_p((V_1(\tau), W_1(\tau)), (V_2(\tau), W_2(\tau))) \le K_6 \tilde{d}^J_p((\tilde{V}_1, \tilde{W}_1), (\tilde{V}_2, \tilde{W}_2))$$

for every $\tau \in [-\infty, \infty]$.

If initial data belongs to $\mathcal{Y}_p$ for $p \ge 1$, then the following propositions hold.

Proposition 1. *Assume that A is injective. Let* $T \in [-\infty, \infty]$. *Let* $J \ge 1$ *and* $p \ge 1$. *Let* ε_6 *and* K_5 *be the constants in Proposition A. Let* $(\tilde{V}, \tilde{W}) \in \mathcal{Y}^J_p(\varepsilon)$ *with* $0 < \varepsilon \le \varepsilon_6$. *Then there exists a solution* $(C, (V, W)) \in BC^1(\mathbb{R}) \times C^1(\mathbb{R}; D(A^{J/2}) \times D(A^{J/2}))$ *of* (3.8)–(3.10) *such that*

(3.24) $$C \in \mathcal{C}_{p,K_5\varepsilon},$$

(3.25) $$(V(\tau), W(\tau)) \in \mathcal{Y}^J_p(K_5\varepsilon) \quad \text{for every } \tau \in \mathbb{R},$$

(3.26) $$(V, W) \in C(\mathbb{R}; \mathcal{Y}^J_p).$$

The uniqueness holds in the class

$$(C, (V, W)) \in BC^1(\mathbb{R}) \times C^1(\mathbb{R}; D(A^{J/2}) \times D(A^{J/2})) \quad \text{if} \quad T \ne \pm\infty,$$

$$(C,(V,W)) \in BC^1(\mathbb{R}) \times C^1(\mathbb{R}; D(A^{J/2}) \times D(A^{J/2})), \quad C' \in L^1(\mathbb{R})$$

if $T = \pm\infty$. *Furthermore the following hold. Put*

$$U(\tau) := A^{-1/2}(V(\tau) - W(\tau)) \quad \textit{for} \quad \tau \in \mathbb{R}.$$

Then $U \in C^1(\mathbb{R}; D(A^{(J+1)/2})$, *and the limit*

$$U_{\pm\infty} = \lim_{\tau\to\pm\infty} U(\tau) \textit{ in } D(A^{(J+1)/2})$$

exists. The limits $C_{\pm\infty}$, $V_{\pm\infty}$ *and* $W_{\pm\infty}$ *shown by Proposition A and* $U_{\pm\infty}$ *satisfy the following:*

(3.27) $$\lim_{\tau\to\pm\infty} \langle\tau\rangle^{p-1}\|U(\tau) - U_{\pm\infty}\|_{J+1} = 0,$$

(3.28) $$\left|\int_0^{\pm\infty} \langle\tau\rangle^{p-1}\|V(\tau) - V_{\pm\infty}\|_J d\tau\right| + \left|\int_0^{\pm\infty} \langle\tau\rangle^{p-1}\|W(\tau) - W_{\pm\infty}\|_J d\tau\right| < \infty,$$

(3.29) $$\left|\int_0^{\pm\infty} \langle\tau\rangle^{p-2}\|U(\tau) - U_{\pm\infty}\|_{J+1} d\tau\right| < \infty \quad \textit{if} \quad p \geq 2,$$

(3.30) $$V_{\pm\infty} - W_{\pm\infty} = A^{1/2}U_{\pm\infty}.$$

(3.31) $$(V_{\pm\infty}, W_{\pm\infty}) \in \mathcal{Y}_p^J(K_5\varepsilon),$$

(3.32) $$\int_0^{\infty} \langle\tau\rangle^{p-1}|C(\tau) - C_{+\infty}|d\tau + \int_{-\infty}^{0} \langle\tau\rangle^{p-1}|C(\tau) - C_{-\infty}|d\tau < \infty.$$

Proposition 2. *Let* $J \geq 1$, $p \geq 1$ *and* $\varepsilon > 0$. *Let* ε_7 *be a positive constant given by Propositions B. There exist a positive constant* K_7 *such that the following assertion holds. Let* $(\tilde{V}_j, \tilde{W}_j) \in \mathcal{Y}_p^J(\varepsilon_7)$, *and assume that* $(C_j,(V_j,W_j)) \in C^1(\mathbb{R}) \times \cap_{j=0,1} C^j(\mathbb{R}; D(A^{(J-j)/2}) \times D(A^{(J-j)/2}))$ *are solutions of* (3.8)–(3.10) *with* $(\tilde{V}, \tilde{W}) = (\tilde{V}_j, \tilde{W}_j)$ *for* $j = 1, 2$, *satisfying*

(3.33) $$m_0 \leq C(\tau) \leq m_1 \quad \textit{for every } \tau \in \mathbb{R}.$$

Assume moreover that $C \in \mathcal{C}_p$ *if* $T = \pm\infty$. *Then*

(3.34) $$\tilde{d}^J_{Y_p}((V_1(\tau), W_1(\tau)), (V_2(\tau), W_2(\tau))) \leq K_7 \tilde{d}^J_{Y_p}((\tilde{V}_1, \tilde{W}_1), (\tilde{V}_2, \tilde{W}_2))$$

for every $\tau \in [-\infty, \infty]$.

4. Proof of Propositions

Since we sometimes use the following calculations, we describe them as a lemma.

Lemma 3. *Let* Z *be a Banach space with norm* $\|\cdot\|_Z$. *Let* p *be an nonnegative number. If* $f \in C^1(\mathbb{R}; Z)$ *satisfies* $|t|^p\|f'(t)\|_Z \in L^1(\mathbb{R})$, *then the limit* $\lim_{t\to\pm\infty} f(t) = \alpha_{\pm\infty}$ *exists in* Z *and satisfies the following.*

(4.1) $$\lim_{t\to\pm\infty} |t|^p\|f(t) - \alpha_{\pm\infty}\|_Z = 0.$$

Furthermore, if $p \geq 1$ *and* $\langle t\rangle^{p-1}t\|f'(t)\|_Z \in L^1(\mathbb{R})$, *then*

(4.2) $$\int_t^{\infty} \langle\sigma\rangle^{p-1}\|f(\sigma) - \alpha_{+\infty}\|_Z d\sigma \leq \int_t^{\infty} \langle\sigma\rangle^{p-1}|\sigma|\big\|f'(\sigma)\big\|_Z d\sigma < \infty$$

for every $t \geq 0$, *and*

$$\int_{-\infty}^{t} \langle\sigma\rangle^{p-1}\|f(\sigma)-\alpha_{-\infty}\|_Z d\sigma \leq \int_{-\infty}^{t} \langle\sigma\rangle^{p-1}|\sigma|\left\|f'(\sigma)\right\|_Z d\sigma < \infty$$

for every $t \leq 0$.

Proof. We only prove lemma in the case $t \geq 0$. The case $t \leq 0$ is calculated in the same way. Let S be an arbitrary nonnegative number of $\mathbb{R}$. Then

$$|\tau|^p\|f(S)-f(\tau)\|_Z \leq |\tau|^p \int_{\tau}^{S} \|f'(\sigma)\|_Z d\sigma \leq \int_{\tau}^{S} |\sigma|^p\|f'(\sigma)\|_Z d\sigma$$

for every $0 \leq \tau \leq S < \infty$. Hence, there exists a limit $\lim_{t\to+\infty} f(t)$ in Z satisfying (4.1).

For every $t \geq 0$, we have by Fubini's Thoerem

$$\int_{t}^{\infty} \langle\tau\rangle^{p-1}\|f(\tau)-\alpha_{+\infty}\|_Z d\tau \leq \int_{t}^{\infty} \langle\tau\rangle^{p-1} \int_{\tau}^{\infty} \left\|f'(\sigma)\right\|_Z d\sigma d\tau$$
$$= \int_{t}^{\infty}\int_{t}^{\sigma} \langle\tau\rangle^{p-1}\left\|f'(\sigma)\right\|_Z d\tau d\sigma \leq \int_{t}^{\infty} \langle\sigma\rangle^{p-1}\sigma\left\|f'(\sigma)\right\|_Z d\sigma < \infty,$$

which implies (4.2). □

Proof of Proposition 1.

(Step 1) By Propositions A and (0.4), we see that $\langle t\rangle^p q \in L^1(\mathbb{R})$ and $\sup_{\tau\in\mathbb{R}}(\|V(\tau)\|_J + \|W(\tau)\|_J) < \infty$. Hence, using the fact that (V, W) satisfies (3.8), we have

$$\langle\tau\rangle^p(\left\|V'(\tau)\right\|_J + \left\|W'(\tau)\right\|_J) \in L^1(\mathbb{R}).$$

Thus by Lemma 3 with $Z = D(A^{J/2})$, we obtain (3.28).

(Step 2) We show that there is $U(\tau) \in D(A^{1/2})$ such that

$$V(\tau) - W(\tau) = A^{1/2}U(\tau)(\in R(A^{1/2})) \quad \text{for every} \quad \tau \in \mathbb{R}, \tag{4.3}$$

which satisfies $U \in C^1(\mathbb{R}; H)$. Kajitani [9, Proposition 2.4] proved this fact for a second order differential operator A which defines non-negative definite self-adjoint operator on $L^2(\mathbb{R}^n)$, by using the differential equation of $V(\tau)-W(\tau)$ together with the expression $e^{i\tau A^{1/2}} = \cos(\tau A^{1/2}) + i\sin(\tau A^{1/2})$ and the boundedness of $(\tau A^{1/2})^{-1}\sin(\tau A^{1/2})$. By using this idea, we prove (4.3). From (3.8), it follows that

$$\frac{d}{d\tau}(V(\tau)-W(\tau)) = q(\tau)e^{2i\tau A^{1/2}}(V(\tau)-W(\tau)) - 2iq(\tau)\sin(2\tau A^{1/2})V(\tau) \tag{4.4}$$

for every $\tau \in \mathbb{R}$. Hence,

$$\frac{d}{d\tau}\left(\exp\left(-\int_{T}^{\tau} q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)(V(\tau)-W(\tau))\right)$$
$$= -2i\exp\left(-\int_{T}^{\tau} q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)q(\tau)\sin(2\tau A^{1/2})V(\tau)$$

for every $\tau \in \mathbb{R}$. Here, $\exp\left(-\int_T^\tau q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)$ is meaningful by spectral decomposition. Thus,

$$V(\tau) - W(\tau) = \exp\left(\int_T^\tau q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)(V(T) - W(T)) - 2i\int_T^\tau \exp\left(\int_s^\tau q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)q(s)\sin(2sA^{1/2})V(s)ds. \tag{4.5}$$

By the assumption that $(V(T), W(T)) = (\tilde{V}, \tilde{W}) \in \mathcal{Y}_p$, there is $\tilde{U} \in D(A^{1/2})$ such that $V(T) - W(T) = A^{1/2}\tilde{U}$. By (3.2), the assumption $q(s)s \in L^1(\mathbb{R})$ and the closeness of the operator $A^{1/2}$, we see that the second term of the right-hand side of (4.5) belongs to $R(A^{1/2})$. Hence, $V(\tau) - W(\tau)$ is expressed as

$$V(\tau) - W(\tau) = A^{1/2}U(\tau),$$

where

$$\begin{aligned} U(\tau) := {} & \exp\left(\int_T^\tau q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)\tilde{U} \\ & - 4i\int_T^\tau \exp\left(\int_s^\tau q(\sigma)e^{2i\sigma A^{1/2}}d\sigma\right)q(s)s\left((2sA^{1/2})^{-1}\sin(2sA^{1/2})\right)V(s)ds \\ \in {} & D(A^{1/2}) \end{aligned} \tag{4.6}$$

for every $\tau \in \mathbb{R}$, and $U \in L^\infty(\mathbb{R}; H)$. We show that $U \in C^1(\mathbb{R}; H)$. We easily see that the first term of (4.6) belongs to $C^1(\mathbb{R}; H)$. The inequality

$$\begin{aligned} & \left\|A^{-1/2}\sin(2(s+h)A^{1/2})f - A^{-1/2}\sin(2sA^{1/2})\right\|_{\mathcal{L}(H)} \\ & = 2\left\|\cos((2s+h)A^{1/2})\sin(hA^{1/2})A^{-1/2}\right\|_{\mathcal{L}(H)} \le 2|h| \quad \text{for } h \in \mathbb{R} \end{aligned}$$

implies that $A^{-1/2}\sin(2sA^{1/2})$ is a norm continuous operator on H with respect to s. Using this fact together with the facts that $\langle t\rangle q \in L^1(\mathbb{R})$ and $V \in C(\mathbb{R}; H)$, we see that the second term of the right-hand side of (4.6) also belongs to $C^1(\mathbb{R}; H)$, and thus we conclude that $U \in C^1(\mathbb{R}; H)$. This fact together with (3.14) and (3.15) implies (3.25) and (3.26).

(Step 3) Last we consider the asymptotic behavior. Equation (4.4) yields

$$\frac{d}{d\tau}U(\tau) = q(\tau)e^{2i\tau A^{1/2}}U(\tau) - 4iq(\tau)\tau(2\tau A^{1/2})^{-1}\sin(2\tau A^{1/2})V(\tau)$$

for every $\tau \in \mathbb{R}$. Then, by the uniform boundedness of $V(\tau)$ and $U(\tau)$, the fact that $\tau\langle\tau\rangle^{p-1}q(\tau) \in L^1(\mathbb{R})$ and the boundedness of the operator $(2\tau A^{1/2})^{-1}\sin(2\tau A^{1/2})$ on H uniformly to τ, we can use Lemma 3 with $Z = H$ to obtain the existence of $U_{\pm\infty} = \lim_{\tau\to\pm\infty} U(\tau)$ in H satisfying (3.27) and (3.29) with $J = 0$. This fact, (3.18) and (3.28) imply (3.27) and (3.29) for general J. By the closeness of the operator $A^{1/2}$ in H, we see that $U_{\pm\infty} \in D(A^{1/2})$ and satisfies (3.30).

Since (3.14) and (3.16) hold, Fatou's lemma implies $(V_\infty, W_\infty) \in \mathcal{X}_p^J(K_5\varepsilon)$, which together with (3.30) yields (3.31).

Lemma 3 together with (3.24) implies (3.32). □

Proof of Proposition 2. By using the expression (4.6), we easily see that

$$\begin{aligned}
&\|U_1(\tau) - U_2(\tau)\|_J \\
&\le e^{\|q_1\|_{L^1(\mathbb{R})}}\left\|\tilde{U}_1 - \tilde{U}_2\right\|_J + e^{\max\{\|q_1\|_{L^1(\mathbb{R})},\|q_2\|_{L^1(\mathbb{R})}\}}\|q_1 - q_2\|_{L^1(\mathbb{R})}\left\|\tilde{U}_2\right\|_J \\
&\qquad + 4e^{\max\{\|q_1\|_{L^1(\mathbb{R})},\|q_2\|_{L^1(\mathbb{R})}\}}\|q_1 - q_2\|_{L^1(\mathbb{R})}\|sq_1\|_{L^1(\mathbb{R})} \sup_{s\in\mathbb{R}} \|V_1(s)\|_J \\
&\qquad + 4e^{\|q_2\|_{L^1(\mathbb{R})}}\|s(q_1 - q_2)\|_{L^1(\mathbb{R})} \sup_{s\in\mathbb{R}} \|V_1(s)\|_J \\
&\qquad + 4e^{\|q_2\|_{L^1(\mathbb{R})}}\|sq_2\|_{L^1(\mathbb{R})} \sup_{s\in\mathbb{R}} \|V_1(s) - V_2(s)\|_J
\end{aligned}$$

for every $\tau \in \mathbb{R}$, where $U_j = A^{-1/2}(V_j - W_j)$ and $\tilde{U}_j = A^{-1/2}(\tilde{V}_j - \tilde{W}_j)$. This inequality, (3.23) and (3.25) yield

$$\begin{aligned}
&\|U_1(\tau) - U_2(\tau)\|_J \\
&\le e^{\max\{\|q_1\|_{L^1(\mathbb{R})},\|q_2\|_{L^1(\mathbb{R})}\}}\Big(\left\|\tilde{U}_1 - \tilde{U}_2\right\|_J + \|q_1 - q_2\|_{L^1(\mathbb{R})}\left\|\tilde{U}_2\right\|_J \\
&\qquad + 4K_5\varepsilon_7(\|q_1 - q_2\|_{L^1(\mathbb{R})}\|sq_1\|_{L^1(\mathbb{R})} + \|s(q_1 - q_2)\|_{L^1(\mathbb{R})}) \\
&\qquad + 4K_6\|sq_2\|_{L^1(\mathbb{R})}\tilde{d}_p^J((\tilde{V}_1, \tilde{W}_1), (\tilde{V}_2, \tilde{W}_2))\Big),
\end{aligned}$$

which together with Lemma D, (3.22), (3.23) and (3.24) yields (3.34). □

5. Proof of Theorem 1

Proof of Theorem 1. Let ε_1 and ε_7 be positive constants given by Theorem A and Proposition B, respectively. Let K denote various constants independent of t. Assume that $\varepsilon_4 \le \min\{1, \varepsilon_1, \varepsilon_7/(2K_{10})\}$, where

(5.1) $$K_{10} = \max\{\frac{1}{\sqrt{m_0}}, \sqrt{m_1}\}(\ge 1).$$

(Step 1) We give a solution u of (0.1). Let $(\phi_0, \psi_0) \in Y_p(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_4]$. By Lemma 1, we have

(5.2) $$(\tilde{V}, \tilde{W}) := G_0(m(\|A^{1/2}\phi_0\|^2), \phi_0, \psi_0) \in \mathcal{Y}_p^J(2K_{10}\varepsilon) \subset \mathcal{Y}_p^J(\varepsilon_7).$$

Then by Propositions A and 1, the equation (3.8)–(3.10) with $(\tilde{V}, \tilde{W})$ defined by (5.2) has a solution

(5.3) $$(C, (V, W)) \in \mathcal{C}_{p,1} \times C^1(\mathbb{R}; D(A^{J/2}) \times D(A^{J/2})),$$

satisfying

$$\begin{aligned}
&(V, W) \in C(\mathbb{R}, \mathcal{Y}_p(2K_5K_{10}\varepsilon)) \\
&U := A^{-1/2}(V - W) \in C^1(\mathbb{R}, D(A^{(J+1)/2}).
\end{aligned}$$

Furtheremore, the limits

(5.4) $$\begin{aligned}
&C_{\pm\infty} = \lim_{\tau\to\pm\infty} C(\tau) \in \mathbb{R}, \\
&V_{\pm\infty} = \lim_{\tau\to\pm\infty} V(\tau) \quad \text{and} \quad W_{\pm\infty} = \lim_{\tau\to\pm\infty} W(\tau) \quad \text{in } D(A^{J/2}), \\
&U_{\pm\infty} = \lim_{\tau\to\pm\infty} U(\tau) \text{ in } D(A^{(J+1)/2}),
\end{aligned}$$

exist and satisfy (3.18), (3.19), (3.27)–(3.30),

$$(V_{\pm\infty}, W_{\pm\infty}) \in \mathcal{Y}_p^J(2K_5K_{10}\varepsilon), \tag{5.5}$$

and (3.32). Let $t(\tau)$ be the function defined by (3.11). By the fact that $C \in \mathcal{C}_{p,1}$, it holds that

$$\frac{1}{m_1} \leq \frac{dt}{d\tau}(\tau) = \frac{1}{C(\tau)} \leq \frac{1}{m_0}. \tag{5.6}$$

Then, $t(\tau)$ is a strictly increasing function on $\mathbb{R}$. Let $\tau(t)$ be the inverse of $t(\tau)$, and define

$$c(t) = C(\tau(t)). \tag{5.7}$$

Here we note that

$$\tau(t) = \int_0^t c(s)ds, \quad \text{for} \quad t \in \mathbb{R}, \tag{5.8}$$

and the assumption that $C \in \mathcal{C}_p$, (3.13) and (5.6) yield

$$c \in \mathcal{C}_p. \tag{5.9}$$

Since

$$\begin{aligned} &e^{-i\tau(t)A^{1/2}}V(\tau(t)) - e^{i\tau(t)A^{1/2}}W(\tau(t)) \\ &= e^{-i\tau(t)A^{1/2}}(V(\tau(t)) - W(\tau(t))) + (e^{-i\tau(t)A^{1/2}} - e^{i\tau(t)A^{1/2}})W(\tau(t)) \\ &= e^{-i\tau(t)A^{1/2}}A^{1/2}U(\tau(t)) - 2i\sin(\tau(t)A^{1/2})W(\tau(t)) \in R(A^{1/2}), \end{aligned}$$

we can define $u(t)$ by

$$u(t) := \frac{i}{2\sqrt{C(\tau(t))}} e^{-B(t)/2} A^{-1/2}\left(e^{-i\tau(t)A^{1/2}}V(\tau(t)) - e^{i\tau(t)A^{1/2}}W(\tau(t))\right) \tag{5.10}$$

Then we have

$$u(t) = \frac{i}{2\sqrt{C(\tau(t))}} e^{-B(t)/2}\left(e^{-i\tau(t)A^{1/2}}U(\tau(t)) - 2iA^{-1/2}\sin(\tau(t)A^{1/2})W(\tau(t))\right). \tag{5.11}$$

Since $(C, (V, W))$ is the solution of (3.8)–(3.10) with $T = 0$, we see that

$$\begin{aligned} u'(t) &= \frac{\sqrt{C(\tau(t))}}{2} e^{-B(t)/2}\left(e^{-i\tau(t)A^{1/2}}V(\tau(t)) + e^{i\tau(t)A^{1/2}}W(\tau(t))\right), \\ c(t) &= C(\tau(t)) = m(\|A^{1/2}u(t)\|), \end{aligned}$$

and $u(t) \in \cap_{j=0,1,2} C^{2-j}(\mathbb{R} : D(A^{(2-j)/2}))$ is a solution of (0.1). By the uniqueness of the solution which is stated in Theorem A, the solution given by Theorem A equals u given by the formula (5.11).

(Step 2) We express the solution v of $(LE : C_{\pm\infty})$. The formulas (5.3), (5.4) and (5.6) together with Lemma 3 with $Z = \mathbb{R}$ imply (1.11) and (1.12), and we can define

$$a_{\pm\infty} := \int_0^{\pm\infty} (c(t) - C_{\pm\infty})\, dt = \int_0^{\pm\infty} (C(\tau) - C_{\pm\infty}) \frac{1}{C(\tau)} d\tau \in \mathbb{R}, \tag{5.12}$$

with

$$|a_{\pm\infty}| \leq \frac{1}{m_0}\left|\int_0^{\pm\infty} |C(\tau) - C_{\pm\infty}|\, d\tau\right| \tag{5.13}$$
$$\leq \left|\frac{1}{m_0}\int_0^{\pm\infty} |\sigma C'(\sigma)| d\sigma\right| \leq \frac{1}{m_0}|C'|_1 \leq \frac{1}{m_0}.$$

Here we used the assumption that $C \in \mathcal{C}_{1,1}(\mathbb{R})$ in the last inequality above. Using Lemma 3 with $Z = \mathbb{R}$ again, we have

$$|\tau(t) - (C_{-\infty}t + a_{-\infty})| = \left|\int_{-\infty}^{t} (c(s) - C_{-\infty}) ds\right| \leq \int_{-\infty}^{t} |sc'(s)| ds, \tag{5.14}$$

for every $t < 0$. Thus,

$$\langle t\rangle^{p-1}|\tau(t) - (C_{-\infty}t + a_{-\infty})| \leq \int_{-\infty}^{t} \langle\sigma\rangle^{p}|c'(\sigma)| d\sigma$$

for every $t < 0$, which together with (5.9) implies

$$\lim_{t\to-\infty} \langle t\rangle^{p-1}|\tau(t) - (C_{-\infty}t + a_{-\infty})| = 0. \tag{5.15}$$

In the case $p \geq 2$, it follows from (5.14),

$$\int_{-\infty}^{0} \langle t\rangle^{p-2}|\tau(t) - (C_{-\infty}t + a_{-\infty})| dt \leq \int_{-\infty}^{0}\int_{-\infty}^{t} \langle t\rangle^{p-2}|sc'(s)| ds dt \tag{5.16}$$
$$= \int_{-\infty}^{0}\int_{s}^{0} \langle t\rangle^{p-2}|sc'(s)| dt ds \leq \frac{1}{p-1}\int_{-\infty}^{0} \langle s\rangle^{p-1}|sc'(s)| ds.$$

Hence by (5.9), we have

$$\int_{-\infty}^{0} \langle t\rangle^{p-2}|\tau(t) - (C_{-\infty}t + a_{-\infty})| dt < \infty. \tag{5.17}$$

By (3.30) of Proposition 1, we have

$$e^{-i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}V_{\pm\infty} - e^{i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}W_{\pm\infty} \tag{5.18}$$
$$= e^{-i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}A^{1/2}U_{\pm\infty}$$
$$- 2iA^{1/2}A^{-1/2}\sin((C_{\pm\infty}t + a_{\pm\infty})A^{1/2})W_{\pm\infty})$$
$$\in R(A^{1/2}).$$

with $U_{\pm\infty} \in D(A^{1/2})$. Thus, we can define function $v_{\pm}(t)$ by

$$v_{\pm}(t) := \frac{i}{2C_{\pm\infty}^{1/2}} e^{-B_{\pm\infty}/2}A^{-1/2} \tag{5.19}$$
$$\times\left(e^{-i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}V_{\pm\infty} - e^{i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}W_{\pm\infty}\right)$$

Then we have

$$v_{\pm}(t) = \frac{1}{2C_{\pm\infty}^{1/2}} e^{-B_{\pm\infty}/2}\left(ie^{-i(C_{\pm\infty}t+a_{\pm\infty})A^{1/2}}U_{\pm\infty} - 2iA^{-1/2}\sin((C_{\pm\infty}t + a_{\pm\infty})A^{1/2})W_{\pm\infty}\right), \tag{5.20}$$

where $B_{\pm\infty} = \int_0^{\pm\infty} b(s)ds$. Then, we easily see that

$$v_\pm(t) \in \bigcap_{j=0,1,2} C^j(\mathbb{R}; D(A^{(J+1-j)/2})),$$

and satisfies (LE:$C_{\pm\infty}$).

(Step 3) We prove formulas (1.14) and (1.15) for $t \to -\infty$ and (1.17). Formula (1.13) follows from (1.14). The formulas (1.14) and (1.15) for $t \to +\infty$ and (1.18) are proved in the same way. By definitions (5.10) and (5.19),

$$\begin{aligned} A^{1/2}(u(t) - v_-(t)) = {} & \frac{i}{2\sqrt{c(t)}} e^{-B(t)/2} \\ & \times \Big(e^{-i\tau(t)A^{1/2}}(V(\tau(t)) - V_{-\infty}) + (e^{-i\tau(t)A^{1/2}} - e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})V_{-\infty} \\ & \quad - e^{i\tau(t)A^{1/2}}(W(\tau(t)) - W_{-\infty}) - (e^{i\tau(t)A^{1/2}} - e^{i(C_{-\infty}t+a_{-\infty})A^{1/2}})W_{-\infty} \Big) \\ & + \frac{1}{2}\left(\frac{ie^{-B(t)/2}}{\sqrt{c(t)}} - \frac{ie^{-B_{-\infty}/2}}{\sqrt{C_{-\infty}}} \right) \cdot \left(e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}} V_{-\infty} - e^{i(C_{-\infty}t+a_{-\infty})A^{1/2}} W_{-\infty} \right). \end{aligned}$$

Let k be a nonnegative integer such that $k \le J$. Since $c(t) \ge m_0$, we have by the intermediate theorem that

$$\begin{aligned} (5.21) \qquad & \left| \frac{e^{-B(t)/2}}{\sqrt{c(t)}} - \frac{e^{-B_{-\infty}/2}}{\sqrt{C_{-\infty}}} \right| \\ & \le e^{-B(t)/2} \left| \frac{1}{\sqrt{c(t)}} - \frac{1}{\sqrt{C_{-\infty}}} \right| + \frac{1}{\sqrt{C_{-\infty}}} \left| e^{-B(t)/2} - e^{-B_{-\infty}/2} \right| \\ & = \frac{e^{-B(t)/2}|c(t) - C_{-\infty}|}{\sqrt{C_{-\infty}c(t)}(\sqrt{C_{-\infty}} + \sqrt{c(t)})} + \frac{e^{-B(\tilde{t})/2}}{\sqrt{C_{-\infty}}} \left| \int_{-\infty}^t b(s)ds \right| \quad (\exists \tilde{t} \in (-\infty, t)) \\ & \le \frac{e^{\|b\|_{L^1}/2}}{m_0^{1/2}} \left(\int_{-\infty}^t |b(s)|ds + \frac{1}{m_0}|c(t) - C_{-\infty}| \right). \end{aligned}$$

Thus, we have

$$\begin{aligned} (5.22) \qquad & \left\| A^{1/2}(u(t) - v_-(t)) \right\|_k \\ & \le \frac{e^{\|b\|_{L^1}/2}}{2m_0^{1/2}} \Big[\|V(\tau(t)) - V_{-\infty}\|_k + \|W(\tau(t)) - W_{-\infty}\|_k \\ & \qquad + \left\| (e^{-i\tau(t)A^{1/2}} - e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})V_{-\infty} \right\|_k \\ & \qquad + \left\| (e^{i\tau(t)A^{1/2}} - e^{i(C_{-\infty}t+a_{-\infty})A^{1/2}})W_{-\infty} \right\|_k \\ & \qquad + \left(\int_{-\infty}^t |b(s)|ds + \frac{1}{m_0}|C(\tau(t)) - C_{-\infty}| \right) (\|V_{-\infty}\|_k + \|W_{-\infty}\|_k) \Big]. \end{aligned}$$

We shall estimate the term $\left\| (e^{-i\tau(t)A^{1/2}} - e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})V_{-\infty} \right\|_k$ in the inequality above. Let $k \in [0, J-1]$ and let f be an arbitrary element of $D(A^{(k+1)/2})$. Using the equality $(d/ds)e^{-isA^{1/2}}f = -ie^{-isA^{1/2}}A^{1/2}f$ and that the operator $e^{iaA^{1/2}}$ is unitary on $D(A^{k/2})$, we have

$$\begin{aligned} (5.23) \qquad \left\| (e^{-i\tau(t)A^{1/2}} - e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})f \right\|_k & = \left\| \int_{C_{-\infty}t+a_{-\infty}}^{\tau(t)} e^{-isA^{1/2}} A^{1/2} f ds \right\|_k \\ & \le |\tau(t) - (C_{-\infty}t + a_{-\infty})| \left\| A^{1/2} f \right\|_k. \end{aligned}$$

Hence (5.15) implies

$$\lim_{t\to-\infty}\langle t\rangle^{p-1}\left\|(e^{-i\tau(t)A^{1/2}}-e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})f\right\|_k=0. \tag{5.24}$$

Since operators $e^{-i\tau(t)A^{1/2}}$ and $e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}}$ are bounded in $D(A^{k/2})$, and since $D(A^{(k+1)/2})$ is dense in $D(A^{k/2})$, it follows that (5.24) holds also for every $f\in D(A^{k/2})$ in the case $p=1$. Thus, by the fact that $V_{-\infty}\in D(A^{J/2})$, we see that

$$\lim_{t\to-\infty}\left\|(e^{-i\tau(t)A^{1/2}}-e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})V_{-\infty}\right\|_J=0,$$
$$\lim_{t\to-\infty}\langle t\rangle^{p-1}\left\|(e^{-i\tau(t)A^{1/2}}-e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})V_{-\infty}\right\|_{J-1}=0.$$

In the same way, we have

$$\lim_{t\to-\infty}\left\|(e^{i\tau(t)A^{1/2}}-e^{i(C_{-\infty}t+a_{-\infty})A^{1/2}})W_{-\infty}\right\|_J=0,$$
$$\lim_{t\to-\infty}\langle t\rangle^{p-1}\left\|(e^{i\tau(t)A^{1/2}}-e^{i(C_{-\infty}t+a_{-\infty})A^{1/2}})W_{-\infty}\right\|_{J-1}=0.$$

By using these convergences, (3.18), (3.19) together with (3.13) and (1.10), we derive the following estimates from (5.22)

$$\lim_{t\to-\infty}\|A^{1/2}(u(t)-v_-(t))\|_J=0,$$
$$\lim_{t\to-\infty}\langle t\rangle^{p-1}\|A^{1/2}(u(t)-v_-(t))\|_{J-1}=0.$$

In the same way, we can prove that

$$\lim_{t\to-\infty}\|u'(t)-v_-'(t)\|_J=\lim_{t\to-\infty}\langle t\rangle^{p-1}\|u'(t)-v_-'(t)\|_{J-1}=0.$$

Thus, the proof of (1.14) and (1.15) for $t\to-\infty$ is complete if we show that

$$\lim_{t\to-\infty}\langle t\rangle^{p-1}\|u(t)-v_-(t)\|=0. \tag{5.25}$$

From equalities (5.11) and (5.20), it follows that

$$\begin{aligned}
u(t)-v_-(t)=&\frac{i}{2\sqrt{C(\tau(t))}}e^{-B(t)/2}\\
&\times\Big[e^{-i\tau(t)A^{1/2}}(U(\tau(t))-U_{-\infty})+(e^{-i\tau(t)A^{1/2}}-e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}})U_{-\infty}\\
&\quad-2i\sin(\tau(t)A^{1/2})A^{-1/2}(W(\tau(t))-W_{-\infty})\\
&\quad+4iA^{-1/2}\sin\Big((\tau(t)-(C_{-\infty}t+a_{-\infty}))A^{1/2}/2\Big)\\
&\qquad\times\cos\Big((\tau(t)+(C_{-\infty}t+a_{-\infty}))A^{1/2}/2\Big)W_{-\infty}\Big]\\
&+\left(\frac{ie^{-B(t)/2}}{2\sqrt{c(t)}}-\frac{ie^{-B_{-\infty}/2}}{2\sqrt{C_{-\infty}}}\right)\\
&\quad\times\Big(e^{-i(C_{-\infty}t+a_{-\infty})A^{1/2}}U_{-\infty}-2i\sin((C_{-\infty}t+a_{\pm\infty})A^{1/2})A^{-1/2}W_{-\infty}\Big).
\end{aligned}$$

Hence, by (5.21), (5.23), (3.2) and $\left\|\cos(sA^{1/2})\right\|_{\mathcal{L}(H)}\le 1$, we obtain

$$\begin{aligned}
&\|u(t) - v_-(t)\| \\
&\le \frac{e^{\|b\|_{L^1}/2}}{2m_0^{1/2}} \Big[\|U(\tau(t)) - U_{-\infty}\| + |\tau(t) - (C_{-\infty}t + a_{-\infty})| \|A^{1/2}U_{-\infty}\| \\
&\quad + 2|\tau(t)| \|W(\tau(t)) - W_{-\infty}\| + 2\,|(\tau(t) - (C_{-\infty}t + a_{-\infty}))|\, \|W_{-\infty}\| \\
&\quad + \left(\int_{-\infty}^{t} |b(s)| ds + \frac{1}{m_0}|c(t) - C_{-\infty}| \right) \Big(\|U_{-\infty}\| + 2|C_{-\infty}t + a_{\pm\infty}| \|W_{-\infty}\| \Big) \Big].
\end{aligned}$$

Thus, by using (3.13), we have

$$\begin{aligned}
\text{(5.26)}\qquad &\langle t\rangle^k \|u(t) - v_-(t)\| \\
&\le K \Big[\langle \tau(t)\rangle^k \|U(\tau(t)) - U_{-\infty}\| + \langle \tau(t)\rangle^{k+1} \|W(\tau(t)) - W_{\pm\infty}\| \\
&\quad + \langle t\rangle^k |\tau(t) - (C_{-\infty}t + a_{-\infty})| (\|A^{1/2}U_{-\infty}\| + \|W_{-\infty}\|) \\
&\quad + \left(\langle t\rangle^{k+1} \int_{-\infty}^{t} |b(s)| ds + \langle \tau(t)\rangle^{k+1} |C(\tau(t)) - C_{-\infty}| \right) \Big(\|U_{-\infty}\| + \|W_{-\infty}\| \Big) \Big].
\end{aligned}$$

By (1.10), (3.18), (3.19), (3.27) and (5.15), we see that the right-hand side of (5.26) with $k = p - 1$ tends to 0 as $t \to -\infty$. Thus, (5.25) holds, which completes the proof of (1.14) and (1.15).

Assume that $p \ge 2$. By Fubini's Theorem together with (1.10), we have

$$\begin{aligned}
\text{(5.27)}\qquad \int_{-\infty}^{0} \langle t\rangle^{p-1} \int_{-\infty}^{t} |b(s)| ds dt &= \int_{-\infty}^{0} \int_{s}^{0} \langle t\rangle^{p-1} |b(s)| dt ds \\
&\le \int_{-\infty}^{0} |s| \langle s\rangle^{p-1} |b(s)| ds < \infty.
\end{aligned}$$

Inequalities (5.23) and (5.17) imply

$$\int_{-\infty}^{0} \langle t\rangle^{p-2} \left\| (e^{-i\tau(t)A^{1/2}} - e^{-i(C_{-\infty}t + a_{-\infty})A^{1/2}}) V_{-\infty} \right\|_{J-1} dt < \infty.$$

In the same way, we have

$$\int_{-\infty}^{0} \langle t\rangle^{p-2} \left\| (e^{i\tau(t)A^{1/2}} - e^{i(C_{-\infty}t + a_{-\infty})A^{1/2}}) W_{-\infty} \right\|_{J-1} dt < \infty.$$

By these inequalities, (5.27), (3.28) and (3.32) together with (5.6), we derive the following from (5.22);

$$\text{(5.28)}\qquad \int_{-\infty}^{0} \langle t\rangle^{p-2} \|A^{1/2}(u(t) - v_-(t))\|_{J-1} dt < \infty.$$

In the same way, we can prove

$$\text{(5.29)}\qquad \int_{-\infty}^{0} \langle t\rangle^{p-2} \|u'(t) - v_-'(t))\|_{J-1} dt < \infty.$$

By using (3.28), (3.29), (3.32), (5.17) and (5.27), we derive the following from (5.26);

$$\int_{-\infty}^{0} \langle t\rangle^{p-2} \|u(t) - v_-(t)\| dt < \infty,$$

which together with (5.28) and (5.29) implies (1.17).

Uniqueness of the solution $v_\pm$ is clear.

(Step 4) We prove the continuity of the mapping $\Upsilon_\pm$. Formula (5.19) implies

$$\begin{aligned}\Upsilon_\pm(\phi_0,\psi_0) :&= (\phi_\pm,\psi_\pm) = (v_\pm(0), \frac{dv_\pm}{dt}(0))\\ &= \exp(-B_{\pm\infty}/2)\mathcal{G}_0\left(C_{\pm\infty},\left(e^{-ia_{\pm\infty}A^{1/2}}V_{\pm\infty}, e^{ia_{\pm\infty}A^{1/2}}W_{\pm\infty}\right)\right).\end{aligned}$$

By Lemma 2 (i) together with (5.5) and (5.13), we have

$$\left(e^{-ia_{\pm\infty}A^{1/2}}V_{\pm\infty}, e^{ia_{\pm\infty}A^{1/2}}W_{\pm\infty}\right) \in \mathcal{Y}_p^J(K_{11}\varepsilon)$$

with $K_{11} := 2^{p/4+1}\langle\frac{1}{m_0}\rangle^{p/2}K_5K_{10}$. Hence by (i) of Lemma 1, we have

$$(\phi_\pm,\psi_\pm) \in Y_p^J(K_2^\pm\varepsilon) \text{ with } K_2^\pm = \exp(-B_{\pm\infty}/2)K_{10}K_{11},$$

and therefore we obtain $\Upsilon_\pm(Y_p^J(\varepsilon)) \subset Y_p^J(K_2^\pm\varepsilon)$.

We show the continuity of the mapping $\Upsilon_\pm$. By the definition of $(\tilde{V},\tilde{W})$ in (5.2), Lemma 1 (ii) together with the continuity of m, we easily see that the mapping

$$(\phi_0,\psi_0) \mapsto (\tilde{V},\tilde{W})$$

is continuous from $Y_p^J(\varepsilon)$ to $\mathcal{Y}_p^J(2K_{10}\varepsilon) \subset \mathcal{Y}_p^J(\varepsilon_7)$. By Propositions 1 and 2,

$$(\tilde{V},\tilde{W}) \mapsto (C(\cdot),(V_{\pm\infty},W_{\pm\infty}))$$

is continuous from $\mathcal{Y}_p^J(\varepsilon_7)$ to $\mathcal{C}_{p,K_5\varepsilon_7} \times \mathcal{Y}_p^J(K_5\varepsilon_7)$. Thus, if we prove the continuity of the mapping

$$\Phi : C(\cdot) \mapsto a_{\pm\infty}$$

from $\mathcal{C}_{p,K_5\varepsilon_7}$ to $\mathbb{R}$, Lemma 2 (ii) guarantees the continuity of the mapping

$$(C(\cdot),(V_{\pm\infty},W_{\pm\infty})) \mapsto \left(C_{\pm\infty},\left(e^{-ia_{\pm\infty}A^{1/2}}V_{\pm\infty}, e^{ia_{\pm\infty}A^{1/2}}W_{\pm\infty}\right)\right)$$

from $\mathcal{C}_{p,K_5\varepsilon_7} \times \mathcal{Y}_p^J(K_5\varepsilon_7)$ to $[m_0,m_1] \times \mathcal{Y}_p^J(K_{11}\varepsilon_7)$. Then Lemma 1 (ii) guarantees the continuity of the mapping

$$\left(C_{\pm\infty},\left(e^{-ia_{\pm\infty}A^{1/2}}V_{\pm\infty}, e^{ia_{\pm\infty}A^{1/2}}W_{\pm\infty}\right)\right) \mapsto (\phi_\pm,\psi_\pm)$$

from $[m_0,m_1] \times \mathcal{Y}_p^J(K_{11}\varepsilon_7)$ to $\mathcal{Y}_p^J(K_2^\pm\varepsilon_7)$, and therefore the continuity of $\Upsilon_\pm$ follows.

Now we prove the continuity of the mapping Φ. Let $C_j \in \mathcal{C}_{p,K_5\varepsilon_7}$, and let $C_{j,\infty}$ and a_j be the numbers defined by (5.4) and (5.12) with $C = C_j$ ($j = 1,2$). Then $a_{j,\pm\infty}$ is expressed as

$$\begin{aligned}a_{j,\pm\infty} &= \int_0^{\pm\infty}\left(1-\frac{C_{j,\pm\infty}}{C_j(\tau)}\right)d\tau\\ &= -\int_0^{\pm\infty} C_{j,\pm\infty}\left(\int_\tau^{\pm\infty}\left(\frac{1}{C_j(\sigma)}\right)' d\sigma\right)d\tau\\ &= -\int_0^{\pm\infty}\int_0^{\sigma}\frac{C_{j,\pm\infty}C_j'(\sigma)}{C_j(\sigma)^2}d\tau d\sigma\\ &= -\int_0^{\pm\infty}\sigma C_{j,\pm\infty}\frac{C_j'(\sigma)}{C_j(\sigma)^2}d\sigma,\end{aligned} \tag{5.30}$$

where we used Fubini's Theorem in the third step. Thus, we have

$$a_{1,\pm\infty} - a_{2,\pm\infty} = -\int_0^{\pm\infty} \sigma\left(C_{1,\pm\infty}\frac{C_1'(\sigma)}{C_1(\sigma)^2} - C_{2,\pm\infty}\frac{C_2'(\sigma)}{C_2(\sigma)^2}\right)d\sigma.$$

Since

$$|C_{1,-\infty}\frac{C_1'(\sigma)}{C_1(\sigma)^2} - C_{2,-\infty}\frac{C_2'(\sigma)}{C_2(\sigma)^2}| \le |\frac{C_1'(\sigma)}{C_1(\sigma)^2}||C_{1,-\infty} - C_{2,-\infty}|$$
$$+ \frac{C_{2,-\infty}}{C_1(\sigma)^2}|C_1'(\sigma) - C_2'(\sigma)| + \frac{C_{2,-\infty}|C_2'(\sigma)|}{C_1(\sigma)^2C_2(\sigma)^2}(C_1(\sigma) + C_2(\sigma))|C_1(\sigma) - C_2(\sigma)|,$$

and $C_j \in \mathcal{C}_{1,1}$, we have

$$\begin{aligned}|a_{1,-\infty} - a_{2,-\infty}| \le & \frac{|C_1'|_1}{m_0^2}\|C_1 - C_2\|_{L^\infty} \\ & + \frac{m_1}{m_0^2}|C_1' - C_2'|_1 + \frac{2m_1^2}{m_0^4}\|C_1 - C_2\|_{L^\infty}|C_2'|_1 \\ \le & \left(\frac{1+m_1}{m_0^2} + \frac{2m_1^2}{m_0^4}\right)\|C_1 - C_2\|_{\mathcal{C}_p}.\end{aligned} \tag{5.31}$$

In the same way, we see that $|a_{1,+\infty} - a_{2,+\infty}|$ is dominated by the right-hand side of (5.31). Hence we have proved the continuity of Φ, and thus we conclude that the continuity of the mapping $\Upsilon_\pm$ holds. □

6. Proof of Theorems 2 and 3

Proof of Theorem 2. Let K_5, ε_7 and K_7 be positive constants given by Propositions A, B and 2, respectively, and K_{10} be the number defined by (5.1). Put

$$K_{12}^\pm = 2e^{B_{\pm\infty}/2}K_{10},\ K_{13} = 2^{p/4}\left\langle\frac{m_1}{m_0^2}\right\rangle^{p/2},\ K_3^\pm = K_5K_{10}K_{12}^\pm K_{13}, \tag{6.1}$$

where $B_{\pm\infty} = \int_0^{\pm\infty} b(s)ds$. Let ε_5 be a positive constant satisfying

$$K_{10}\varepsilon_5 < \min\{\sqrt{Lm_0}, \frac{m_0}{\sqrt{m_2}}\} \quad \text{and} \quad \max\{K_{12}^+, K_{12}^-\}K_{13}\varepsilon_5 \le \varepsilon_7. \tag{6.2}$$

Assume that $\varepsilon \le \varepsilon_5$, and let $(\phi,\psi) \in \tilde{Y}_p^J(\varepsilon)$. We prove the statements of Theorem 2 in the case "−". The case "+" is proved in the same way.

(Step 1) In Proof (Step 1) of [15, Theorem 3] (see also Proof of [11, Theorem 1.3]), we proved the existence of c_∞ satisfying (1.9) and the Lipschitz continuity of the mapping $(\phi,\psi) \mapsto c_\infty$ from $D(A^{1/4}) \times H \to \mathbb{R}$. Especially c_∞ is determined uniquely, and satisfies

$$m_0 \le c_\infty \le m_1. \tag{6.3}$$

(Step 2) We express the solution of $(LE : c_\infty)$. Put

$$(V_\infty, W_\infty) := G_0(c_\infty, (\phi,\psi)), \quad U_\infty := -2ic_\infty^{1/2}\phi = A^{-1/2}(V_\infty - W_\infty). \tag{6.4}$$

Then by (6.3) and Lemma 1, we have

$$(V_\infty, W_\infty) \in \mathcal{Y}_p^J(2K_{10}\varepsilon). \tag{6.5}$$

Since $V_\infty - W_\infty = A^{1/2}U_\infty \in R(A^{1/2})$, we can define

$$(6.6)\qquad v(t) := \frac{i}{2c_\infty^{1/2}} A^{-1/2}\left(e^{-ic_\infty tA^{1/2}}V_\infty - e^{ic_\infty tA^{1/2}}W_\infty\right), \\ = \frac{i}{2c_\infty^{1/2}}\left(e^{-ic_\infty tA^{1/2}}U_\infty - 2iA^{-1/2}\sin(c_\infty tA^{1/2})W_\infty\right).$$

Then we easily see that $v(t)$ is the unique solution of $(LE : c_\infty)$ with initial value

$$(v(0), v'(0)) = (\phi, \psi).$$

(Step 3) We show the existence of a solution $(u_-, \partial u_-/\partial t)$ of (0.1) satisfying (1.22) and (1.23). Convergence (1.19) follows from (1.22). In view of (6.1) and (6.2) and (6.5), Propositions A and 1 with $T = -\infty$ imply that there uniquely exists a solution

$$(C(\tau), (V(\tau), W(\tau))) \in \mathcal{C}_{p,1} \times (C^1(\mathbb{R}; D(A^{J/2}) \times D(A^{J/2})) \cap C(\mathbb{R}; \mathcal{Y}_p^J))$$

of (3.8), (3.9) and

$$(6.7)\qquad (V_{-\infty}, W_{-\infty}) = \lim_{\tau\to-\infty}(V(\tau), W(\tau)) = e^{B_{-\infty}/2}(V_\infty, W_\infty)(\in \tilde{\mathcal{Y}}_p^J(K_{12}^-\varepsilon) \subset \tilde{\mathcal{Y}}_p^J(\varepsilon_7)),$$

where $\tilde{\mathcal{Y}}_p^J = \mathcal{Y}_p^J \cap \tilde{\mathcal{X}}$ and the limit is taken in $D(A^{J/2}) \times D(A^{J/2})$, and the solution satisfies the following:

$$(6.8)\qquad (V(\tau), W(\tau)) \in \mathcal{Y}_p^J(K_5K_{12}^-\varepsilon) \quad \text{for every } \tau \in [-\infty, \infty],$$

and the limit $C_{-\infty} = \lim_{\tau\to-\infty} C(\tau)(\geq m_0)$ exists and satisfies (3.21). Then in the same way as in the proof of [15, Theorem 3], we can prove

$$(6.9)\qquad c_\infty = C_{-\infty} = \lim_{\tau\to-\infty} C(\tau),$$

by observing that both c_∞ and $C_{-\infty}$ are unique solution of

$$\lambda = m\left(\frac{\|V_\infty\|^2 + \|W_\infty\|^2}{4\lambda}\right).$$

Let $t(\tau)$ and $c(t)$ be the functions defined by (3.11) and (5.7). In the same way as in the definition of $a_{\pm\infty}$ (see (5.12) and (5.30)), together with using (6.9), we can define

$$(6.10)\qquad T_0 = T_0(C) := -\frac{1}{c_\infty}\int_{-\infty}^0 (c(s) - c_\infty)\,ds \\ = \int_{-\infty}^0 \left(\frac{1}{C(\tau)} - \frac{1}{c_\infty}\right)d\tau = \int_{-\infty}^0 \frac{\tau C'(\tau)}{C(\tau)^2}d\tau.$$

From the last equality, (5.8) and the fact that $C \in \mathcal{C}_{p,1}$, it follows that

$$(6.11)\qquad |T_0| \leq \frac{1}{m_0^2}\int_{-\infty}^0 |\tau C'(\tau)|d\tau \leq \frac{1}{m_0^2}|C'|_1 \leq \frac{1}{m_0^2}, \\ |\tau(-T_0)| = \left|\int_{-T_0}^0 c(s)ds\right| \leq m_1|T_0| \leq \frac{m_1}{m_0^2}.$$

Equalities (5.8) and (6.10) yield

$$\tau(t-T_0)-c_\infty t=(\tau(t-T_0)-c_\infty(t-T_0))-c_\infty T_0=\int_{-\infty}^{t-T_0}(c(s)-c_\infty)ds.$$

Thus in the same way as in the proof of (5.15) and (5.17), we obtain

$$\lim_{t\to-\infty}\langle t\rangle^{p-1}|\tau(t-T_0)-c_\infty t|=0, \tag{6.12}$$

$$\int_{-\infty}^{0}\langle t\rangle^{p-2}|\tau(t-T_0)-c_\infty t|dt<\infty \quad \text{if } p\geq 2. \tag{6.13}$$

Let u^* be a function defined by (5.10), that is,

$$\begin{aligned} u^*(t):&=\frac{i}{2\sqrt{c(t)}}e^{-B(t)/2}A^{-1/2}\left(e^{-i\tau(t)A^{1/2}}V(\tau(t))-e^{i\tau(t)A^{1/2}}W(\tau(t))\right)\\ &=\frac{i}{2\sqrt{C(\tau(t))}}e^{-B(t)/2}\Big(e^{-i\tau(t)A^{1/2}}U(\tau(t))\\ &\qquad -2iA^{-1/2}\sin(\tau(t)A^{1/2})W(\tau(t))\Big), \end{aligned} \tag{6.14}$$

which is a solution of (0.1) satisfying $u^*\in\bigcap_{j=0}^{1}C^j(\mathbb{R};D(A^{(J+1-j)/2}))$. Put

$$u_-(t)=u^*(t-T_0). \tag{6.15}$$

Then it is easy to see that u_- is a solution of (0.1).

Since $C(\tau(t))=m(\|A^{1/2}u^*(t)\|^2)$, the equality (6.9) implies

$$\lim_{t\to-\infty}m(\|A^{1/2}u(t)\|^2)=\lim_{t\to-\infty}m(\|A^{1/2}u^*(t)\|^2)=c_\infty,$$

that is (1.21) for $t\to-\infty$ holds. The fact $C\in\mathcal{C}_p$ implies (1.20).

By definition (6.15) and expressions (6.6) and (6.14), we have

$$\begin{aligned} u_-(t)-v(t)&=\frac{i}{2\sqrt{C(\tau(t-T_0))}}e^{-B(t-T_0)/2}\\ &\quad\times\Big[e^{-i\tau(t-T_0)A^{1/2}}(U(\tau(t-T_0))-e^{B_\infty/2}U_\infty)\\ &\quad+(e^{-i\tau(t-T_0)A^{1/2}}-e^{-iC_{-\infty}tA^{1/2}})e^{B_\infty/2}U_\infty\\ &\quad-2iA^{-1/2}\sin(\tau(t-T_0)A^{1/2})(W(\tau(t))-e^{B_\infty/2}W_\infty)\\ &\quad+4iA^{-1/2}\sin\left((\tau(t-T_0)-C_{-\infty}t)A^{1/2}/2\right)\\ &\qquad\times\cos\left((\tau(t-T_0)+C_{-\infty}t)A^{1/2}/2\right)e^{B_\infty/2}W_\infty\Big]\\ &\quad+\left(\frac{ie^{-B(t-T_0)/2}}{2\sqrt{c(t-T_0)}}-\frac{ie^{-B_{-\infty}/2}}{2\sqrt{C_{-\infty}}}\right)e^{B_\infty/2}\\ &\quad\times\left(e^{-iC_{-\infty}tA^{1/2}}U_\infty-2iA^{-1/2}\sin(C_{-\infty}tA^{1/2})W_\infty\right), \end{aligned}$$

Then using (6.12) and (6.13), we can prove (1.22)–(1.25) for $t\to-\infty$, in the same way as in the proof of Theorem 1.

(Step 4) We prove the uniqueness of the solution satisfying (1.19) and (1.20) in the case "- ". Let u_- be the solution constructed above and put $u_1=u_-$. Let $c(t)$, $\tau(t)$, $C(\tau)$, $V(\tau)$ and $W(\tau)$ be the functions defined in (Step 3). Let $u_2\in C^1(\mathbb{R};\mathcal{H}_{1/2})$ be an arbitrary solution of (0.1) satisfying

$$(\mathcal{A}^{1/2}u_2, u_2') \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; D(A^{(1-j)/2}) \times D(A^{(1-j)/2})),$$

(1.19) for $t \to -\infty$ and (1.20). We show that $u_1 = u_2$. Let $c_j(t), \tau_j(t), t_j(\tau), C_j(\tau), V_j(\tau), W_j(\tau)$ be the functions defined by (3.3), (3.4) and (3.5) for $u = u_j$ $(j = 1, 2)$. Then $C_j(\tau) = m(\|\mathcal{A}^{1/2}u_j(t_j(\tau))\|^2) \in C^1(\mathbb{R})$, $(V_j, W_j) \in C^1(\mathbb{R}; D(A^{1/2}) \times D(A^{1/2}))$, and $(C_j, (V_j, W_j))$ satisfies (3.8)–(3.9) and $C_j'(\tau) \in L^1(\mathbb{R})$. Since (V, W) satisfies (3.5) with $u = u^*$ and

$$\begin{aligned}(6.16)\qquad C(\tau(t - T_0)) &= c(t - T_0) = m(\|A^{1/2}u^*(t - T_0)\|^2)\\ &= m(\|A^{1/2}u_1(t)\|^2) = c_1(t) = C_1(\tau_1(t)),\\ \tau(t - T_0) - \tau(-T_0) &= \int_{-T_0}^{t-T_0} m(\|A^{1/2}u^*(s)\|^2)ds\\ &= \int_0^t m(\|A^{1/2}u_1(s)\|^2)ds = \tau_1(t),\end{aligned}$$

we see that

$$\begin{aligned}(6.17)\qquad & V(\tau(t - T_0))\\ &= c(t - T_0)^{-1/2}e^{i\tau(t-T_0)A^{1/2}+B(t-T_0)/2}\\ &\qquad \times \left(u^{*\prime}(t - T_0) - ic(t - T_0)A^{1/2}u^*(t - T_0)\right)\\ &= c_1(t)^{-1/2}e^{i\tau_1(t)A^{1/2}+B(t)/2}e^{i\tau(-T_0)A^{1/2}+\int_t^{t-T_0} b(s)ds/2}\\ &\qquad \times \left(u_1'(t) - ic_1(t)A^{1/2}u_1(t)\right)\\ &= e^{-\int_{t-T_0}^t b(s)ds/2}e^{i\tau(-T_0)A^{1/2}}V_1(\tau_1(t)).\end{aligned}$$

In the same way, we have

$$(6.18)\qquad W(\tau(t - T_0)) = e^{-\int_{t-T_0}^t b(s)ds/2}e^{-i\tau(-T_0)A^{1/2}}W_1(\tau_1(t)).$$

By (6.7) and the assumption $b \in L^1(\mathbb{R})$, letting $t \to -\infty$ in the equations (6.17) and (6.18), we see that the limits

$$(6.19)\qquad (V_{1,-\infty}, W_{1,-\infty}) := \lim_{\tau\to-\infty}(V_1(\tau), W_1(\tau))$$

exists in $D(A^{J/2}) \times D(A^{J/2})$ with

$$(6.20)\qquad (V_{1,-\infty}, W_{1,-\infty}) = (e^{-i\tau(-T_0)A^{1/2}}V_{-\infty}, e^{i\tau(-T_0)A^{1/2}}W_{-\infty}).$$

Hence, by Lemma 2 (i) together with (6.7) and (6.11), we have

$$(6.21)\qquad (V_{1,-\infty}, W_{1,-\infty}) \in \mathcal{Y}_p^J(K_{12}^-K_{13}\varepsilon),$$

where K_{12}^- and K_{13} are constants defined by (6.1). Equality (1.19) with $u_- = u_j$ $(j = 1, 2)$ for the same v implies

$$(6.22)\qquad \lim_{t\to-\infty}\left(\|A^{1/2}(u_1(t) - u_2(t))\| + \|u_1'(t) - u_2'(t)\|\right) = 0.$$

Since $\lim_{\tau\to-\infty} C(\tau)$ exists, (6.16) implies that $\lim_{\tau\to-\infty} C_1(\tau)$ exists. Hence by (6.22), (6.23) $$\lim_{\tau\to-\infty} C_1(\tau) = \lim_{\tau\to-\infty} m(\|A^{1/2}u_1(t_1(\tau))\|^2) = \lim_{t\to-\infty} m(\|A^{1/2}u_1(t)\|^2) = \lim_{t\to-\infty} m(\|A^{1/2}u_2(t)\|^2) = \lim_{\tau\to-\infty} m(\|A^{1/2}u_2(t_2(\tau))\|^2) = \lim_{\tau\to-\infty} C_2(\tau).$$

By the equality $\lim_{\tau\to-\infty} B(t_1(\tau)) = B_{-\infty} = \lim_{\tau\to-\infty} B(t_2(\tau))$, (6.19), (6.21)– (6.23), we see that the limit $\lim_{\tau\to-\infty}(V_2(\tau), W_2(\tau)) := (V_{2,-\infty}, W_{2,-\infty})$ exists in $D(A^{J/2}) \times D(A^{J/2})$ with

$$(V_{2,-\infty}, W_{2,-\infty}) = (V_{1,-\infty}, W_{1,-\infty}) \in \mathcal{Y}_p^J(K_{12}^- K_{13}\varepsilon) \subset \mathcal{Y}_p^J(\varepsilon_7).$$

Hence, $(C_1(\tau), (V_1(\tau), W_1(\tau)))$ and $(C_2(\tau), (V_2(\tau), W_2(\tau)))$ are solutions of (3.8)–(3.10) with the same data at $T = -\infty$. Hence by Proposition 2, we have

$$(C_1(\tau), (V_1(\tau), W_1(\tau))) = (C_2(\tau), (V_2(\tau), W_2(\tau)))$$

for every $\tau \in \mathbb{R}$, and therefore $u_1(t) = u_2(t)$ for every $t \in \mathbb{R}$.

(Step 5) We prove the continuity of the wave operator. By the construction above,

$$W_-(\phi, \psi) = (\phi_0, \psi_0) = \left(u^*(-T_0), \frac{du^*}{dt}(-T_0)\right) = \mathcal{G}_0\left(C(\tau(-T_0)), (e^{-i\tau(-T_0)A^{1/2}} V(\tau(-T_0)), e^{i\tau(-T_0)A^{1/2}} W(\tau(-T_0))\right),$$

where u^*, τ, T_0, V, W are functions and numbers defined in (Step 3). By the procedure above together with (6.8), (6.11) and Lemmas 1 (i) and 2 (i), we see that the operator W_- is decomposed into the following four operators:

$$W_-(\phi, \psi) = \mathcal{G}_0 \circ \Phi_3 \circ \Phi_2 \circ \Phi_1(\phi, \psi),$$

with

$$\Phi_1 : (\phi, \psi) \mapsto (c_\infty, (V_{-\infty}, W_{-\infty})) := (c_\infty, e^{B_{-\infty}/2} G_0(c_\infty, (\phi, \psi)))$$
$$\tilde{Y}_p^J(\varepsilon) \to [m_0, m_1] \times \tilde{\mathcal{Y}}_p^J(K_{12}^-\varepsilon),$$
$$\Phi_2 : (c_\infty, (V_{-\infty}, W_{-\infty})) \mapsto (C, (V, W))$$
$$[m_0, m_1] \times \tilde{\mathcal{Y}}_p^J(K_{12}^-\varepsilon) \to C_{p,K_5K_{12}^-\varepsilon} \times BC(\mathbb{R} : \mathcal{Y}_p^J(K_5 K_{12}^-\varepsilon)),$$
$$\Phi_3 : (C, (V, W))$$
$$\mapsto (C(\tau(-T_0)), (e^{-i\tau(-T_0)A^{1/2}} V(\tau(-T_0)), e^{i\tau(-T_0)A^{1/2}} W(\tau(-T_0))))$$
$$C_{p,K_5K_{12}^-\varepsilon} \times BC(\mathbb{R} : \mathcal{Y}_p^J(K_5 K_{12}^-\varepsilon)) \to [m_0, m_1] \times \mathcal{Y}_p^J(K_5 K_{12}^- K_{13}\varepsilon),$$
$$\mathcal{G}_0 : [m_0, m_1] \times \mathcal{Y}_p^J(K_5 K_{12}^- K_{13}\varepsilon) \to Y_p^J(K_3^-\varepsilon).$$

Here $BC(\mathbb{R} : \mathcal{Y}_p^J(K_5 K_{12}^-\varepsilon))$ denotes a metric space of all $\mathcal{Y}_p^J(K_5 K_{12}^-\varepsilon)$ valued bounded continuous functions, where the distance of two element (V_j, W_j) for $j = 1, 2$ is defined by

$$\sup_{\tau\in\mathbb{R}} d_{\mathcal{Y}_p^J}((V_1(\tau), W_1(\tau)), (V_2(\tau), W_2(\tau))).$$

The function τ is the inverse of $t(\tau)$ which is defined by (3.11), and T_0 is the number defined by (6.10), and both depend on C. We write $\tau = \tau_C$ and $T_0 = T_0(C)$, if we need to represent the dependence on C.

By Lemma 1 (ii) together with the continuity of the mapping $(\phi, \psi) \mapsto c_\infty$ which is shown in (Step 1), we easily see that the mapping Φ_1 is continuous.

Propositions 2 implies the continuity of the mapping Φ_2.

We prove the continuity of the mapping Φ_3. For this purpose, we first show that

$$\Psi_0 : C \mapsto \tau(-T_0) = \tau_C(-T_0(C)) := \Psi_0(C)$$

is continuous from $\mathcal{C}_p$ to $\mathbb{R}$. Let $C_j \in \mathcal{C}_p$ and put $T_j := -T_0(C_j)$, $\tau_j = \tau_{C_j}$ for $j = 1, 2$. By the definition of T_0 together with the fact $c_\infty = \lim_{\tau\to-\infty} C(\tau)$, we obtain the following, in the same way as in the proof of (5.31):

$$|T_1 - T_2| \leq K\|C_1 - C_2\|_{\mathcal{C}_p}. \tag{6.24}$$

By (3.11), we have

$$\begin{aligned} T_1 - T_2 = t_1(\tau_1(T_1)) - t_2(\tau_2(T_2)) &= \int_0^{\tau_1(T_1)} \frac{1}{C_1(\sigma)} d\sigma - \int_0^{\tau_2(T_2)} \frac{1}{C_2(\sigma)} d\sigma \\ &= \int_{\tau_2(T_2)}^{\tau_1(T_1)} \frac{1}{C_1(\sigma)} d\sigma - \int_0^{\tau_2(T_2)} \frac{C_1(\sigma) - C_2(\sigma)}{C_1(\sigma)C_2(\sigma)} d\sigma. \end{aligned}$$

Since $m_0 \leq C_j(\sigma) \leq m_1$ for every $\sigma \in \mathbb{R}$, the above formula yields

$$\begin{aligned} \frac{1}{m_1}|\tau_1(T_1) - \tau_2(T_2)| &\leq \left| \int_{\tau_2(T_2)}^{\tau_1(T_1)} \frac{1}{C_1(\sigma)} d\sigma \right| \\ &\leq |T_1 - T_2| + \frac{1}{m_0^2} \int_{\mathbb{R}} |(C_1 - C_2)(\sigma)| d\sigma. \end{aligned} \tag{6.25}$$

This inequality together with (6.24) yields

$$|\Psi_0(C_1) - \Psi_0(C_2)| = |\tau_1(T_1) - \tau_2(T_2)| \leq K\|C_1 - C_2\|_{\mathcal{C}_p}, \tag{6.26}$$

which implies the continuity of Ψ_0. Thus

$$\begin{aligned} &|C_1(\Psi_0(C_1)) - C_2(\Psi_0(C_2))| \\ &\leq |C_1(\Psi_0(C_1)) - C_1(\Psi_0(C_2))| + |C_1(\Psi_0(C_2)) - C_2(\Psi_0(C_2))| \\ &\leq \left\|C_1'\right\|_{L^\infty} |\Psi_0(C_1) - \Psi_0(C_2)| + \|C_1 - C_2\|_{L^\infty} \leq K\|C_1 - C_2\|_{\mathcal{C}_p}, \end{aligned} \tag{6.27}$$

which implies the continuity of the mapping

$$\Psi_1 : C \mapsto C(\tau(-T_0)) = C(\Psi_0(C))$$

from $\mathcal{C}_p \to \mathbb{R}$. Next we prove that the mapping

$$\begin{aligned} \Psi_2 :&(C, (V, W)) \mapsto (V(\Psi_0(C)), W(\Psi_0(C))) \\ &\mathcal{C}_{p,K_5K_{12}^-\varepsilon} \times C(\mathbb{R} : \mathcal{Y}_p^J(K_5K_{12}^-\varepsilon)) \to \mathcal{Y}_p^J \end{aligned}$$

is continuous at every point $(C_1, (V_1, W_1)) \in \mathcal{C}_{p,K_5K_{12}^-\varepsilon} \times C(\mathbb{R} : \mathcal{Y}_p^J(K_5K_{12}^-\varepsilon))$. We have

$$\begin{aligned} &\tilde{d}^J_{\mathcal{Y}_p}((V_1(\Psi_0(C_1)), W_1(\Psi_0(C_1))), (V(\Psi_0(C)), W(\Psi_0(C)))) \\ &\leq \tilde{d}^J_{\mathcal{Y}_p}((V_1(\Psi_0(C_1)), W_1(\Psi_0(C_1))), (V_1(\Psi_0(C)), W_1(\Psi_0(C)))) \\ &\quad + \tilde{d}^J_{\mathcal{Y}_p}((V_1(\Psi_0(C)), W_1(\Psi_0(C))), (V(\Psi_0(C)), W(\Psi_0(C)))) \end{aligned} \tag{6.28}$$

for $(C, (V, W)) \in \mathcal{C}_{p,K_5K_{12}^-\varepsilon} \times C(\mathbb{R} : \mathcal{Y}_p^J(K_5K_{12}^-\varepsilon))$. Assume that

$$\|C - C_1\|_{\mathcal{C}_p} + \sup_{\tau\in\mathbb{R}} d^J_{\mathcal{Y}_p}((V(\tau), W(\tau)), (V_1(\tau), W_1(\tau))) \to 0.$$

Then (6.26) and the assumption $(V_1, W_1) \in C(\mathbb{R}, \mathcal{Y}_p^J)$ imply that the first-term of the right-hand side of (6.28) tends to 0. It is clear that the second term of the right-hand side of (6.28) tends to 0. Hence the continuity Ψ_2 holds. Combining the continuities of Ψ_0, Ψ_2 and Φ_0 (see Lemma 2 (ii)) together, we obtain that the mapping

$$\begin{aligned}\Psi_3 : (C, (V, W)) \mapsto & (e^{-i\Psi_0(C)A^{1/2}} V(\Psi_0(C)), e^{i\Psi_0(C)A^{1/2}} W(\Psi_0(C))) \\ &= \Phi_0(-\Psi_0(C), \Psi_2(C, (V, W)))\end{aligned}$$

is continuous at the point $(C_1, (V_1, W_1))$ from $\mathcal{C}_{p,K_5K_{12}^-\varepsilon} \times C(\mathbb{R}; \mathcal{Y}_p^J(K_5K_{12}^-\varepsilon))$ to $\mathcal{X}_p^J$. Continuities of Ψ_1 and Ψ_3 imply the continuity of Φ_3.

By definition and Proposition A, we see that $W_-(\phi, \psi) \in \tilde{Y}_p^J(K_3^-\varepsilon)$. The operator $\mathcal{G}_0$ is continuous by Lemma 1 (ii), which completes the proof of the continuity of the wave operator W_-.

(Step 6) Assume that $0 < \varepsilon \le \min\{\varepsilon_4/K_3^\pm, \varepsilon_5/K_2^\pm\}$. We check the last assertion in the case " $-$ ". Let $(\phi_0, \psi_0) \in \tilde{Y}_p(\varepsilon)$. Then by Theorem 1, there exist a unique solution $u(t)$ of (0.1)–(0.2) with limit $\lim_{t\to-\infty} m(\|A^{1/2}u(t)\|^2) =: C_{-\infty}$, and a unique solution v_- of (LE:$C_{-\infty}$) and (1.13). Furthermore,

$$\begin{aligned}\Upsilon_- : (u(0), u'(0)) = (\phi_0, \psi_0) \mapsto (\phi_-, \psi_-) &= (v_-(0), v_-'(0)) \\ &\in \tilde{Y}_p^J(K_2^-\varepsilon) \subset \tilde{Y}_p^J(\varepsilon_5).\end{aligned}$$

Since u and v_- satisfy (1.19) for $-$ with $u_- = u$ and $v = v_-$, the uniqueness of the solution of (0.1) and (1.19), which was proved in (Step 4), implies

$$W_-(v_-(0), v_-'(0)) = (u(0), u'(0)).$$

Hence,

$$W_-\Upsilon_-(\phi_0, \psi_0) = W_-(v_-(0), v_-'(0)) = (u(0), u'(0)) = (\phi_0, \psi_0).$$

This means that $W_-\Upsilon_-$ is the identity mapping on $\tilde{Y}_p^J(\varepsilon)$. In the same way, we can check that Υ_-W_- is the identity mapping on $\tilde{Y}_p^J(\varepsilon)$, and thus the assertion follows. We can also check the case " + " in the same way. □

Proof of Theorem 3. Assume that $0 < \varepsilon \le \min\{\varepsilon_4/K_3^-, \varepsilon_5\}$. Then by Theorems 1 and 2, the operator $S = W_+^{-1}W_-$ is a homeomorphism from $\tilde{Y}_p^J(\varepsilon)$ to $S(\tilde{Y}_p^J(\varepsilon))(\subset \tilde{Y}_p^J(K_2^+K_3^-\varepsilon))$. □

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