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INVARIANCE OF AN ENDO-CLASS UNDER THE ESSENTIALLY TAME JACQUET-LANGLANDS CORRESPONDENCE

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Abstract

Let F be a non-Archimedean local field with a finite residue field. We prove that the conjecture, presented by Broussous, Sécherre, and Stevens, is verified in the essetially tame case, that is, that the Jacquet-Langlands correspondence, which was explicitly described by Bushnell and Henniart, preserves an endo-class for irreducible essentially tame representations of inner forms of $\mathrm{GL}_n(F)$, $n \geq 1$, of parametric degree n. Moreover we give explicitly a parameter set for such representations of an inner form G of $\mathrm{GL}_n(F)$ which contain simple characters belonging to an endo-class.

Introduction

Let F be a non-Archimedean local field with a finite residue field k_F of characteristic p. For a positive integer n, let A be a simple central F-algebra of dimension n^2, and let $G = A^\times$ be the multiplicative group of A.

In [10], [18], and [2], it was proved that there exists a canonical bijection, referred to as the *Jacquet-Langlands correspondence*, between the sets of equivalence classes of irreducible essentially square-integrable representations of $\mathrm{GL}_n(F)$ and $G = A^\times$. We will denote this correspondence by $\boldsymbol{j}$.

In a series of papers by Bushnell and Henniart [13], [5], [6], and Silberger and Zink [22], [23], an explicit description of the Jacquet-Langlands correspondence was given by using the structure theory of types, which was begun by Bushnell and Kutzko [9] and developed by Grabitz, Silberger, and Zink [12] and by Sécherre and Stevens [19], [20], [21].

The notion of an *endo-equivalence class* (in short, *endo-class*) over F was introduced by Bushnell and Henniart [4] for $\mathrm{GL}_n(F)$, and it was generalized to the inner form $A^\times$ by Broussous, Sécherre, and Stevens [3]. In the latter article, associated with any essentially square-integrable representation of $G = A^\times$, an endo-class over F was defined, and it was conjectured that this endo-class is invariant under the Jacquet-Langlands correspondence. We will refer to this as the BSS conjecture. Moreover, it was noted that the BSS conjecture is verified for the Jacquet-Langlands correspondence $\boldsymbol{j}$, described by [22], [23], for irreducible essentially square-integrable representations of G which have level zero.

Recently, Imai and Tsushima [15] showed that the BSS conjeture is also verified for simple epipelagic representations of G which have positive level.

In [8], the notion of *essentially tame* for irreducible cuspidal representations of G was

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defined, and the concept of *parametric degree* for irreducible cuspidal representations of $\mathrm{GL}_n(F)$ was generalized to those of G. The Jacquet-Langlands correspondence $\boldsymbol{j}$, which is described in Theorem 1.1 below, induces a canonical bijection, denoted by $\boldsymbol{j}_A$, between the set, denoted by $\mathcal{A}_n^{\mathrm{et}}(F)$, of equivalence classes of irreducible essentially tame representations of $\mathrm{GL}_n(F)$, of parametric degree n and the set, denoted by $\mathcal{A}_m^{\mathrm{et}}(D)$ of those of G. An explicit description of this correspondence $\boldsymbol{j}_A : \mathcal{A}_n^{\mathrm{et}}(F) \simeq \mathcal{A}_m^{\mathrm{et}}(D)$ was given by using the parameterization of the latter set in terms of *admissible pairs* over F of degree n, which was introduced by Howe [14].

In this paper, by using the results of Sécherre [19] and Bushnell and Henniart [8], we prove that the BSS conjecture is also verified for the essentially tame Jacquet-Langlands correspondence $\boldsymbol{j}_A$, that is, an endo-class $\boldsymbol{\Theta}$ determines the subsets $\mathcal{A}_n^{\mathrm{et}}(F, \boldsymbol{\Theta})$ and $\mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta})$ of $\mathcal{A}_n^{\mathrm{et}}(F)$ and $\mathcal{A}_m^{\mathrm{et}}(D)$, respectively, and $\boldsymbol{j}_A$ induces a bijection

$$\boldsymbol{j}_{A,\boldsymbol{\Theta}} : \mathcal{A}_n^{\mathrm{et}}(F, \boldsymbol{\Theta}) \simeq \mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta})$$

(see Theorem 2.1 and Corollary 2.2). We see that a parameter set for $\mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta})$ associated with the endo-class $\boldsymbol{\Theta}$ is given as follows:

$$\mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta}) \simeq (X(k_E^\times)_{k_{E_0}-\mathrm{reg}}/\langle\sigma_0\rangle) \times \mathbb{C}^\times,$$

where for certain unramified extension E/E_0 associated with $\boldsymbol{\Theta}$, $X(k_E^\times)_{k_{E_0}-\mathrm{reg}}/\langle\sigma_0\rangle$ denotes the set of $\mathrm{Gal}(k_E/k_{E_0})$-orbits in the set of k_{E_0}-regular characters of $k_E^\times$ (see Theorem 2.9 for the details). Using this result, the parameter set for equivalence classes of irreducible essentially tame cuspidal representations of level zero and the correspondence are explicitly described (see Propositions 2.10 and Corollary 2.11). Moreover, we prove that the parameter sets for equivalence classes of irreducible essentially tame and totally ramified representations of $\mathrm{GL}_n(F)$ and G associated with a single endo-class become both $X(k_F^\times) \times \mathbb{C}^\times$, and the correspondence is also described by this parameter set (see Proposition 2.12, Corollary 2.13, and Theorem 2.14).

This paper is organized as follows. In Section 1, we recall the definitions of the Jacquet-Langlands correspondence [10], [18], [2], the simple types in G [9], [20], and the endo-class [4], [3]. Next, we recall the parameterization of the equivalence classes of irreducible essentially tame cuspidal representations of G of parametric degree n in terms of the admissible pairs of degree n and the description of the Jacquet-Langlands correspondence through this parameterization, which was established by [8]. In Section 2, we state the main theorem (Theorem 2.1), which states that the Jacquet-Langlands correspondence preserves an endo-class for the equivalence classes of essentially tame representations of G of parametric degree n, and we determine explicitly the parameterization of equivalence classes of such representations associated with an endo-class. Next, we see the examples not only in the level zero case, but also in the essentially tame and totally ramified case. In Section 3, we provide a proof of Theorem 2.1.

1. Essentially tame representations

We recall the results of Bushnell and Henniart [7],[8] for the essentially tame Jacquet-Langlands correspondence for inner forms of $\mathrm{GL}_n(F)$.

1.1. Preliminaries. Let F be a non-Archimedean local field with a finite residue field of characteristic p. For a finite, commutative, or non-commutative field extension K of F, we denote by $\mathfrak{o}_K$ its ring of integers, by $\mathfrak{p}_K$ the maximal ideal of $\mathfrak{o}_K$, and by k_K its residue field. We write as $U_K = \mathfrak{o}_K^\times$ the multiplicative group of $\mathfrak{o}_K$ and $U_K^1 = 1 + \mathfrak{p}_K$.

Let A be a simple central F-algebra of dimension n^2, $n \geq 1$. Then, there exists a central F-division algebra D of dimension d^2, $d \geq 1$, such that $A \simeq \mathrm{M}_m(D)$, where m is an integer and $n = md$. We identify $A = \mathrm{M}_m(D)$ through the isomorphism, and we set $G = A^\times = \mathrm{GL}_m(D)$.

In this paper, all representations of a totally disconnected locally compact group are assumed to be smooth with complex coefficients.

An irreducible representation π of G is referred to as *essentially square-integrable* if there exists a character χ of G such that $\pi\chi(:= \pi \otimes (\chi \circ \mathrm{Nrd}))$ is unitary and has a nonzero coefficient which is square-integrable on G/Z, where Nrd denotes the reduced norm in A and Z is the center of G. We denote by $\mathcal{A}_m^{(2)}(D)$ the set of equivalence classes of essentially square-integrable representations of $G = \mathrm{GL}_m(D)$. In particular, $\mathcal{A}_n^{(2)}(F)$ denotes the set of equivalence classes of essentially square-integrable representations of $\mathrm{GL}_n(F)$.

Theorem 1.1 ([10], [18], [2]). *There exists a canonical bijection, referred to as the Jacquet-Langlands correspondence,*

$$\boldsymbol{j} : \mathcal{A}_n^{(2)}(F) \simeq \mathcal{A}_m^{(2)}(D)$$

which is uniquely characterized by the character relation

$$\mathrm{tr}\, \pi(g) = (-1)^{n-m} \mathrm{tr}\, \boldsymbol{j}(\pi)(g')$$

for any elliptic regular elements $g \in \mathrm{GL}_n(F), g' \in \mathrm{GL}_m(D)$ that have the same reduced characteristic polynomial.

A representation π of G is referred to as *cuspidal* if there exists a nonzero coefficient which is compactly supported modulo the center Z of G. Then, the set $\mathcal{A}_m^{(2)}(D)$ contains the set, denoted by $\mathcal{A}_m^{(0)}(D)$, of equivalence classes of irreducible cuspidal representations of G.

Let π be an irreducible representation of G. We say that the representation π has *level zero* if for a maximal $\mathfrak{o}_F$-order $\mathfrak{A}$ of A with the Jacobson radical $\mathfrak{P}$, π has a nonzero $U^1(\mathfrak{A})$-fixed vector, where we set $U^1(\mathfrak{A}) = 1 + \mathfrak{P}$, and otherwise, we say that the representation π has *positive level*.

In [9], [19], [20], [21], it was proved that $\pi \in \mathcal{A}_m^{(0)}(D)$ contains a pair (J, λ) in G that is referred to as a *maximal simple type* and which consists of a compact open subgroup J of G and an irreducible representation λ of J. The pair (J, λ) can be constructed as follows. If the representation π has level zero, then we have

(1) $J = U = U(\mathfrak{A}) = \mathfrak{A}^\times$ for a maximal $\mathfrak{o}_F$-order $\mathfrak{A}$ of A;
(2) $\lambda = \sigma$ is an irreducible representation of $J = U$ that is trivial on $U^1(\mathfrak{A})$ and inflated from a cuspidal representation $\overline{\sigma}$ of the finite group $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq \mathrm{GL}_m(k_D)$.

Such a pair $(J, \lambda) = (U, \sigma)$ is referred to as a maximal simple type in G *of level zero*.

Suppose that the representation $\pi \in \mathcal{A}_m^{(0)}(D)$ has positive level. Then, there exists a quadraple $[\mathfrak{A}, \ell, 0, \beta]$ in A, referred to as a *simple stratum*. Here, $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order of A, ℓ is a positive integer, β is an element of A which satisfies $\beta \in \mathfrak{A}^{-\ell} \backslash \mathfrak{A}^{-\ell+1}$ and generates a subfield $F[\beta]$ over F, and an integer $k_0(\beta, \mathfrak{A})$, referred to as the *critical exponent* of

$[\mathfrak{A}, \ell, 0, \beta]$, is defined and is negative. Let $E = F[\beta]$, and let $B = C_A(E)$, the centralizer of E in A. Then, $\mathfrak{A}$ satisfies $x\mathfrak{A}x^{-1} = \mathfrak{A}$, for any $x \in E^\times$, and $\mathfrak{B} = \mathfrak{A} \cap B$ is a maximal $\mathfrak{o}_E$-order in B. There exists a central E-division algebra D_E such that $B \simeq \mathrm{M}_f(D_E)$ for some integer $f > 0$.

Attached to the simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A, two $\mathfrak{o}_F$-lattices $\mathfrak{J} = \mathfrak{J}(\beta, \mathfrak{A})$, $\mathfrak{H} = \mathfrak{H}(\beta, \mathfrak{A})$ of A are defined, and the groups $J = J(\beta, \mathfrak{A}), H = H(\beta, \mathfrak{A})$ are defined by

$$J = J(\beta, \mathfrak{A}) = \mathfrak{J} \cap U(\mathfrak{A}),\ H = H(\beta, \mathfrak{A}) = \mathfrak{H} \cap U(\mathfrak{A}),$$

respectively. Furthermore, the normal subgroups J^1, H^1 of J, H are defined by

$$J^1 = J^1(\beta, \mathfrak{A}) = \mathfrak{J} \cap U^1(\mathfrak{A}),\ H^1 = H^1(\beta, \mathfrak{A}) = \mathfrak{H} \cap U^1(\mathfrak{A}),$$

respectively. Then, we have $J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_f(k_{D_E})$.

Let us fix an additive character

$$\psi_F : F \longrightarrow \mathbb{C}^\times$$

which is trivial on $\mathfrak{p}_F$ but not on $\mathfrak{o}_F$. We will fix this character to be ψ_F. The set of *simple characters* of the group $H^1 = H^1(\beta, \mathfrak{A})$, denoted by $\mathcal{C}(\mathfrak{A}, \beta, \psi_F)$, is defined through the character ψ_F. In order to prove the main theorem, we shall recall the definition in detail in Section 3.1. For a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$, it can be proved that there exists a unique irreducible representation of $J^1 = J^1(\beta, \mathfrak{A})$, denoted by $\eta = \eta(\theta)$, which contains θ.

A pair such as (J, λ) above, attached to the simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A, can be constructed as follows. There exist a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ and a maximal simple type $(U(\mathfrak{B}), \sigma_{\mathfrak{B}})$ in B such that

$$\lambda = \kappa \otimes \sigma$$

where

(1) the representation κ is an extension to J of the representation $\eta = \eta(\theta)$ which is intertwined by all of $B^\times$;
(2) the representation $\sigma = \sigma_{\mathfrak{B}}$ is the inflation to J of a cuspidal representation $\overline{\sigma}_{\mathfrak{B}}$ of the finite group $J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_f(k_{D_E})$.

Such a pair (J, λ) is said to be a maximal simple type in G *of positive level*, and attached to the simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A.

A maximal simple type (U, σ) in G of level zero can be regarded as a simple type attached to a null stratum $[\mathfrak{A}, 0, 0, 0]$ in A [20, Remarque 4.1]. In this case, from $\beta = 0$, we have $E = F, B = A, \mathfrak{B} = \mathfrak{A}$, and $\overline{\sigma}_{\mathfrak{B}} = \overline{\sigma}$.

1.2. Essentially tame cuspidal representations. Let $\pi \in \mathcal{A}_m^{(0)}(D)$, and let (J, λ) be a maximal simple type in G which is contained in π and is attached to a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A. Then, by definition, we have $\lambda = \kappa \otimes \sigma$ (resp. $\lambda = \sigma$) when (J, λ) is of positive level (resp. level zero), where σ is the inflation of the representation $\overline{\sigma}_{\mathfrak{B}}$ (resp. $\overline{\sigma}$) of the finite group $J/J^1 \simeq \mathrm{GL}_f(k_{D_E})$, as before. Here, we set $f = m, E = F$, and $D_E = D$ in the level zero case. Thus, the Galois group $\Gamma = \mathrm{Gal}(k_{D_E}/k_E)$ of the extension k_{D_E}/k_E acts naturally on J/J^1. We denote by w_0 the number of distinct Γ-conjugates of $\overline{\sigma}_{\mathfrak{B}}$ (resp. $\overline{\sigma}$). Following [8], we define the invariant $\delta(\pi)$, referred to as the *parametric degree* of the representation π, by

$$\delta(\pi) = f w_0[E : F].$$

More generally, the parametric degree $\delta(\pi)$ for $\pi \in \mathcal{A}_m^{(2)}(D)$ is defined by $\delta(\pi) = \delta(\pi_0)$ for a representation π_0 in the cuspidal support of π (see [8, Section 2.8]).

For $\pi \in \mathcal{A}_m^{(2)}(D)$, we denote by $t(\pi)$ the number of unramified characters χ of $F^\times$ such that $\pi \otimes (\chi \circ \mathrm{Nrd}) \simeq \pi$. It was proved in [8] that $\delta(\pi)|n$ and $t(\pi)|\delta(\pi)$.

A representation $\pi \in \mathcal{A}_m^{(2)}(D)$ is referred to as *essentially tame* if the characteristic p of the residue field k_F of F does not divide $\delta(\pi)/t(\pi)$. We denote by $\mathcal{A}_m^{\mathrm{et}}(D)$ the set of $\pi \in \mathcal{A}_m^{(2)}(D)$ which are essentially tame with $\delta(\pi) = n$. We note that all the representations of $\mathcal{A}_m^{\mathrm{et}}(D)$ are cuspidal.

The set $\mathcal{A}_m^{\mathrm{et}}(D)$ is characterized as follows.

Proposition 1.2 ([8, Section 2.8, Lemma])**.** *Let $\pi \in \mathcal{A}_m^{(0)}(D)$. We have that π is essentially tame if and only if either*

(1) *π contains a maximal simple type of level zero, or*
(2) *π has positive level and contains a maximal simple type in G attached to a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A such that the field extension $F[\beta]/F$ is tamely ramified.*

By [8, Section 2.8, Corollary 2], the Jacquet-Langlands correspondence $\boldsymbol{j}$, described in Theorem 1.1, induces a canonical bijection

$$\boldsymbol{j}_A : \mathcal{A}_n^{\mathrm{et}}(F) \simeq \mathcal{A}_m^{\mathrm{et}}(D). \qquad (1.1)$$

1.3. Endo-classes. Let D be a central F-division algebra of dimension d^2, $d \geq 1$, and set $A = \mathrm{M}_m(D)$, $m \geq 1$.

Bushnell and Henniart [4] introduced the notion of an *endo equivalence class* (in short, *endo-class*) *over* F of a simple character for $\mathrm{GL}_n(F)$ attached to a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in $\mathrm{M}_n(F)$, where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order. We denote by $\mathcal{E}(F)$ the set of endo-classes over F of simple characters for $\mathrm{GL}_n(F)$, where $n \geq 1$ varies. This notion was generalized to a simple character for $\mathrm{GL}_m(D)$ attached to a simple stratum $[\Lambda, \ell, 0, \beta]$ in $A = \mathrm{M}_m(D)$, where Λ is an $\mathfrak{o}_D$-lattice sequence [19], [21], [3]. We denote by $\widetilde{\mathcal{E}}(F)$ the set of endo-classes over F of simple characters for $\mathrm{GL}_m(D)$, where $n = md \geq 1$ varies. Then, from [3, Corollary 8.2], it follows that $\mathcal{E}(F)$ is a subset of $\widetilde{\mathcal{E}}(F)$.

For a simple character $\theta \in \mathcal{C}(\Lambda, \beta, \psi_F)$ attached to a simple stratum $[\Lambda, \ell, 0, \beta]$ in $A = \mathrm{M}_m(D)$, we denote by $\mathcal{E}_F(\theta)$ the endo-class over F defined by the pair $([\Lambda, \ell, 0, \beta], \theta)$. This generalizes $\mathcal{E}_F(\theta)$ for $\mathrm{GL}_n(F)$, which was defined by [7, Notation and background], to one for $\mathrm{GL}_m(D)$.

For a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ of A, we have the null stratum $[\mathfrak{A}, 0, 0, 0]$ in A and the attached compact subgroup $H^1 = H^1(\mathfrak{A}, 0) = U^1(\mathfrak{A})$ of $G = A^\times = \mathrm{GL}_m(D)$. Since the trivial character $1_{U^1(\mathfrak{A})}$ of the group $U^1(\mathfrak{A})$ can be regarded as a simple character of H^1, we adjoin to $\widetilde{\mathcal{E}}(F)$ a trivial element $\boldsymbol{\Theta}_0$, which may be regarded as the endo-class $\mathcal{E}_F(1_{U^1(\mathfrak{A})})$ for such an arbitrary hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ of A.

Let χ be a non-trivial character of the group $U_F^1 = 1 + \mathfrak{p}_F$. The character χ can be also regarded as a simple character of $\mathcal{C}(\mathfrak{o}_F, c, \psi_F)$ attached to some simple stratum $[\mathfrak{o}_F, -v_F(c), 0, c]$ in $F = \mathrm{End}_F(F)$, where $v_F(c)$ denotes the F-valuation of $c \in F$ (see [7, Section 1.2]). Thus, we can form $\mathcal{E}_F(\chi) \in \mathcal{E}(F)$. From [4, Corollary (9.13)], for a finite, tamely ramified extension K/F, there exists a canonical surjection

$$\mathrm{R}_{K/F} : \mathcal{E}(K) \longrightarrow \mathcal{E}(F)$$

that is transitive in the extension K/F.

1.4. Admissible pairs. By Howe and Moy [14], [17], Bushnell and Henniart [7] proved that the set $\mathcal{A}_n^{\mathrm{et}}(F)$, defined in Section 1.2, has a canonical one-to-one correspondence with the set of *admissible pairs over* F, denoted by $(E/F, \xi)$, which consist of tamely ramified extensions E/F of degree n and certain characters of $E^\times$. They generalized this result for $\mathrm{GL}_n(F)$ to that for $\mathrm{GL}_m(D)$ and described explicitly the Jacquet-Langlands correspondence $\boldsymbol{j}_A$ of (1.1) through this parameterization [8].

We recall the definition of admissible character [14].

Definition 1.3 [14], [17], [7], [8]. Let E/F be a finite, tamely ramified extension.

(i) Let χ be a character of U_E^1. A pair $(E/F, \chi)$ is referred to as an *admissible 1-pair* if χ does not factor through the norm map $\mathrm{N}_{E/K}$ for any field K, where $F \subset K \subsetneq E$.

(ii) Let ξ be a character of $E^\times$. A pair $(E/F, \xi)$ is referred to as an *admissible pair over* F if

(1) ξ does not factor through $\mathrm{N}_{E/K}$ for any field K, where $F \subset K \subsetneq E$.
(2) if $\xi|U_E^1$ factors through $\mathrm{N}_{E/K}$ for a field K, where $F \subset K \subsetneq E$, then E/K is unramified.

The *degree* of an admissible pair $(E/F, \xi)$ over F is defined by the extension degree $[E : F]$ of E/F. Let $(E/F, \eta)$ and $(E'/F, \eta')$ be pairs such as those in Definition 1.3. We say that $(E/F, \eta)$ and $(E'/F, \eta')$ are *F-isomorphic* if there exists an F-isomorphism $\alpha : E \longrightarrow E'$ such that $\eta = \eta' \circ \alpha$. We write $P^{(1)}(F)$ (resp. $P_n(F)$) for the set of F-isomorphism classes of admissible 1-pairs (resp. admissible pairs over F and of degree n).

Hereinafter, $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ means that $\delta(\pi) = n$ and π is an essentially tame representation of $G = \mathrm{GL}_m(D)$. Attached to each $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, an ($F$-isomorphism class of) admissible pair $(E/F, \xi)$ in $P_n(F)$ can be constructed as follows [7],[8].

Suppose that $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ and that π has level zero. Then, there exists a maximal simple type (U, σ) in G of level zero which is contained in π, and from this maximal simple type, we obtain an admissible pair $(E/F, \xi)$ over F and of degree n that satisfies

(1) E/F is an unramified extension of degree n;
(2) the character ξ of $E^\times$ is *tamely ramified*, that is, $\xi|U_E^1 = 1$.

Thus, by [3, Section 9.2], we may set the endo-class $\boldsymbol{\Theta}(\pi)$ of the representation π to be the trivial endo-class $\boldsymbol{\Theta}_0$ defined as above.

We next assume that $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ has positive level. Then, there exist a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A and a maximal simple type (J, λ) in G attached to it such that $\lambda|H^1(\beta, \mathfrak{A})$ contains a simple character $\theta \in C(\mathfrak{A}, \beta, \psi_F)$. Again using [3, Section 9.2], we set

$$\boldsymbol{\Theta}(\pi) = \mathcal{E}_F(\theta).$$

The maximal simple type (J, λ) above can be extended to a pair $(\boldsymbol{J}, \Lambda)$, referred to as an *extended maximal simple type*, so that π is compactly induced from Λ to G as follows [8, Section 4.3].

(1) $\boldsymbol{J} = \boldsymbol{J}_B(\sigma_{\mathfrak{B}})J^1$, where $\boldsymbol{J}_B(\sigma_{\mathfrak{B}})$ denotes the normalizer of the maximal simple type

$(U(\mathfrak{B}), \sigma_{\mathfrak{B}})$ in $B = C_A(\beta)$, as in Section 1.1;

(2) Λ is an extension of λ.

Let $E_0 = F[\beta]$. Recall that $B = C_A(E_0)$ and $\mathfrak{B} = B \cap \mathfrak{A}$. There exists an admissible 1-pair $(E_0/F, \xi_0)$ such that

$$\theta|U^1(\mathfrak{B}) = \xi_0 \circ \mathrm{Nrd}_B,$$

and E/E_0 is an unramified extension in B such that $\mathfrak{B}$ is *E-pure*, that is, $x\mathfrak{B}x^{-1} = \mathfrak{B}$ for $x \in E^\times$, and $[E : F] = n$.

We now choose a prime element ϖ_F in $\mathfrak{o}_F$ for the following discussion.

Lemma 1.4 ([8, Section 4.3, Lemma])**.** *There exists a unique character ξ_w of $E^\times$ that satisfies*

(1) $\xi_w|U_E^1 = \xi_0 \circ \mathrm{N}_{E/E_0}$*;*

(2) $\xi_w(\varpi_F) = 1$*;*

(3) ξ_w *has finite, p-power order.*

The representation Λ of the extended maximal simple type $(\boldsymbol{J}, \Lambda)$ can be decomposed into a tensor product $\Lambda = \Lambda_w \otimes \Lambda_t$ of representations Λ_w and Λ_t that satisfy

(1) Λ_w is a representation of $\boldsymbol{J}$ to which can be extended the unique irreducible representation $\eta(\theta)$ of J^1 containing θ;

(2) Λ_t is trivial on J^1 and induces a maximal simple type $(\boldsymbol{J} \cap B, \Lambda_t|\boldsymbol{J} \cap B)$ in $B^\times$ of level zero.

From the representation Λ_t above, as in the level zero case, we obtain an admissible pair $(E/E_0, \xi_t)$ over E_0, where ξ_t is tamely ramified. Hence, we can define the character ξ of $E^\times$ by

$$\xi = \xi_t \cdot \xi_w$$

from the above characters ξ_w and ξ_t. Then, it is easy to prove that $(E/F, \xi)$ is an admissible pair over F and of degree n. Thus, we obtain the map $\pi \mapsto (E/F, \xi)$. This pair $(E/F, \xi)$ is said to be *attached to* π.

Theorem 1.5 ([8, Section 6, Parameterization Theorem])**.** *Assume that $(E/F, \xi)$ is in $P_n(F)$. Then, there exists a unique $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ such that $(E/F, \xi)$ is F-isomorphic to an admissible pair attached to π. In such a case, let us write*

$$\pi = \pi_D(\xi).$$

Then, the map $(E/F, \xi) \mapsto \pi_D(\xi)$ induces a bijection

$$P_n(F) \simeq \mathcal{A}_m^{\mathrm{et}}(D).$$

We note that $\pi_D(\xi)$ in this theorem is ${}_D\Pi_\xi$ in the notation of [8]. This theorem was first proved for $\mathrm{GL}_n(F)$ [5], and then it was generalized to $\mathrm{GL}_m(D)$ [8]. The map $\pi \mapsto (E/F, \xi)$ above is the inverse of the map $(E/F, \xi) \mapsto \pi_D(\xi)$ of Theorem 1.5.

Theorem 1.6 ([8, Section 6, First Comparison Theorem])**.** *Assume that $(E/F, \xi)$ is in $P_n(F)$. Then, there exists a unique tamely ramified character, denoted by $\nu = \nu(D, \xi)$, of $E^\times$ with $\nu^2 = 1$ such that $(E/F, \xi\nu) \in P_n(F)$ and*

$$\pi_D(\xi) = \boldsymbol{j}_A(\pi_F(\xi\nu)).$$

We note that the values of the character $\nu = \nu(D, \xi)$ of Theorem 1.6, which is ${}_D\nu_\xi$ in the notation of [8], are completely determined [8, Sections 6 and 7].

2. Invariance of an endo-class under $\boldsymbol{j}_A$

2.1. Conjecture 9.5 of Broussous, Sécherre and Stevens. Assume that $(E/F, \xi) \in P_n(F)$. If $\xi|U_E^1 = 1$, then the representations $\pi_F(\xi\nu)$ and $\pi_D(\xi)$ in Theorem 1.6 both have level zero. Thus, we obtain

$$\boldsymbol{\Theta}(\pi_F(\xi\nu)) = \boldsymbol{\Theta}(\pi_D(\xi)) = \boldsymbol{\Theta}_0,$$

which shows that Conjecture 9.5 of [3] is true. This is stated in the Introduction of [3] (cf. [16]).

We will assume that $\xi|U_E^1 \neq 1$. Then [7, Section 2.3], there exists a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in $A = \mathrm{M}_n(F)$ and a simple character $\theta \in \mathcal{C}(\mathfrak{A}, \beta, \psi_F)$ such that

(1) there exists the minimal subextension $F \subset E_0 \subset E$ such that $\xi|U_E^1 = \xi_0 \circ \mathrm{N}_{E/E_0}$ for some admissible 1-pair $(E_0/F, \xi_0)$;
(2) $E_0 = F[\beta]$;
(3) $\mathcal{E}_F(\theta) = \mathrm{R}_{E_0/F}(\mathcal{E}_{E_0}(\xi_0)) \in \mathcal{E}(F)$.

From this character ξ_0 of $E_0^\times$, through the fixed prime element ϖ_F, we obtain the character ξ_w of $E^\times$ as defined by Lemma 1.4. Set $\xi_t = \xi_w^{-1}\xi$. Then, $(E/E_0, \xi_t)$ is an admissible pair over E_0 and ξ_t is tamely ramified. We define the subgroup $\boldsymbol{J}$ of $A^\times = \mathrm{GL}_n(F)$ by

$$\boldsymbol{J} = E^\times J(\beta, \mathfrak{A}).$$

From $(E/E_0, \xi_t)$, we obtain an irreducible representation Λ_t of $\boldsymbol{J}$ that is trivial on $J^1 = J^1(\beta, \mathfrak{A})$ and an extension Λ_w to $\boldsymbol{J}$ of the irreducible representation $\eta(\theta)$ of $J^1(\beta, \mathfrak{A})$ that contains θ such that $\Lambda = \Lambda_w \otimes \Lambda_t$ forms an extended maximal simple type $(\boldsymbol{J}, \Lambda)$ in $\mathrm{GL}_n(F)$ and

$$\pi_F(\xi) = \text{c-Ind}_{\boldsymbol{J}}^{\mathrm{GL}_n(F)}\Lambda.$$

From [7, Section 2.3, Theorem], we obtain

$$\boldsymbol{\Theta}(\pi_F(\xi)) = \mathcal{E}_F(\theta) = \mathrm{R}_{E_0/F}(\mathcal{E}_{E_0}(\xi_0)).$$

We will denote the last endo-class of the above equalities as $\boldsymbol{\Theta}_{\xi_0}$. Let

$$\boldsymbol{\Theta}_\xi = \mathrm{R}_{E/F}(\mathcal{E}_E(\xi|U_E^1)).$$

Then, from $\xi|U_E^1 = \xi_0 \circ \mathrm{N}_{E/E_0}$, we have $\boldsymbol{\Theta}_\xi = \boldsymbol{\Theta}_{\xi_0}$. Thus, we obtain

$$\boldsymbol{\Theta}(\pi_F(\xi)) = \boldsymbol{\Theta}_\xi = \boldsymbol{\Theta}_{\xi_0}. \tag{2.1}$$

Since the character $\nu = \nu(D, \xi)$ given by Theorem 1.6 is tamely ramified, $(\xi\nu)|U_E^1 = \xi|U_E^1$. Thus, we obtain

$$\boldsymbol{\Theta}(\pi_F(\xi\nu)) = \boldsymbol{\Theta}_{\xi\nu} = \boldsymbol{\Theta}_\xi. \tag{2.2}$$

Now we can state the following main result.

Theorem 2.1. *Let $(E/F,\xi) \in P_n(F)$ with $\xi|U_E^1 \neq 1$, and $(E_0/F,\xi_0)$ be the corresponding admissible 1-pair as above. Then, for $\pi_D(\xi) \in \mathcal{A}_m^{\mathrm{et}}(D)$, we have*

$$\boldsymbol{\Theta}(\pi_D(\xi)) = \boldsymbol{\Theta}_\xi = \boldsymbol{\Theta}_{\xi_0}.$$

We shall prove this theorem in the next section. In this section, we will assume that this theorem holds.

Corollary 2.2. *For $\pi \in \mathcal{A}_n^{\mathrm{et}}(F)$, we have*

$$\boldsymbol{\Theta}(\boldsymbol{j}_A(\pi)) = \boldsymbol{\Theta}(\pi).$$

Proof. This follows from Equations (2.1) and (2.2) and from Theorems 1.6 and 2.1. □

This implies that the Conjecture 9.5 of [3] is verified for the essentially tame representations of inner forms of $\mathrm{GL}_n(F)$. Given $\boldsymbol{\Theta}$ an endo-class over F, set

$$\mathcal{A}_m^{\mathrm{et}}(D,\boldsymbol{\Theta}) = \{\pi \in \mathcal{A}_m^{\mathrm{et}}(D) : \boldsymbol{\Theta}(\pi) = \boldsymbol{\Theta}\}.$$

If $\mathcal{A}_m^{\mathrm{et}}(D,\boldsymbol{\Theta})$ is non-empty, then the Jacquet-Langlands correspondence $\boldsymbol{j}_A$ of (1.1) induces a canonical bijection

$$\boldsymbol{j}_{A,\boldsymbol{\Theta}} : \mathcal{A}_n^{\mathrm{et}}(F,\boldsymbol{\Theta}) \simeq \mathcal{A}_m^{\mathrm{et}}(D,\boldsymbol{\Theta}). \tag{2.3}$$

2.2. Parameterization for $\mathcal{A}_m^{\mathrm{et}}(D,\boldsymbol{\Theta})$. We give a parameterization for $\mathcal{A}_m^{\mathrm{et}}(D,\boldsymbol{\Theta})$ and describe the correspondence $\boldsymbol{j}_{A,\boldsymbol{\Theta}}$.

Lemma 2.3. *Let $E/F, E'/F$ be finite field extensions, and let E_0/F be a subextension of E/F. If there exists an F-isomorphism $\alpha : E \longrightarrow E'$, then we have*

$$\alpha \circ \mathrm{N}_{E/E_0}(x) = \mathrm{N}_{E'/\alpha(E_0)}(\alpha(x))$$

for any $x \in E$.

Proof. This is elementary. □

Let $(E/F,\xi) \in P_n(F)$. Then, from [8, Section 4.1, Lemma], there exists an admissible 1-pair $(E_0/F,\xi_0)$ with $E_0 \subset E$ and $\xi|U_E^1 = \xi_0 \circ \mathrm{N}_{E/E_0}$, where E/E_0 is unramified. For this pair $(E/E_0,\xi_0)$, let ξ_w be the unique character of $E^\times$ defined by the conditions of Lemma 1.4, and set $\xi_t = \xi_w^{-1}\xi$. Then, $(E/E_0,\xi_t)$ is an admissible pair over E_0 and ξ_t is tamely ramified, as stated in Section 1.4. Hence, we obtain the decomposition $\xi = \xi_w\xi_t$.

Proposition 2.4. *Let $(E/F,\xi)$ be an admissible pair over F of degree n such that ξ can be decomposed into $\xi = \xi_w\xi_t$, as above. Let $(E'/F,\xi')$ be another pair in $P_n(F)$, with $\xi' = \xi'_w\xi'_t$, where $\xi'_w = \xi'_0 \circ \mathrm{N}_{E'/E'_0}$ on $U_{E'}^1$ for some $(E'_0/F,\xi'_0) \in P^{(1)}(F)$, as above. Suppose that there exists an F-isomorphism $\alpha_0 : (E'_0/F,\xi'_0) \simeq (E_0/F,\xi_0)$. Then, there exists an F-isomorphism $\alpha : E' \longrightarrow E$ that extends α_0 such that for some admissible pair $(E/E_0,\vartheta)$ with ϑ tamely ramified,*

$$\alpha : (E'/F,\xi') \simeq (E/F,\xi_w\vartheta)$$

holds.

Proof. Through the F-isomorphism α_0, we see that E'/E'_0 and E/E_0 are unramified extensions of the same degree. Thus, there exists an F-isomorphism $\alpha : E' \longrightarrow E$ that extends α_0. Define the characters ξ_w^o and ϑ of $E^\times$ by

$$\xi_w^o = \xi'_w \circ \alpha^{-1}, \ \vartheta = \xi'_t \circ \alpha^{-1},$$

respectively. Since $\alpha^{-1}(U_E^1) = U_{E'}^1$, we have

$$\xi_w^o(x) = \xi'_0 \circ \mathrm{N}_{E'/E'_0}(\alpha^{-1}(x))$$

for any $x \in U_E^1$. By Lemma 2.3, we obtain

$$\xi_w^o(x) = \xi'_0 \circ \alpha_0^{-1} \circ \mathrm{N}_{E/E_0}(x) = \xi_0 \circ \mathrm{N}_{E/E_0}(x) = \xi_w(x)$$

for any $x \in U_E^1$. The equality $\xi'_w(\varpi_F) = \xi_w(\varpi_F) = 1$ follows immediately, and for integers $a \geq 1$, we have

$$(\xi_w^o)^{p^a} = (\xi'_w \circ \alpha^{-1})^{p^a} = 1.$$

Hence, we obtain the equlity $\xi_w^o = \xi_w$ as a character of $E^\times$. By definition, $\vartheta = \xi'_t \circ \alpha^{-1}$ is tamely ramified. Since $(E'/E'_0, \xi'_t)$ is admissible, again from Lemma 2.3, it follows that $(E/E_0, \vartheta)$ is also admissible, through the F-isomorphism α. Consequently, we obtain

$$\xi' \circ \alpha^{-1} = (\xi'_w \xi'_t) \circ \alpha^{-1} = (\xi'_w \circ \alpha^{-1})(\xi'_t \circ \alpha^{-1}) = \xi_w \vartheta,$$

which implies that α is the F-isomorphism $(E'/F, \xi') \simeq (E/F, \xi_w \vartheta)$. □

We denote by $X_t(E^\times)$ the set of tamely ramified (quasi-)characters of $E^\times$.

Proposition 2.5 (cf. [8, Section 4.3, Remark]). *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\mathbf{\Theta} = \mathbf{\Theta}(\pi)$. Let $\xi = \xi_w \xi_t$ and $\xi_w = \xi_0 \circ \mathrm{N}_{E/E_0}$ on U_E^1, for some $(E_0/F, \xi_0) \in P^{(1)}(F)$, as in Proposition 2.4. Denote by $\Xi(E_0)$ the set of $\vartheta \in X_t(E^\times)$ such that $(E/E_0, \vartheta)$ is admissible. Then, the set $\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$ is given by*

$$\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta}) = \{\pi_D(\xi_w \vartheta) : \vartheta \in \Xi(E_0)\}.$$

Proof. The inclusion $\supset$ is clear. We prove the converse inclusion $\subset$. Assume that $\pi' \in \mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$, that is, $\mathbf{\Theta}(\pi') = \mathbf{\Theta}$. Then, from Theorem 1.5, there exists a unique $(E'/F, \xi') \in P_n(F)$ such that $\pi' \simeq \pi_D(\xi')$. We may identify $\pi' = \pi_D(\xi')$. For the character ξ', we have $\xi'|U_{E'}^1 = \xi'_0 \circ \mathrm{N}_{E'/E'_0}$, for some $(E'_0/F, \xi'_0) \in P^{(1)}(F)$, where E'_0/F is a subextension of E'/F. From Theorem 2.1, we have

$$\mathbf{\Theta} = \mathbf{\Theta}_\xi = \mathbf{\Theta}_{\xi_0}$$

and $\mathbf{\Theta}_{\xi'} = \mathbf{\Theta}(\pi')$. By assumption, we thus obtain

$$\mathbf{\Theta}_{\xi'} = \mathbf{\Theta}(\pi') = \mathbf{\Theta} = \mathbf{\Theta}_{\xi_0}.$$

Hence, from [7, Theorem 1.3], there exists an F-isomorphism

$$\alpha_0 : (E'_0/F, \xi'_0) \simeq (E_0/F, \xi_0),$$

and from Proposition 2.4, there exist an F-isomorphism $\alpha : E' \simeq E$ that extends α_0 and an admissible pair $(E/E_0, \vartheta)$ with ϑ tamely ramified such that

$$\alpha : (E'/F, \xi') \simeq (E/F, \xi_w\vartheta).$$

Consequently, Theorem 1.5 implies that

$$\pi' = \pi_D(\xi') \simeq \pi_D(\xi_w\vartheta).$$

□

The Jacquet-Langlands correspondence $\boldsymbol{j}_{A,\boldsymbol{\Theta}}$ of (2.3) can be described as follows.

Proposition 2.6. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\pi)$. Let $\xi = \xi_w\xi_t$ and $\xi_w = \xi_0 \circ \mathrm{N}_{E/E_0}$ on U_E^1, for some $(E_0/F, \xi_0) \in P^{(1)}(F)$, as in Proposition 2.4. Then, there exists a unique tamely ramified character $\nu_{\boldsymbol{\Theta}}$ of $E^\times$ which depends only on the endo-class $\boldsymbol{\Theta}$ and satisfies $\nu_{\boldsymbol{\Theta}}^2 = 1$, where the Jacquet-Langlands correspondence $\boldsymbol{j}_{A,\boldsymbol{\Theta}}$ of (2.3) can be described by*

$$\boldsymbol{j}_{A,\boldsymbol{\Theta}}(\pi_F(\xi_w\vartheta\nu_{\boldsymbol{\Theta}})) = \pi_D(\xi_w\vartheta)$$

for any $\vartheta \in \Xi(E_0)$.

Proof. For a representation $\pi_D(\xi_w\vartheta) \in \mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta})$, let $\nu_{\boldsymbol{\Theta}} = \nu(D, \xi_w\vartheta)$ be as in Theorem 1.6. Then, from Proposition 2.5 and Corollary 1 to Second Comparison Theorem of [8, Section 7.1], the character $\nu_{\boldsymbol{\Theta}}$ does not depend on the choice of $\xi_w\vartheta$ and depends only on the endo-class $\boldsymbol{\Theta}$. The remainder of this proposition was proved in Theorem 1.6. □

In the following discussions, we retain the situation of Proposition 2.5. Set

$$\mathcal{G}_0 = \{\alpha \in \mathrm{Aut}(E|F) : \alpha(E_0) = E_0 \text{ and } \xi_0 \circ \alpha = \xi_0\},$$

where $\mathrm{Aut}(E|F)$ denotes the group of F-automorphisms of the field E. Let $\vartheta \in \Xi(E_0)$ and $\alpha \in \mathcal{G}_0$. Then, $(E/E_0, \vartheta \circ \alpha)$ is an admissible pair and $\vartheta \circ \alpha$ is tamely ramified. Thus, by the definition of $\Xi(E_0)$ in Proposition 2.5, we obtain $\vartheta \circ \alpha \in \Xi(E_0)$, which implies that $\mathcal{G}_0$ acts on $\Xi(E_0)$ by $(\alpha, \vartheta) \mapsto \vartheta \circ \alpha$. We denote by $\Xi(E_0)/\mathcal{G}_0$ the set of $\mathcal{G}_0$-orbits in $\Xi(E_0)$.

Proposition 2.7. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\pi)$. Let $\xi = \xi_w\xi_t$ and $\xi_w = \xi_0 \circ \mathrm{N}_{E/E_0}$ on U_E^1 for some $(E_0/F, \xi_0) \in P^{(1)}(F)$, as in Proposition 2.5. Then, the map $\vartheta \mapsto \pi_D(\xi_w\vartheta)$ induces a canonical bijection*

$$\Xi(E_0)/\mathcal{G}_0 \simeq \mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta}).$$

Proof. Proposition 2.5 shows that the map $\vartheta \mapsto \pi_D(\xi_w\vartheta)$ induces the surjection $\Xi(E_0) \longrightarrow \mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta})$. Let $\vartheta, \vartheta' \in \Xi(E_0)$ satisfy $\pi_D(\xi_w\vartheta) \simeq \pi_D(\xi_w\vartheta')$. Then, from Theorem 2.1, it follows that

$$\boldsymbol{\Theta}(\pi_D(\xi_w\vartheta)) = \boldsymbol{\Theta}_{\xi_0} = \boldsymbol{\Theta}(\pi_D(\xi_w\vartheta')).$$

From Proposition 2.4, we obtain an F-isomorphism $\alpha_0 : (E_0/F, \xi_0) \simeq (E_0/F, \xi_0)$ and an F-isomorphism $\alpha : (E/F, \xi_w\vartheta) \simeq (E/F, \xi_w\vartheta')$ that extends α_0. Thus, we obtain

$$\xi_w\vartheta' = (\xi_w\vartheta) \circ \alpha = (\xi_w \circ \alpha)(\vartheta \circ \alpha).$$

Through the F-isomorphism α_0, we see that $\xi_w = \xi_w \circ \alpha$, similar to the proof of Proposition

2.4. Hence, we obtain $\vartheta' = \vartheta \circ \alpha$ and $\alpha \in \mathcal{G}_0$. □

We describe the set $\Xi(E_0)/\mathcal{G}_0$ in Proposition 2.7 explicitly. Denote by $\boldsymbol{\mu}_E$ the set of roots of unity in E of order relatively prime to p, and let ϖ_E be a prime element of E. We may identify $\boldsymbol{\mu}_E = k_E^\times$ and $E^\times = k_E^\times \times U_E^1 \times \varpi_E^{\mathbb{Z}}$. Thus, the map $\vartheta \mapsto (\vartheta|\boldsymbol{\mu}_E, \vartheta(\varpi_E))$ gives a canonical bijection

$$X_t(E^\times) \simeq X(k_E^\times) \times \mathbb{C}^\times.$$

Lemma 2.8. *Let the notation and assumptions be as in Proposition 2.7, and let $\vartheta \in X_t(E^\times)$. Then, we have $\mathcal{G}_0 = \mathrm{Gal}(E/E_0)$, and $(E/E_0, \vartheta)$ is admissible, that is, $\vartheta \in \Xi(E_0)$ if and only if $\vartheta|\boldsymbol{\mu}_E$ is k_{E_0}-regular.*

Proof. Since $(E_0/F, \xi_0)$ is an admissible 1-pair, by definition, $\xi_0|U_{E_0}^1$ does not factor through $\mathrm{N}_{E_0/K}$, for any field K such that $F \subset K \subsetneq E_0$. Let $\alpha \in \mathcal{G}_0$. Then, by the definition of $\mathcal{G}_0$ above, $\alpha(E_0) = E_0$ and $\xi_0 \circ \alpha = \xi_0$. Thus, by [7, A.1. Lemma], this implies $\alpha|E_0$ is trivial, that is, $\alpha \in \mathrm{Gal}(E/E_0)$. This shows the first assertion.

For the second assertion, since the field extension E/E_0 is unramified, $\vartheta \in \Xi(E_0)$ if and only if $\vartheta \circ \sigma$ are distinct for all $\sigma \in \mathrm{Gal}(E/E_0)$. Equivalently, $\vartheta|\boldsymbol{\mu}_E$ is $\mathrm{Gal}(k_E/k_{E_0})$-regular, referred to as k_{E_0}-regular. Hence, the proof is completed. □

Let σ_0 be a generator of $\mathrm{Gal}(E/E_0)$. We may identify $\mathrm{Gal}(k_E/k_{E_0}) = \mathrm{Gal}(E/E_0) = \langle\sigma_0\rangle$. Denote by $X(k_E^\times)_{k_{E_0}-\mathrm{reg}}$ the set of k_{E_0}-regular characters χ in $X(k_E^\times)$, and by $X(k_E^\times)_{k_{E_0}-\mathrm{reg}}/\langle\sigma_0\rangle$ the set of $\langle\sigma_0\rangle$-orbits in the set $X(k_E^\times)_{k_{E_0}-\mathrm{reg}}$.

Theorem 2.9. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\boldsymbol{\Theta} = \boldsymbol{\Theta}(\pi)$. Let $\xi = \xi_w \xi_t$ and $\xi_w = \xi_0 \circ \mathrm{N}_{E/E_0}$ on U_E^1 for some $(E_0/F, \xi_0) \in P^{(1)}(F)$, as in Proposition 2.5. Then, we have a canonical bijection*

$$\mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta}) \simeq (X(k_E^\times)_{k_{E_0}-\mathrm{reg}}/\langle\sigma_0\rangle) \times \mathbb{C}^\times.$$

Proof. By Lemma 2.8, we have $\Xi(E_0) \simeq X(k_E^\times)_{k_{E_0}-\mathrm{reg}} \times \mathbb{C}^\times$ under the map $\vartheta \mapsto (\vartheta|\boldsymbol{\mu}_E, \vartheta(\varpi_{E_0}))$, and for $\sigma \in \mathcal{G}_0 = \mathrm{Gal}(E/E_0)$, we have

$$(\vartheta \circ \sigma|\boldsymbol{\mu}_E, \vartheta \circ \sigma(\varpi_{E_0})) = ((\vartheta|\boldsymbol{\mu}_E) \circ \overline{\sigma}, \vartheta(\varpi_{E_0})),$$

where we identify $\sigma = \overline{\sigma}$ in $\mathrm{Gal}(E/E_0) = \mathrm{Gal}(k_E/k_{E_0})$. Thus, there exists a canonical bijection between the set $\Xi(E_0)/\mathcal{G}_0$ in Proposition 2.7 and $(X(k_E^\times)_{k_{E_0}-\mathrm{reg}}/\langle\sigma_0\rangle) \times \mathbb{C}^\times$. Hence, the proof is completed. □

We consider the level zero case. Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ have level zero, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Then, from [8, Section 4.2] and Theorem 2.1, it follows that ξ is tamely ramified and $\boldsymbol{\Theta}(\pi) = \boldsymbol{\Theta}_0$. By definition, we have $E_0 = F$ and $\xi_0 = \xi_w = 1$. Thus, we have $\Xi(E_0) = \Xi(F)$ and $\mathcal{G}_0 = \mathrm{Gal}(E/F)$, where we remark that E is an unramified extension field of $E_0 = F$. Hence, by Theorem 2.9, we can re-write [8, Section 4.2, Proposition] as follows.

Proposition 2.10. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ have level zero, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Then, there exists a canonical bijection*

$$\mathcal{A}_m^{\mathrm{et}}(D, \boldsymbol{\Theta}_0) \simeq (X(k_E^\times)_{k_F\text{-reg}}/\langle\sigma_0\rangle) \times \mathbb{C}^\times.$$

We denote by $\pi_D([\chi], c)$ the image in $\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta}_0)$ of an element $([\chi], c)$ in the right-hand side set under the bijection given in Proposition 2.10, where $[\chi]$ denotes the $\langle\sigma_0\rangle$-orbit of the element $\chi \in X(k_E^\times)_{k_F\text{-reg}}$.

We remark that the parameter set in Proposition 2.10 is equal to $\mathrm{Gal}(F_n/F)\backslash X_t'(F_n^\times)$ contained in the parameter set $\mathscr{T}_0^n$ for equivalence classes of level zero essentially square-integrable representations of an inner form of $\mathrm{GL}_n(F)$ obtained by [23, Theorem 1].

Corollary 2.11. *The Jacquet-Langlands correspondence $\boldsymbol{j}_{A,\mathbf{\Theta}_0}$ of (2.3) for the level zero case is described by*

$$\boldsymbol{j}_{A,\mathbf{\Theta}_0}(\pi_F([\chi], c)) = \pi_D([\chi], c)$$

for any $(\chi, c) \in X(k_E^\times)_{k_F\text{-reg}} \times \mathbb{C}^\times$.

Proof. It follows from the last statement of the proof of [8, Section 6, First Comparison Theorem] (see [8, Section 6.8]) that the tamely ramified character $\nu_{\mathbf{\Theta}_0}$ defined by Proposition 2.6 is trivial. Thus, the corollary follows from Theorem 1.6 and Proposition 2.10. □

We next consider the totally ramified case.

Proposition 2.12. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\mathbf{\Theta} = \mathbf{\Theta}(\pi)$. Suppose that $t(\pi) = 1$, that is, E/F is totally ramified. Then, every representation in $\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$ is totally ramified, and there exists a canonical bijection*

$$\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta}) \simeq X(k_F^\times) \times \mathbb{C}^\times.$$

Proof. Since E/F is totally ramified, we have $E = E_0$, $\mathcal{G}_0 = (1)$, and $k_E = k_F$. Thus, the assertion follows immediately from Theorem 2.9. □

We denote by $\pi_D(\chi, c)$ the image in $\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$ of $(\chi, c) \in X(k_F^\times) \times \mathbb{C}^\times$ under the bijection given in Proposition 2.12, as above.

Corollary 2.13. *Let $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$, and let $(E/F, \xi) \in P_n(F)$ be attached to π. Let $\mathbf{\Theta} = \mathbf{\Theta}(\pi)$. Suppose that $t(\pi) = 1$. Then, the Jacquet-Langlands correspondence $\boldsymbol{j}_{A,\mathbf{\Theta}}$ of (2.3) is described by*

$$\boldsymbol{j}_{A,\mathbf{\Theta}}(\pi_F(\chi, (-1)^{n-m}c)) = \pi_D(\chi, c)$$

for any $(\chi, c) \in X(k_F^\times) \times \mathbb{C}^\times$.

Proof. The tamely ramified character $\nu = \nu_{\mathbf{\Theta}}$ of $E^\times$ defined by Proposition 2.6 can be expressed as follows [8, Section 5.2, Theorem]:

$$\nu(x) = (-1)^{(n-m)v_E(x)}, \ x \in E^\times.$$

Thus, the corollary follows directly from Proposition 2.6. □

Theorem 2.14. *Let the notation and assumptions be as in Corollary 2.13. Then, the Jacquet-Langlands correspondence $\boldsymbol{j}_{A,\mathbf{\Theta}}$ on the set $\mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$ of (2.3) can be re-written by*

$$\boldsymbol{j}_{A,\mathbf{\Theta}}(\pi_F(\chi, (-1)^{n-1}c)) = \pi_D(\chi, (-1)^{m-1}c)$$

for any $(\chi, c) \in X(k_F^\times) \times \mathbb{C}^\times$.

Proof. For a representation $\pi_D(\chi, c) \in \mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$ for $(\chi, c) \in X(k_F^\times) \times \mathbb{C}^\times$, we replace this representation by $\pi_D(\chi, (-1)^{m-1}c)$. Then, from Corollary 2.13, we obtain

$$\pi_D(\chi, (-1)^{m-1}c) = \boldsymbol{j}_{A,\mathbf{\Theta}}(\pi_F(\chi, (-1)^{n-m}(-1)^{m-1}c)),$$

which shows this theorem. □

In Theorem 2.14, we choose the parameterization

$$(\chi, c) \mapsto \pi_D(\chi, (-1)^{m-1}c) : X(k_F^\times) \times \mathbb{C}^\times \simeq \mathcal{A}_m^{\mathrm{et}}(D, \mathbf{\Theta})$$

as defined by Imai and Tsushima [15].

3. Proof of Theorem 2.1

We recall the definitions and the notation of [19], and using them, we prove Theorem 2.1.

3.1. Simple character. Let A be a simple central F-algebra, and let V be a simple left A-module. Let $D = \mathrm{End}_A(V)^{\mathrm{op}}$ be the opposite of the central F-division algebra $\mathrm{End}_A(V)$ with $\dim_F(D) = d^2, d \geq 1$. Then, V can be regarded as a right D-vector space, and there exists a canonical isomorphism of F-algebras between A and $\mathrm{End}_D(V)$. Hereinafter, we identify $A = \mathrm{End}_D(V)$ through this isomorphism.

Let L/F be an unramified field extension in D with $[L : F] = d$. Following [19, 2.2.2], we set

$$\overline{A} = A \otimes_F L$$

and $\overline{V} = V \otimes_L L$. Then, $\overline{V}$ is isomorphic to V as L-vector spaces. We have $\overline{A} = \mathrm{End}_L(\overline{V})$. We regard $\overline{V}$ as a left $\overline{A}$-module.

Let $[\mathfrak{A}, \ell, 0, \beta]$ be a simple stratum in $A = \mathrm{End}_D(V)$. Let $\Lambda = (\Lambda_k)_{k\in\mathbb{Z}}$ be a strict $\mathfrak{o}_D$-lattice sequence in V with $\mathfrak{A} = \mathfrak{P}_0(\Lambda)$, where for $a \in \mathbb{Z}$, we define

$$\mathfrak{P}_a(\Lambda) = \{x \in A : x\Lambda_k \subset \Lambda_{k+a},\ k \in \mathbb{Z}\}.$$

Set $\overline{\Lambda} = \Lambda \otimes_{\mathfrak{o}_L} \mathfrak{o}_L = (\Lambda_k \otimes_{\mathfrak{o}_L} \mathfrak{o}_L)_{k\in\mathbb{Z}}$. Then, $\overline{\Lambda}$ can be identified with Λ as $\mathfrak{o}_L$-lattice sequences through the identification $\overline{V} = V$. Then, we have $\mathfrak{P}_a(\overline{\Lambda}) = \mathfrak{P}_a(\Lambda) \otimes_{\mathfrak{o}_F} \mathfrak{o}_L, a \in \mathbb{Z}$. In particular, we have $\overline{\mathfrak{A}} = \mathfrak{P}_0(\overline{\Lambda}) = \mathfrak{A} \otimes_{\mathfrak{o}_F} \mathfrak{o}_L$.

We may identify $\beta = \beta \otimes 1 \in \overline{A} = A \otimes_F L$. Then, $[\overline{\mathfrak{A}}, \ell, 0, \beta]$ is a stratum in $\overline{A}$. This is not always simple. Set $E = F[\beta]$ and $\overline{E} = E \otimes_F L$. Then, the E-algebra $\overline{E}$ is decomposed into the sum of simple components

$$\overline{E} = E^1 \oplus \cdots \oplus E^s$$

for the integer $s = \gcd(f(E|F), d)$. Here, $f(E|F)$ denotes the residue degree of E/F. Each E^i is an unramified extension of E such that $[E^i : E] = [L : F]/s$, and it is also a finite field extension of L.

For the character ψ_F as defined in Section 1.1, we fix an additive character

$$\psi_L : L \longrightarrow \mathbb{C}^\times,$$

trivial on $\mathfrak{p}_L$ but not on $\mathfrak{o}_L$, that extends ψ_F.

Attached to a simple stratum $[\mathfrak{A}, \ell, 0, \beta]$ in A, the families of $\mathfrak{o}_F$-lattices of A

$$\mathfrak{J}^k = \mathfrak{J}^k(\beta, \mathfrak{A}),\ \mathfrak{H}^k = \mathfrak{H}^k(\beta, \mathfrak{A}),\ k \geq 0$$

are defined, and the families of compact open subgroups of $A^\times$

$$J^k = J^k(\beta, \mathfrak{A}) = \mathfrak{J} \cap U^k(\mathfrak{A}),\ H^k = H^k(\beta, \mathfrak{A}) = \mathfrak{H} \cap U^k(\mathfrak{A}),\ k \geq 0$$

are defined (see [19, Proposition 3.42]), where we set $U^k(\mathfrak{A}) = 1 + \mathfrak{P}^k$. Write $J = J^0$ and $H = H^0$. Attached to the stratum $[\overline{\mathfrak{A}}, \ell, 0, \beta]$ in $\overline{A}$, the $\mathfrak{o}_L$-lattices $\overline{\mathfrak{J}}^k = \mathfrak{J}^k(\beta, \overline{\mathfrak{A}}), \overline{\mathfrak{H}}^k = \mathfrak{H}^k(\beta, \overline{\mathfrak{A}}), k \geq 0$, of $\overline{A}$ are defined similarly. From [19, Section 3.1.3], it follows that these $\mathfrak{o}_L$-lattices have the same properties as those of $\mathfrak{J}^k, \mathfrak{H}^k$. The subgroups of $\overline{G} = \overline{A}^\times$

$$\overline{J}^k = J^k(\beta, \overline{\mathfrak{A}}),\ \overline{H}^k = H^k(\beta, \overline{\mathfrak{A}}),\ k \geq 0$$

are also defined. Set $\Gamma = \mathrm{Gal}(L/F)$. Then, Γ acts naturally on $\overline{A} = A \otimes_F L$ and so on $\overline{J}^k, \overline{H}^k$, $k \geq 0$. For a Γ-set X, we denote by X^Γ the set of Γ-fixed elements in X. Then, we have $(\overline{J}^k)^\Gamma = J^k$, $(\overline{H}^k)^\Gamma = H^k, k \geq 0$.

With respect to the characters ψ_F, ψ_L, the set of *quasi-simple characters* of $H^1(\beta, \overline{\mathfrak{A}})$

$$Q(\overline{\mathfrak{A}}, \beta, \psi_L) = Q(\overline{\mathfrak{A}}, 0, \beta)$$

is defined by [19, Definition 3.22], and then the set of *simple characters* $C(\mathfrak{A}, \beta, \psi_F)$ is defined by

$$C(\mathfrak{A}, \beta, \psi_F) = \{\theta | H^1(\beta, \mathfrak{A}) : \theta \in Q(\overline{\mathfrak{A}}, \beta, \psi_L)\}$$

(see [19, Definition 3.45]).

Assume that for $\beta \in A$, the F-algebra $F[\beta]$ is a field, and set $E = F[\beta]$. As in [9, (1.2)], we write $\mathrm{A}(E) = \mathrm{End}_F(E)$ and

$$\mathfrak{A}(E) = \mathrm{End}^0_{\mathfrak{o}_F}(\{\mathfrak{p}^i_E : i \in \mathbb{Z}\}) = \{x \in \mathrm{A}(E) : x\mathfrak{p}^i_E \subset \mathfrak{p}^i_E, \text{ for all } i \in \mathbb{Z}\}.$$

Then, $[\mathfrak{A}(E), \ell, 0, \beta]$, with $\ell = -v_E(\beta)$, is a pure stratum in $\mathrm{A}(E)$. We write $k_F(\beta) = k_0(\beta, \mathfrak{A}(E))$ for the critical exponent of $[\mathfrak{A}(E), \ell, 0, \beta]$. Moreover, we assume that $k_F(\beta) < 0$. Then, $[\mathfrak{A}(E), \ell, 0, \beta]$ becomes a simple stratum in $\mathrm{A}(E)$. We can then write

$$C_F(\beta, \psi_F) = C(\mathfrak{A}(E), \beta, \psi_F)$$

which is defined by [9, (3.2)].

Let L/F and $\overline{E} = E \otimes_F L$ be as above. Set $\mathrm{A}(\overline{E}) = \mathrm{End}_L(\overline{E})$ and $\mathfrak{A}(\overline{E}) = \mathrm{End}^0_{\mathfrak{o}_L}(\{\mathfrak{p}^i_{\overline{E}} : i \in \mathbb{Z}\})$. Following the decomposition $\overline{E} = E^1 \oplus \cdots \oplus E^s$ as before, we have $\mathfrak{p}^i_{\overline{E}} = \mathfrak{p}^i_{E^1} \oplus \cdots \oplus \mathfrak{p}^i_{E^s}$, $i \in \mathbb{Z}$. Then, $[\mathfrak{A}(\overline{E}), \ell, 0, \beta]$ is a stratum in $A(\overline{E})$ and is not always simple. As in [19, 3.3.3], we set

$$C_L(\beta, \psi_L) = Q(\mathfrak{A}(\overline{E}), \beta, \psi_L).$$

Let $\beta = \beta^1 \oplus \cdots \oplus \beta^s \in \overline{E} = E^1 \oplus \cdots \oplus E^s$, with $\beta^i \in E^i$ for $1 \leq i \leq s$. Then, $[\mathfrak{A}(E^i), \ell, 0, \beta^i]$ is a simple stratum in $\mathrm{A}(E^i)$, for $1 \leq i \leq s$. Thus, we have

$$C_L(\beta^i, \psi_L) = Q(\mathfrak{A}(E^i), \beta^i, \psi_L) = C(\mathfrak{A}(E^i), \beta^i, \psi_L).$$

From [19, p. 385], there exists a canonical bijection

$$\varphi^L_\beta : C_L(\beta, \psi_L) \simeq C_L(\beta^1, \psi_L) \times \cdots \times C_L(\beta^s, \psi_L).$$

3.2. Transfer. Assume that $[\mathfrak{A}, \ell, 0, \beta]$ is a simple stratum in A. Let $[\overline{\mathfrak{A}}, \ell, 0, \beta]$ be the corresponding stratum in $\overline{A}$, as above. Let $\overline{\Lambda}$ be the strict $\mathfrak{o}_L$-lattice sequence in $\overline{V} = V \otimes_L L$, as defined above, such that $\overline{\mathfrak{A}} = \mathfrak{P}_0(\overline{\Lambda})$. From the decomposition $\overline{E} = E^1 \oplus \cdots \oplus E^s$, the left $\overline{E}$-module $\overline{V}$ is decomposed into the sum of simple components

$$\overline{V} = V^1 \oplus \cdots \oplus V^s,$$

where V^i is an E^i-vector space, for $1 \leq i \leq s$. Since E^i/L is a finite field extension, V^i can be regarded as a left L-vector space. Thus, we can set

$$\overline{A}^i = \mathrm{End}_L(V^i),$$

and

$$\overline{\Lambda}^i = \overline{\Lambda}|V^i.$$

Moreover, we set $\overline{\mathfrak{A}}^i = \mathfrak{P}_0(\overline{\Lambda}^i)$ in $\overline{A}^i$. Then, $[\overline{\mathfrak{A}}^i, \ell, 0, \beta^i]$ is a simple stratum in $\overline{A}^i$ (cf. [3, Section 6.2]). From [19, Corollary 3.34], it follows that there exists a bijection

$$\varphi_{\overline{\mathfrak{A}},\beta} : Q(\overline{\mathfrak{A}}, \beta, \psi_L) \simeq \prod_{i=1}^{s} C(\overline{\mathfrak{A}}^i, \beta^i, \psi_L).$$

We define the transfer

$$\tau_{\overline{\mathfrak{A}},\beta} : C_L(\beta, \psi_L) \simeq Q(\overline{\mathfrak{A}}, \beta, \psi_L)$$

by the following commutative diagram

$$\begin{array}{ccc} C_L(\beta, \psi_L) & \xrightarrow{\tau_{\overline{\mathfrak{A}},\beta}} & Q(\overline{\mathfrak{A}}, \beta, \psi_L) \\ {\scriptstyle \varphi^L_\beta}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_{\overline{\mathfrak{A}},\beta}} \\ \prod_{i=1}^s C_L(\beta^i, \psi_L) & \xrightarrow[\prod_i \tau_{\overline{\mathfrak{A}}^i,\beta^i}]{} & \prod_{i=1}^s C(\overline{\mathfrak{A}}^i, \beta^i, \psi_L). \end{array}$$

We now note that each transfer

$$\varphi_{\overline{\mathfrak{A}}^i,\beta^i} : C_L(\beta^i, \psi_L) = C(\mathfrak{A}(E^i), \beta^i, \psi_L) \longrightarrow C(\overline{\mathfrak{A}}^i, \beta^i, \psi_L)$$

is defined by [9, (3.6.1)] for L-split groups.

Theorem 3.1 ([19, Theorem 3.53])**.** *Let the notation and assumptions be as above. Then, there exists a bijection*

$$\tau_{\mathfrak{A},\beta} : C_F(\beta, \psi_F) \longrightarrow C(\mathfrak{A}, \beta, \psi_F)$$

such that the diagram

$$\begin{array}{ccc} C_L(\beta, \psi_L) & \xrightarrow{\tau_{\overline{\mathfrak{A}},\beta}} & Q(\overline{\mathfrak{A}}, \beta, \psi_L) \\ {\scriptstyle \mathrm{res}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}} \\ C_F(\beta, \psi_F) & \xrightarrow[\tau_{\mathfrak{A},\beta}]{} & C(\mathfrak{A}, \beta, \psi_F) \end{array}$$

is commutative.

3.3. Proof of Theorem 2.1. We now prove Theorem 2.1.

We assume that $\pi \in \mathcal{A}_m^{\mathrm{et}}(D)$ has positive level. Let $(E/F, \xi)$ be an admissible pair over F that has degree n and is attached to π. By assumption, we have $\xi|U_E^1 \neq 1$. Here, we may set $\pi = \pi_D(\xi)$. On the other hand, there exists a pair $([\mathfrak{A}, \ell, 0, \beta], \theta)$ such that $[\mathfrak{A}, \ell, 0, \beta]$ is a simple stratum in A and $\theta \in C(\mathfrak{A}, \beta, \psi_F)$ is contained in $\pi|H^1(\beta, \mathfrak{A})$. Thus, we obtain

$$\mathbf{\Theta}(\pi) = \mathcal{E}_F(\theta) \in \widetilde{\mathcal{E}}(F),$$

as in Section 1.4. Set $E_0 = F[\beta]$. Then, again from the arguments of Section 1.4, there exists a character ξ_0 of $U_{E_0}^1$ which satisfies

$$\theta|U_E^1 = \xi|U_E^1 = \xi_0 \circ \mathrm{N}_{E/E_0}.$$

We note that $H^1(\beta, \mathfrak{A}) \cap E = U_E^1$. Moreover, $(E_0/F, \xi_0)$ is an admissible 1-pair. From Theorem 3.1, it follows that there exists a unique simple character $\theta_0 \in C_F(\beta, \psi_F)$ such that

$$\theta = \tau_{\mathfrak{A},\beta}(\theta_0).$$

Thus, from [3, Definitions 1.5 and 1.10], we obtain

$$\mathbf{\Theta}(\pi) = \mathcal{E}_F(\theta) = \mathcal{E}_F(\theta_0)$$

in $\widetilde{\mathcal{E}}(F)$. Again from Theorem 3.1, there exist $\overline{\theta}_0 \in C_L(\beta, \psi_L)$ and $\overline{\theta} \in Q(\overline{\mathfrak{A}}, \beta, \psi_L)$ such that

$$\overline{\theta}_0|H^1(\beta, \mathfrak{A}(E)) = \theta_0, \ \overline{\theta}|H^1(\beta, \mathfrak{A}) = \theta$$

and moreover,

$$\overline{\theta} = \tau_{\overline{\mathfrak{A}},\beta}(\overline{\theta}_0).$$

In order to prove Theorem 2.1, it is enough to prove

$$\overline{\theta}_0(x) = \overline{\theta}(x), \ x \in U_E^1. \tag{3.1}$$

In fact, this implies that

$$\theta_0(x) = \theta(x) = \xi(x) = \xi_0(\mathrm{N}_{E/E_0}(x)), \ x \in U_E^1.$$

We apply the arguments of [7, Section 2.3] to $\mathcal{E}_F(\theta_0) \in \mathcal{E}(F)$, so that

$$\mathbf{\Theta}(\pi) = \mathcal{E}_F(\theta) = \mathcal{E}_F(\theta_0) = \mathbf{\Theta}_{\xi_0} = \mathbf{\Theta}_{\xi},$$

which is the desired result of Theorem 2.1.

We will prove Equation (3.1). For the quasi-simple characters $\overline{\theta}_0, \overline{\theta}$, set

$$\varphi_\beta^L(\overline{\theta}_0) = (\overline{\theta}_0^i) \in \prod_{i=1}^s C_L(\beta^i, \psi_L),$$

$$\varphi_{\overline{\mathfrak{A}},\beta}(\overline{\theta}) = (\overline{\theta}^i) \in \prod_{i=1}^s C(\overline{\mathfrak{A}}^i, \beta^i, \psi_L).$$

By the commutative diagram before Theorem 3.1, for each i, $1 \leq i \leq s$, we have

$$\overline{\theta}^i = \tau_{\overline{\mathfrak{A}}^i,\beta^i}(\overline{\theta}_0^i). \tag{3.2}$$

For a fixed i, we have

$$\overline{\theta}_0^i = \overline{\theta}_0 | H^1(\beta^i, \mathfrak{A}(E^i)),\ \overline{\theta}^i = \overline{\theta} | H^1(\beta^i, \overline{\mathfrak{A}}^i).$$

Since we have

$$U_E^1 \subset U_{E^i}^1 \subset H^1(\beta^i, \mathfrak{A}(E^i)) \cap H^1(\beta^i, \overline{\mathfrak{A}}^i),$$

from Equation (3.2) and [9, (3.6.1)], we hence obtain Equation (3.1). The proof of Theorem 2.1 is completed.

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