Kato, K., Kobayashi, M. and Ito, S.
Osaka J. Math.
54 (2017), 209–228

REMARK ON CHARACTERIZATION OF WAVE FRONT SET BY WAVE PACKET TRANSFORM

Keiichi KATO, Masaharu KOBAYASHI and Shingo ITO

(Received July 22, 2015, revised January 18, 2016)

Abstract
In this paper, we give characterizations of usual wave front set and wave front set in H^s in terms of wave packet transform without any restriction on basic wave packet, which give complete answers of the question raised by G. B. Folland.

1. Introduction

In this paper, we discuss on characterization of wave front sets in terms of the wave packet transform. The wave front set is introduced by L. Hörmander [6], which is one of the main tools of microlocal analysis in C^∞ category. The wave front set of a distribution is a set of singular points of the distribution in phase space. Such an idea to classify the singularities of generalized functions "microlocally" has been introduced by M. Sato, J. Bros and D. Iagolnitzer and L. Hörmander independently in the 1970s (see M. Sato-T. Kawai-M. Kashiwara [15], L. Hörmander [7], [8], F. Trèves [16]).

The wave packet transform has been introduced by A. Córdoba-C. Fefferman [1]. Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$. Then the wave packet transform $W_\phi f(x,\xi)$ of f with the wave packet generated by a function ϕ is defined by

$$W_\phi f(x,\xi) = \int_{\mathbb{R}^n} \overline{\phi(y-x)} f(y) e^{-iy\cdot\xi} dy.$$

We call the above ϕ basic wave packet in this paper. Wave packet transform is called short time Fourier transform or windowed Fourier transform in several literatures. We refer to [5] for more details.

G. B. Folland introduced the characterization of C^∞ wave front set in terms of wave packet transform with a positive symmetric Schwartz's function as basic wave packet ([3, Theorem 3.22]). He proposed a question whether the same conclusion is still valid without the restriction of basic wave packet. T. Ōkaji [12, Theorem 2.2] has given a partial answer with a Schwartz's function ϕ satisfying $\int x^\alpha \phi dx \neq 0$ for some multi-indices α as basic wave packet. Using such basic wave packet, he has also given a sufficient condition and a necessary condition which imply that a point in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ belongs in H^s wave front set. But he does not give a characterization of H^s wave front set in terms of wave packet transform. In this paper, using wave packet transform, we give characterizations for C^∞ wave front set and H^s wave front set without any restriction on basic wave packets. The definitions of the C^∞ wave front set $WF(\cdot)$ and H^s wave front set $WF_{H^s}(\cdot)$ are given in

2010 Mathematics Subject Classification. Primary 35A18; Secondary 42B10.

Definitions 1.1 and 1.2. Our main results are the following theorems.

Theorem 1.1. *Let $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. The following conditions are equivalent.*

(i) $(x_0,\xi_0) \notin WF(u)$.

(ii) There exist $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, a neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ and $a \geq 1$ there exists a constant $C_{N,a}$ satisfying

$$|W_{\phi_\lambda} u(x,\lambda\xi)| \leq C_{N,a}\lambda^{-N}$$

for all $\lambda \geq 1$, $x \in K$ and $\xi \in \Gamma$ with $a^{-1} \leq |\xi| \leq a$, where $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\varphi(\lambda^{\frac{1}{2}}x)$.

(iii) There exist a neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$, $a \geq 1$ and $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ there exists a constant $C_{N,a,\phi}$ satisfying

$$|W_{\phi_\lambda} u(x,\lambda\xi)| \leq C_{N,a,\phi}\lambda^{-N}$$

for all $\lambda \geq 1$, $x \in K$ and $\xi \in \Gamma$ with $a^{-1} \leq |\xi| \leq a$, where $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\varphi(\lambda^{\frac{1}{2}}x)$.

Theorem 1.2. *Let $s \in \mathbb{R}$, $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. The following conditions are equivalent.*

(i) $(x_0,\xi_0) \notin WF_{H^s}(u)$.

(ii) There exist $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, a neighborhood K of x_0 and a neighborhood V of ξ_0 such that

$$\int_1^\infty \lambda^{n-1+2s} \int_V \int_K |W_{\phi_\lambda} u(x,\lambda\xi)|^2 dx d\xi d\lambda < \infty, \tag{1.1}$$

where $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\varphi(\lambda^{\frac{1}{2}}x)$.

(iii) There exist a neighborhood K of x_0 and a neighborhood V of ξ_0 such that

$$\int_1^\infty \lambda^{n-1+2s} \int_V \int_K |W_{\phi_\lambda} u(x,\lambda\xi)|^2 dx d\xi d\lambda < \infty \tag{1.2}$$

for all $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, where $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\varphi(\lambda^{\frac{1}{2}}x)$.

Remark 1.3. We would like to emphasize the fact that basic wave packets in above theorems have no restriction. For the proof of Theorem 1.1, in [3] it has been assumed that ϕ is positive and symmetric and in [12] it has been assumed that ϕ satisfies $\int_{\mathbb{R}^n} x^\alpha \phi(x)dx \neq 0$ for some α (see Theorem A below). For the proof of Theorem 1.2, P. Gérard [4, Proposition 1.1] has shown it only when $\phi(x)$ is a Gaussian function $e^{-\frac{|x|^2}{2}}$ (see also J. M. Delort [2, Theorem 1.2]) and in [12] it has been assumed that ϕ satisfies $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$ and they make a loss with respect to the order of λ when they show (i) implies (iii) (see Theorem B below).

Remark 1.4. In Theorem 1.1 and 1.2, we can replace $\lambda^{\frac{bn}{2}}\phi(\lambda^b x)$ $(0 < b < 1)$ for $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\varphi(\lambda^{\frac{1}{2}}x)$ without any change of our proof. This fact plays an important role in our application given in Section 1.5.

Let us begin with a brief review of the history of characterizations of wave front sets by the wave packet transform. The C^∞ wave front set is defined as follows.

Definition 1.1. (C^∞ wave front set) Let $(x_0,\xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. We say that (x_0, ξ_0) is not in the C^∞ wave front set of u, denoted by $(x_0, \xi_0) \notin WF(u)$, if and only if there exist a function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ there exists $C_N > 0$ satisfying

$$|\mathcal{F}[\chi f](\xi)| \le C_N(1 + |\xi|)^{-N} \tag{1.3}$$

for $\xi \in \Gamma$.

Concerning the C^∞ wave front set, following characterization is known.

Theorem A. *(G. B. Folland* [3, Theorem 3.22] *and T. Ōkaji* [12, Theorem 2.2]*). Let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} x^\alpha \phi(x)dx \neq 0$ for some multi-indices α and put $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\phi(\lambda^{\frac{1}{2}}x)$. Then, $(x_0, \xi_0) \notin WF(u)$ if and only if there exist a neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ and for all $a \ge 1$ there exists a constant $C_{N,a}$ satisfying*

$$|W_{\phi_\lambda} u(x, \lambda\xi)| \le C_{N,a}\lambda^{-N}$$

for $\lambda \ge 1$, $x \in K$ and $\xi \in \Gamma$ with $a^{-1} \le |\xi| \le a$.

T. Ōkaji [12] also discuss the H^s wave front set. The H^s wave front set is defined as follows.

Definition 1.2. (H^s wave front set) Let $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. We say that (x_0, ξ_0) is not in the H^s wave front set of u, denoted by $(x_0, \xi_0) \notin WF_{H^s}(u)$, if and only if there exist a conic neighborhood Γ of ξ_0 and a function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near x_0 such that

$$\|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|_{L^2(\Gamma)} < \infty, \tag{1.4}$$

where $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

Concerning the H^s wave front set, following characterization is known.

Theorem B. *(T. Ōkaji* [12, Theorem 2.4]*) Let $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Suppose that $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \phi(x)dx \neq 0$ and put $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\phi(\lambda^{\frac{1}{2}}x)$. If there exist a neighborhood K of x_0 and a neighborhood V of ξ_0 such that*

$$\int_1^\infty \lambda^{n-1+2s} \int_V \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi d\lambda < \infty \tag{1.5}$$

then $(x_0, \xi_0) \notin WF_{H^s}(u)$. Conversely, if $(x_0, \xi_0) \notin WF_{H^s}(u)$ then there exist a neighborhood K of x_0 and a neighborhood V of ξ_0 such that

$$\int_1^\infty \lambda^{n-1+2s-\varepsilon} \int_V \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi d\lambda < \infty \tag{1.6}$$

for all $\varepsilon > 0$.

Remark 1.5. There is a gap between (1.5) and (1.6). In Theorem 1.2, we remove this gap and weaken the condition of ϕ.

Remark 1.6. S. Pilipović, N. Teofanov and J. Toft [13], [14] treat wave front set in Fourier-Lebesgue space in terms of short time Fourier transform, which is called wave packet transform in this paper.

This paper is organized as follows. In Section 2, we prepare several propositions and lemmas. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.1. In Section 5, we give an application of our characterization of wave front set.

2. Preliminary

2.1. Notations. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ stands $\{y \in \mathbb{R}^n \,|\, |y - x| < r\}$. $\mathcal{F}[f](\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$ is the Fourier transform of f. For a subset A of $\mathbb{R}^n$, we denote the complement of A by A^c, the set of all interior points of A by A° and the closure of A by $\overline{A}$. $\mathbf{1}_A$ is the characteristic function of A, that is, $\mathbf{1}_A(x) = 1$ for $x \in A$ and $\mathbf{1}_A(x) = 0$ for $x \in A^c$. Throughout this paper, C, C_i, C' and C'_i $(i = 1, 2, 3, \ldots)$ serve as positive constants, if the precise value of which is not needed and C_ϕ denote positive constants depending on ϕ.

2.2. Key Proposition and Lemmas. First, we prepare the following proposition and lemma. Proposition 2.1 and Lemma 2.3 are used in Section 4. Proposition 2.2, Lemma 2.4, 2.5 and 2.6 are used in Section 3. Although Propositions 2.1, 2.2 and Lemma 2.3 are easy to prove by the standard method, we give the proof for reader's convenience in Appendix.

Proposition 2.1. *Let $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Assume that $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\chi \equiv 1$ near x_0. Then, the following conditions are equivalent.*

(i) There exists a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ there exists $C_N > 0$ satisfying

$$|\mathcal{F}[\chi u](\xi)| \leq C_N(1 + |\xi|)^{-N}$$

for $\xi \in \Gamma$.

(ii) There exists a neighborhood V of ξ_0 such that for all $N \in \mathbb{N}$ there exists $C_N > 0$ satisfying

$$|\mathcal{F}[\chi u](\lambda\xi)| \leq C_N(1 + \lambda|\xi|)^{-N}$$

for $\xi \in V$ and $\lambda \geq 1$.

Proposition 2.2. *Let $s \in \mathbb{R}$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Assume that $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\chi \equiv 1$ near x_0. Then, the following conditions are equivalent.*

(i) There exists a conic neighborhood Γ of ξ_0 such that

$$\|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|_{L^2(\Gamma)} < \infty. \tag{2.1}$$

(ii) There exists a neighborhood V of ξ_0 such that

$$\int_1^\infty \lambda^{n-1+2s} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V)} d\lambda < \infty. \tag{2.2}$$

Lemma 2.3. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$, $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near x_0. Suppose that neighborhoods V and V' of ξ_0 satisfy $\overline{V'} \subset V$ and $\|\mathcal{F}[\chi u](\lambda\xi)\|_{L^2(V)} < \infty$*

for $\lambda \geq 1$. *Then, for all* $\zeta \in \mathcal{S}(\mathbb{R}^n)$ *and* $N \in \mathbb{N}$ *there exist* $C_{N,\zeta} > 0$ *satisfying*

$$\int_{V'} |\mathcal{F}[\zeta\chi u](\lambda\xi)|^2 d\xi \leq C_{N,\zeta}\left(\int_V |\mathcal{F}[\chi u](\lambda\xi)|^2 d\xi + \lambda^{-N}\right). \tag{2.3}$$

Lemma 2.4. *Let* $u \in \mathcal{S}'(\mathbb{R}^n)$, $\phi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$, $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ *and* $\chi \in C_0^\infty(\mathbb{R}^n)$ *with* $\chi \equiv 1$ *near* x_0. *Suppose that K is a neighborhood of* x_0 *and satisfies* $\overline{K} \subset \{x \in \mathbb{R}^n | \chi(x) = 1\}^\circ$. *Then, there exists* $m \in \mathbb{N}$ *(which depends on only u) such that for all* $\mu \in \mathbb{R}$ *there exists* $C_\mu > 0$ *satisfying*

$$\mathbf{1}_K(x)\left|W_{\phi_\lambda}[(1-\chi)u](x, \lambda\xi)\right| \leq C_\mu \lambda^{-\mu}\langle\xi\rangle^m, \tag{2.4}$$

where $\phi_\lambda(x) = \lambda^{\frac{n}{4}}\phi(\lambda^{\frac{1}{2}}x)$.

Proof. Since $u \in \mathcal{S}'(\mathbb{R}^n)$, the structure theorem of $\mathcal{S}'$ (see, for example [11, Theorem 2.14]) yields that there exist $l, m \in \mathbb{N}$ and functions $f_\alpha \in L^2(\mathbb{R}^n)$ such that

$$u(y) = \langle y\rangle^l \sum_{|\alpha|\leq m} \partial^\alpha f_\alpha(y),$$

where α denotes multi-indices. Put $g(x, y) = \overline{\phi_\lambda(y - x)}\{1 - \chi(y)\}\langle y\rangle^l$ and take $N \in \mathbb{N}$ satisfying $4N \geq 2m + n + 1$. Since $(1 - \Delta_y)^N e^{-iy\cdot(\lambda\xi-\eta)} = \langle\lambda\xi - \eta\rangle^{2N} e^{-iy\cdot(\lambda\xi-\eta)}$ for $N \in \mathbb{N}$, we have, by Schwarz's inequality and integration by parts,

$$\begin{aligned}
\left|W_{\phi_\lambda}[(1-\chi)u](x, \lambda\xi)\right| &\leq \sum_{|\alpha|\leq m}\left|\int_{\mathbb{R}^n} g(x,y)\partial^\alpha f_\alpha(y)e^{-i\lambda y\cdot\xi}dy\right| \\
&\leq \sum_{|\alpha|\leq m}\int_{\mathbb{R}^n}\left|\mathcal{F}[g(x,\cdot)](\lambda\xi - \eta)\eta^\alpha \mathcal{F}[f_\alpha](\eta)\right|d\eta \\
&\leq \sum_{|\alpha|\leq m}\|\mathcal{F}[f_\alpha]\|_{L^2}\left(\int_{\mathbb{R}^n}\left|\eta^\alpha\int_{\mathbb{R}^n} g(x,y)e^{-iy\cdot(\lambda\xi-\eta)}dy\right|^2 d\eta\right)^{\frac{1}{2}} \\
&\leq (2\pi)^{\frac{n}{2}}\sum_{|\alpha|\leq m}\|f_\alpha\|_{L^2}\left\{\int_{\mathbb{R}^n}|\eta|^{2|\alpha|}\left(\int_{\mathbb{R}^n}\frac{|(1-\Delta_y)^N g(x,y)|}{\langle\lambda\xi-\eta\rangle^{2N}}dy\right)^2 d\eta\right\}^{\frac{1}{2}} \\
&\leq (2\pi)^{\frac{n}{2}}\sum_{|\alpha|\leq m}\|f_\alpha\|_{L^2}\left(\int_{\mathbb{R}^n}\frac{|\eta|^{2|\alpha|}}{\langle\lambda\xi-\eta\rangle^{4N}}d\eta\right)^{\frac{1}{2}}\int_{\mathbb{R}^n}|(1-\Delta_y)^N g(x,y)|dy.
\end{aligned} \tag{2.5}$$

From the fact that

$$|\eta|^{2m} \leq 2^{2m-1}(|\eta - \lambda\xi|^{2m} + |\lambda\xi|^{2m}) \leq 2^{2m-1}(\langle\eta - \lambda\xi\rangle^{2m} + \langle\lambda\xi\rangle^{2m}),$$

it follows that

$$\int_{\mathbb{R}^n}\frac{|\eta|^{2|\alpha|}}{\langle\lambda\xi-\eta\rangle^{4N}}d\eta \leq \int_{\mathbb{R}^n}\langle\lambda\xi\rangle^{2m}\frac{|\eta|^{2m}}{\langle\lambda\xi\rangle^{2m}\langle\lambda\xi-\eta\rangle^{4N}}d\eta \leq C_1\lambda^{2m}\langle\xi\rangle^{2m}. \tag{2.6}$$

On the other hand,

$$\int_{\mathbb{R}^n}|(1-\Delta_y)^N g(x,y)|dy \tag{2.7}$$

$$\leq \sum_{|\beta_1|+|\beta_2|+|\beta_3|\leq 2N} C_{\beta_1,\beta_2,\beta_3} \int_{\mathbb{R}^n} |\partial_y^{\beta_1}\phi_\lambda(y-x)\cdot\partial_y^{\beta_2}\{1-\chi(y)\}\cdot\partial_y^{\beta_3}\langle y\rangle^l|dy$$

holds. If $x \in K$ and $y \in \mathrm{supp}\{\partial_y^{\beta_2}(1-\chi(y))\}$ then there exist $C_2 > 0$ and $C_2' > 0$ such that $|y-x| \geq C_2$ and thus $|y-x| \geq C_2'\langle y-x\rangle$, which shows that

$$|\partial_y^{\beta_3}\langle y\rangle^l| \leq C_3\langle y\rangle^l \leq C_3'\langle y-x\rangle^M\langle x\rangle^l = C_M|y-x|^M\langle x\rangle^l$$

for $M \geq l$. Thus we have

$$\begin{aligned}(2.8)\qquad &\mathbf{1}_K(x)\int_{\mathbb{R}^n}|\partial_y^{\beta_1}\phi_\lambda(y-x)\cdot\partial_y^{\beta_2}\{1-\chi(y)\}\cdot\partial_y^{\beta_3}\langle y\rangle^l|dy\\ &\leq C_M'\,\mathbf{1}_K(x)\lambda^{\frac{n}{4}+\frac{|\beta_1|}{2}-\frac{M}{2}}\int_{\mathbb{R}^n}|(\partial_y^{\beta_1}\phi)(\lambda^{\frac12}(y-x))|\,(\lambda^{\frac12}|y-x|)^M\langle x\rangle^l dy\\ &\leq C_M''\lambda^{N-\frac{M}{2}-\frac{n}{4}}.\end{aligned}$$

From (2.5), (2.6), (2.7) and (2.8) we obtain

$$(2.9)\qquad \mathbf{1}_K(x)\left|W_{\phi_\lambda}[(1-\chi)u](x,\lambda\xi)\right| \leq C_M'''\lambda^{m+N-\frac{M}{2}-\frac{n}{4}}\langle\xi\rangle^m.$$

Thus, if we take M sufficiently large, we obtain (2.4). □

Lemma 2.5. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \geq 1$ and $\delta, k > 0$. Set $A = \{\eta \in \mathbb{R}^n \,|\, |\eta| \geq \delta\lambda^{\frac34}\}$. Then, for all $q > 0$ there exists $C > 0$ such that*

$$(2.10)\qquad \int_A \left|\langle\lambda\eta\rangle^k\mathcal{F}[\phi](\lambda^{-\frac12}\eta)\right|^2 d\eta \leq C\lambda^{-q}$$

for all $\lambda \geq 1$.

Proof. If $\eta \in A$, then $|\lambda^{-\frac12}\eta| \geq \delta\lambda^{\frac14}$ and thus

$$|\mathcal{F}[\phi](\lambda^{-\frac12}\eta)| \leq \frac{|\lambda^{-\frac12}\eta|^p|\mathcal{F}[\phi](\lambda^{-\frac12}\eta)|}{\delta^p\lambda^{\frac{p}{4}}}$$

for all $p > 0$. Since $\lambda \geq 1$, simple calculation yields that $\langle\lambda\eta\rangle^2 \leq \lambda^3\langle\lambda^{-\frac12}\eta\rangle^2$. Thus, we have

$$\begin{aligned}\int_A\left|\langle\lambda\eta\rangle^k\mathcal{F}[\phi](\lambda^{-\frac12}\eta)\right|^2 d\eta &\leq \int_A \lambda^{3k}\langle\lambda^{-\frac12}\eta\rangle^{2k}\frac{|\lambda^{-\frac12}\eta|^{2p}|\mathcal{F}[\phi](\lambda^{-\frac12}\eta)|^2}{\delta^{2p}\lambda^{\frac{p}{2}}}d\eta\\ &= \lambda^{3k-\frac{p}{2}+\frac{n}{2}}\delta^{-2p}\|\langle\cdot\rangle^k|\cdot|^p\mathcal{F}[\phi]\|^2_{L^2(\mathbb{R}^n)}\end{aligned}$$

by the change of variables. Therefore, if we take p sufficiently large then we obtain the desired result. □

Lemma 2.6. *Let $k \in \mathbb{N}$, $\chi \in C_0^\infty(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$, $f \in C^\infty(\mathbb{R}^{2n})$ and all derivatives of f be bounded. Put*

$$(2.11)\qquad F_{\alpha,\beta}(\eta,\xi) = \iint_{\mathbb{R}^{2n}}\chi(y)\partial_x^\alpha f(x,y)\partial_x^\beta(\psi_\lambda(y-x))e^{-ix\cdot\eta-iy\cdot\xi}dxdy$$

for multi-indices α,β, where $\psi_\lambda(x) = \lambda^{\frac{n}{4}}\psi(\lambda^{\frac12}x)$. Then,

$$(2.12)\qquad \|\langle\xi'\rangle^k F_{\alpha,\beta}(\eta,\lambda\xi-\xi')\|^2_{L^2_{\xi'}(\mathbb{R}^n)} \leq C\lambda^{3k+2+|\beta|}\langle\xi\rangle^{2k}.$$

Proof. Let $N = [\frac{2k+n}{4}] + 1$. Since $(1-\Delta_y)^N e^{-iy\cdot(\lambda\xi-\xi')} = \langle\lambda\xi-\xi'\rangle^{2N} e^{-iy\cdot(\lambda\xi-\xi')}$, we have, by integration by parts,

$$\begin{aligned}
&|F_{\alpha,\beta}(\eta,\lambda\xi-\xi')| \\
&\le \frac{1}{\langle\lambda\xi-\xi'\rangle^{2N}} \iint_{\mathbb{R}^{2n}} \Big|(1-\Delta_y)^N\{\chi(y)\partial_x^\alpha f(x,y)\partial_x^\beta(\psi_\lambda(y-x))\}\Big| dxdy \\
&\le \frac{C}{\langle\lambda\xi-\xi'\rangle^{2N}} \sum_{|\gamma_1|+|\gamma_2|+|\gamma_3|\le 2N} \iint_{\mathbb{R}^{2n}} \Big|\partial_y^{\gamma_1}\chi(y)\cdot\partial_y^{\gamma_2}\partial_x^\alpha f(x,y)\cdot\partial_y^{\gamma_3}\partial_x^\beta(\psi_\lambda(y-x))\Big| dxdy \\
&\le \frac{C'}{\langle\lambda\xi-\xi'\rangle^{2N}} \sum_{|\gamma_1|+|\gamma_3|\le 2N} \lambda^{\frac{n}{4}+\frac{|\gamma_3|}{2}+\frac{|\beta|}{2}} \iint_{\mathbb{R}^{2n}} |\partial_y^{\gamma_1}\chi(y)\cdot(\partial^{\gamma_3+\beta}\psi)(\lambda^{\frac{1}{2}}(y-x))| dxdy \\
&= \frac{C'}{\langle\lambda\xi-\xi'\rangle^{2N}} \sum_{|\gamma_1|+|\gamma_3|\le 2N} \lambda^{-\frac{n}{4}+\frac{|\gamma_3|}{2}+\frac{|\beta|}{2}} \int_{\mathbb{R}^n} |\partial_y^{\gamma_1}\chi(y)| dy \int_{\mathbb{R}^n} |\partial^{\gamma_3+\beta}\psi(x)| dx \\
&\le \frac{C''}{\langle\lambda\xi-\xi'\rangle^{2N}} \lambda^{-\frac{n}{4}+N+\frac{|\beta|}{2}}.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\|\langle\xi'\rangle^k F_{\alpha,\beta}(\eta,\lambda\xi-\xi')\|^2_{L^2_{\xi'}(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle\xi'\rangle^{2k} |F_{\alpha,\beta}(\eta,\lambda\xi-\xi')|^2 d\xi' \\
&\le C\lambda^{-\frac{n}{2}+2N+|\beta|}\langle\lambda\xi\rangle^{2k} \int_{\mathbb{R}^n} \frac{\langle\xi'\rangle^{2k}}{\langle\lambda\xi-\xi'\rangle^{4N}\langle\lambda\xi\rangle^{2k}} d\xi'.
\end{aligned}$$

From the fact that $\langle\xi'\rangle^{2k} \le 2^{2k-1}(\langle\lambda\xi-\xi'\rangle^{2k} + \langle\lambda\xi\rangle^{2k})$, we have

$$\frac{\langle\xi'\rangle^{2k}}{\langle\lambda\xi-\xi'\rangle^{4N}\langle\lambda\xi\rangle^{2k}} \le \frac{2^{2k}}{\langle\lambda\xi-\xi'\rangle^{4N-2k}}.$$

Thus, we obtain

$$\|\langle\xi'\rangle^k F_{\alpha,\beta}(\eta,\lambda\xi-\xi')\|^2_{L^2_{\xi'}(\mathbb{R}^n)} \le C\lambda^{-\frac{n}{2}+2[\frac{2k+n}{4}]+2+|\beta|+2k}\langle\xi\rangle^{2k} \le C\lambda^{3k+2+|\beta|}\langle\xi\rangle^{2k}.$$

□

3. Proof of Theorem 1.2

Trivially, (iii) yields (ii). So, in order to prove Theorem 1.2, we show that (i) implies (iii) in Proposition 3.1 and (ii) implies (i) in Proposition 3.2.

Proposition 3.1. *Under the same assumption as in Theorem 1.2, (i) implies (iii).*

Proof. By (i) and Proposition 2.2, there exist a neighborhood V_1 of ξ_0 and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near x_0 such that

$$\int_1^\infty \lambda^{n-1+2s}\|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V_1)} d\lambda < \infty.$$

Take $d > 0$ satisfying $\overline{B(\xi_0, 2d)} \subset V_1$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$, $V = B(\xi_0, d)$ and K be a neighborhood of x_0 satisfying $\overline{K} \subset \{x \in \mathbb{R}^n \,|\, \chi(x) = 1\}^\circ$. In order to prove (3.10), we divide $W_{\phi_\lambda}u(x,\lambda\xi)$ into two parts:

$$W_{\phi_\lambda}u(x,\lambda\xi) = W_{\phi_\lambda}[\chi u](x,\lambda\xi) + W_{\phi_\lambda}[(1-\chi)u](x,\lambda\xi).$$

From Lemma 2.4, it follows that for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\int_V \int_K |W_{\phi_\lambda}[(1-\chi)u](x, \lambda\xi)|^2 dx d\xi \le C_N \lambda^{-N}. \tag{3.1}$$

Let $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ satisfy $\widetilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. Applying Taylor's theorem to $\widetilde{\chi}(x)$, we have

$$\widetilde{\chi}(x) = \widetilde{\chi}(y) + \sum_{1\le|\alpha|\le L} \frac{\partial_x^\alpha \widetilde{\chi}(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=L+1} (x-y)^\alpha R_\alpha(x,y), \tag{3.2}$$

where

$$R_\alpha(x,y) = \frac{L+1}{\alpha!} \int_0^1 \partial_x^\alpha \widetilde{\chi}(y+\theta(x-y))(1-\theta)^L d\theta.$$

By Plancherel's theorem and (3.2), we have

$$\begin{aligned}\int_V \int_K |W_{\phi_\lambda}[\chi u](x, \lambda\xi)|^2 dx d\xi &\le \int_V \int_{\mathbb{R}^n} |\widetilde{\chi}(x) W_{\phi_\lambda}[\chi u](x, \lambda\xi)|^2 dx d\xi \\ &= (2\pi)^{-n} \int_V \int_{\mathbb{R}^n} \left| \mathcal{F}_{x\to\eta}\left[\widetilde{\chi}(x) W_{\phi_\lambda}[\chi u](x,\lambda\xi)\right](\eta)\right|^2 d\eta d\xi \\ &\le C \int_V \int_{\mathbb{R}^n} (|I_1|^2 + |I_2|^2) d\eta d\xi,\end{aligned}$$

where

$$I_1 = \iint_{\mathbb{R}^{2n}} \overline{\phi_\lambda(y-x)} \chi(y) u(y) e^{-i\lambda y\cdot\xi - ix\cdot\eta} dy dx \tag{3.3}$$

and

$$I_2 = \sum_{|\alpha|=L+1} \iint_{\mathbb{R}^{2n}} R_\alpha(x,y)(x-y)^\alpha \overline{\phi_\lambda(y-x)} \chi(y) u(y) e^{-i\lambda y\cdot\xi - ix\cdot\eta} dy dx. \tag{3.4}$$

Here, we use $\widetilde{\chi}\chi = \chi$ and $(\partial_x^\alpha \widetilde{\chi})\chi \equiv 0$ for $|\alpha| \ge 1$.

First, we consider I_1. By Fubini's theorem and the change of variables, we have

$$I_1 = \lambda^{-\frac{n}{4}} \overline{\mathcal{F}[\phi](\lambda^{-\frac{1}{2}}\eta)} \mathcal{F}[\chi u](\eta + \lambda\xi).$$

Let $B = B(0, d\lambda^{\frac{3}{4}})$. If $\eta \in B$, $\xi \in V$ and $\eta + \lambda\xi = \lambda\xi'$ then $\xi' \in B(\xi_0, 2d)$. So, we have

$$\begin{aligned}\int_V \int_B |I_1|^2 d\eta d\xi &= \int_B \int_V |\lambda^{-\frac{n}{4}} \mathcal{F}[\phi](\lambda^{-\frac{1}{2}}\eta)|^2 |\mathcal{F}[\chi u](\eta+\lambda\xi)|^2 d\xi d\eta \\ &\le \int_B |\lambda^{-\frac{n}{4}} \mathcal{F}[\phi](\lambda^{-\frac{1}{2}}\eta)|^2 d\eta \int_{B(\xi_0,2d)} |\mathcal{F}[\chi u](\lambda\xi')|^2 d\xi' \\ &\le (2\pi)^n \|\phi\|^2_{L^2(\mathbb{R}^n)} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V_1)}.\end{aligned} \tag{3.5}$$

If $\eta \in B^c$ and $\xi \in V$ then $\langle \eta + \lambda\xi \rangle \le C\langle \lambda\eta \rangle$ holds. Since $u \in \mathcal{S}'(\mathbb{R}^n)$, there exists $k \in \mathbb{N}$ such that $\langle \xi \rangle^{-k} \mathcal{F}[\psi u](\xi) \in L^2(\mathbb{R}^n)$ for all $\psi \in C_0^\infty(\mathbb{R}^n)$. Thus, the change of variables and Lemma 2.5 yield that for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\int_V \int_{B^c} |I_1|^2 d\eta d\xi \le C \int_{B^c} \int_V \frac{\langle \lambda\eta \rangle^{2k}}{\langle \eta + \lambda\xi \rangle^{2k}} \left| \lambda^{-\frac{n}{4}} \mathcal{F}[\phi](\lambda^{-\frac{1}{2}}\eta) \mathcal{F}[\chi u](\eta+\lambda\xi) \right|^2 d\xi d\eta \tag{3.6}$$

$$\leq C\lambda^{-\frac{3n}{2}} \int_{B^c} |\langle \lambda\eta\rangle^k \mathcal{F}[\phi](\lambda^{-\frac{1}{2}}\eta)|^2 d\eta \int_{\mathbb{R}^n} \left| \frac{\mathcal{F}[\chi u](\xi)}{\langle \xi \rangle^k} \right|^2 d\xi$$
$$\leq C_N \lambda^{-N}.$$

Next, we consider I_2. Let $M \in \mathbb{N}$ and $4M \geq n+1$. Since $(1-\Delta_x)^M e^{-ix\cdot\eta} = \langle \eta \rangle^{2M} e^{-ix\cdot\eta}$, we have, by integration by parts and Schwarz's inequality,

$$\text{(3.7)} \qquad |I_2| \leq \sum_{|\alpha|=L+1} \left| \iint_{\mathbb{R}^{2n}} \frac{(1-\Delta_x)^M \{R_\alpha(x,y)(x-y)^\alpha \overline{\phi_\lambda(y-x)}\}}{\langle \eta \rangle^{2M}} \chi(y)u(y)e^{-ix\cdot\eta - i\lambda y\cdot\xi} dy dx \right|$$
$$\leq \frac{1}{\langle \eta \rangle^{2M}} \sum_{|\alpha|=L+1} \int_{\mathbb{R}^n} |G_\alpha(\eta, \lambda\xi - \xi')\mathcal{F}[\chi u](\xi')| d\xi'$$
$$\leq \frac{1}{\langle \eta \rangle^{2M}} \left\| \frac{\mathcal{F}[\chi u]}{\langle \cdot \rangle^k} \right\|_{L^2(\mathbb{R}^n)} \sum_{|\alpha|=L+1} \|\langle \xi' \rangle^k G_\alpha(\eta, \lambda\xi - \xi')\|_{L^2_{\xi'}(\mathbb{R}^n)},$$

where

$$G_\alpha(\eta, \xi) = \iint_{\mathbb{R}^{2n}} \widetilde{\chi}(y)(1-\Delta_x)^M \{R_\alpha(x,y)(x-y)^\alpha \overline{\phi_\lambda(y-x)}\} e^{-ix\cdot\eta - iy\cdot\xi} dx dy.$$

Put $g(x) = (-x)^\alpha \overline{\phi(x)}$ and $g_\lambda(x) = \lambda^{\frac{n}{4}} g(\lambda^{\frac{1}{2}}x)$. Since

$$|G_\alpha(\eta,\xi)| = \lambda^{-\frac{|\alpha|}{2}} \left| \iint_{\mathbb{R}^{2n}} \widetilde{\chi}(y)(1-\Delta_x)^M \{R_\alpha(x,y) g_\lambda(y-x)\} e^{-ix\cdot\eta - iy\cdot\xi} dx dy \right|$$
$$\leq \lambda^{-\frac{L+1}{2}} \sum_{|\beta|+|\gamma|\leq 2M} C_{\beta,\gamma} \left| \iint_{\mathbb{R}^{2n}} \widetilde{\chi}(y)\partial_x^\beta R_\alpha(x,y) \partial_x^\gamma (g_\lambda(y-x)) e^{-ix\cdot\eta - iy\cdot\xi} dx dy \right|,$$

we have, by Lemma 2.6,

$$\text{(3.8)} \qquad \|\langle \xi' \rangle^k G_\alpha(\eta, \lambda\xi - \xi')\|^2_{L^2_{\xi'}(\mathbb{R}^n)} \leq C\langle \xi \rangle^{2k} \lambda^{-(L+1)+3k+2+2M}.$$

Since we can take L arbitrarily, (3.7) and (3.8) yield that for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\text{(3.9)} \qquad \int_V \int_{\mathbb{R}^n} |I_2|^2 d\eta d\xi \leq C_N \lambda^{-N}.$$

Combining (3.1), (3.5), (3.6) and (3.9), we have

$$\text{(3.10)} \qquad \int_V \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi \leq C_N \left(\|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V_1)} + \lambda^{-N} \right)$$

for all $N \in \mathbb{N}$ and all $\lambda \geq 1$. Taking $N = n + [2s] + 2$, we obtain

$$\int_1^\infty \lambda^{n-1+2s} \int_V \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi d\lambda$$
$$\leq C \int_1^\infty \lambda^{n-1+2s} \left(\|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V_1)} + \lambda^{-n-[2s]-2} \right) d\lambda < \infty.$$

□

Proposition 3.2. *Under the same assumption as in Theorem 1.2, (ii) implies (i).*

Proof. From the assumption (ii), there exist $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, a neighborhood K of x_0 and a neighborhood V_1 of ξ_0 such that

$$\int_1^\infty \lambda^{n-1+2s} \int_{V_1} \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi d\lambda < \infty. \tag{3.11}$$

Since $u \in \mathcal{S}'(\mathbb{R}^n)$, there exists $k \in \mathbb{N}$ such that $\|\langle\xi\rangle^{-k}\mathcal{F}[\psi u](\xi)\|_{L^2(\mathbb{R}^n)} < \infty$ for all $\psi \in C_0^\infty(\mathbb{R}^n)$. So, simple calculation yields that there exists a neighborhood V_2 of ξ_0 such that

$$\lambda^{n-2k} \int_{V_2} |\mathcal{F}[\psi u](\lambda\xi)|^2 d\xi \le C_\psi < \infty. \tag{3.12}$$

Let $\widetilde{\chi} \in C_0^\infty(\mathbb{R}^n)$ with $\widetilde{\chi} \equiv 1$ near x_0, $\mathrm{supp}\widetilde{\chi} \subset K^\circ$ and $0 \le \widetilde{\chi} \le 1$ and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\overline{K} \subset \{x \in \mathbb{R}^n \,|\, \chi(x) = 1\}^\circ$ and take $d > 0$ satisfying $B(\xi_0, d) \subset V_1 \cap V_2$. From Proposition 2.2 and (3.11), it is enough to show that there exist $\lambda_0 \ge 1$ and a neighborhood V of ξ_0 such that

$$\int_V |\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi \le C \int_{B(\xi_0,d)} \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi + C'\lambda^{-M} \tag{3.13}$$

for $\lambda \ge \lambda_0$ and $M > n+2s$. Put $V' = B(\xi_0, \frac{d}{2})$. Since $(A_1+A_2+A_3+A_4)^2 \le 4(A_1^2+A_2^2+A_3^2+A_4^2)$ for $A_1, A_2, A_3, A_4 \in \mathbb{R}$, we have, by (3.2) and Plancherel's theorem,

$$\begin{aligned}
&\int_{V'} \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi \\
&\ge \int_{V'} \int_{\mathbb{R}^n} |\widetilde{\chi}(x) W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi \\
&= (2\pi)^{-n} \int_{V'} \int_{\mathbb{R}^n} \left|\mathcal{F}_{x\to\eta}\Big[\widetilde{\chi}(x) W_{\phi_\lambda}[\chi u](x, \lambda\xi) + \widetilde{\chi}(x) W_{\phi_\lambda}[(1-\chi)u](x, \lambda\xi)\Big](\eta)\right|^2 d\eta d\xi \\
&= (2\pi)^{-n} \int_{V'} \int_{\mathbb{R}^n} |I_1 + I_2 + I_3 + I_4|^2 d\eta d\xi \\
&\ge (2\pi)^{-n} \int_{V'} \int_{\mathbb{R}^n} \left(\frac{1}{4}|I_1|^2 - |I_2|^2 - |I_3|^2 - |I_4|^2\right) d\eta d\xi,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \iint_{\mathbb{R}^{2n}} \widetilde{\chi}(y)\overline{\phi_\lambda(y-x)} u(y) e^{-i\lambda y\cdot\xi - ix\cdot\eta} dy dx, \\
I_2 &= \sum_{1\le|\alpha|\le L} \iint_{\mathbb{R}^{2n}} \frac{\partial_x^\alpha \widetilde{\chi}(y)}{\alpha!} (x-y)^\alpha \overline{\phi_\lambda(y-x)} u(y) e^{-i\lambda y\cdot\xi - ix\cdot\eta} dy dx, \\
I_3 &= \sum_{|\alpha|=L+1} \iint_{\mathbb{R}^{2n}} R_\alpha(x,y)(x-y)^\alpha \overline{\phi_\lambda(y-x)} \chi(y) u(y) e^{-i\lambda y\cdot\xi - ix\cdot\eta} dy dx
\end{aligned}$$

and

$$I_4 = \mathcal{F}_{x\to\eta}[\widetilde{\chi}(x) W_{\phi_\lambda}[(1-\chi)u](x, \lambda\xi)](\eta).$$

Here, we use $(\partial_x^\alpha \widetilde{\chi})\chi = \partial_x^\alpha \widetilde{\chi}$ for any α. Thus, we have

$$\frac{1}{4} \int_{V'} \int_{\mathbb{R}^n} |I_1|^2 d\eta d\xi \le (2\pi)^n \int_{V'} \int_K |W_{\phi_\lambda} u(x, \lambda\xi)|^2 dx d\xi + \int_{V'} \int_{\mathbb{R}^n} |I_2|^2 d\eta d\xi \tag{3.14}$$

$$+\int_{V'}\int_{\mathbb{R}^n}|I_3|^2 d\eta d\xi+\int_{V'}\int_{\mathbb{R}^n}|I_4|^2 d\eta d\xi.$$

Let $N \in \mathbb{N}$ with $N > 4s+4k$, $\delta = \frac{d}{2(2N-1)}$. Since $I_1 = \overline{\mathcal{F}[\phi_\lambda](\eta)}\mathcal{F}[\widetilde{\chi}u](\eta+\lambda\xi)$, we have, by the change of variables and Fubini's theorem,

$$\begin{aligned}\int_{V'}\int_{\mathbb{R}^n}|I_1|^2 d\eta d\xi &\geq \int_{V'}\int_{B(0,\delta\lambda^{\frac{3}{4}})}|I_1|^2 d\eta d\xi\\ &= \int_{B(0,\delta\lambda^{\frac{3}{4}})}|\mathcal{F}[\phi_\lambda](\eta)|^2\int_{V'}|\mathcal{F}[\widetilde{\chi}u](\eta+\lambda\xi)|^2 d\xi d\eta\\ &= \int_{B(0,\delta\lambda^{\frac{3}{4}})}|\mathcal{F}[\phi_\lambda](\eta)|^2\int_{\Omega_{\lambda,\eta,\xi}}|\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi d\eta\\ &\geq \int_{B(0,\delta\lambda^{\frac{3}{4}})}|\mathcal{F}[\phi_\lambda](\eta)|^2 d\eta\int_{B(\xi_0,\frac{d}{2}-\delta)}|\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi,\end{aligned}$$

where $\Omega_{\lambda,\eta,\xi} = \{\xi+\frac{\eta}{\lambda}\,|\,\xi\in V'\}$. We note that if $\xi\in V'$, $\eta\in B(0,\delta\lambda^{\frac{3}{4}})$ and $\lambda\geq 1$ then $B(\xi_0,\frac{d}{2}-\delta)\subset\Omega_{\lambda,\eta,\xi}$. Since

$$\int_{B(0,\delta\lambda^{\frac{3}{4}})}|\mathcal{F}[\phi_\lambda](\eta)|^2 d\eta \longrightarrow \|\phi\|_{L^2}^2 \qquad (\lambda\to\infty),$$

there exists $\lambda_0 \geq 1$ such that

$$\int_{B(0,\delta\lambda^{\frac{3}{4}})}|\mathcal{F}[\phi_\lambda](\eta)|^2 d\eta \geq \frac{1}{2}\|\phi\|_{L^2}^2 > 0$$

for $\lambda\geq\lambda_0$. Therefore we obtain

$$\frac{1}{2}\|\phi\|_{L^2}^2\int_{B(\xi_0,\frac{d}{2}-\delta)}|\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi \leq \int_{V'}\int_{\mathbb{R}^n}|I_1|^2 d\eta d\xi \tag{3.15}$$

for $\lambda\geq\lambda_0$.

Let $M > n+2s$. By Fubini's theorem and the change of variables, we have

$$I_2 = \sum_{1\leq|\alpha|\leq L}\frac{\lambda^{-\frac{|\alpha|}{2}-\frac{n}{4}}}{\alpha!}\overline{\mathcal{F}[(-y)^\alpha\phi(y)](\lambda^{-\frac{1}{2}}\eta)}\mathcal{F}[(\partial^\alpha\widetilde{\chi})u](\eta+\lambda\xi).$$

Thus, in the similar calculation as (3.5) and (3.6), it follows that

$$\int_{V'}\int_{\mathbb{R}^n}|I_2|^2 d\eta d\xi \leq C\sum_{1\leq|\alpha|\leq L}\lambda^{-|\alpha|}\left\{\|\mathcal{F}[(\partial^\alpha\widetilde{\chi})u](\lambda\xi)\|_{L^2(B(\xi_0,\frac{d}{2}+\delta))}^2+\lambda^{-M}\right\}. \tag{3.16}$$

Taking L sufficiently large, then in the same way as (3.9) we have

$$\int_{V'}\int_{\mathbb{R}^n}|I_3|^2 d\eta d\xi \leq C\lambda^{-M}. \tag{3.17}$$

From Lemma 2.4, we have

$$\int_{V'}\int_{\mathbb{R}^n}|I_4|^2 d\eta d\xi \leq C\lambda^{-M}. \tag{3.18}$$

By (3.14), (3.15), (3.16), (3.17) and (3.18), we obtain

$$\int_{B(\xi_0,\frac{d}{2}-\delta)} |\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi \le C\Big(\int_{B(\xi_0,\frac{d}{2})}\int_K |W_{\phi_\lambda}u(x,\lambda\xi)|^2 dx d\xi + \sum_{1\le|\alpha|\le L}\lambda^{-|\alpha|}\int_{B(\xi_0,\frac{d}{2}+\delta)} |\mathcal{F}[(\partial^\alpha\widetilde{\chi})u](\lambda\xi)|^2 d\xi + \lambda^{-M}\Big). \tag{3.19}$$

If we replace $\int_{B(\xi_0,\frac{d}{2}-\delta)} |\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi$ with $\int_{B(\xi_0,\frac{d}{2}+\delta)} |\mathcal{F}[(\partial^\alpha\widetilde{\chi})u](\lambda\xi)|^2 d\xi$ in the left hand side of (3.19), then similar calculation yields that

$$\int_{B(\xi_0,\frac{d}{2}+\delta)} |\mathcal{F}[(\partial^\alpha\widetilde{\chi})u](\lambda\xi)|^2 d\xi \le C\Big(\int_{B(\xi_0,\frac{d}{2}+2\delta)}\int_K |W_{\phi_\lambda}u(x,\lambda\xi)|^2 dx d\xi + \sum_{1\le|\beta|\le L}\lambda^{-|\beta|}\int_{B(\xi_0,\frac{d}{2}+3\delta)} |\mathcal{F}[(\partial^{\alpha+\beta}\widetilde{\chi})u](\lambda\xi)|^2 d\xi + \lambda^{-M}\Big). \tag{3.20}$$

Applying (3.20) to the second term of the right hand side of (3.19), we have

$$\int_{B(\xi_0,\frac{d}{2}-\delta)} |\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi \le C\Big(\int_{B(\xi_0,\frac{d}{2}+2\delta)}\int_K |W_{\phi_\lambda}u(x,\lambda\xi)|^2 dx d\xi + \sum_{\substack{1\le|\alpha|\le L\\ 1\le|\beta|\le L}}\lambda^{-|\alpha|-|\beta|}\int_{B(\xi_0,\frac{d}{2}+3\delta)} |\mathcal{F}[(\partial^{\alpha+\beta}\widetilde{\chi})u](\lambda\xi)|^2 d\xi + \lambda^{-M}\Big). \tag{3.21}$$

Continuing in this fashion, we obtain

$$\begin{aligned}
&\int_{B(\xi_0,\frac{d}{2}-\delta)} |\mathcal{F}[\widetilde{\chi}u](\lambda\xi)|^2 d\xi \\
&\le C\Big(\int_{B(\xi_0,\frac{d}{2}+N\delta)}\int_K |W_{\phi_\lambda}u(x,\lambda\xi)|^2 dx d\xi \\
&\qquad + \lambda^{-N}\sum_{\substack{1\le|\alpha_j|\le L\\ 1\le j\le N}}\int_{B(\xi_0,\frac{d}{2}+(2N-1)\delta)} |\mathcal{F}[(\partial^{\alpha_1+\cdots+\alpha_N}\widetilde{\chi})u](\lambda\xi)|^2 d\xi + \lambda^{-M}\Big).
\end{aligned}$$

Since $B(\xi_0,\frac{d}{2}+N\delta)\subset B(\xi_0,\frac{d}{2}+(2N-1)\delta)\subset B(\xi_0,d)$, $M>n+2s$ and $N>2k+2s$, we obtain (3.13) by (3.12) as $V=B(\xi_0,\frac{d}{2}-\delta)$. □

4. Proof of Theorem 1.1

Trivially, (iii) yields (ii). So, in order to prove Theorem 1.1, we show that (i) implies (iii) in Proposition 4.1 and (ii) implies (i) in Proposition 4.2.

Proposition 4.1. *Under the same assumption as in Theorem 1.1, (i) implies (iii).*

Proof. By (i), there exist $\chi\in C_0^\infty(\mathbb{R}^n)$ with $\chi\equiv 1$ near x_0 and a conic neighborhood Γ of ξ_0 such that for all $N\in\mathbb{N}$ there exists $C_N>0$ such that

$$|\mathcal{F}[\chi u](\xi)|\le C_N(1+|\xi|)^{-N} \tag{4.1}$$

for $\xi\in\Gamma$. Thus, it follows that for all $N\in\mathbb{N}$ and $a\ge 1$ there exists $C_{N,a}>0$ such that

$$|\mathcal{F}[\chi u](\lambda\xi)|\le C_N(1+\lambda|\xi|)^{-N}\le C_{N,a}\lambda^{-N} \tag{4.2}$$

for $\lambda \geq 1$ and $\xi \in V = \{\xi \in \Gamma \,|\, (3a)^{-1} \leq |\xi| \leq 3a\}$.

Let K_1 and K_2 be a neighborhood of x_0 satisfying $\overline{K_1} \subset \{x \in \mathbb{R}^n | \chi(x) = 1\}^\circ$ and $\overline{K_2} \subset K_1$ and let Γ_1 and Γ_2 be a conic neighborhood of ξ_0 satisfying $\overline{\Gamma_1} \subset \Gamma$ and $\overline{\Gamma_2} \subset \Gamma_1$. Put $V_1 = \{\xi \in \Gamma_1 \,|\, (2a)^{-1} \leq |\xi| \leq 2a\}$ and $V_2 = \{\xi \in \Gamma_2 \,|\, a^{-1} \leq |\xi| \leq a\}$. Let $\chi_1 \in C_0^\infty(K_1)$ satisfy $\chi_1 \equiv 1$ on K_2 and let $\chi_2 \in C_0^\infty(V_1)$ satisfy $\chi_2 \equiv 1$ on V_2. Then, by the fundamental theorem of calculus, we have

$$\begin{aligned}
&|\mathbf{1}_{K_2}(x)\mathbf{1}_{V_2}(\xi)W_{\phi_\lambda}u(x,\lambda\xi)| \\
&\leq \left| \int_{-\infty}^{\xi_n} \cdots \int_{-\infty}^{\xi_1} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \partial_\eta^\tau \partial_z^\tau \Big\{ \chi_1(z)\chi_2(\eta) W_{\phi_\lambda}u(z,\lambda\eta) \Big\} dz_1 \cdots dz_n d\eta_1 \cdots d\eta_n \right| \\
&\leq \iint_{\mathbb{R}^{2n}} \Big| \partial_\eta^\tau \partial_z^\tau \{\chi_1(z)\chi_2(\eta) W_{\phi_\lambda}u(z,\lambda\eta)\} \Big| dz d\eta \\
&\leq \sum_{0\leq\alpha\leq\tau} \sum_{0\leq\beta\leq\tau} C_{\alpha,\beta} \int_{V_1} \int_{K_1} |\partial_z^\alpha \partial_\eta^\beta [W_{\phi_\lambda}u(z,\lambda\eta)]| dz d\eta,
\end{aligned}$$

where $\tau = (1, 1, \ldots, 1) \in (\mathbb{N} \cup \{0\})^n$. Since

$$\begin{aligned}
\partial_{z_j}(W_{\phi_\lambda}u(z,\lambda\eta)) &= -\lambda^{\frac{1}{2}} W_{(\partial_{z_j}\phi)_\lambda}u(z,\lambda\eta), \\
\partial_{\eta_j}(W_{\phi_\lambda}u(z,\lambda\eta)) &= -i\lambda W_{\phi_\lambda}[y_j u](z,\lambda\eta)
\end{aligned}$$

and Schwarz's inequality, we have

$$\begin{aligned}
(4.3)\quad &|\mathbf{1}_{K_2}(x)\mathbf{1}_{V_2}(\xi)W_{\phi_\lambda}u(x,\lambda\xi)| \\
&\leq \sum_{0\leq\alpha\leq\tau} \sum_{0\leq\beta\leq\tau} C_{\alpha,\beta} \lambda^{\frac{|\alpha|}{2}+|\beta|} \left(\int_{V_1} \int_{K_1} |W_{(\partial_z^\alpha\phi)_\lambda}[y^\beta u](z,\lambda\eta)|^2 dz d\eta \right)^{\frac{1}{2}}.
\end{aligned}$$

In the same way to obtain (3.10), we have, by Lemma 2.3, for all $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$(4.4)\qquad \int_{V_1} \int_{K_1} |W_{(\partial_x^\alpha\phi)_\lambda}[y^\beta u](x,\lambda\xi)|^2 dx d\xi \leq C_N \left(\int_V |\mathcal{F}[\chi u](\lambda\xi)|^2 d\xi + \lambda^{-N} \right).$$

From (4.2), (4.3) and (4.4), we obtain the desired result. □

Proposition 4.2. *Under the same assumption as in Theorem 1.1, (ii) implies (i).*

Proof. From the assumption (ii), there exist $\phi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, a neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ and $a \geq 1$ there exists $C_{N,a} > 0$ satisfying

$$(4.5)\qquad |W_{\phi_\lambda}u(x,\lambda\xi)| \leq C_{N,a}\lambda^{-N}$$

for all $\lambda \geq 1$, $x \in K$ and $\xi \in \Gamma$ with $a^{-1} \leq |\xi| \leq a$. Take $b > 1$ satisfying that $\xi_0 \in \{\xi \in \Gamma \,|\, b^{-1} \leq |\xi| \leq b\}$. Put $V_1 = \{\xi \in \Gamma \,|\, b^{-1} \leq |\xi| \leq b\}$. In the same way to obtain (3.13), there exist a neighborhood V_2 of ξ_0 and $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ near x_0 and $\mathrm{supp}\chi \subset K^\circ$ such that for all $N \in \mathbb{N}$

$$(4.6)\qquad \int_{V_2} |\mathcal{F}[\chi u](\lambda\xi)|^2 d\xi \leq C_N \left(\int_{V_1} \int_K |W_{\phi_\lambda}u(x,\lambda\xi)|^2 dx d\xi + \lambda^{-N} \right).$$

So, we have, by (4.5),

$$\int_{V_2} |\mathcal{F}[\chi u](\lambda\xi)|^2 d\xi \le C_N \lambda^{-N}. \tag{4.7}$$

Let V_3 and V_4 be a neighborhood of ξ_0 satisfying $\overline{V_3} \subset V_2$ and $\overline{V_4} \subset V_3$ and let $\chi_1 \in C_0^\infty(V_3)$ satisfy $\chi_1 \equiv 1$ on V_4. Then, by the fundamental theorem of calculus and Schwarz's inequality, we have

$$\begin{aligned} |\mathbf{1}_{V_4}(\xi)\mathcal{F}[\chi^2 u](\lambda\xi)| &\le \left| \int_{-\infty}^{\xi_n} \cdots \int_{-\infty}^{\xi_1} \partial_\eta^\tau \left\{\chi_1(\eta)\mathcal{F}[\chi^2 u](\lambda\eta)\right\} d\eta_1 \cdots d\eta_n \right| \\ &\le \int_{\mathbb{R}^n} \left|\partial_\eta^\tau \{\chi_1(\eta)\mathcal{F}[\chi^2 u](\lambda\eta)\}\right| d\eta \\ &\le \sum_{0\le\alpha\le\tau} C_\alpha \int_{V_3} |\partial_\eta^\alpha \{\mathcal{F}[\chi^2 u](\lambda\eta)\}| d\eta \\ &\le \sum_{0\le\alpha\le\tau} C'_\alpha \left(\int_{V_3} |\partial_\eta^\alpha \{\mathcal{F}[\chi^2 u](\lambda\eta)\}|^2 d\eta \right)^{\frac{1}{2}}, \end{aligned}$$

where $\tau = (1, 1, \ldots, 1) \in (\mathbb{N} \cup \{0\})^n$. Since $\partial_{\eta_j}\{\mathcal{F}[\chi^2 u](\lambda\eta)\} = -i\lambda\mathcal{F}[y_j\chi^2 u](\lambda\eta)$, we have, by Lemma 2.3 and (4.7), for all $N \in \mathbb{N}$

$$\begin{aligned} |\mathbf{1}_{V_4}(\xi)\mathcal{F}[\chi^2 u](\lambda\xi)| &\le \sum_{0\le\alpha\le\tau} C_\alpha \lambda^{|\alpha|} \left(\int_{V_3} |\mathcal{F}[y^\alpha \chi^2 u](\lambda\eta)|^2 d\eta \right)^{\frac{1}{2}} \\ &\le \sum_{0\le\alpha\le\tau} C'_{\alpha,N} \lambda^{|\alpha|} \left(\int_{V_2} |\mathcal{F}[\chi u](\lambda\eta)|^2 d\eta + \lambda^{-N} \right)^{\frac{1}{2}} \\ &\le C_N \lambda^{n-\frac{N}{2}}. \end{aligned}$$

If $\xi \in V_4$ and $\lambda \ge 1$ then simple calculation yields that $\lambda^{-N} \le C_N(1 + \lambda|\xi|)^{-N}$. Thus, we have

$$|\mathcal{F}[\chi^2 u](\lambda\xi)| \le C_N(1 + \lambda|\xi|)^{-N} \quad \text{for} \quad \xi \in V_4.$$

From Proposition 2.1, we obtain the desired result. □

5. Application

In this section, we give an application of Theorem 1.1 (See K. Kato and S. Ito [10] for more details). We consider the Schrödinger equation with the potential $V(t, x)$:

$$\begin{cases} i\partial_t u + \frac{1}{2}\triangle u - V(t, x)u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{5.1}$$

where $i = \sqrt{-1}$, $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$ and $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$. We assume that $V(t, x)$ is a real valued function in $C^\infty(\mathbb{R} \times \mathbb{R}^n)$ and there exists a constant ρ such that $0 \le \rho < 2$ and for all multi-indices α,

$$\left|\partial_x^\alpha V(t, x)\right| \le C(1 + |x|)^{\rho-|\alpha|}$$

holds for some $C > 0$ and for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Let $\phi_0(x) \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $\phi(t, x) =$

$U(t)\phi_0(x) = \mathcal{F}^{-1}_{\xi\to x}[e^{-\frac{it|\xi|^2}{2}}\mathcal{F}\phi_0(\xi)](x)$. The initial value problem (5.1) is transformed by the wave packet transform with the basic wave packet $\phi(t,x)$ to

$$(5.2)\qquad \begin{cases} \left(i\partial_t + i\xi\cdot\nabla_x - i\nabla_x V(t,x)\cdot\nabla_\xi - \frac{1}{2}|\xi|^2 - \widetilde{V}(t,x)\right) \\ \qquad\qquad \times W_{\phi(t,\cdot)}[u(t,\cdot)](x,\xi) = R[\phi,u](t,x,\xi), \\ W_{\phi(0,\cdot)}[u(0,\cdot)](x,\xi) = W_{\phi_0}u_0(x,\xi), \end{cases}$$

where $\widetilde{V}(t,x) = V(t,x) - \nabla_x V(t,x)\cdot x$ and

$$R[\phi,u](t,x,\xi) = \sum_{|\alpha|=2}\frac{2}{\alpha!}\int_{\mathbb{R}^n}\overline{\phi(t,y-x)} \times\left(\int_0^1 \partial_x^\alpha V(t,x+\theta(y-x))(1-\theta)d\theta\right)(y-x)^\alpha u(t,y)e^{-iy\cdot\xi}dy$$

(for the deduction of (5.2), see [9]). By the method of characteristics, we have the integral equation

$$(5.3)\quad W_{\phi(t,\cdot)}[u(t,\cdot)](x,\xi) = e^{-iA(t,x,\xi)}W_{\phi_0}u_0(x(0;t,x,\xi),\xi(0;t,x,\xi)) - i\int_0^t e^{-iB(s;t,x,\xi)}R[\phi,u](s,x(s;t,x,\xi),\xi(s;t,x,\xi))|ds,$$

where

$$A(t,x,\xi) = \int_0^t\{\frac{1}{2}|\xi(s;t,x,\xi)|^2 + \widetilde{V}(s,x(s;t,x,\xi))\}ds,$$

$$B(s;t,x,\xi) = \int_s^t\{\frac{1}{2}|\xi(\tau;t,x,\xi)|^2 + \widetilde{V}(\tau,x(\tau;t,x,\xi))\}d\tau$$

and $x(s;t,x,\xi)$, $\xi(s;t,x,\xi)$ are the solutions of

$$\begin{cases}\dot{x}(s) &= \xi(s),\ x(t) = x,\\ \dot{\xi}(s) &= -\nabla_x V(s,x(s)),\ \xi(t)=\xi.\end{cases}$$

Fix $t_0\in\mathbb{R}$ and put $\phi_\lambda(t,x) = U(t)(\lambda^{\frac{bn}{2}}\phi_0(\lambda^b x))$ for $0<b<1$. In (5.3), substituting $x(t;t_0,x,\lambda\xi)$, $\xi(t;t_0,x,\lambda\xi)$ and $\phi_\lambda(-t_0,x)$ for x,ξ and $\phi_0(x)$ respectively, we have

$$(5.4)\quad |W_{\phi_\lambda(t-t_0,\cdot)}u(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi))| \le |W_{\phi_\lambda(-t_0,\cdot)}u_0(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi))| + \left|\int_0^t |R[\phi_\lambda,u](s,x(s;t_0,x,\lambda\xi),\xi(s;t_0,x,\lambda\xi))|ds\right|.$$

Here, we use the fact that

$$x(s;t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)) = x(s;t_0,x,\lambda\xi),$$
$$\xi(s;t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi)) = \xi(s;t_0,x,\lambda\xi)$$

and $U(t)\phi_\lambda(-t_0,x) = \phi_\lambda(t-t_0,x)$. Let $a>0$, $(x_0,\xi_0)\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, K be a neighborhood of x_0 and Γ be a conic neighborhood of ξ_0. If we assume that $b=\min(\frac{2-\rho}{4},\frac{1}{4})$ and

$$|W_{\phi_\lambda(-t_0,\cdot)}u_0(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi))| \le C_{N,a,\phi_0}\lambda^{-N}$$

for all $\lambda \ge 1$, $x \in K$ and $\xi \in \Gamma \cap \{\xi \,|\, a^{-1} \le |\xi| \le a\}$ then we can show that

$$|W_{\phi_\lambda(t-t_0,\cdot)}u(t,x(t;t_0,x,\lambda\xi),\xi(t;t_0,x,\lambda\xi))| \le C'_{N,a,\phi_0}\lambda^{-N} \tag{5.5}$$

for all $\lambda \ge 1$, $x \in K$ and $\xi \in \Gamma \cap \{\xi \,|\, a^{-1} \le |\xi| \le a\}$ (see [10] for details). In (5.5), taking $t = t_0$, we have

$$|W_{\phi_\lambda(0,\cdot)}u(t_0,x,\lambda\xi)| \le C'\lambda^{-N}$$

for all $\lambda \ge 1$, $x \in K$ and $\xi \in \Gamma \cap \{\xi \,|\, a^{-1} \le |\xi| \le a\}$. Thus, we obtain $(x_0,\xi_0) \notin WF(u(t_0,x))$ by Theorem 1.1. From the equivalency of Theorem 1.1, we can show the converse in the same way. Owing to the fact, we can prove the following theorem.

Theorem 5.1. (K. Kato and S. Ito [10, Theorem 1.2]) *Take $b = \min(\frac{2-\rho}{4}, \frac{1}{4})$. Let $u_0(x) \in L^2(\mathbb{R}^n)$ and $u(t,x)$ be a solution of* (5.1) *in $C(\mathbb{R}; L^2(\mathbb{R}^n))$ and fix $t_0 \in \mathbb{R}$. Then, $(x_0,\xi_0) \notin WF(u(t_0,x))$ if and only if there exist a neighborhood K of x_0, a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$, for all $a \ge 1$ and for all $\phi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$, there exists a constant $C_{N,a,\phi_0} > 0$ satisfying*

$$|W_{\phi_\lambda(-t_0,\cdot)}u_0(x(0;t_0,x,\lambda\xi),\xi(0;t_0,x,\lambda\xi))| \le C_{N,a,\phi_0}\lambda^{-N}$$

for $\lambda \ge 1$, $a^{-1} \le |\xi| \le a$ and $(x,\xi) \in K \times \Gamma$, where $\phi_\lambda(t,x) = U(t)(\lambda^{\frac{bn}{2}}\phi_0(\lambda^b x))$.

Appendix A.

Proof of Proposition 2.1. First, under the assumption (i), we show (ii). From the assumption (i), we have

$$|\mathcal{F}[\chi u](\lambda\xi)| \le C_N(1+\lambda|\xi|)^{-N}$$

for $\lambda \ge 1$ and $\xi \in \Gamma$. Let V be a neighborhood of ξ_0 satisfying $\overline{V} \subset \Gamma$. If $\lambda \ge 1$ and $\xi \in V$ then $\lambda\xi \in \Gamma$. Thus, we have (ii).

Conversely, we assume (ii). Put $\Gamma_1 = \{\lambda\xi \,|\, \xi \in V, \lambda > 0\}$ and $\Gamma_2 = \{\lambda\xi \,|\, \xi \in V, \lambda \ge 1\}$. By the assumption, we have

$$\mathbf{1}_{\Gamma_2}(\xi)|\mathcal{F}[\chi u](\xi)| \le C_N(1+|\xi|)^{-N}.$$

Let $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ satisfy $\chi \equiv 1$ on $\Gamma_1 \cap \Gamma_2^c$. Then, we have

$$\mathbf{1}_{\Gamma_1\cap\Gamma_2^c}(\xi)|\mathcal{F}[\chi u](\xi)| \le |\chi_1(\xi)\mathcal{F}[\chi u](\xi)| \le C_N(1+|\xi|)^{-N},$$

where $C_N = \sup_{\xi\in\mathbb{R}^n}|(1+|\xi|)^N\chi_1(\xi)\mathcal{F}[\chi u](\xi)|$. Thus, we obtain the desired result. □

Proof of Proposition 2.2. First, we show (i) implies (ii). Let Γ be a conic neighborhood of ξ_0 satisfying (2.1). Take $\delta > 0$ with $|\xi_0| - \delta > 0$. Let $V = \{\xi \in \Gamma \,|\, |\xi_0| - \delta \le |\xi| \le |\xi_0| + \delta\}$, $S^{n-1} = \{\xi \in \mathbb{R}^n \,|\, |\xi| = 1\}$ and $A = S^{n-1} \cap \Gamma$. By the change of variables $\xi = r\sigma$, where $r > 0$ and $\sigma \in A$, we have

$$I \equiv \int_1^\infty \lambda^{n-1+2s}\|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V)}d\lambda$$

$$= \int_1^\infty \lambda^{n-1+2s} \int_{|\xi_0|-\delta}^{|\xi_0|+\delta} \int_A r^{n-1} |\mathcal{F}[\chi u](\lambda r\sigma)|^2 d\sigma dr d\lambda$$
$$\leq C \int_{|\xi_0|-\delta}^{|\xi_0|+\delta} \int_1^\infty \int_A r(r\lambda)^{n-1+2s} |\mathcal{F}[\chi u](\lambda r\sigma)|^2 d\sigma d\lambda dr,$$

where $C = \max((|\xi_0| - \delta)^{-2s-1}, (|\xi_0| + \delta)^{-2s-1})$. Again by the change of variables $\lambda r = \lambda'$ and $r = r'$, we have

$$I \leq C \int_{|\xi_0|-\delta}^{|\xi_0|+\delta} \int_r^\infty \int_A \lambda^{n-1+2s} |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda dr$$
$$\leq 2\delta C \int_{|\xi_0|-\delta}^\infty \int_A \lambda^{n-1+2s} |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda$$
$$\leq 2\delta C' \int_0^\infty \int_A \lambda^{n-1}(1 + \lambda^2|\sigma|^2)^s |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda.$$

By the change of variables $\xi = \lambda\sigma$, we obtain

$$I \leq C \int_\Gamma \langle\xi\rangle^{2s} |\mathcal{F}[\chi u](\xi)|^2 d\xi < \infty.$$

Therefore we obtain (2.2).

Next, we show (ii) implies (i). Let V be a neighborhood of ξ_0 satisfying (2.2). Take $d > 0$ such that $|\xi_0| - d > 0$ and $B(\xi_0, d) \subset V$ and put $\Gamma_d = \{\lambda\xi \mid \xi \in B(\xi_0, d), \lambda > 0\}$. Let Γ be a conic neighborhood of ξ_0 satisfying $\overline{\Gamma} \subset \Gamma_d$. We divide $\|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|_{L^2(\Gamma)}$ into two parts:

$$\text{(A.1)} \qquad \|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|^2_{L^2(\Gamma)} = \|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|^2_{L^2(\Gamma\cap\{|\xi|\leq 1\})} + \|\langle\xi\rangle^s \mathcal{F}[\chi u](\xi)\|^2_{L^2(\Gamma\cap\{|\xi|\geq 1\})}$$
$$\equiv I_1 + I_2.$$

Since $u \in \mathcal{S}'(\mathbb{R}^n)$, there exists $k \in \mathbb{N}$ such that $\langle\xi\rangle^{-k}\mathcal{F}[\chi u](\xi) \in L^2(\mathbb{R}^n)$. Thus, we have

$$\text{(A.2)} \qquad I_1 = \int_{\Gamma\cap\{|\xi|\leq 1\}} \langle\xi\rangle^{2s+2k} \langle\xi\rangle^{-2k} |\mathcal{F}[\chi u](\xi)|^2 d\xi$$
$$\leq 2^{|s|+k} \int_{\mathbb{R}^n} \langle\xi\rangle^{-2k} |\mathcal{F}[\chi u](\xi)|^2 d\xi \leq C.$$

Let $A' = \{\xi \in \Gamma \mid |\xi| = 1\}$ and take $d' > 0$ satisfying

$$V' = \{\xi \in \Gamma \mid |\xi_0| - d' \leq |\xi| \leq |\xi_0| + d'\} \subset B(\xi_0, d).$$

By the change of variables $\xi = \lambda\sigma$, we have

$$I_2 = \int_1^\infty \int_{A'} \lambda^{n-1}(1 + \lambda^2)^s |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda$$
$$\leq C \int_1^\infty \lambda^{n-1+2s} \int_{A'} |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda$$
$$= \frac{C}{2\delta'} \int_{|\xi_0|-d'}^{|\xi_0|+d'} \int_1^\infty \lambda^{n-1+2s} \int_{A'} |\mathcal{F}[\chi u](\lambda\sigma)|^2 d\sigma d\lambda d\theta.$$

Again by the change of variables $\lambda = \lambda'\theta'$ and $\theta = \theta'$, we have

$$I_2 \leq C \int_{|\xi_0|-d'}^{|\xi_0|+d'} \int_{\frac{1}{\theta}}^\infty \int_{A'} \lambda^{n-1+2s} \theta^{n+2s} |\mathcal{F}[\chi u](\lambda\theta\sigma)|^2 d\sigma d\lambda d\theta$$

$$\le C' \int_{\frac{1}{|\xi_0|+d'}}^{\infty} \lambda^{n-1+2s} \int_{|\xi_0|-d'}^{|\xi_0|+d'} \int_{A'} \theta^{n-1} |\mathcal{F}[\chi u](\lambda\theta\sigma)|^2 d\sigma d\theta d\lambda,$$

where $C' = C \times \max((|\xi_0| - d')^{1+2s}, (|\xi_0| + d')^{1+2s})$. By the change of variables $\xi = \theta\sigma$, we have

$$\text{(A.3)} \qquad I_2 \le C \int_{\frac{1}{|\xi_0|+d'}}^{\infty} \lambda^{n-1+2s} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V')} d\lambda$$
$$\le C\Big(\int_{1}^{\infty} \lambda^{n-1+2s} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V')} d\lambda + \Big| \int_{\frac{1}{|\xi_0|+d'}}^{1} \lambda^{n-1+2s} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V')} d\lambda \Big| \Big).$$

Since $\langle\xi\rangle^{-k} \mathcal{F}[\chi u](\xi) \in L^2(\mathbb{R}^n)$, we have

$$\text{(A.4)} \quad \Big| \int_{\frac{1}{|\xi_0|+d'}}^{1} \lambda^{n-1+2s} \|\mathcal{F}[\chi u](\lambda\xi)\|^2_{L^2(V')} d\lambda \Big|$$
$$\le C \Big| \int_{\frac{1}{|\xi_0|+d'}}^{1} \lambda^{2s} (1 + \lambda^2 (|\xi_0| + d')^2)^k \|\langle\xi\rangle^{-k} \mathcal{F}[\chi u](\xi)\|^2_{L^2(\mathbb{R}^n)} d\lambda \Big| \le C'.$$

From (A.1), (A.2), (A.3) and (A.4), we obtain (2.1). □

Proof of Lemma 2.3. Simple calculation yields that

$$\int_{V'} |\mathcal{F}[\zeta\chi u](\lambda\xi)|^2 d\xi$$
$$\le C\lambda^{2n} \Big(\int_{V'} \Big| \int_{V} \mathcal{F}[\zeta](\lambda\xi - \lambda\eta) \mathcal{F}[\chi u](\lambda\eta) d\eta \Big|^2 d\xi + \int_{V'} \Big| \int_{V^c} \mathcal{F}[\zeta](\lambda\xi - \lambda\eta) \mathcal{F}[\chi u](\lambda\eta) d\eta \Big|^2 d\xi \Big)$$
$$= C\lambda^{2n} (I_1 + I_2).$$

By Young's inequality and the change of variables, we have

$$I_1 \le \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} \mathcal{F}[\zeta](\lambda\xi - \lambda\eta) \mathbf{1}_V(\eta) \mathcal{F}[\chi u](\lambda\eta) d\eta \Big|^2 d\xi$$
$$\le \lambda^{-2n} \|\zeta\|_{L^1} \int_{V} |\mathcal{F}[\chi u](\lambda\xi)|^2 d\xi.$$

Since $u \in \mathcal{S}'(\mathbb{R}^n)$, there exists $k \in \mathbb{N}$ such that $\langle\xi\rangle^{-k} \mathcal{F}[\chi u](\xi) \in L^1(\mathbb{R}^n)$. If $\xi \in V'$ and $\eta \in V^c$ then there exists $C > 0$ such that $|\xi - \eta| \ge C$. So, we have

$$I_2 = \int_{V'} \Big| \int_{V^c} \langle\lambda(\xi - \eta)\rangle^k (\lambda|\xi - \eta|)^N \mathcal{F}[\zeta](\lambda\xi - \lambda\eta) \times \frac{\mathcal{F}[\chi u](\lambda\eta)}{\langle\lambda\eta\rangle^k} \times \frac{\langle\lambda\eta\rangle^k}{\langle\lambda\xi\rangle^k \langle\lambda(\xi - \eta)\rangle^k} \times \langle\lambda\xi\rangle^k \times \frac{1}{\lambda^N |\xi - \eta|^N} d\eta \Big|^2 d\xi$$
$$\le \frac{C' \lambda^{2k-2N}}{C^{2N}} \int_{V'} \langle\xi\rangle^{2k} d\xi \Big| \int_{\mathbb{R}^n} \frac{\mathcal{F}[\chi u](\lambda\eta)}{\langle\lambda\eta\rangle^k} d\eta \Big|^2$$

$$\leq C_N \lambda^{2k-2n-2N}.$$

Therefore, we obtain (2.3). □

ACKNOWLEDGEMENTS. The authors grateful to the referee for several valuable comments and suggestions. Especially, an improvement of the proof of Proposition 3.1 is done thanks to referee's remarks. The first author is partially supported by JSPS Grant-in-Aid for scientific research, Basic Research C # 25400183. The second author is partially supported by JSPS Grant-in Aid for Young Scientists B # 16K17606. The third author is partially supported by JSPS Grant-in Aid for Young Scientists B # 15K17576.

References

[1] A. Córdoba and C. Fefferman: *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations **3** (1978), 979–1005.
[2] J.M. Delort: F.B.I. transformation. Second microlocalization and semilinear caustics. Lecture Notes in Math. 1522. Springer-Verlag, Berlin, (1992).
[3] G.B. Folland: Harmonic analysis in phase space, Ann. of Math. Studies No.122, Princeton Univ. Press, Princeton, NJ, (1989).
[4] P. Gérard: *Moyennisation et régularité deux-microlocale*, Ann. Sci. École Norm. Sup. **23** (1990), 89–121.
[5] K. Gröchenig: Foundations of Time-Frequency Analysis, Birkhäuser Boston (2001).
[6] L. Hörmander: *Fourier integral operators, I*, Acta Math. **127** (1971), 79–183.
[7] L. Hörmander: *Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients*, Comm. Pure Appl. Math. **24** (1971), 671–704.
[8] L. Hörmander: The Analysis of Linear Partial Differential Operators I, II, III, IV, Springer-Verlag, Berlin, (1983), (1985).
[9] K. Kato, M. Kobayashi and S. Ito: *Representation of Schrödinger operator of a free particle via short time Fourier transform and its applications*, Tohoku Math. Journal, **64** (2012), 223–231.
[10] K. Kato and S. Ito: *Singularities for solutions to time dependent Schrödinger equations with sub-quadratic potential*, SUT Journal of Math., **50** (2014), 383–398.
[11] S. Mizohota: The theory of partial differential equations, Cambridge University Press, New York, (1973).
[12] T. Ōkaji: *A note on the wave packet transforms*, Tsukuba J. Math. **25** (2001), 383–397.
[13] S. Pilipović, N. Teofanov and J. Toft: *Wave-front sets in Fourier Lebesgue spaces*, Rend. Semin. Mat. Univ. Politec. Torino **66** (2008), 299–319.
[14] S. Pilipović, N. Teofanov and J. Toft: *Micro-Local Analysis with Fourier Lebesgue Spaces. Part I*, J. Fourier Anal. Appl. **17** (2011), 374-407.
[15] M. Sato, T. Kawai and M. Kashiwara: *Hyperfunctions and pseudodifferential equations*, In: Lecture Notes in Math. 287, Springer-Verlag, New York (1973), 265–529.
[16] F. Trèves: Introduction to pseudodifferential and Fourier integral operators. Vol. 1 and Vol. 2. The University Series in Mathematics. Plenum Press, New York-London, (1980).

Keiichi Kato
Department of Mathematics
Faculty of Science
Tokyo University of Science
Kagurazaka 1–3, Shinjuku-ku
Tokyo 162–8601
Japan
e-mail: kato@ma.kagu.tus.ac.jp

Masaharu Kobayashi
Department of Mathematics
Hokkaido University
Kita 10, Nishi 8, Kita-Ku, Sapporo
Hokkaido 060–0810
Japan
e-mail: m-kobayashi@math.sci.hokudai.ac.jp

Shingo Ito
College of Liberal Arts and Sciences
Kitasato University, Kitasato 1–15–1
Minami-ku, Sagamihara
Kanagawa 252–0373
Japan
e-mail: singoito@kitasato-u.ac.jp