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WILLMORE-LIKE FUNCTIONALS FOR SURFACES IN 3-DIMENSIONAL THURSTON GEOMETRIES

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Abstract

We find analogues of the Willmore functional for each of the Thurston geometries with 4–dimensional isometry group such that the CMC–spheres in these geometries are critical points of these functionals.

1. Introduction

Let M be a closed orientable surface and $f\colon M \to N$ be an immersion of M into a 3–dimensional Riemannian manifold N. Set:

$$\mathcal{W}(f) = \int_M \left(H^2 + \overline{K}\right) d\mu,$$

where H is the mean curvature of the immersed surface, the value of $\overline{K}$ at a point $p \in M$ is defined as the sectional curvature of the 2–plane $f_*(T_pM)$ in N, $d\mu$ is the area element of the induced metric on M. We will refer to the functional $\mathcal{W}$ as the Willmore functional. It is known that $\mathcal{W}(f)$ is a conformal invariant [19].

In the 3–dimensional space forms $\mathbb{R}^3, \mathbb{H}^3$ and $\mathbb{S}^3$ the functional $\mathcal{W}$ enjoys the property that the CMC spheres are critical points of $\mathcal{W}$; recall that in the 3–dimensional space forms the CMC spheres are exactly the round spheres by the Hopf theorem. However, this property for the Willmore functional $\mathcal{W}$ fails to hold in the other 3–dimensional Thurston geometries.

In this paper we will introduce the Willmore–like functionals for the certain family of Riemannian manifolds $E(k,\tau)$ that include the model spaces for all Thurston geometries with 4–dimensional group of isometries, i.e., the products $\mathbb{S}^2\times\mathbb{R}$ and $\mathbb{H}^2\times\mathbb{R}$, the Heisenberg group Nil and the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$. The functionals to be introduced in these geometries have the form:

$$\int_M \left(H^2 + \alpha\overline{K} + \beta\right) d\mu, \tag{1.1}$$

where α and β are some constants that depend on k and τ. In the case of the Heisenberg group Nil (for $k = 0$ and $\tau = \frac{1}{2}$) the functional:

$$\int_M \left(H^2 + \frac{1}{4}\overline{K} - \frac{1}{16}\right) d\mu$$

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was obtained in [5] based on the Weierstrass representation for surfaces in Nil. Then it was shown [6] that the CMC spheres in Nil are critical points of this functional. In the case of the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ (for $k = -1$ and $\tau = -\frac{1}{2}$) it was shown that for the functional:

$$\int_M \left(H^2 + \frac{1}{4}\overline{K} - \frac{5}{16}\right) d\mu,$$

the minimum among the rotationally invariant spheres is attained exactly at the CMC spheres [4].

The main result of the paper is the following theorem.

Theorem 1. *The CMC spheres in $E(k,\tau)$ are critical points of the following Willmore–like functional:*

$$E(f) = \int_M \left(H^2 + \frac{1}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4}\right) d\mu. \tag{1.2}$$

In addition to Theorem 1 we will prove the following theorem.

Theorem 2. *For rotationally invariant spheres in $E(k,\tau)$ the functional $E(f)$ attains its minimum exactly at the CMC spheres.*

Remark 1. We note that:

$$E(f) = \mathcal{W}(f) + \int_M \left(-\frac{3}{4}\overline{K} + \frac{k}{4} - \frac{\tau^2}{4}\right) d\mu.$$

Remark 2. It can be seen that Theorem 1 agrees with the results obtained earlier for Nil [6] and the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ [4]. In addition, the functional E for the case $\mathbb{S}^2 \times \mathbb{R}$ ($k = 1$ and $\tau = 0$) coincides up to a constant factor with the functional [1] $\int_M (4H^2 + \overline{K} + 1)$ mentioned in [6, § 6.2].

The structure of the remaining part of this paper is as follows. In § 2 we give the description of the Riemannian manifolds $E(k,\tau)$. In § 3 we review the characterizations of the CMC spheres in these manifolds. In § 4 we give the details of the proof of Theorem 1. In § 5 we give the details of the proof of Theorem 2.

2. The Riemannian manifods $E(k,\tau)$

The model spaces for the four Thurston geometries: Nil, $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ belong to the family of Riemannian 3–manifolds $E(k,\tau)$, $k \in \mathbb{R}, \tau \in \mathbb{R}$ that are as follows. If $k \geqslant 0$ then $E(k,\tau)$ is $\mathbb{R}^3$ with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{k}{4}(x^2 + y^2)\right)^2} + \left(dz + \frac{\tau(ydx - xdy)}{1 + \frac{k}{4}(x^2 + y^2)}\right)^2. \tag{2.1}$$

If $k < 0$ then $E(k,\tau)$ is the product $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) \times \mathbb{R}$ with the metric (2.1), where $\mathrm{D}^2(\frac{2}{\sqrt{-k}}) = \{(x,y) \,|\, x^2 + y^2 < \frac{4}{-k}\}$. The family $E(k,\tau)$ is also referred to as Bianchi–Cartan–Vranceanu

[1] In [6, § 6.2] the term H should be considered as $2H$.

family [3, 9]. The projection of $E(k, \tau)$ onto the 2–dimensional domain of constant curvature k given by the map $(x, y, z) \mapsto (x, y)$ is a Riemannian fibration. The fibres of such a fiber bundle are geodesics and its unitary tangent vectors $\frac{\partial}{\partial z}$ form a Killing vector filed; this field is also referred to as the vertical vector field. The parameter k is called the base curvature and τ the bundle curvature.

If $k = -1, \tau = 0$ then $E(k, \tau)$ is the product $\mathbb{H}^2 \times \mathbb{R}$. If $k = 1, \tau = 0$ then $E(k, \tau)$ is obtained from the product $\mathbb{S}^2 \times \mathbb{R}$ by removing one fibre. If $k = 0, \tau \neq 0$ then $E(k, \tau)$ is the Heisenberg group Nil with the left–invariant metric determined by the parameter τ. If $k < 0, \tau \neq 0$ then $E(k, \tau)$ is the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ with the left–invariant metric determined by the parameters k and τ. For the case $k > 0, \tau \neq 0$, the manifolds $E(k, \tau)$ are obtained from the covering of the Berger spheres by removing one fibre.

For more details we refer the reader to [16, 8]. We will need the following proposition.

Proposition 1. *The sectional curvature of a 2–plane in* $E(k, \tau)$ *equals:*

$$\overline{K} = \tau^2 + (k - 4\tau^2)\nu^2, \tag{2.2}$$

where ν is the scalar product of a unit normal vector to the plane and the vertical vector $\xi = \frac{\partial}{\partial z}$ with respect to the metric (2.1).

Proof. The identity (2.2) can be obtained directly from the general formula for the Riemann curvature tensor of $E(k, \tau)$ shown in [8, Proposition 2.1]. □

3. The CMC spheres in $E(k, \tau)$

The rotationally invariant CMC surfaces in the products $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$ and the Heisenberg group Nil were described in [12, 14] and [7, 11, 17] respectively. The case of the Lie group $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ and the Berger spheres were studied in [15] and [18]. In order to describe rotationally invariant CMC surfaces in $E(k, \tau)$ we will follow the approach used in [11, 17] for $E(0, \frac{1}{2})$.

For the cylindrical coordinates ρ, θ, z in $E(k, \tau)$ such that $x = \rho \cos \theta$, $y = \rho \sin \theta$, $z = z$ the metric (2.1) has the form:

$$ds^2 = \frac{1}{(1 + \frac{k}{4}\rho^2)^2} d\rho^2 + \frac{\rho^2 + \tau^2\rho^4}{(1 + \frac{k}{4}\rho^2)^2} d\theta^2 - \frac{2\tau\rho^2}{1 + \frac{k}{4}\rho^2} dz d\theta + dz^2. \tag{3.1}$$

We note that $\rho \in [0, R)$, where $R = \frac{2}{\sqrt{-k}}$ if $k < 0$ and $R = \infty$ if $k \geqslant 0$.

The group SO(2) acts on $E(k, \tau)$ by rotations $\theta \mapsto \theta + const$ around z–axis. The rotations are isometries and the factor–space $E(k, \tau)/\mathrm{SO}(2)$ is the 2–dimensional domain $\mathrm{B}(k, \tau) = \{(u, v) | u \in [0, R), v \in \mathbb{R}\}$ with the metric:

$$d\widetilde{s}^2 = \frac{1}{(1 + \frac{k}{4}u^2)^2} du^2 + \frac{1}{1 + \tau^2 u^2} dv^2, \tag{3.2}$$

so the projection $E(k, \tau) \to \mathrm{B}(k, \tau)$ is a Riemannian submersion.

For a given rotationally invariant surface we define by $\gamma(s) = (u(s), v(s))$ its projection onto $\mathrm{B}(k, \tau)$, where s is a natural parameter with respect to the metric (3.2). Let σ be the angle between $\dot{\gamma}$ and $\frac{\partial}{\partial u}$. It can be verified (cf. [11, eq. 2]) that for the metric (3.2) the geodesic curvature of $\gamma(s)$ equals:

$$\widetilde{k} = \dot{\sigma} - \frac{\tau^2 u(1 + \frac{k}{4}u^2)}{(1 + \tau^2 u^2)} \sin\sigma. \tag{3.3}$$

The mean curvature of a rotationally invariant surface is given by the reduction theorem (cf. [11, p. 178]) as follows:

$$H = \frac{1}{2}\left(\widetilde{k} - \frac{\partial}{\partial n} \ln\mu\right), \tag{3.4}$$

where $n = (-(1 + \frac{k}{4}u^2)\sin\sigma, \sqrt{1 + \tau^2 u^2}\cos\sigma)$ is a normal vector in $\mathrm{B}(k, \tau)$ to $\gamma(s)$ and $\mu = \frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2}$ is the factor of the volume form for an SO(2) orbit with respect to the metric (3.1). From (3.3) and (3.4) we obtain:

$$H = \frac{1}{2}\left(\dot{\sigma} + \left(\frac{1}{u} - k\frac{u}{4}\right)\sin\sigma\right). \tag{3.5}$$

Thus, we obtain that for a profile $\gamma(s) = (u(s), v(s))$ of a rotationally invariant CMC surface the following system of ODE is satisfied:

$$\begin{cases} \dot{u} = \left(1 + \frac{k}{4}u^2\right)\cos\sigma, \\ \dot{v} = \sqrt{1 + \tau^2 u^2}\sin\sigma, \\ \dot{\sigma} = 2H - \left(\frac{1}{u} - k\frac{u}{4}\right)\sin\sigma. \end{cases} \tag{3.6}$$

It can be straightforwardly verified that the system (3.6) has the following first integral:

$$J = \frac{u}{1 + \frac{k}{4}u^2}(\sin\sigma - Hu). \tag{3.7}$$

Then we have the following proposition.

Proposition 2. *If $k \leqslant 0$ then for any H such that $H^2 > \frac{-k}{4}$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k, \tau)$; moreover, if $H^2 \leqslant \frac{-k}{4}$ then there exists no CMC sphere of constant mean curvature H in $E(k, \tau)$. If $k > 0$ then for any $H \neq 0$ there exists a rotationally invariant CMC sphere of constant mean curvature H in $E(k, \tau)$. For every rotationally invariant CMC sphere in $E(k, \tau)$ the first integral* (3.7) *vanishes: $J = 0$. The CMC spheres in $E(k, \tau)$ are unique up to isometries.*

Proof. The proof is based on an analysis of a qualitative behavior of the solutions of (3.6) depending on the values of J and H. Such an analysis is straightforward and it was done for $E(0, \frac{1}{2})$ in [11, 17]; the case of $E(-1, -\frac{1}{2})$ was shown in [4]. The uniqueness of the CMC spheres was proved in [1, 2]. Also, see [10] for the complete proof of Proposition 2. □

By Proposition 2 we obtain that on a rotationally invariant CMC sphere in $E(k, \tau)$ the following equality holds:

$$\sin\sigma = Hu. \tag{3.8}$$

Remark 3. Although there exists no a minimal sphere in $E(k,\tau)$, such spheres exist for $\mathbb{S}^2\times\mathbb{R}$ and the Berger spheres. We recall that for $k>0$ the manifolds $E(k,\tau)$ are obtained from the corresponding homogeneous manifolds by removing one fibre.

4. The proof of Theorem 1

For an immersion $f: M\to E(k,\tau)$ of a closed orientable surface M into $E(k,\tau)$ set:

$$E_{\alpha,\beta}(f)=\int_M\left(H^2+\alpha\overline{K}+\beta\right)d\mu. \tag{4.1}$$

Let $F: M\times[0,1]\to E(k,\tau)$ be a normal variation of the immersion f, i.e., $F(p,0)=f(p)$ for all $p\in M$ and $\frac{\partial F(p,t)}{\partial t}=\varphi n$, where n is the unit normal vector field to M and the velocity φ is a smooth function on M. We will denote by δ the operator $\frac{\partial}{\partial t}|_{t=0}$. We will need the following proposition.

Proposition 3. *Under a normal variation with the velocity φ the following identities hold:*

$$\delta d\mu=-2H\varphi d\mu, \tag{4.2}$$

$$\delta n=-\nabla\varphi, \tag{4.3}$$

$$2\delta H=\Delta\varphi+(4H^2-2K_e+\mathrm{Ric}(n,n))\varphi, \tag{4.4}$$

where ∇ is the gradient and Δ is the Laplace–Beltrami operator on M, K_e is the extrinsic Gauss curvature.

Proof. The proof is standard, one may look it up in [13]. □

It follows from Proposition 1 that the term $\mathrm{Ric}(n,n)$ equals $k-2\tau^2-(k-4\tau^2)\nu^2$. Therefore we have that:

$$2\delta H=\Delta\varphi+(4H^2-2K_e+k-2\tau^2-(k-4\tau^2)\nu^2)\varphi. \tag{4.5}$$

Let T be the projection of the vertical field ξ on M, i.e., $T=\xi-\nu n$. By (4.3) we have that $\delta\nu=\delta\langle n,\xi\rangle=-\langle\nabla\varphi,\xi\rangle$. Therefore we obtain:

$$\int_M\nu\delta\nu d\mu=\int_M\mathrm{div}(\nu T)\varphi d\mu. \tag{4.6}$$

By Proposition 1 we have:

$$E_{\alpha,\beta}(f)=\int_M(H^2+\alpha(k-4\tau^2)\nu^2+\beta+\alpha\tau^2)d\mu. \tag{4.7}$$

Then by (4.5),(4.6) and (4.2) we obtain:

$$\begin{aligned}\delta E_{\alpha,\beta}(f)=\int_M(\Delta H+H(2H^2-2K_e-(1+2\alpha)(k-4\tau^2)\nu^2+\\ +k-2\tau^2-2\beta-2\alpha\tau^2)+2\alpha(k-4\tau^2)\mathrm{div}(\nu T))\varphi d\mu.\end{aligned} \tag{4.8}$$

Therefore the Euler–Lagrange equation of the functional (4.1) is as follows:

$$\Delta H + H(2H^2 - 2K_e - (1+2\alpha)(k-4\tau^2)\nu^2 + \\ + k - 2\tau^2 - 2\beta - 2\alpha\tau^2) + 2\alpha(k-4\tau^2)\mathrm{div}(\nu T) = 0. \tag{4.9}$$

By the Gauss theorem we obtain $K_e = K - \overline{K} = K - (k-4\tau^2)\nu^2 - \tau^2$, where K is the intrinsic Gauss curvature. Then (4.9) can be rewritten as follows:

$$\Delta H + H(2H^2 - 2K + (1-2\alpha)(k-4\tau^2)\nu^2 + \\ + k - 2\beta - \alpha\tau^2) + 2\alpha(k-4\tau^2)\mathrm{div}(\nu T) = 0. \tag{4.10}$$

Consider a CMC sphere in $E(k,\tau)$. For the coordinates on this sphere we choose θ and s; recall that θ is an angle from the cylindrical coordinate system in $E(k,\tau)$ and s is the natural parameter on the projection $\gamma(s) = (u(s), v(s))$ of this sphere onto $\mathrm{B}(k,\tau)$. By (3.1), for these coordinates the metric on a CMC sphere is as follows:

$$\frac{\rho^2 + \tau^2\rho^4}{(1+\frac{k}{4}\rho^2)^2} d\theta^2 + ds^2. \tag{4.11}$$

By (3.2), (3.6) and (4.11) it can be straightforwardly verified that on a CMC sphere in $E(k,\tau)$:

$$\nu = \frac{\cos\sigma}{\sqrt{1+\tau^2u^2}}, \tag{4.12}$$

$$K = -\frac{1+\frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds^2} \frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2}, \tag{4.13}$$

$$\mathrm{div}(\nu T) = \frac{1+\frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds} \frac{u\cos\sigma\sin\sigma}{(1+\frac{k}{4}u^2)\sqrt{1+\tau^2u^2}}. \tag{4.14}$$

Put $\alpha = \frac{1}{4}$ and $\beta = \frac{k}{4} - \frac{\tau^2}{4}$. Substituting (4.12),(4.13) and (4.14) into the left hand side of (4.10) we obtain that it equals:

$$2H^3 + H\left(2\frac{1+\frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds^2} \frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2} + \frac{1}{2}(k-4\tau^2)\frac{\cos^2\sigma}{1+\tau^2u^2} + \frac{k}{2}\right) + \\ + \frac{1}{2}(k-4\tau^2)\frac{1+\frac{k}{4}u^2}{u\sqrt{1+\tau^2u^2}} \frac{d}{ds} \frac{u\cos\sigma\sin\sigma}{(1+\frac{k}{4}u^2)\sqrt{1+\tau^2u^2}} \tag{4.15}$$

Using the equation (3.8) and the system (3.6), it can be verified that the expression (4.15) vanishes on a CMC sphere. Theorem 1 is proved.

5. The proof of Theorem 2

Let us substitute (2.2) into the formula (1.2) for the functional $E(f)$. Then we have:

$$E(f) = \int_M \left(H^2 + \left(\frac{k}{4} - \tau^2\right)\nu^2 + \frac{k}{4}\right) d\mu. \tag{5.1}$$

Let us consider a rotationally invariant sphere in $E(k,\tau)$ defined by a curve $\gamma(s) = (u(s), v(s)) \subset \mathrm{B}(k,\tau)$. By (3.5) we can represent H^2 as follows:

$$H^2 = \frac{1}{4}\left(\dot\sigma - \left(\frac{1}{u} + k\frac{u}{4}\right)\sin\sigma\right)^2 + \left(\frac{\dot\sigma \sin\sigma}{u} - k\frac{\sin^2\sigma}{4}\right). \tag{5.2}$$

For a rotationally invariant surface we have:

$$\nu^2 = \frac{\cos^2\sigma}{1 + \tau^2 u^2}, \tag{5.3}$$

$$d\mu = \frac{u\sqrt{1+\tau^2u^2}}{1 + \frac{k}{4}u^2} ds. \tag{5.4}$$

Substituting (5.2), (5.3) and (5.4) into (5.1) we obtain:

$$\begin{aligned} E(f) = 2\pi \int_\gamma \frac{1}{4}\left(\dot\sigma - \left(\frac{1}{u} + k\frac{u}{4}\right)\sin\sigma\right)^2 \frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2} ds + \\ 2\pi \int_\gamma \left(\frac{\dot\sigma\sin\sigma}{u} - k\frac{\sin^2\sigma}{4} + \left(\frac{k}{4} - \tau^2\right)\frac{\cos^2\sigma}{1+\tau^2u^2} + \frac{k}{4}\right)\frac{u\sqrt{1+\tau^2u^2}}{1+\frac{k}{4}u^2} ds. \end{aligned} \tag{5.5}$$

By (3.6) we have: $\cos\sigma = \frac{\dot u}{1+\frac{k}{4}u^2}$. Substituting this into the integrand of the second summand in (5.5) we obtain:

$$-\frac{\ddot u\sqrt{1+\tau^2u^2}}{(1+\frac{k}{4}u^2)^2} + \frac{u\dot u^2}{(1+\frac{k}{4}u^2)^3}\left(\frac{3}{4}k\sqrt{1+\tau^2u^2} + \left(\frac{k}{4} - \tau^2\right)\frac{1}{\sqrt{1+\tau^2u^2}}\right). \tag{5.6}$$

It can be verified that the expression (5.6) is equal to $\frac{d}{ds}\left[-\frac{\dot u\sqrt{1+\tau^2u^2}}{(1+\frac{k}{4}u^2)^2}\right]$. Therefore, for a rotationally invariant sphere the second summand in (5.5) is equal to 4π.

The first summand in (5.5) is nonnegative. It vanishes iff the following holds:

$$\dot\sigma - \left(\frac{1}{u} + k\frac{u}{4}\right)\sin\sigma = 0. \tag{5.7}$$

It follows from (3.6) that (5.7) holds iff a rotationally invariant sphere is CMC. Theorem 2 is proved.

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