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EFFECTS OF RANDOMIZATION ON ASYMPTOTIC PERIODICITY OF NONSINGULAR TRANSFORMATIONS

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Abstract

It is known that the Perron–Frobenius operators of piecewise expanding C^2 transformations possess an asymptotic periodicity of densities. On the other hand, external noise or measurement errors are unavoidable in practical systems; therefore, all realistic mathematical models should be regarded as random iterations of transformations. This paper aims to discuss the effects of randomization on the asymptotic periodicity of densities.

1. Introduction

It is known that if $T : [0,1) \to [0,1)$ is a piecewise expanding C^2 transformation, then its corresponding Perron–Frobenius operator $\mathcal{L}_T$, which we define in Section 2, exhibits an asymptotic periodicity of densities. That is, there exist probability density functions $g_{i,j,T} \in L^1([0,1))$ and functionals $\lambda_{i,j,T}(\cdot)$ on $L^1([0,1))$ $(1 \le i \le s(T),\ 1 \le j \le r(i,T))$ satisfying the following conditions (see Definition 2.7 in Section 2):

(i) $g_{i,j,T} \cdot g_{k,l,T} = 0$ for all $(i,j) \neq (k,l)$;

(ii) For every $i \in \{1,2,\dots,s(T)\}$, $\{g_{i,j,T}\}_{j=1}^{r(i,T)}$ are periodic points of $\mathcal{L}_T$: $\mathcal{L}_T(g_{i,j,T}) = g_{i,j+1,T}$ $(1 \le j \le r(i,T)-1)$ and $\mathcal{L}_T(g_{i,r(i,T),T}) = g_{i,1,T}$ hold;

(iii) $\lim_{n\to\infty} \left\|(\mathcal{L}_T)^n(f - \sum_{i=1}^{s(T)} \sum_{j=1}^{r(i,T)} \lambda_{i,j,T}(f) g_{i,j,T})\right\|_{L^1([0,1))} = 0$ for $f \in L^1([0,1))$.

Recall that the asymptotic periodicity of $\mathcal{L}_T$ describes the ergodic properties of the transformation T (see, for example, [1]). If we define $A_{i,j} = \{x \in [0,1); g_{i,j,T}(x) > 0\}$ and $A_i = \bigcup_{j=1}^{r(i,T)} A_{i,j}$, then the asymptotic periodicity of densities for $\mathcal{L}_T$ [(i)–(iii)] implies that T exhibits the following asymptotic periodicity [1],[7],[9]:

(a) For every $i \in \{1,2,\dots,s(T)\}$, A_i is T-invariant [*i.e.*, $T(A_i) = A_i$], and the restriction $T_i \equiv T|_{A_i}$ is ergodic with respect to the Lebesgue measure m;

(b) For every $i \in \{1,2,\dots,s(T)\}$, there exists a T-invariant measure μ_i on A_i that is equivalent to $m|_{A_i}$;

(c) For $B \equiv [0,1) \setminus \bigcup_{i=1}^{s(T)} A_i$, we have that $T^{-1}(B) \subset B$, and $\lim_{n\to\infty} m(T^{-n}(B)) = 0$;

(d) For every $i \in \{1,2,\dots,s(T)\}$, we have that for the power $T_i^* \equiv (T_i)^{r(i,T)}$ of the transformation T_i, $T_i^*(A_{i,j}) = A_{i,j}$ $(1 \le j \le r(i,T))$ holds, and T_i^* is an exact endomorphism on $A_{i,j}$

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$(1 \le j \le r(i,T))$. Further, for every $i \in \{1, 2, \ldots, s(T)\}$, the transformation T_i permutes $\{A_{i,j}\}_{j=1}^{r(i,T)}$ cyclically: $T_i(A_{i,j}) = A_{i,j+1}$ $(1 \le j \le r(i,T) - 1)$ and $T_i(A_{i,r(i,T)}) = A_{i,1}$ hold.

The above argument outlines the asymptotic periodicity of a single transformation T. On the other hand, external noise, measurement errors, or inaccuracy are unavoidable in practical systems. Therefore, every realistic mathematical model should be regarded as a number of random iterations of transformations T_y $(y \in Y)$:

$$T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} x,$$

which is the first coordinate of the iterations of the skew product transformation

$$S^n(x, \omega) = (T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} x, \sigma^n \omega)$$

defined in Subsection 2.2 [3]. In this paper, we consider only the case in which transformations T_{ω_i} are independently chosen. Then, under some assumptions, the skew product transformation $S (\equiv S^1)$ is known to have asymptotic periodicity in the sense given above. In this regard, it must be noted that the skew product transformation S can be regarded as a random transformation. This paper is concerned with the effects of this type of randomization on the asymptotic periodicity.

This paper is organized as follows. In Section 2, we review the necessary concepts and results from the general theory of Perron–Frobenius operators, as well as those relating to the random iterations of nonsingular transformations. In Section 3, our main results are presented. In Section 4, we discuss a sufficient condition for the assumption of our main results. In Section 5, we present some examples with numerical experiments.

2. Preliminaries

In Subsection 2.1, we define the Perron–Frobenius operator and state its basic properties that are necessary for our discussion. In Subsection 2.2, we review the necessary concepts and results from the theory of random iterations of transformations.

Although most of the results in this section are already well known or can be easily seen, we give some of their proofs for completeness.

2.1. Perron–Frobenius operators. Let $(X, \mathcal{F}, m)$ be a probability space and $T : X \to X$ be an m-nonsingular transformation, *i.e.*, a measurable transformation, satisfying $m(T^{-1}(A)) = 0$ for $A \in \mathcal{F}$ with $m(A) = 0$. Further, we denote the set of p-th integrable functions on X with respect to the measure m as $L^p(m) \equiv L^p(X, \mathcal{F}, m)$ $(p \in [1, \infty])$. Then, we define the Perron–Frobenius operator corresponding to $(X, \mathcal{F}, m, T)$ as follows.

DEFINITION 2.1. The Perron–Frobenius operator $\mathcal{L}_T$ on $L^1(m)$ is defined as

$$\mathcal{L}_T f \equiv \frac{dm_f}{dm}, \quad \text{where } m_f(A) = \int_{T^{-1}(A)} f(x) dm(x).$$

The Perron–Frobenius operator $\mathcal{L}_T : L^1(m) \to L^1(m)$ is characterized by the following well-known proposition:

Proposition 2.2. *For $f \in L^1(m)$, $\mathcal{L}_T f$ is the unique element in $L^1(m)$ satisfying*

$$\int_X (\mathcal{L}_T f)(x)h(x)dm(x) = \int_X f(x)h(Tx)dm(x)$$

for every $h \in L^\infty(m)$.

As an operator on $L^1(m)$, $\mathcal{L}_T$ has the following properties, which are easily shown from Proposition 2.2:

Proposition 2.3. *The operator $\mathcal{L}_T$ on $L^1(m)$ is positive, bounded, and linear, and it has the following properties:*

(1) $\mathcal{L}_T$ preserves integrals; i.e., $\int_X (\mathcal{L}_T f)(x)dm(x) = \int_X f(x)dm(x)$ holds for $f \in L^1(m)$;
(2) For $f \in L^1(m)$, we have the inequality $|(\mathcal{L}_T f)(x)| \leq (\mathcal{L}_T|f|)(x)$ (m-a.e.);
(3) $\mathcal{L}_T$ is a contraction; i.e., $\|\mathcal{L}_T f\|_{L^1(m)} \leq \|f\|_{L^1(m)}$ holds for $f \in L^1(m)$;
(4) $(\mathcal{L}_T)^n = \mathcal{L}_{T^n}$ holds, where $\mathcal{L}_{T^n}$ represents the Perron–Frobenius operator corresponding to T^n;
(5) For $g \in L^\infty(m)$ and $f \in L^1(m)$, we have $g \cdot \mathcal{L}_T f = \mathcal{L}_T((g \circ T)f)$, where $(g \circ T)(x) \equiv g(Tx)$;
(6) $\mathcal{L}_T f = f$ if and only if $f(x)dm(x)$ is T-invariant.

By applying Proposition 2.3, we can obtain Propositions 2.4 and 2.5.

Proposition 2.4. *Assume that $\mathcal{L}_T f = g$ holds for some nonnegative functions $f, g \in L^1(m)$. Then*

$$T^{-1}\{g > 0\} \supset \{f > 0\} \quad (m\text{-a.e.});$$

i.e.,

$$m\Big(\{f > 0\} \setminus T^{-1}\{g > 0\}\Big) = 0$$

is satisfied.

Proof. Using Proposition 2.3 (5) and the assumption that $\mathcal{L}_T f = g$, we can show that

$$(\mathcal{L}_T f)(x) = g(x) = 1_{\{g>0\}}(x)g(x) = 1_{\{g>0\}}(x)(\mathcal{L}_T f)(x) = \mathcal{L}_T(1_{T^{-1}\{g>0\}}f)(x).$$

We have $\int_X f(x)dm(x) = \int_X 1_{T^{-1}\{g>0\}}(x)f(x)dm(x)$, as $\mathcal{L}_T$ preserves integrals. Therefore, the inequality $f(x) - 1_{T^{-1}\{g>0\}}(x)f(x) \geq 0$ ($x \in X$) shows that $f(x) = 1_{T^{-1}\{g>0\}}(x)f(x)$ (m-a.e. x). This completes the proof. □

Proposition 2.5. *Let $f \in L^1(m)$ be nonnegative, and let $A \in \mathcal{F}$. If $\mathcal{L}_T f = f$ and $T^{-1}(A) \supset A$, then $\mathcal{L}_T(f 1_A) = f 1_A$.*

Proof. Using the given assumptions and Proposition 2.3 (5), we have that

$$1_A(x)f(x) = 1_A(x)(\mathcal{L}_T f)(x) = \mathcal{L}_T(f 1_{T^{-1}(A)})(x) \geq \mathcal{L}_T(f 1_A)(x), \quad (m\text{-a.e. } x). \tag{2.1}$$

By combining inequality (2.1) and the fact that $\int_X \{1_A(x)f(x) - \mathcal{L}_T(1_A f)(x)\}\, dm(x) = 0$, we obtain $\mathcal{L}_T(f 1_A) = f 1_A$. □

If $\lim_{n\to\infty}(\mathcal{L}_T)^n f = g$ holds, then the limit set of $T^n\{f \neq 0\}$ is the support of g. That is, we have the following proposition.

Proposition 2.6. *Assume that*

$$\lim_{n\to\infty} \|(\mathcal{L}_T)^n f - g\|_{L^1(m)} = 0 \tag{2.2}$$

holds for some nonnegative functions $f, g \in L^1(m)$. *Then*

$$m\Big(\{f > 0\} \setminus \bigcup_{n=0}^{\infty} T^{-n}\{g > 0\}\Big) = 0. \tag{2.3}$$

Proof. First, we prove that the equation

$$\lim_{n\to\infty} m\left(\{f > \varepsilon\} \setminus T^{-n}\{g > 0\}\right) = 0 \tag{2.4}$$

holds for any $\varepsilon > 0$. In fact, we clearly have the estimate

$$\begin{aligned}
\|(\mathcal{L}_T)^n f - (\mathcal{L}_T)^n \left(f 1_{T^{-n}\{g>0\}}\right)\|_{L^1(m)} &= \int_X ((\mathcal{L}_T)^n \left(f - f 1_{T^{-n}\{g>0\}}\right))(x) dm(x) \\
&= \int_X f(x)\left(1 - 1_{T^{-n}\{g>0\}}(x)\right) dm(x) \\
&\geq \varepsilon \int_{\{f>\varepsilon\}} \left(1 - 1_{T^{-n}\{g>0\}}(x)\right) dm(x) \\
&= \varepsilon\, m\left(\{f > \varepsilon\} \setminus T^{-n}\{g > 0\}\right). \qquad (2.5)
\end{aligned}$$

On the other hand, the inequality

$$\begin{aligned}
&\|(\mathcal{L}_T)^n f - (\mathcal{L}_T)^n \left(f 1_{T^{-n}\{g>0\}}\right)\|_{L^1(m)} \\
&\quad \leq \|(\mathcal{L}_T)^n f - g\|_{L^1(m)} + \|(\mathcal{L}_T)^n \left(f 1_{T^{-n}\{g>0\}}\right) - g 1_{\{g>0\}}\|_{L^1(m)} \\
&\quad = \|(\mathcal{L}_T)^n f - g\|_{L^1(m)} + \|1_{\{g>0\}} \cdot ((\mathcal{L}_T)^n f - g)\|_{L^1(m)} \\
&\quad \leq 2\|(\mathcal{L}_T)^n f - g\|_{L^1(m)},
\end{aligned}$$

together with assumption (2.2), shows that

$$\lim_{n\to\infty} \|(\mathcal{L}_T)^n f - (\mathcal{L}_T)^n \left(f 1_{T^{-n}\{g>0\}}\right)\|_{L^1(m)} = 0. \tag{2.6}$$

Therefore, the convergence in (2.4) follows from (2.5) and (2.6).

By applying assumption (2.2), we have that $\mathcal{L}_T g = g$. As a result, Proposition 2.4 shows that $T^{-n}\{g > 0\} \supset T^{-(n-1)}\{g > 0\}$ (m-a.e.) for $n \geq 1$. Therefore, equation (2.4) implies

$$m\left(\{f > \varepsilon\} \setminus \bigcup_{n=0}^{\infty} T^{-n}\{g > 0\}\right) = 0 \tag{2.7}$$

for any $\varepsilon > 0$. This proves our result (2.3). □

Let us define the asymptotic periodicity of $\mathcal{L}_T$ as follows.

Definition 2.7. $\mathcal{L}_T$ is asymptotically periodic if there exist positive integers $s(T)$ and $r(i, T)$ ($1 \leq i \leq s(T)$), probability density functions $g_{i,j,T} \in L^1(m)$, and bounded linear functionals $\lambda_{i,j,T}(\cdot)$ on $L^1(m)$ ($1 \leq i \leq s(T), 1 \leq j \leq r(i, T)$) such that

(i) $g_{i,j,T} \cdot g_{k,l,T} = 0$ for all $(i, j) \neq (k, l)$;

(ii) $\mathcal{L}_T(g_{i,j,T}) = g_{i,j+1,T}$ $(1 \leq j \leq r(i,T)-1)$ and $\mathcal{L}_T(g_{i,r(i,T),T}) = g_{i,1,T}$ hold for all i;

(iii) $\lim_{n\to\infty} \left\|(\mathcal{L}_T)^n(f - \sum_{i=1}^{s(T)}\sum_{j=1}^{r(i,T)} \lambda_{i,j,T}(f)g_{i,j,T})\right\|_{L^1(m)} = 0$ for any $f \in L^1(m)$.

Using the above propositions and assuming the asymptotic periodicity of $\mathcal{L}_T$, we can show the asymptotic periodicity of limit sets for T.

Proposition 2.8. *Suppose that $\mathcal{L}_T$ is asymptotically periodic. Then, if we denote $g_{i,T}(x) \equiv \dfrac{1}{r(i,T)} \sum_{j=1}^{r(i,T)} g_{i,j,T}(x)$ $(x \in X)$, $A_i \equiv \{x \in X; g_{i,T}(x) > 0\}$, and $A_{i,j} \equiv \{x \in X; g_{i,j,T}(x) > 0\}$, T has the following asymptotic periodicity:*

(a) For every $i \in \{1,2,\dots,s(T)\}$, A_i is T-invariant [i.e., $T(A_i) = A_i$], and $d\mu_i(x) \equiv g_{i,T}(x)dm(x)$ is an ergodic T-invariant probability measure on A_i;

(b) For $B \equiv X \setminus \bigcup_{i=1}^{s(T)} A_i$, we have that $T^{-1}(B) \subset B$ and $\lim_{n\to\infty} m(T^{-n}(B)) = 0$;

(c) For every power $T_i^* \equiv (T_i)^{r(i,T)}$ of $T_i \equiv T|_{A_i}$, where $i \in \{1,2,\dots,s(T)\}$, we have that $T_i^*(A_{i,j}) = A_{i,j}$ $(1 \leq j \leq r(i,T))$ holds, and T_i^* is an exact endomorphism on $A_{i,j}$ $[1 \leq j \leq r(i,T)]$. Further, for every $i \in \{1,2,\dots,s(T)\}$, T_i permutes $\{A_{i,j}\}_{j=1}^{r(i,T)}$ cyclically: $T_i(A_{i,j}) = A_{i,j+1}$ $(1 \leq j \leq r(i,T)-1)$ and $T_i(A_{i,r(i,T)}) = A_{i,1}$ hold.

The following key proposition is established on the basis of the ergodicity of each A_i, where $\{A_1, A_2, \cdots, A_{s(T)}, B\}$ is the disjoint decomposition of X given in Proposition 2.8.

Proposition 2.9. *Let $\{A_1, A_2, \cdots, A_{s(T)}, B\}$ be a measurable partition given in Proposition* 2.8*, and let $A \in \mathcal{F}$ with $m(A) > 0$ and $T^{-1}(A) \supset A$. Then,*

$$A_i \cap A = \emptyset \quad \textit{or} \quad A_i \subset A \quad \textit{for} \quad i = 1,2,\dots,s(T), \quad \textit{and} \quad A \cap \bigcup_{i=1}^{s(T)} A_i \neq \emptyset.$$

Proof. Recall that $g_{i,T}(x)$ is a probability density function of an ergodic, T-invariant measure μ_i on A_i. From Proposition 2.5, we obtain that $\mathcal{L}_T(g_{i,T} \cdot 1_A) = g_{i,T} \cdot 1_A$. The ergodicity of μ_i allows us to state that either $g_{i,T} \cdot 1_A = 0$ or $g_{i,T} \cdot 1_A = g_{i,T}$ holds for $1 \leq i \leq s(T)$. Therefore, either $A_i \cap A = \emptyset$ or $A_i \subset A$ holds for $1 \leq i \leq s(T)$. If we assume that $A \cap \bigcup_{i=1}^{s(T)} A_i = \emptyset$ holds, then we have that $A \subset B$. Under this assumption, we have that $T^{-n}(A) \supset A$ for $n \in \mathbb{N}$; hence, $m(T^{-n}(B)) \geq m(T^{-n}(A)) \geq m(A) > 0$ for $n \in \mathbb{N}$. This contradicts the condition that $\lim_{n\to\infty} m(T^{-n}(B)) = 0$. □

2.2. Random iteration. To formulate our main results, we need to introduce several further concepts. In this subsection, we define the random iteration of m-nonsingular transformations.

[I] Let Y be a complete separable metric space, $\mathcal{B}(Y)$ be its topological Borel field, and η be a probability measure on $(Y, \mathcal{B}(Y))$. Further, define $\Omega \equiv \Pi_{i=1}^{\infty} Y$, and let us write $\mathcal{B}(\Omega)$ for the topological Borel field of Ω. We insert the product measure $P \equiv \Pi_{i=1}^{\infty}\eta$ on $(\Omega, \mathcal{B}(\Omega))$.

[II] Let $(X, \mathcal{F}, m)$ be a probability space and $(T_y)_{y\in Y}$ be a family of m-nonsingular transformations on X, such that the mapping $(x,y) \to T_y x$ is measurable.

To study the behavior of the random iterations, we consider the skew product transformation $S : X \times \Omega \to X \times \Omega$, defined as

$$S(x, \omega) \equiv (T_{\omega_1} x, \sigma\omega), \quad (x, \omega) \in X \times \Omega, \tag{2.8}$$

where ω_1 represents the first coordinate of $\omega = (\omega_i)_{i=1}^{\infty}$, and $\sigma : \Omega \to \Omega$ is the shift transformation to the left, which is defined as $\sigma((\omega_i)_{i=1}^{\infty}) = (\omega_{i+1})_{i=1}^{\infty}$. Note that, for $n \in \mathbb{N}$, we have

$$S^n(x, \omega) = (T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} x, \sigma^n \omega). \tag{2.9}$$

Therefore, we can consider the random iteration of m-nonsingular transformations as $\pi S^n(x, \omega)$, writing $\pi : X \times \Omega \to X$ for the projection on X. Under these settings, Morita [4],[5],[6] investigated the existence of invariant measures and their mixing properties. His method is also useful for our purpose.

Because $(T_y)_{y \in Y}$ are m-nonsingular transformations, S is a nonsingular transformation on $(X \times \Omega, \mathcal{F} \times \mathcal{B}(\Omega), m \times P)$. Therefore, we can define the Perron–Frobenius operator $\mathcal{L}_S : L^1(m \times P) \to L^1(m \times P)$ corresponding to S as

$$\iint_{X \times \Omega} h(x, \omega)(\mathcal{L}_S f)(x, \omega) dm(x) dP(\omega) = \iint_{X \times \Omega} f(x, \omega) h(S(x, \omega)) dm(x) dP(\omega)$$

for $h \in L^\infty(m \times P)$, where $L^p(m \times P) \equiv L^p(X \times \Omega, \mathcal{F} \times \mathcal{B}(\Omega), m \times P)$ for $p \in [1, \infty]$. Lemma 4.1 in [6] can be rewritten as follows:

Proposition 2.10. *(i) If $(\mathcal{L}_S f)(x, \omega) = \lambda f(x, \omega)$ holds for $|\lambda| = 1$, then f does not depend on ω.*

(ii) For every $f \in L^1(m)$, we have

$$(\mathcal{L}_S f)(x, \omega) = \int_Y (\mathcal{L}_{T_y} f)(x) \eta(dy), \quad (m \times P\text{-a.e.}); \tag{2.10}$$

hence, $\mathcal{L}_S f \in L^1(m)$.

Proposition 2.10 allows us to consider $\mathcal{L}_S$ as an operator on $L^1(m)$. Then, we have the following key proposition:

Proposition 2.11. *Assume that $(\mathcal{L}_S f)(x, \omega) = g(x)$ $(m \times P\text{-a.e.})$ holds for some nonnegative functions f, $g \in L^1(m)$. Then, there exists a set $Y_0 \in \mathcal{B}(Y)$, with $\eta(Y_0) = 1$, such that for every $\omega_1 \in Y_0$,*

$$(T_{\omega_1})^{-1}\{g > 0\} \supset \{f > 0\} \ (m\text{-a.e.}); \quad \text{i.e.,} \quad m\Big(\{f > 0\} \setminus (T_{\omega_1})^{-1}\{g > 0\}\Big) = 0$$

is satisfied.

Proof. By applying Proposition 2.4, we obtain the equation

$$(m \times P)\Big(\{(x, \omega) \in X \times \Omega;\ f(x) > 0\} \setminus S^{-1}\{(x, \omega) \in X \times \Omega;\ g(x) > 0\}\Big) = 0.$$

Fubini's theorem implies that there exists a set $\Omega_0 \in \mathcal{B}(\Omega)$, with $P(\Omega_0) = 1$, such that

$$m\Big(\{x \in X; f(x) > 0\} \setminus (T_{\omega_1})^{-1}\{x \in X; g(x) > 0\}\Big) = 0$$

holds for $(\omega_i)_{i=1}^{\infty} \in \Omega_0$. Define $Y_0 \equiv \widetilde{\pi}_1(\Omega_0)$, where $\widetilde{\pi}_1 : \Omega \to Y$ is the projection given by $\widetilde{\pi}_1((\omega_i)_{i=1}^{\infty}) = \omega_1$ for $(\omega_i)_{i=1}^{\infty} \in \Omega$. Then we obtain our proposition. □

3. Main results

In this section, we state our main results using the notation defined in Subsection 2.2. To state our main results, we assume the asymptotic periodicity of $\mathcal{L}_S$. Then there exist probability density functions $g_{i,j,S} \in L^1(m)$ and functionals $\lambda_{i,j,S}(\cdot)$ on $L^1(m)$ $(1 \leq i \leq s(S),\ 1 \leq j \leq r(i, S))$ satisfying the conditions of Definition 2.7. We denote $s(S)$, $r(i, S)$, $g_{i,j,S}$, and $\lambda_{i,j,S}(\cdot)$ by $\widehat{s}$, $\widehat{r}(i)$, $\widehat{g}_{i,j}$, and $\widehat{\lambda}_{i,j}(\cdot)$, respectively. Note that $\widehat{s}$ is the number of ergodic components of $\mathcal{L}_S$, and $\widehat{r}(i)$ is the number of cycles of each ergodic component $(1 \leq i \leq \widehat{s})$. It is also meaningful to consider a sufficient condition for the above assumption, which will be discussed in Section 4.

Let Y_1 denote the set of parameters $y \in Y$ such that $\mathcal{L}_{T_y}$ is asymptotically periodic; that is, for $y \in Y_1$, there exist probability density functions $g_{i,j,T_y} \in L^1(m)$ and functionals $\lambda_{i,j,T_y}(\cdot)$ on $L^1(m)$ $(1 \leq i \leq s(T_y),\ 1 \leq j \leq r(i, T_y))$ satisfying the conditions of Definition 2.7. Note also that $s(T_y)$ is the number of ergodic components of $\mathcal{L}_{T_y}$, and $r(i, T_y)$ is the period of cycles of each ergodic component $(1 \leq i \leq s(T_y))$. If $\mathcal{L}_S$ is asymptotically periodic, $\eta(Y_1)$ is positive in many cases; this is also confirmed by the examples given in Section 5. Under the above notation, Proposition 2.11 can be rewritten as follows:

Proposition 3.1. *Suppose that $\mathcal{L}_S$ is asymptotically periodic. Then there exists a set $Y_0 \in \mathcal{B}(Y)$, with $\eta(Y_0) = 1$, such that*

$$\{\widehat{g}_{i,j} > 0\} \subset T_y^{-1}\{\widehat{g}_{i,j+1} > 0\}\ (1 \leq j \leq \widehat{r}(i) - 1) \text{ and } \{\widehat{g}_{i,\widehat{r}(i)} > 0\} \subset T_y^{-1}\{\widehat{g}_{i,1} > 0\}$$

hold for $y \in Y_0$.

Remark 3.2. Define $\widehat{g}_i \equiv \dfrac{1}{\widehat{r}(i)} \displaystyle\sum_{j=1}^{\widehat{r}(i)} \widehat{g}_{i,j}$ $(1 \leq i \leq \widehat{s})$ and $g_{i,T_y} \equiv \dfrac{1}{r(i, T_y)} \displaystyle\sum_{j=1}^{r(i,T_y)} g_{i,j,T_y}$ $(1 \leq i \leq s(T_y))$. Then, $\widehat{g}_i$ and g_{i,T_y} are the densities of the ergodic invariant probabilities of S and T_y, respectively.

We are now in a position to state the first main result of this paper.

Theorem 3.3. *Suppose that $\mathcal{L}_S$ is asymptotically periodic. Let Y_0 be the set as in Proposition 3.1. Then, for every $i \in \{1, 2, \ldots, \widehat{s}\}$ and $y \in Y_0 \cap Y_1$, we have the following statements.*

(1) We have that $\{\widehat{g}_i > 0\} \cap \bigcup_{k=1}^{s(T_y)} \{g_{k,T_y} > 0\} \neq \emptyset$ holds. Moreover, either $\{\widehat{g}_i > 0\} \supset \{g_{k,T_y} > 0\}$ or $\{\widehat{g}_i > 0\} \cap \{g_{k,T_y} > 0\} = \emptyset$ holds for every $k \in \{1, 2, \ldots, s(T_y)\}$. This means that $\widehat{s}$ is not greater than $s(T_y)$.

(2) For $k_0 \in \{1, 2, \ldots, s(T_y)\}$ satisfying $\{\widehat{g}_i > 0\} \supset \{g_{k_0,T_y} > 0\}$, $\widehat{r}(i)$ is a divisor of $r(k_0, T_y)$.

Proof. (1) Note that $\{\widehat{g}_i > 0\} = \bigcup_{j=1}^{\widehat{r}(i)} \{\widehat{g}_{i,j} > 0\}$. Then, Proposition 3.1 shows that $T_y^{-1}\{\widehat{g}_i > 0\} \supset \{\widehat{g}_i > 0\}$. Hence, we easily obtain the first statement, which follows from Proposition 2.9.

(2) For simplicity, let us write $\widehat{g}_{k,r} \equiv \widehat{g}_{k,j}$ for $r = l \cdot \widehat{r}(k) + j$ $(1 \leq k \leq \widehat{s}, 1 \leq j \leq \widehat{r}(k), l \in \mathbb{N})$ and $g_{k,r,T_y} \equiv g_{k,j,T_y}$ for $r = l \cdot r(k, T_y) + j$ $(1 \leq k \leq s(T_y), 1 \leq j \leq r(k, T_y), l \in \mathbb{N})$. Then, from Proposition 3.1, we have that $\{\widehat{g}_{k,j} > 0\} \subset T_y^{-1}\{\widehat{g}_{k,j+1} > 0\}$ $(1 \leq k \leq \widehat{s}, j \in \mathbb{N})$.

Because $g_{k_0,j+1,T_y} = \mathcal{L}_{T_y}(g_{k_0,j,T_y})$ holds for $j \in \mathbb{N}$, we have that

$$1_{\{\widehat{g}_{i,j+1}>0\}} g_{k_0,j+1,T_y} = 1_{\{\widehat{g}_{i,j+1}>0\}} \mathcal{L}_{T_y}(g_{k_0,j,T_y}) = \mathcal{L}_{T_y}(1_{T_y^{-1}\{\widehat{g}_{i,j+1}>0\}} g_{k_0,j,T_y}).$$

From the assumption that $\{\widehat{g}_i > 0\} \supset \{g_{k_0,T_y} > 0\}$, we have $g_{k_0,j,T_y} = 1_{\{\widehat{g}_i>0\}} g_{k_0,j,T_y}$. Note that the sets $\{\widehat{g}_{i,j} > 0\}$ $(j = 1, 2, \ldots, \widehat{r}(i))$ are mutually disjoint. Then, we can obtain that $T_y^{-1}\{\widehat{g}_{i,j+1} > 0\} \cap \{\widehat{g}_i > 0\} = \{\widehat{g}_{i,j} > 0\}$. Hence, we have that

$$\mathcal{L}_{T_y}(1_{T_y^{-1}\{\widehat{g}_{i,j+1}>0\}} g_{k_0,j,T_y}) = \mathcal{L}_{T_y}(1_{T_y^{-1}\{\widehat{g}_{i,j+1}>0\}} 1_{\{\widehat{g}_i>0\}} g_{k_0,j,T_y}) = \mathcal{L}_{T_y}(1_{\{\widehat{g}_{i,j}>0\}} g_{k_0,j,T_y}).$$

This implies that

$$1_{\{\widehat{g}_{i,j+1}>0\}} g_{k_0,j+1,T_y} = \mathcal{L}_{T_y}(1_{\{\widehat{g}_{i,j}>0\}} g_{k_0,j,T_y}) \quad \text{for } j \in \mathbb{N}. \tag{3.1}$$

Recall the assumption that $\{\widehat{g}_i > 0\} \supset \{g_{k_0,T_y} > 0\}$. Then, by renumbering, we can assume that $1_{\{\widehat{g}_{i,1}>0\}} g_{k_0,1,T_y}$ is a nontrivial function. Thus, we obtain a sequence of nontrivial nonnegative functions $h_j \equiv 1_{\{\widehat{g}_{i,j}>0\}} g_{k_0,j,T_y}$ $(j \in \mathbb{N})$ satisfying the equations $h_{j+1} = \mathcal{L}_{T_y} h_j$ $(j \in \mathbb{N})$ and $h_j \cdot h_l = 0$ for $1 \leq j < l \leq N$, where N denotes the least common multiple of $\widehat{r}(i)$ and $r(k_0, T_y)$. It clearly follows that

$$\sum_{j=1}^{N} h_j = \sum_{j=1}^{N} 1_{\{\widehat{g}_{i,j}>0\}} g_{k_0,j,T_y} \leq \sum_{r=1}^{\widehat{r}(i)} \sum_{j=1}^{r(k_0,T_y)} 1_{\{\widehat{g}_{i,r}>0\}} g_{k_0,j,T_y} = \sum_{j=1}^{r(k_0,T_y)} g_{k_0,j,T_y}.$$

Therefore, we obtain the estimate

$$\begin{aligned} N \int_X h_1(x) dm(x) &= \sum_{j=0}^{N-1} \int_X ((\mathcal{L}_{T_y})^j h_1)(x) dm(x) \\ &= \int_X \sum_{j=1}^{N} h_j(x) dm(x) \\ &\leq \int_X \sum_{j=1}^{r(k_0,T_y)} g_{k_0,j,T_y}(x) dm(x) \\ &= r(k_0, T_y) \int_X g_{k_0,1,T_y}(x) dm(x). \end{aligned}$$

If $N > r(k_0, T_y)$ holds, then we have that

$$\int_X h_1(x) dm(x) \leq \frac{r(k_0, T_y)}{N} \int_X g_{k_0,1,T_y}(x) dm(x) < \int_X g_{k_0,1,T_y}(x) dm(x).$$

This shows that $\{h_1 > 0\} \subsetneq \{g_{k_0,1,T_y} > 0\}$; *i.e.*, $m(\{g_{k_0,1,T_y} > 0\} \setminus \{h_1 > 0\}) > 0$. Because $(\mathcal{L}_{T_y})^{nN}(h_1) = h_1$ and $(\mathcal{L}_{T_y})^{nN}(g_{i,j,T_y}) = g_{i,j,T_y}$ hold for $n \in \mathbb{N}$, we have that

$$\Big\|(\mathcal{L}_{T_y})^{nN}(h_1 - \sum_{k=1}^{s(T_y)} \sum_{j=1}^{r(k,T_y)} \lambda_{k,j,T_y}(h_1) g_{k,j,T_y})\Big\|_{L^1(m)}$$

$$
\begin{aligned}
&= \Big\| h_1 - \sum_{k=1}^{s(T_y)} \sum_{j=1}^{r(k,T_y)} \lambda_{k,j,T_y}(h_1) g_{k,j,T_y} \Big\|_{L^1(m)} \\
&= \| h_1 - \lambda_{k_0,1,T_y}(h_1) g_{k_0,1,T_y} \|_{L^1(m)} + \Big\| \sum_{(k,j)\neq(k_0,1)} \lambda_{k,j,T_y}(h_1) g_{k,j,T_y} \Big\|_{L^1(m)} \\
&\geq \| h_1 - \lambda_{k_0,1,T_y}(h_1) g_{k_0,1,T_y} \|_{L^1(m)} \\
&= \int_X (1_{\{h_1>0\}}(x) + 1_{\{h_1=0\}}(x)) \big| h_1(x) - \lambda_{k_0,1,T_y}(h_1) g_{k_0,1,T_y}(x) \big| dm(x) \\
&= |1 - \lambda_{k_0,1,T_y}(h_1)| \int_X h_1(x)\, dm(x) \\
&\quad + |\lambda_{k_0,1,T_y}(h_1)| \int_X 1_{\{g_{k_0,1,T_y}>0,\ h_1=0\}}(x) g_{k_0,1,T_y}(x) dm(x).
\end{aligned}
$$

If $\lambda_{k_0,1,T_y}(h_1) = 1$ holds, the right-hand side of the above inequality is

$$
\int_X 1_{\{g_{k_0,1,T_y}>0,\ h_1=0\}}(x) g_{k_0,1,T_y}(x) dm(x),
$$

which is strictly positive. If $\lambda_{k_0,1,T_y}(h_1) \neq 1$ holds, we have

$$
|1 - \lambda_{k_0,1,T_y}(h_1)| \int_X h_1(x)\, dm(x) > 0.
$$

Therefore, there exists a positive constant a such that

$$
\Big\| (\mathcal{L}_{T_y})^{nN} (h_1 - \sum_{k=1}^{s(T_y)} \sum_{j=1}^{r(k,T_y)} \lambda_{k,j,T_y}(h_1) g_{k,j,T_y}) \Big\|_{L^1(m)} \geq a > 0
$$

holds for every $n \in \mathbb{N}$. This contradicts the fact that

$$
\lim_{n\to\infty} \Big\| (\mathcal{L}_{T_y})^{nN} (h_1 - \sum_{k=1}^{s(T_y)} \sum_{j=1}^{r(k,T_y)} \lambda_{k,j,T_y}(h_1) g_{k,j,T_y}) \Big\|_{L^1(m)} = 0
$$

for $y \in Y_1$. Therefore, we have $N = r(k_0, T_y)$. This implies that $\widehat{r}(i)$ is a divisor of $r(k_0, T_y)$. □

When the identity map I_d on X is chosen with positive probability, Proposition 3.1 can be applied to show that the transformation S is exact on $\{\widehat{g}_i > 0\}$ $(1 \leq i \leq \widehat{s})$:

Theorem 3.4. *Suppose that $\mathcal{L}_S$ is asymptotically periodic and $\eta(\{y \in Y;\ T_y = I_d\}) > 0$ is satisfied. Then, for every $i \in \{1, 2, \ldots, \widehat{s}\}$, $\widehat{r}(i) = 1$ holds.*

Proof. Assume that $\widehat{r}(i) \geq 2$ holds for some $1 \leq i \leq \widehat{s}$. Then, we have that $\mathcal{L}_S(\widehat{g}_{i,1}) = \widehat{g}_{i,2}$ and $\widehat{g}_{i,1} \cdot \widehat{g}_{i,2} = 0$. It follows from Proposition 3.1 that $\{\widehat{g}_{i,1} > 0\} \subset I_d^{-1}\{\widehat{g}_{i,2} > 0\}$. This contradicts the fact that $\widehat{g}_{i,1} \cdot \widehat{g}_{i,2} = 0$. □

4. A sufficient condition for a skew product transformation to have asymptotic periodicity of densities

In this section, we discuss a sufficient condition for a skew product of one-dimensional transformations to have asymptotic periodicity of densities. Let $I \equiv [0,1]$ be the unit interval, $\mathcal{F} \equiv \mathcal{B}([0,1])$ be the Borel field, and m be the Lebesgue measure on $(I, \mathcal{F})$. We consider a family $(T_y)_{y \in Y}$ of m-nonsingular transformations on $[0,1]$, where Y is a complete separable metric space equipped with a probability measure η on $(Y, \mathcal{B}(Y))$.

For $f : [0,1] \to \mathbb{C}$, we denote the total variation of f on $[0,1]$ by $var(f)$. It is known that $V \equiv \{f \in L^1([0,1]); v(f) < \infty\}$ is a non-closed subspace of $L^1([0,1])$, where $v(f) = \inf\{var(\tilde{f}); \tilde{f}$ is a version of $f\}$ for $f \in L^1([0,1])$. On the other hand, letting

$$\|f\|_V \equiv \|f\|_{L^1([0,1])} + v(f) \quad \text{for } f \in V,$$

we can easily prove that $(V, \|\cdot\|_V)$ is a Banach space, and that the inequality $\|fg\|_V \leq 2\|f\|_V\|g\|_V$ holds for $f, g \in V$ (cf. [8]).

Definition 4.1. Let $\mathcal{D}_\infty$ be the set of all transformations $T : [0,1] \to [0,1]$ satisfying the following conditions:

(1) There is a countable partition $\{I_j\}_j$ of I by disjoint intervals such that the restriction $T|_{I_j}$ of T to I_j can be extended to a monotonic C^2-function on the closure $\bar{I}_j$ for each j, and the collection $\{J_j \equiv T(I_j)\}_j$ consists of a finite number of different subintervals;

(2) $\gamma(T) \equiv \inf_{x \in [0,1]} |T'(x)| > 0$ holds.

We state the following inequality for a single transformation $T \in \mathcal{D}_\infty$, which was established by Rousseau-Egele [8].

Proposition 4.2. *Assume that $T \in \mathcal{D}_\infty$ and that the corresponding partition $\{I_j\}_j$ and $\gamma(T)$ from Definition 4.1 are given. Then, we have the following inequality:*

$$v(\mathcal{L}_T f) \leq \alpha(T) v(f) + \beta(T)\|f\|_{L^1([0,1])}, \quad (f \in V),$$
$$\textit{where } \ \alpha(T) \equiv \frac{2}{\gamma(T)} \ \textit{ and } \ \beta(T) \equiv \sup_j \left\{\frac{1}{m(I_j)}\right\} + \sup_j \left\{\frac{\sup_{x \in I_j} |(T_j^{-1})''(x)|}{\inf_{x \in I_j} |(T_j)'(x)|}\right\}.$$

We now consider the skew product transformation S of $(T_y)_{y \in Y} \subset \mathcal{D}_\infty$. The following proposition enables us to give a sufficient condition for $\mathcal{L}_S$ to have asymptotic periodicity.

Proposition 4.3. *Assume that $(T_y)_{y \in Y} \subset \mathcal{D}_\infty$ is given and that the inequalities*

$$\int_Y \int_Y \cdots \int_Y \alpha(T_{y_{n_0}} \circ T_{y_{n_0-1}} \circ \cdots \circ T_{y_1}) \eta(dy_{n_0}) \eta(dy_{n_0-1}) \cdots \eta(dy_1) < 1 \tag{4.1}$$

$$\int_Y \int_Y \cdots \int_Y \beta(T_{y_{n_0}} \circ T_{y_{n_0-1}} \circ \cdots \circ T_{y_1}) \eta(dy_{n_0}) \eta(dy_{n_0-1}) \cdots \eta(dy_1) < \infty \tag{4.2}$$

hold for some $n_0 \in \mathbb{N}$. Then, there exist real numbers $\alpha \in (0,1)$ and $\beta \in (0,\infty)$ satisfying

$$v((\mathcal{L}_S)^{n_0} f) \leq \alpha v(f) + \beta \|f\|_{L^1([0,1])}, \quad (f \in V). \tag{4.3}$$

Using inequality (4.3), the theorem of Ionescu-Tulcea and Marinescu [2] showed the quasi-compactness and asymptotic periodicity of $\mathcal{L}_S$. Note that the families of transformations in Section 5 satisfy inequalities (4.1) and (4.2); thus, $\mathcal{L}_S$ is asymptotically periodic.

5. Numerical examples

This section uses examples to demonstrate our main results from Section 3. We consider the unit interval $X \equiv [0,1]$, Borel field $\mathcal{F} \equiv \mathcal{B}([0,1])$, and Lebesgue measure m on $(X,\mathcal{F})$. Further, in this section, we employ the initial density function $f_0(x) = 2x$ for $x \in [0,1]$, the complete separable metric space $Y = \{y_1, y_2\} \subset \mathbb{R}$ ($y_1 \neq y_2$), and the probability measure η on Y satisfying $\eta(\{y_1\}) = \eta(\{y_2\}) = 1/2$. Thus, the Perron–Frobenius operator $\mathcal{L}_S f$ is obtained as follows:

$$(\mathcal{L}_S f)(x) = \frac{1}{2}\left\{(\mathcal{L}_{T_{y_1}} f)(x) + (\mathcal{L}_{T_{y_2}} f)(x)\right\},\ x \in X.$$

Because $\eta(\{y_i\}) > 0$ ($i = 1,2$), it follows that $Y_0 = \{y_1, y_2\}$.

Example 1. For $m_0 \in \mathbb{N}$, we define the subintervals J_k ($1 \le k \le m_0$) as follows:

$$J_k \equiv \left[\frac{k-1}{m_0}, \frac{k}{m_0}\right),\ (1 \le k \le m_0 - 1),\ \text{and } J_{m_0} \equiv \left[1 - \frac{1}{m_0}, 1\right].$$

We consider the transformation R_3 on X given by $R_3 x \equiv 3x \pmod 1$. Then, we define the transformation $R^\tau : X \to X$ as

$$R^\tau x \equiv \frac{1}{m_0} R_3(m_0 x - k + 1) + \frac{\tau_k - 1}{m_0},\ \text{for } x \in J_k\ (k \in \{1,2,\dots,m_0\}),$$

where $\tau = (\tau_1, \dots, \tau_{m_0})$ is a permutation of the set $\{1,2,\dots,m_0\}$. Further, the Perron–Frobenius operator $\mathcal{L}_{R^\tau} f$ is obtained as follows:

$$\begin{aligned}(\mathcal{L}_{R^\tau} f)(x) =& \frac{1}{3}\left\{ f\left(\frac{x}{3} + \frac{k-1}{m_0} - \frac{\tau_k - 1}{3m_0}\right) + f\left(\frac{x}{3} + \frac{k-1}{m_0} - \frac{\tau_k - 2}{3m_0}\right)\right. \\ &\left. + f\left(\frac{x}{3} + \frac{k-1}{m_0} - \frac{\tau_k - 3}{3m_0}\right)\right\} \text{ for } x \in J_{\tau_k} = R^\tau(J_k)\ (1 \le k \le m_0).\end{aligned}$$

The graph of $\{R^\tau x;\ x \in [0,1]\}$ for $\tau = (2,1,6,4,3,5)$ is shown in Fig 1.

For $T_{y_1} = R^{(3,4,2,1)}$ and $T_{y_2} = R^{(4,5,6,2,3,1)}$

Here, we consider the skew product S of the transformations $T_{y_1} = R^{(3,4,2,1)}$ and $T_{y_2} = R^{(4,5,6,2,3,1)}$. Then, $((\mathcal{L}_{T_y})^n f_0)(\cdot)$ ($y = y_1, y_2$) have the property of asymptotic periodicity, and we obtain the following:

$$\begin{aligned}&Y_1 = \{y_1, y_2\},\ s(T_{y_1}) = s(T_{y_2}) = 1,\ r(1, T_{y_1}) = 4,\ r(1, T_{y_2}) = 6, \\ &g_{1,j_1,T_{y_1}}(x) = 4 \times 1_{\left[\frac{j_1-1}{4}, \frac{j_1}{4}\right]}(x),\ (1 \le j_1 \le 4), \\ &g_{1,j_2,T_{y_2}}(x) = 6 \times 1_{\left[\frac{j_2-1}{6}, \frac{j_2}{6}\right]}(x),\ (1 \le j_2 \le 6).\end{aligned}$$

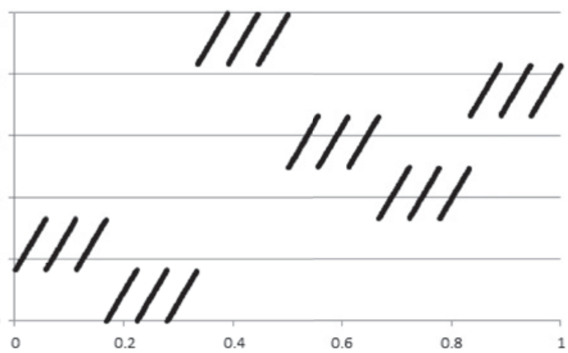

Fig. 1. $\{R^\tau x;\ x \in [0,1]\}$ for $\tau = (2,1,6,4,3,5)$.

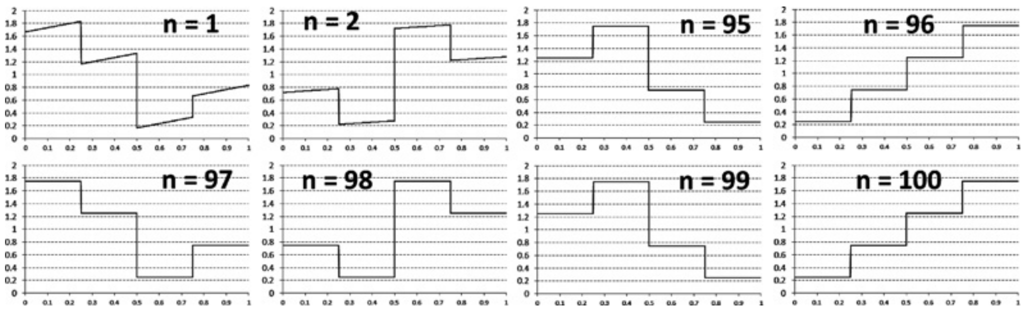

Fig. 2. Results of $((\mathcal{L}_{R^\tau})^n f_0)(\cdot)$ for τ= $(3,4,2,1)$, $n = 1$, 2, 95, 96, 97, 98, 99, and 100.

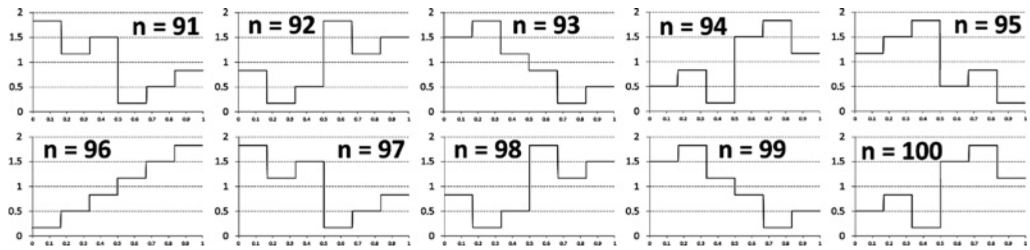

Fig. 3. Results of $((\mathcal{L}_{R^\tau})^n f_0)(\cdot)$ for $\tau = (4,5,6,2,3,1)$ and $91 \le n \le 100$.

The graphs of $((\mathcal{L}_{T_y})^n f_0)(\cdot)$ $(y = y_1, y_2)$, and $((\mathcal{L}_S)^n f_0)(\cdot)$ are shown in Figs 2, 3, and 4, respectively. Further, $((\mathcal{L}_S)^n f_0)(\cdot)$ has the property of asymptotic periodicity, as follows:

$$\widehat{s} = 1,\ \widehat{r}(1) = 2,\ \widehat{g}_{1,j}(x) = 2 \times 1_{\left[\frac{j-1}{2}, \frac{j}{2}\right]}(x),\ (1 \le j \le 2).$$

Note, therefore, that

$$g_{1,T_{y_1}}(x) = g_{1,T_{y_2}}(x) = \widehat{g}_1(x) = 1_{[0,1]}(x)$$

holds, and $\widehat{r}(1) = 2$ is a divisor of $r(1, T_{y_1}) = 4$ and $r(1, T_{y_2}) = 6$, which corresponds to the results given in Theorem 3.3. Because $f_0(x) = 2x$ is a monotone increasing function, we can expect that $0 < \lambda_{1,j,T_y}(f_0) < \lambda_{1,j+1,T_y}(f_0)$ holds for $j = 1, \ldots, r(1, T_y) - 1$ $(y \in \{y_1, y_2\})$,

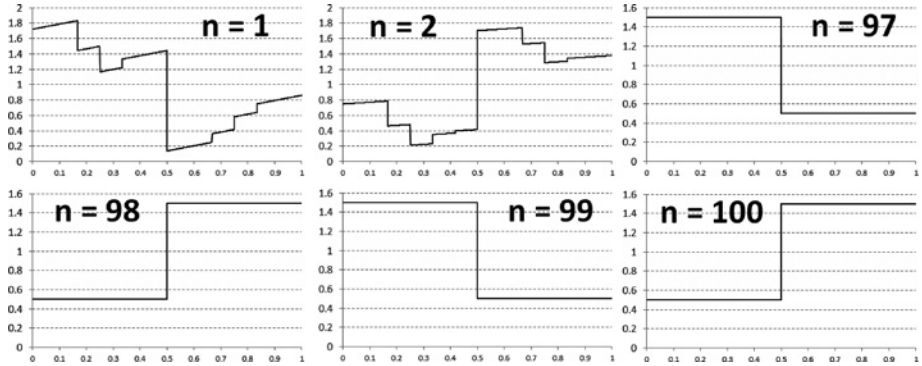

Fig.4. Results of $((\mathcal{L}_S)^n f_0)(\cdot)$, with $T_{y_1} = R^{(3,4,2,1)}$ and $T_{y_2} = R^{(4,5,6,2,3,1)}$, for $n = 1, 2, 97, 98, 99$, and 100.

and that $0 < \widehat{\lambda}_{1,1}(f_0) < \widehat{\lambda}_{1,2}(f_0)$ holds. Actually, we can confirm these tendencies in Fig 2 $[n = r(1, T_{y_1}) \times 25 = 100]$, Fig 3 $[n = r(1, T_{y_2}) \times 16 = 96]$, and Fig 4 $[n = \widehat{r}(1) \times 50 = 100]$.

For $T_{y_1} = R^{(2,1,6,4,3,5)}$ and $T_{y_2} = I_d$

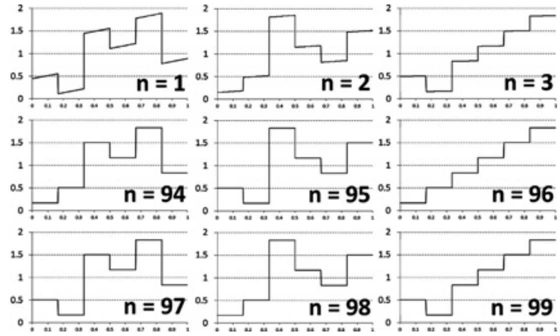

Fig.5. Results of $((\mathcal{L}_{R^\tau})^n f_0)(\cdot)$ for $\tau = (2, 1, 6, 4, 3, 5)$, $n = 1, 2, 3, 94, 95, 96, 97, 98$ and 99.

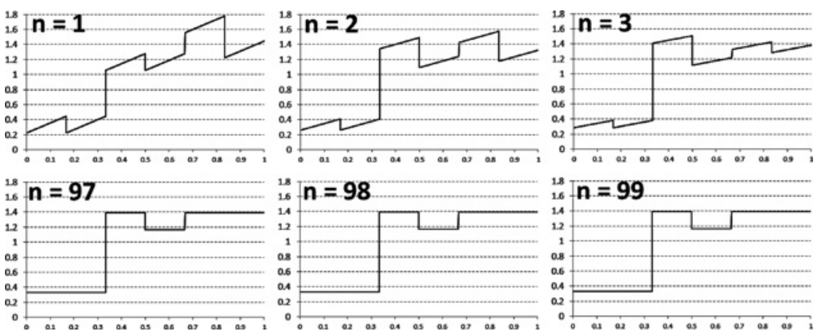

Fig.6. Results of $((\mathcal{L}_S)^n f_0)(\cdot)$, with $T_{y_1} = R^{(2,1,6,4,3,5)}$ and $T_{y_2} = I_d$, for $n = 1, 2, 3, 97, 98$, and 99.

Here, we consider the skew product S of the transformations $T_{y_1} = R^{(2,1,6,4,3,5)}$ and $T_{y_2} = I_d$. Then, the Perron–Frobenius operator $\mathcal{L}_S f$ is obtained as follows:

$$(\mathcal{L}_S f)(x) = \frac{1}{2}(\mathcal{L}_{R^{(2,1,6,4,3,5)}} f)(x) + \frac{1}{2} f(x).$$

Because $I_d(x) = x$ $(x \in [0,1])$ is not expanding, $\mathcal{L}_{I_d}$ does not have the property of asymptotic periodicity, and $Y_1 = \{y_1\}$ holds. The graphs of $((\mathcal{L}_{R^{(2,1,6,4,3,5)}})^n f_0)(\cdot)$ and $((\mathcal{L}_S)^n f_0)(\cdot)$ are shown in Figs 5 and 6. Then, we can confirm the following result:

$$\begin{aligned}
&s(T_{y_1}) = \widehat{s} = 3,\ r(1, T_{y_1}) = 2,\ r(2, T_{y_1}) = 3,\ r(3, T_{y_1}) = 1,\\
&\widehat{r}(1) = 1,\ \widehat{r}(2) = 1,\ \widehat{r}(3) = 1,\\
&g_{1,1,T_{y_1}}(x) = 6 \times 1_{[0,\frac{1}{6}]}(x),\ g_{1,2,T_{y_1}}(x) = 6 \times 1_{[\frac{1}{6},\frac{2}{6}]}(x),\\
&g_{2,1,T_{y_1}}(x) = 6 \times 1_{[\frac{2}{6},\frac{3}{6}]}(x),\ g_{2,2,T_{y_1}}(x) = 6 \times 1_{[\frac{4}{6},\frac{5}{6}]}(x),\ g_{2,3,T_{y_1}}(x) = 6 \times 1_{[\frac{5}{6},1]}(x),\\
&g_{3,1,T_{y_1}}(x) = 6 \times 1_{[\frac{3}{6},\frac{4}{6}]}(x),\\
&\widehat{g}_{1,1}(x) = 3 \times 1_{[0,\frac{1}{3}]}(x),\ \widehat{g}_{2,1}(x) = 2 \times 1_{[\frac{2}{6},\frac{3}{6}]\cup[\frac{4}{6},1]}(x),\ \widehat{g}_{3,1}(x) = 6 \times 1_{[\frac{3}{6},\frac{4}{6}]}(x).
\end{aligned}$$

Note, therefore, that

$$\begin{aligned}
&g_{1,T_{y_1}}(x) = \widehat{g}_1(x) = 3 \times 1_{[0,\frac{1}{3}]}(x),\\
&g_{2,T_{y_1}}(x) = \widehat{g}_2(x) = 2 \times 1_{[\frac{2}{6},\frac{3}{6}]\cup[\frac{4}{6},1]}(x),\\
&g_{3,T_{y_1}}(x) = \widehat{g}_3(x) = 6 \times 1_{[\frac{3}{6},\frac{4}{6}]}(x)
\end{aligned}$$

hold, and that $\widehat{r}(i) = 1$ $(i = 1, 2, 3)$ corresponds to the result given in Theorem 3.4.

Example 2. For a constant $a \in (0, 1/3)$, we define the disjoint subintervals I_k $(1 \le k \le 9)$ as

$$\begin{aligned}
&I_k \equiv [c_{k-1}, c_k)\ (1 \le k \le 8) \text{ and } I_9 \equiv [c_8, c_9],\\
&\quad \text{where } c_0 = 0,\ c_k = \begin{cases} c_{k-1} + a, & (k = 1, 5, 9),\\ c_{k-1} + \frac{1-3a}{6}, & (k = 2, 3, 4, 6, 7, 8). \end{cases}
\end{aligned}$$

Note that $X = [0,1] = \bigcup_{k=1}^{9} I_k$ holds. Then, we define the transformation $Q^{(a)} : X \to X$ as

$$Q^{(a)}x = \begin{cases} c_1 + D^{1,4}_{k-1,k}(x - c_{k-1}), & x \in I_k\ (1 \le k \le 4),\\ c_1 + D^{1,8}_{k-1,k}(x - c_{k-1}), & x \in I_k\ (k = 5),\\ c_5 + D^{5,8}_{k-1,k}(x - c_{k-1}), & x \in I_k\ (6 \le k \le 9), \end{cases} \qquad \text{where } D^{k_3,k_4}_{k_1,k_2} = \frac{c_{k_4} - c_{k_3}}{c_{k_2} - c_{k_1}}.$$

The graph of $\{Q^{(a)}x;\ x \in [0,1]\}$ for $a = 0.05$ is shown in Fig 7. Further, we define the transformation $Q^{(a,b)}$ as $Q^{(a,b)}x = Q^{(a)}x + b$ $(x \in X)$ for $b \in [-a, a]$.

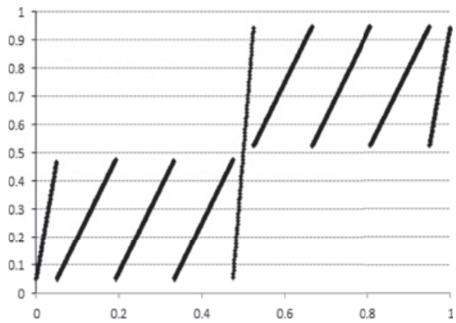

Fig. 7. $\{Q^{(a)}(x); x \in [0,1]\}$ for $a = 0.05$.

Then, $Q^{(a,b)}$ is considered to be the perturbed transformation of $Q^{(a)}$, and the Perron–Frobenius operator $\mathcal{L}_{Q^{(a,b)}} f$ is obtained as follows [(i)–(v)]:

(i) $(\mathcal{L}_{Q^{(a,b)}} f)(x) = D^{0,1}_{1,4} f(c_0)$ for $x = c_1 + b$;

(ii) $(\mathcal{L}_{Q^{(a,b)}} f)(x) = D^{4,5}_{1,8} f(c_4 + D^{4,5}_{1,8}(x - b - c_1)) + \sum_{k=1}^{4} D^{k-1,k}_{1,4} f(c_{k-1} + D^{k-1,k}_{1,4}(x - b - c_1))$ for $x \in (c_1 + b, c_4 + b]$;

(iii) $(\mathcal{L}_{Q^{(a,b)}} f)(x) = D^{4,5}_{1,8} f(c_4 + D^{4,5}_{1,8}(x - b - c_1))$ for $x \in (c_4 + b, c_5 + b]$;

(iv) $(\mathcal{L}_{Q^{(a,b)}} f)(x) = D^{4,5}_{1,8} f(c_4 + D^{4,5}_{1,8}(x - b - c_1)) + \sum_{k=6}^{9} D^{k-1,k}_{5,8} f(c_{k-1} + D^{k-1,k}_{5,8}(x - b - c_5))$ for $x \in (c_5 + b, c_8 + b]$;

(v) $(\mathcal{L}_{Q^{(a,b)}} f)(x) = 0$ for $x \in [0, c_1 + b) \cup (c_8 + b, 1]$.

Setting $\widehat{a} \equiv \dfrac{a(1-3a)}{2(1-2a)}$, we then obtain the following properties:

(A) $((Q^{(a,b)})^n x_0)_{n=0}^{\infty} \subset [0, c_4 + \widehat{a}]$ for $x_0 \in [0, c_4 + \widehat{a}]$ and $b \le \widehat{a}$;

(B) $((Q^{(a,b)})^n x_0)_{n=0}^{\infty} \subset [c_5 - \widehat{a}, 1]$ for $x_0 \in [c_5 - \widehat{a}, 1]$ and $b \ge -\widehat{a}$.

For $a = 0.15$, $b_1 = a/4\ (< \widehat{a})$, and $b_2 = -3a/4\ (< -\widehat{a})$, we consider the transformations $T_{y_1} = Q^{(a,b_1)}$, $T_{y_2} = Q^{(a,b_2)}$, and the corresponding skew product transformation S. The graphs of $((\mathcal{L}_{Q^{(a,b_k)}})^n f_0)(\cdot)$, $(k = 1,2)$, and $((\mathcal{L}_S)^n f_0)(\cdot)$ are shown in Figs 8, 9, and 10, respectively. Then,

$$Y_1 = \{y_1, y_2\},\ s(T_{y_1}) = 2,\ s(T_{y_2}) = \widehat{s} = 1,$$
$$r(1, T_{y_1}) = r(2, T_{y_1}) = 1,\ r(1, T_{y_2}) = 1,\ \widehat{r}(1) = 1$$

are obtained. Further, we can expect, and confirm from Figs 8–10 ($n = 100$), that the densities $g_{i,T_{y_1}}(x)$, $(i = 1,2)$, $g_{1,T_{y_2}}(x)$, and $\widehat{g}_1(x)$ satisfy

$$\{g_{1,T_{y_1}} > 0\} \approx [0.1873, 0.4626], \quad \{g_{2,T_{y_1}} > 0\} \approx [0.6126, 0.8873],$$
$$\{g_{1,T_{y_2}} > 0\} \approx [0.0373, 0.3126], \quad \{\widehat{g}_1 > 0\} \approx [0.0373, 0.4626].$$

Thus, we have that

$$\{g_{1,T_{y_1}} > 0\} \subsetneq \{\widehat{g_1} > 0\},\ \{g_{2,T_{y_1}} > 0\} \cap \{\widehat{g_1} > 0\} = \emptyset,\ \text{ and } \{g_{1,T_{y_2}} > 0\} \subsetneq \{\widehat{g_1} > 0\}$$

hold, which correspond to the result given in Theorem 3.3 (1). Note also that $Q^{(a,b_2)}$ satisfies property (A), and $Q^{(a,b_2)}$ does not satisfy property (B). If $x_0 \in [c_5 - \widehat{a}, 1]$ satisfies $(Q^{(a,b_2)})^{n_0^*} x_0 \in [0, c_4 + \widehat{a}]$ for some $n_0^* \in \mathbb{N}$, then we have that $((Q^{(a,b_2)})^n x_0)_{n=n_0^*}^{\infty} \subset [0, c_4 + \widehat{a}]$. Thus, we can expect, and confirm from Fig 9, that $\lim_{n\to\infty}((\mathcal{L}_{Q^{(a,b_2)}})^n f_0)(x) = 0$ holds for $x \in [c_5 - \widehat{a}, 1]$.

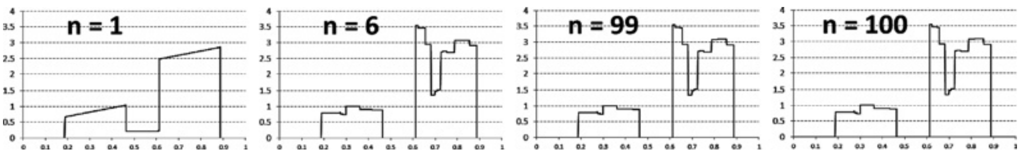

Fig. 8. Results of $((\mathcal{L}_{Q^{(a,b_1)}})^n f_0)(\cdot)$ with $a = 0.15$ and $b_1 = a/4$ ($n = 1, 6, 99$, and 100).

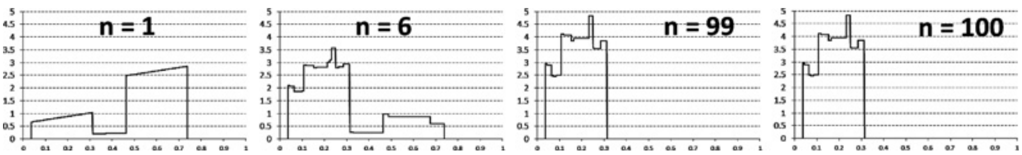

Fig. 9. Results of $((\mathcal{L}_{Q^{(a,b_2)}})^n f_0)(\cdot)$ with $a = 0.15$ and $b_2 = -3a/4$ ($n = 1, 6, 99$, and 100).

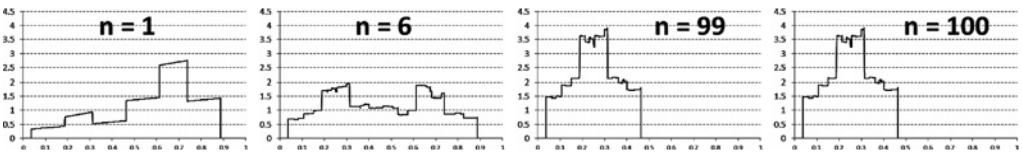

Fig. 10. Results of $((\mathcal{L}_S)^n f_0)(\cdot)$ for $T_{y_k} = Q^{(a,b_k)}$ ($k = 1, 2$) with $a = 0.15$, $b_1 = a/4$, and $b_2 = -3a/4$ ($n = 1, 6, 99$, and 100).

6. Conclusions

In this paper, we studied the effects of randomization on the asymptotic periodicity. We showed that the supports of the ergodic probability densities for random iterations include at least one support of the ergodic probability density for almost all m-nonsingular transformations. This implies that the number of ergodic components of random iterations is not greater than the number of ergodic components of each of the m-nonsingular transformations. We also discussed the period of the limiting densities of random iterations. Our results suggest that even a small noise could change the ergodic properties of the system.

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