SOME FAMILIES OF MINIMAL ELEMENTS FOR A PARTIAL ORDERING ON PRIME KNOTS

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Abstract

We show that all twist knots and certain double twist knots are minimal elements for a partial ordering on the set of prime knots. The keys to these results are presentations of their character varieties using Chebyshev polynomials and a criterion for irreducibility of a polynomial of two variables. These give us an elementary method to discuss the number of irreducible components of the character varieties, which concludes the result essentially.

1. Outline of this research

We research a partial ordering on the set of prime knots by using algebraic sets associated to knot groups, now known as the character varieties of knot groups. The framework of character varieties introduced by Culler and Shalen [4] for a finitely presented group has been giving powerful tools and is now playing important roles in geometry and topology. On the other hand, it is not easy to calculate character varieties and thus to investigate their geometric structures in general, though an underlying idea of character varieties is simple as follows. Let G be a finitely presented group generated by n elements $g_1, \ldots, g_n$. For a representation $\rho \colon G \to \mathrm{SL}_2(\mathbb{C})$, let χ_ρ be the character of ρ, which is the function on G defined by $\chi_\rho(g) := \mathrm{tr}(\rho(g))$ ($\forall g \in G$). Throughout this paper, we simply denote by $\mathrm{tr}(g)$ the trace $\mathrm{tr}(\rho(g))$ for an unspecified representation $\rho \colon G \to \mathrm{SL}_2(\mathbb{C})$. We sometimes omit the brackets in the trace like $\mathrm{tr}(a) = \mathrm{tr}\, a$ for simplicity. By [4] (see also [6]), *the* $\mathrm{SL}_2(\mathbb{C})$*-trace identity*

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}) \quad (\forall A, B \in \mathrm{SL}_2(\mathbb{C}))$$

shows that for any element $g \in G$, $\mathrm{tr}(g)$ is described by a polynomial in $\{\mathrm{tr}(g_i)\}_{1 \le i \le n}$, $\{\mathrm{tr}(g_i g_j)\}_{1 \le i < j \le n}$ and $\{\mathrm{tr}(g_i g_j g_k)\}_{1 \le i < j < k \le n}$. Then the character variety of G, denoted by $X(G)$, is constructed basically by the image of the set $\chi(G)$ of characters of $\mathrm{SL}_2(\mathbb{C})$-representations of G under the map

$$t \colon \chi(G) \to \mathbb{C}^{n + \binom{n}{2} + \binom{n}{3}}, \quad t(\chi_\rho) := (\mathrm{tr}(g_i); \mathrm{tr}(g_i g_j); \mathrm{tr}(g_i g_j g_k)).$$

The resulting set turns out to be a closed algebraic set. By definition, this algebraic set depends on a choice of generators of G (the coordinates of $X(G)$ vary if we change the choice of generating set of G), however, the geometric structures do not depend on that choice up to *bipolynomial map*. Here two algebraic sets V and W in some complex spaces are said to be isomorphic (bipolynomial equivalent) if there exist polynomial maps $f: V \to W$ and $g: W \to V$ such that $g \circ f = id_V$, $f \circ g = id_W$. We call each of f and g an isomorphism or a bipolynomial map. So $X(G)$ is an invariant of G up to isomorphism (bipolynomial equivalence) of algebraic sets.

In this research, we mainly apply *the Chebyshev polynomials $S_n(z)$ ($\forall n \in \mathbb{Z}$) of the second kind* defined by

$$S_0(z) = 1, \quad S_1(z) = z, \quad S_n(z) = zS_{n-1}(z) - S_{n-2}(z),$$

to describe the character varieties. Note that $S_{-n}(z) = -S_{n-2}(z)$ holds for any integer n. $S_n(z)$ naturally appears in the calculations of $X(G)$ since they have the similar property as the $\mathrm{SL}_2(\mathbb{C})$-trace identity. For example, the third relation above exactly coincides with the rule $\mathrm{tr}(z^n) = \mathrm{tr}(z)\,\mathrm{tr}(z^{n-1}) - \mathrm{tr}(z^{n-2})$ coming from the trace identity. The other Chebyshev polynomials $T_n(z)$ used in Section 2 also have the same property. Hence the process of calculations of $X(G)$ using the trace identity can be encoded into the Chebyshev polynomials naturally.

Now we demonstrate a calculation of $X(G)$ using the Chebyshev polynomials $S_n(z)$ in the case where G is *a knot group*. For a knot K in $\mathbb{S}^3$, we denote by $G(K)$ the knot group of K, i.e., the fundamental group of the knot complement $\mathbb{S}^3 - K$. For example, there exist knots, called *2-bridge knots*, each of which is parametrized by a sequence of integers $[a_1, a_2, \ldots, a_r]$ associated to the number of *twists*. (See Fig. 1. For more details, refer to [8].) The knot $K_m = [-2, -m]$ depicted in Fig. 2 is a type of 2-bridge knots, called the *m-twist knot*. Sometimes the sequence $[a_1, a_2, \ldots, a_r]$ is encoded in a rational number p/q ($p > 0$, $-p < q < p$) by the following continued fraction:

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_r}}}.$$

It is shown that p and q may be taken to be coprime and p is always odd. Then we denote by $\mathfrak{b}(p, q)$ the 2-bridge knot with a rational number expression p/q.

Note that for a 2-bridge knot $[a_1, \ldots, a_r]$ the sign of twist $\diagup$ in the white box a_i is positive for i odd, negative for i even.

By *Wirtinger's algorithm* and *Tietze transformations* (reductions of generators and relations), we have the following presentation of the 2-bridge knot group $G(\mathfrak{b}(p, q))$:

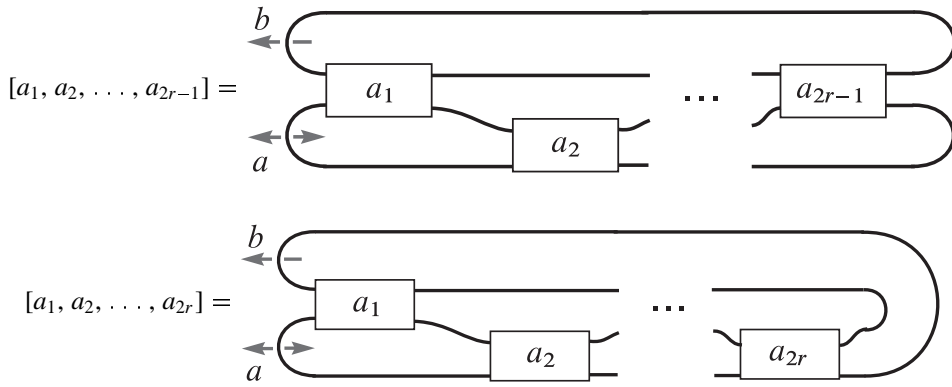$$G(\mathfrak{b}(p, q)) = \langle a, b \mid wa = bw \rangle,$$

Fig. 1. The 2-bridge knots $[a_1, a_2, \ldots, a_{2r-1}]$ and $[a_1, a_2, \ldots, a_{2r}]$ and generators a and b of their knot groups. The orientation of a is chosen so that a and b are conjugate. a_i denotes the number of twists with *sign* in the white box.

where a, b are *meridians* shown in Fig. 1, $w = a^{\varepsilon_1} b^{\varepsilon_2} \cdots a^{\varepsilon_{p-2}} b^{\varepsilon_{p-1}}$ and $\varepsilon_j = (-1)^{\lfloor jq/p \rfloor}$. Here $\lfloor s \rfloor$ denotes the maximal integer m satisfying $m \le s$.

By [10], the character variety $X(\mathfrak{b}(p,q)) := X(G(\mathfrak{b}(p,q)))$ is given as the algebraic set defined by

$$(1.1) \qquad\qquad \operatorname{tr} bwa^{-1} - \operatorname{tr} w = 0.$$

This equation gives us a very powerful tool to calculate the character varieties for 2-bridge knots, on which we heavily rely throughout this paper. For example, in the case where $m = 2n$ $(n > 0)$, Equation (1.1) induces the following presentation of $X(K_{2n}) := X(G(K_{2n}))$. For the twist knot $K_{2n} = \mathfrak{b}(4n+1, -2n)$, we have $w = u^n$ where $u := a^{-1}b^{-1}ab$. Let $x := \operatorname{tr} a = \operatorname{tr} b$, $y := \operatorname{tr} ab^{-1} = \operatorname{tr} y_*$ and $t := t(x,y) = \operatorname{tr} u = \operatorname{tr} a^{-1}b^{-1}ab = y^2 - yx^2 + 2x^2 - 2$, where the presentation of t in x and y is obtained as follows:

$$
\begin{aligned}
\operatorname{tr} a^{-1}b^{-1}ab &= \operatorname{tr} ba^{-1}b^{-1}a \\
&= \operatorname{tr} ba^{-1} \operatorname{tr} b^{-1}a - \operatorname{tr} ba^{-2}b \\
&= (\operatorname{tr} ab^{-1})^2 - \operatorname{tr} ba^{-2} \operatorname{tr} b + \operatorname{tr} a^{-2} \\
&= y^2 - (\operatorname{tr} ba^{-1} \operatorname{tr} a - \operatorname{tr} b) \operatorname{tr} b + (\operatorname{tr} a^{-1})^2 - 2 \\
&= y^2 - (yx - x)x + x^2 - 2 \\
&= y^2 - yx^2 + 2x^2 - 2.
\end{aligned}
$$

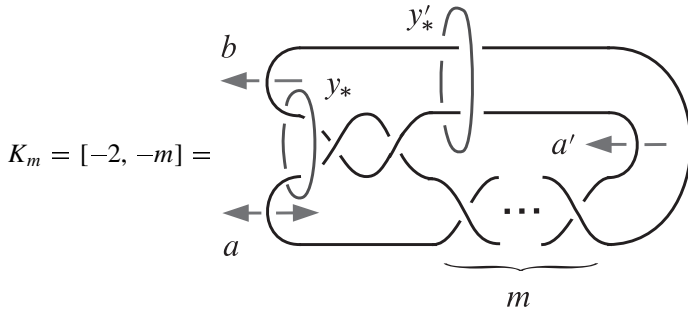We now focus on the Chebyshev polynomials $S_n(t)$. The following lemma is standard,

Fig. 2. The m-twist knot $K_m = [-2, -m] = \mathfrak{b}(2m + 1, -m)$ and loops a, a', b, y_* and y'_*. The orientation of a is given by the right arrow for m odd, by the left arrow for m even.

see e.g. [20, Lemma 2.2].

Lemma 1.1. *Suppose the polynomials f_n ($n \in \mathbb{Z}$) in $\mathbb{C}[x, y]$ satisfy the recurrence relation $f_{n+1} = tf_n - f_{n-1}$. Then $f_n = f_0 S_n(t) - f_{-1} S_{n-1}(t)$ holds.*

Set $f_n := \operatorname{tr} bu^n a^{-1} - \operatorname{tr} u^n$. Then f_n satisfies

$$
\begin{aligned}
f_{n+1} &= \operatorname{tr} bu^{n+1} a^{-1} - \operatorname{tr} u^{n+1} \\
&= \operatorname{tr} a^{-1} bu^{n+1} - \operatorname{tr} u^n \operatorname{tr} u + \operatorname{tr} u^{n-1} \\
&= \operatorname{tr} a^{-1} bu^n \operatorname{tr} u - \operatorname{tr} a^{-1} bu^{n-1} - (\operatorname{tr} u^n)t + \operatorname{tr} u^{n-1} \\
&= t(\operatorname{tr} bu^n a^{-1} - \operatorname{tr} u^n) - (\operatorname{tr} bu^{n-1} a^{-1} - \operatorname{tr} u^{n-1}) \\
&= tf_n - f_{n-1}.
\end{aligned}
$$

So, applying Lemma 1.1 to the defining polynomial $\operatorname{tr} bwa^{-1} - \operatorname{tr} w$, we have

$$
\begin{aligned}
\operatorname{tr} bwa^{-1} - \operatorname{tr} w &= \operatorname{tr} bu^n a^{-1} - \operatorname{tr} u^n \\
&= (\operatorname{tr} ba^{-1} - \operatorname{tr} 1)S_n(t) - (\operatorname{tr} bu^{-1} a^{-1} - \operatorname{tr} u^{-1})S_{n-1}(t) \\
&= (y - 2)S_n(t) - (y - t)S_{n-1}(t) \\
&= (y - 2)(S_n(t) + (y + 1 - x^2)S_{n-1}(t)),
\end{aligned}
$$

since $t - y = (y - 2)(y + 1 - x^2)$. Let $L_n(x, y)$ be the resulting polynomial, that is,

$$
L_n(x, y) = (y - 2)(S_n(t) + (y + 1 - x^2)S_{n-1}(t)).
$$

Then the character variety $X(K_{2n})$ is given by

$$(1.2) \qquad X(K_{2n}) = \{(x, y) \in \mathbb{C} \mid L_n(x, y) = 0\}.$$

Similarly, for the twist knot $K_{2n-1} = \mathfrak{b}(4n - 1, -2n + 1)$, where $n > 0$, let $u := a^{-1}b^{-1}ab$. Then we have $w = u^{n-1}a^{-1}b^{-1}$. In this case, we put $x := \operatorname{tr} a = \operatorname{tr} b$, $y := \operatorname{tr} ab = \operatorname{tr} y_*$ and $t := t(x, y) = \operatorname{tr} u = \operatorname{tr} a^{-1}b^{-1}ab = y^2 - yx^2 + 2x^2 - 2$. As in the case of K_{2n}, it follows from Lemma 1.1 that

$$\begin{aligned}
\operatorname{tr} bwa^{-1} - \operatorname{tr} w &= \operatorname{tr} bu^{n-1}a^{-1}b^{-1}a^{-1} - \operatorname{tr} u^{n-1}a^{-1}b^{-1} \\
&= (\operatorname{tr} ba^{-1}b^{-1}a^{-1} - \operatorname{tr} a^{-1}b^{-1})S_{n-1}(t) \\
&\quad - (\operatorname{tr} bu^{-1}a^{-1}b^{-1}a^{-1} - \operatorname{tr} u^{-1}a^{-1}b^{-1})S_{n-2}(t) \\
&= ((\operatorname{tr} a^{-1})^2 - \operatorname{tr} bab^{-1}a^{-1} - \operatorname{tr} ab)S_{n-1}(t) \\
&\quad - (\operatorname{tr} a^{-2} - \operatorname{tr} b^{-1}a^{-1})S_{n-2}(t) \\
&= (x^2 - t - y)S_{n-1}(t) - (x^2 - 2 - y)S_{n-2}(t) \\
&= (x^2 - y - 2)((y - 1)S_{n-1}(t) - S_{n-2}(t)).
\end{aligned}$$

Let $L'_n(x, y)$ be the above resulting polynomial, that is,

$$L'_n(x, y) = (x^2 - y - 2)((y - 1)S_{n-1}(t) - S_{n-2}(t)).$$

Then we obtain a presentation of the character variety $X(K_{2n-1})$:

$$(1.3) \qquad X(K_{2n-1}) = \{(x, y) \in \mathbb{C}^2 \mid L'_n(x, y) = 0\}.$$

In this way, we can calculate $X(K_m)$ easily by using Equation (1.1).

On the other hand, we can also calculate the character varieties using *the Kauffman bracket skein module* (KBSM), though we need a lot of calculations for $X(K_m)$. (Refer to [1, 16, 17, 18].) For any non-negative integer[1] m, let $R_m(x, y)$ be the polynomial in $\mathbb{C}[x, y]$ defined by

$$R_m(x, y) = (y + 2)\left(S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)\right),$$

and let $\tilde{R}_m(x, y)$ be the second factor of $R_m(x, y)$. It follows from [5] using the KBSM that $X(K_m)$ has the following form:

$$(1.4) \qquad X(K_m) = \{(x, y) \in \mathbb{C}^2 \mid R_m(x, y) = 0\},$$

---

[1]For a negative integer $-m$ ($m > 1$), taking the mirror image of K_{-m} and arranging it, we obtain $X(K_{-m}) = X(K_{m-1})$. In this case, $R_{-m}(x, y)$ will shift to $R_{m-1}(x, y)$.

where $x := -\operatorname{tr} a' = -\operatorname{tr} b$ and $y := -\operatorname{tr} a'b^{-1} = -\operatorname{tr} y'_*$. By definition, the algebraic sets in (1.2) and (1.4) at $m = 2n$, and also the algebraic sets in (1.3) and (1.4) at $m = 2n - 1$ are isomorphic as algebraic sets, however, $R_m(x, y)$ has a nice form to research the number of irreducible components of $X(K_m)$ as follows.

Proposition 1.2 (cf. [2, 12]). *For any positive integer m, $\tilde{R}_m(x, y)$ is irreducible over $\mathbb{C}$. Therefore, $X(K_m)$ consists of two irreducible components.*

Proof.   By the same argument in [14], the factor $\tilde{R}_m(x, y)$

$$\tilde{R}_m(x, y) = S_m(y) - S_{m-1}(y) + x^2 \sum_{i=0}^{m-1} S_i(y)$$

$$= S_m(y) - S_{m-1}(y) + x^2 \frac{S_m(y) - S_{m-1}(y) - 1}{y - 2}$$

cannot be factorized as $(h_1 x + h_2)(h_3 x + h_4)$ where $h_j \in \mathbb{C}[y]$. Moreover, $(h_1 x^2 + h_2)h_3$ where $h_j \in \mathbb{C}[y]$ cannot occur either as a factorization of $\tilde{R}_m(z, y)$, since $S_m(y) - S_{m-1}(y)$ and $(S_m(y) - S_{m-1}(y) - 1)/(y - 2)$ are relatively prime in $\mathbb{C}[y]$. Hence $\tilde{R}_m(x, y)$ is irreducible in $\mathbb{C}[x, y]$, concluding Proposition 1.2.   □

The above method naturally leads us to Proposition 1.3.

Proposition 1.3.   *Suppose $\Phi(x, z) = f(z) + x^2 g(z)$ is a polynomial in $\mathbb{C}[x, z]$ such that $\deg f - \deg g$ is an odd number, and $f(z)$ and $g(z)$ are relatively prime in $\mathbb{C}[z]$. Then $\Phi(x, z)$ is irreducible in $\mathbb{C}[x, z]$.*

Proof.   A basic argument shows Proposition 1.3 likewise. Assume that $\Phi(x, z)$ is reducible in $\mathbb{C}[x, z]$. Since $\gcd(f(z), g(z)) = 1$, we must have

(1.5) $\Phi(x, z) = (h_1(z) + xh_2(z))(h_3(z) + xh_4(z))$,

where h_j's are polynomials in $\mathbb{C}[z]$. Equation (1.5) is equivalent to

$$f(z) = h_1(z)h_3(z), \quad 0 = h_1(z)h_4(z) + h_2(z)h_3(z), \quad g(z) = h_2(z)h_4(z).$$

So it follows that

$$\deg f = \deg h_1 + \deg h_3, \quad \deg h_1 + \deg h_4 = \deg h_2 + \deg h_3,$$
$$\deg g = \deg h_2 + \deg h_4.$$

Hence

$$\deg f - \deg g = (\deg h_1 - \deg h_2) + (\deg h_3 - \deg h_4) = 2(\deg h_1 - \deg h_2)$$

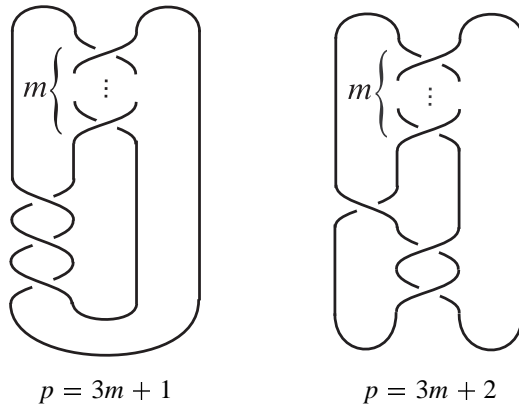$p = 3m + 1$ $p = 3m + 2$

Fig. 3. The 2-bridge knot $\mathfrak{b}(p,3)$ with $p > 3$, $\gcd(p,3) = 1$. Since $\mathfrak{b}(p, 3)$ is a knot, m is even for $p = 3m + 1$ and m is odd for $p = 3m + 2$.

is an even number, a contradiction. This proves Proposition 1.3. $\square$

In fact, the above processes using Equation (1.1) and Proposition 1.3 lead us to a nice way to study the number of irreducible components of the character varieties for 2-bridge knots other than twist knots. For example, we focus on the 2-bridge knot $\mathfrak{b}(p, 3)$, where $p > 3$ and $\gcd(p, 3) = 1$. Since $\mathfrak{b}(p, 3)$ is a knot, p is odd.

As shown in Fig. 3, $\mathfrak{b}(3m + 1, 3)$ is *a twist knot* ($m = 2$) or *a double twist knot*, and $\mathfrak{b}(3m + 2, 3)$ is a twist knot ($m = 1$) or a double twist knot. For the word w in the relation of $G(\mathfrak{b}(p, 3))$, let $\Phi_w(x, z)$ be the polynomial

$$\Phi_w(x, z) = S_d(z) - S_{d-1}(z)$$
$$+ x^2(2 - z)S_{d-l-1}(z)S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor - 1}(z)),$$

where $d = (p - 1)/2$ and $l = \lfloor p/3 \rfloor$. (The original definition of $\Phi_w(x, z)$ will appear in Section 2.)

Theorem 1.4 (Theorem 2.2 in Section 2, cf. [2], [12])**.** $X(\mathfrak{b}(p, 3))$ *with* $p > 3$ *and* $\gcd(p, 3) = 1$ *is presented as the algebraic set defined by* $(z + 2 - x^2)\Phi_w(x, z) = 0$. *Then* $\Phi_w(x, z)$ *is irreducible over* $\mathbb{C}$ *and thus* $X(\mathfrak{b}(p, 3))$ *consists of exactly two irreducible components.*

It is studied in [12] that most double twist knots have exactly two irreducible components in their character varieties. In the proof, some methods in algebraic geometry seem to be applied. On the other hand, our proofs of Theorem 1.4 (i.e., Proposition 2.1 and Theorem 2.2) use only basic calculations on the Chebyshev polynomials $S_n(z)$. This

would indicate more or less an efficiency of the Chebyshev polynomials in the calculations of the character varieties, though the proofs cannot avoid laborious calculations (see Section 2).

The number of irreducible components of $X(K)$ is quite interesting in the sense that these results determine minimal elements for a partial order on the set of prime knots in $\mathbb{S}^3$ defined as follows. Let K and K' be prime knots in $\mathbb{S}^3$ which are *non-trivial*, i.e., they cannot bound embedded disks in $\mathbb{S}^3$. Then we write $K \geq K'$ if there exists an epimorphism (a surjective group homomorphism) from $G(K)$ onto $G(K')$. This defines a partial order on the set of prime knots (refer to [9]). We can apply the following theorem to the partial order $\geq$.

Theorem 1.5 (Theorem 4.4 in [3], cf. Appendix in [14], Corollary 7.1 in [15])**.** *Suppose $K \subset \mathbb{S}^3$ is a hyperbolic knot such that $X(K)$ of K has only two irreducible components. Then $G(K)$ does not surject onto the knot group of any other nontrivial knot.*

Combining Proposition 1.2 and Theorem 1.5, we obtain the following corollary.

Corollary 1.6. *For any positive integer[2] $m > 1$, at which the m-twist knot K_m is hyperbolic, K_m is a minimal element for the partial order $\geq$.*

Note that Corollary 1.6 also holds for $m = 1$ ([9]), where K_1 is the trefoil knot (i.e., a non-hyperbolic knot). The first author has shown this corollary in the case where $2m + 1$ is prime ([14]). Similarly, we can apply Theorems 1.4 and 1.5 to get the following.

Corollary 1.7. *The 2-bridge knot $\mathfrak{b}(p, 3)$ satisfying $p > 3$ and $\gcd(p, 3) = 1$, where it is hyperbolic, is a minimal element for the partial order $\geq$.*

Corollaries 1.6 and 1.7 also show the minimality of twist knots K_m and the 2-bridge knots $\mathfrak{b}(p, 3)$ with respect to the partial order introduced by Silver and Whitten [19] (see also [7]).

To avoid a complicated organization for readers, we do put every laborious calculation in the rest of the paper. So, in the following section, we concentrate our focus on the calculations for
• a special presentation $(z + 2 - x^2)\Phi_w(x, z) = 0$ of $X(\mathfrak{b}(p, 3))$ with $p > 3$ and $\gcd(p, 3) = 1$ using Equation (1.1) (in Subsection 2.1) and,
• a proof of irreducibility of $\Phi_w(x, z) = 0$ using Proposition 1.3 (in Subsection 2.2).

---

[2]According to the property of $R_m(x, y)$ under the mirror image mentioned before, this naturally extends to any negative integer $m < -2$.

2. Character variety of $\mathfrak{b}(p, 3)$ using $S_n(z)$: a proof of Theorem 1.4

The proof of Theorem 1.4 consists of two parts; giving a description $(z + 2 - x^2)\Phi_w(x, z) = 0$ of $X(\mathfrak{b}(p, 3))$ using Equation (1.1) (Proposition 2.1) and the irreducibility of $\Phi_w(x, z)$ using Proposition 1.3 (Theorem 2.2), which describes the main body of $X(\mathfrak{b}(p, 3))$. We state these results first.

For the knot group $G(\mathfrak{b}(p, m)) = \langle a, b \mid wa = bw \rangle$ of the 2-bridge knot $\mathfrak{b}(p, m)$, where $w = a^{\varepsilon_1}b^{\varepsilon_2} \cdots a^{\varepsilon_{p-2}}b^{\varepsilon_{p-1}}$ and $\varepsilon_j = (-1)^{\lfloor jq/p \rfloor}$, let $z = \mathrm{tr}(ab)$ and $d = (p - 1)/2$. Note that $\varepsilon_j = \varepsilon_{p-j}$. In general, it follows from [10] that the polynomial $\mathrm{tr}(bwa^{-1}) - \mathrm{tr}(w)$ (the left hand side of Equation (1.1)), whose zero set coincides with the character variety $X(\mathfrak{b}(p, m))$, is described by

$$\mathrm{tr}(bwa^{-1}) - \mathrm{tr}(w) = (z + 2 - x^2)\Phi_w(x, z),$$

where $\Phi_w(x, z)$ is the polynomial in $\mathbb{C}[x, z]$ defined by

$$\Phi_w(x, z) = \mathrm{tr}\, w - \mathrm{tr}\, w' + \cdots + (-1)^{d-1}\, \mathrm{tr}\, w^{(d-1)} + (-1)^d.$$

Here if u is a word, then u' denotes the word obtained from u by deleting the letter at the beginning and the letter at the end of u. In general, $u^{(j)}$ denotes the element obtained from u by applying the deleting operation j times.

In the case of $\mathfrak{b}(p, 3)$ satisfying $p > 3$ and $\gcd(p, 3) = 1$, we can describe more precisely the polynomial $\Phi_w(x, z)$ using the Chebyshev polynomials $S_n(z)$ as follows.

Proposition 2.1. *For the 2-bridge knot $\mathfrak{b}(p, 3)$ with $p > 3$ and $\gcd(p, 3) = 1$, one has*

$$\begin{aligned}\Phi_w(x, z) = {}& S_d(z) - S_{d-1}(z) \\ &+ x^2(2 - z)S_{d-l-1}(z)S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor - 1}(z)),\end{aligned}$$

where $l = \lfloor p/3 \rfloor$.

This presentation of $\Phi_w(x, z)$ leads us to one of the main results in this paper.

Theorem 2.2. *For the 2-bridge knot $\mathfrak{b}(p, 3)$ with $p > 3$ and $\gcd(p, 3) = 1$, $\Phi_w(x, z)$ is irreducible in $\mathbb{C}[x, z]$.*

In the rest of this section, we first show Proposition 2.1. Then we prove Theorem 2.2 by Propositions 1.3 and 2.1 and Lemmas 2.5 to 2.7 shown below.

2.1. Proof of Proposition 2.1. At first, we consider the general case $\mathfrak{b}(p, m)$. For $j = 1, \ldots, d$, let

$$
w_j := \begin{cases} a^{\varepsilon_j} b^{\varepsilon_{j+1}} \cdots a^{\varepsilon_{2d-j}} b^{\varepsilon_{2d+1-j}}, & \text{if } j \text{ is odd,} \\ b^{\varepsilon_j} a^{\varepsilon_{j+1}} \cdots b^{\varepsilon_{2d-j}} a^{\varepsilon_{2d+1-j}}, & \text{if } j \text{ is even.} \end{cases}
$$

Note that $w_1 = w$ and $w_{j+1} = (w_j)' = w^{(j)}$ are satisfied. Similarly, for $j = 1, \ldots, d$, let

$$
u_j := \begin{cases} w_{j+1} b^{\varepsilon_j}, & \text{if } j \text{ is odd,} \\ w_{j+1} a^{\varepsilon_j}, & \text{if } j \text{ is even,} \end{cases} \qquad v_j := \begin{cases} a^{\varepsilon_j} w_{j+1}, & \text{if } j \text{ is odd,} \\ b^{\varepsilon_j} w_{j+1}, & \text{if } j \text{ is even,} \end{cases}
$$

where we set $w_{d+1} := 1$.

Lemma 2.3. (1) *If $\varepsilon_j = \varepsilon_{j+1}$, then*

$$
\operatorname{tr} w_j = z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2},
$$

$$
x \operatorname{tr} u_j = x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} u_{j+1},
$$

$$
x \operatorname{tr} v_j = x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} v_{j+1}.
$$

(2) *If $\varepsilon_j = -\varepsilon_{j+1}$, then*

$$
\operatorname{tr} w_j = (z - x^2) \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2} + x \operatorname{tr} u_{j+1} + x \operatorname{tr} v_{j+1}.
$$

Proof. For a word u in two letters a and b, let $\overleftarrow{u}$ be the word obtained from u by writing the letters in u in the reversed order. Then, by [10, Lemma 3.2.2] we have $\operatorname{tr} \overleftarrow{u} = \operatorname{tr} u$.

For $M \in \mathrm{SL}_2(\mathbb{C})$, by the Cayley–Hamilton theorem we have $M = (\operatorname{tr} M)I - M^{-1}$. Here I denotes the 2×2 identity matrix. Since $\varepsilon_j = \varepsilon_{p-j}$ holds, we note that w_j is also expressed by

$$
w_j = \begin{cases} a^{\varepsilon_j} b^{\varepsilon_{j+1}} \cdots a^{\varepsilon_{j+1}} b^{\varepsilon_j}, & \text{if } j \text{ is odd,} \\ b^{\varepsilon_j} a^{\varepsilon_{j+1}} \cdots b^{\varepsilon_{j+1}} a^{\varepsilon_j}, & \text{if } j \text{ is even.} \end{cases}
$$

(1) Suppose $\varepsilon_j = \varepsilon_{j+1}$. If j is even, then we have

$$x \operatorname{tr} u_j = x \operatorname{tr}(w_{j+1} a^{\varepsilon_j}) = x \operatorname{tr}(w_{j+1} a^{\varepsilon_{j+1}}) = x \operatorname{tr}((a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}) a^{\varepsilon_{j+1}})$$

$$= x \operatorname{tr}((a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}})(xI - a^{-\varepsilon_{j+1}}))$$

$$= x^2 \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}) - x \operatorname{tr}(b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}})$$

$$= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} u_{j+1},$$

$$x \operatorname{tr} v_j = x \operatorname{tr}(b^{\varepsilon_j} w_{j+1}) = x \operatorname{tr}(b^{\varepsilon_{j+1}} w_{j+1}) = x \operatorname{tr}(b^{\varepsilon_{j+1}}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}))$$

$$= x \operatorname{tr}((xI - b^{-\varepsilon_{j+1}})(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}))$$

$$= x^2 \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}) - x \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}})$$

$$= x^2 \operatorname{tr} w_{j+1} - x \operatorname{tr} v_{j+1},$$

$$\operatorname{tr} w_j = \operatorname{tr}(b^{\varepsilon_j} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}} a^{\varepsilon_j}) = \operatorname{tr}((b^{\varepsilon_{j+1}} a^{\varepsilon_{j+1}}) b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} (b^{\varepsilon_{j+1}} a^{\varepsilon_{j+1}}))$$

$$= \operatorname{tr}((b^{\varepsilon_{j+1}} a^{\varepsilon_{j+1}}) b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} (zI - (b^{\varepsilon_{j+1}} a^{\varepsilon_{j+1}})^{-1}))$$

$$= z \operatorname{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}}) - \operatorname{tr}(b^{\varepsilon_{j+2}} \ldots a^{\varepsilon_{j+2}})$$

$$= z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}.$$

The case of odd j is similar.

(2) Suppose $\varepsilon_j = -\varepsilon_{j+1}$. If j is even, then we have

$$x \operatorname{tr} u_j = x \operatorname{tr}((a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}}) a^{-\varepsilon_{j+1}}) = x \operatorname{tr}(b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}})$$

$$= x \operatorname{tr} u_{j+1},$$

$$x \operatorname{tr} v_j = x \operatorname{tr}(b^{-\varepsilon_{j+1}}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}} b^{\varepsilon_{j+1}})) = x \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+2}} \cdots a^{\varepsilon_{j+2}})$$

$$= x \operatorname{tr} v_{j+1},$$

$$\operatorname{tr} w_j = \operatorname{tr}(a^{-\varepsilon_{j+1}} b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} b^{-\varepsilon_{j+1}})$$

$$= \operatorname{tr}((xI - a^{\varepsilon_{j+1}}) b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} (xI - b^{\varepsilon_{j+1}}))$$

(2.1)
$$= x^2 \operatorname{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}}) + \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}})$$

$$- x \operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}}) - x \operatorname{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}}).$$

We first obtain

(2.2)
$$x^2 \operatorname{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}}) = x^2 \operatorname{tr} \overleftarrow{w_{j+1}} = x^2 \operatorname{tr} w_{j+1}.$$

By the argument for $\operatorname{tr} w_j$ in (1), we get

(2.3)
$$\operatorname{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}}) = z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}.$$

We also have

$$x \, \mathrm{tr}(a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}})$$

(2.4)

$$= x \, \mathrm{tr}((xI - a^{-\varepsilon_{j+1}}) b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}})$$

$$= x^2 \, \mathrm{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}}) - x \, \mathrm{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \ldots b^{\varepsilon_{j+2}})$$

$$= x^2 \, \mathrm{tr} \, \overleftarrow{w_{j+1}} - x \, \mathrm{tr} \, \overleftarrow{u_{j+1}} = x^2 \, \mathrm{tr} \, w_{j+1} - x \, \mathrm{tr} \, u_{j+1}.$$

Similarly, the following holds:

(2.5)
$$x \, \mathrm{tr}(b^{\varepsilon_{j+1}} a^{\varepsilon_{j+2}} \cdots b^{\varepsilon_{j+2}} a^{\varepsilon_{j+1}} b^{\varepsilon_{j+1}}) = x^2 \, \mathrm{tr} \, w_{j+1} - x \, \mathrm{tr} \, v_{j+1}.$$

From Equations (2.1), (2.2), (2.3), (2.4) and (2.5), we get

$$\mathrm{tr} \, w_j = (z - x^2) \, \mathrm{tr} \, w_{j+1} - \mathrm{tr} \, w_{j+2} + x \, \mathrm{tr} \, u_{j+1} + x \, \mathrm{tr} \, v_{j+1}.$$

The case of odd j is similar. This completes the proof of the lemma. □

We apply Lemma 2.3 to describe $\mathrm{tr} \, w_j$ in $\Phi_w(x, z)$ by $S_n(z)$'s. For the 2-bridge knot $\mathfrak{b}(p, 3)$, we can check that $\varepsilon_j = 1$ if $j \leq l$ and $\varepsilon_j = -1$ if $l + 1 \leq j \leq d + 1$, where $l = \lfloor p/3 \rfloor$.

CASE 1: $l + 1 \leq j \leq d$. Since $\varepsilon_j = \varepsilon_{j+1}$, by Lemma 2.3,

$$\mathrm{tr} \, w_j = z \, \mathrm{tr} \, w_{j+1} - \mathrm{tr} \, w_{j+2},$$

$$x \, \mathrm{tr} \, u_j = x^2 \, \mathrm{tr} \, w_{j+1} - x \, \mathrm{tr} \, u_{j+1},$$

$$x \, \mathrm{tr} \, v_j = x^2 \, \mathrm{tr} \, w_{j+1} - x \, \mathrm{tr} \, v_{j+1}.$$

Note that $\mathrm{tr} \, w_d = \mathrm{tr} \, a^{\varepsilon_d} b^{\varepsilon_{d+1}} = \mathrm{tr} \, a^{\varepsilon_d} b^{\varepsilon_d} = z$ and $\mathrm{tr} \, w_{d+1} = \mathrm{tr} \, 1 = 2$. Applying the above equations recursively, we obtain

$$\mathrm{tr} \, w_j = T_{d+1-j}(z),$$

$$x \, \mathrm{tr} \, u_j = x^2(\mathrm{tr} \, w_{j+1} - \mathrm{tr} \, w_{j+2} + \cdots + (-1)^{d-1-j} \, \mathrm{tr} \, w_d) + (-1)^{d-j} x \, \mathrm{tr} \, u_d$$

$$= x^2(T_{d-j}(z) - T_{d-1-j}(z) + \cdots + (-1)^{d-1-j} T_1(z) + (-1)^{d-j}),$$

$$x \, \mathrm{tr} \, v_j = x^2(\mathrm{tr} \, w_{j+1} - \mathrm{tr} \, w_{j+2} + \cdots + (-1)^{d-1-j} \, \mathrm{tr} \, w_d) + (-1)^{d-j} x \, \mathrm{tr} \, v_d$$

$$= x^2(T_{d-j}(z) - T_{d-1-j}(z) + \cdots + (-1)^{d-1-j} T_1(z) + (-1)^{d-j}),$$

where $T_n(z) \, (\forall n \in \mathbb{Z})$ are the Chebyshev polynomials defined by $T_0(z) = 2$, $T_1(z) = z$ and $T_{n+1}(z) = z T_n(z) - T_{n-1}(z)$. In particular,

$$\mathrm{tr} \, w_{l+1} = T_{d-l}(z),$$

$$x \, \mathrm{tr} \, u_{l+1} = x \, \mathrm{tr} \, v_{l+1}$$

$$= x^2(T_{d-1-l}(z) - T_{d-2-l}(z) + \cdots + (-1)^{d-l-2} T_1(z) + (-1)^{d-l-1}).$$

Case 2: $1 \leq j \leq l-1$. Since $\varepsilon_j = \varepsilon_{j+1}$, by Lemma 2.3,

$$\operatorname{tr} w_j = z \operatorname{tr} w_{j+1} - \operatorname{tr} w_{j+2}.$$

By Lemma 1.1, it follows that $\operatorname{tr} w_j = S_{l-j}(z) \operatorname{tr} w_l - S_{l-1-j}(z) \operatorname{tr} w_{l+1}$.

Case 3: $j = l$. Since $\varepsilon_l = -\varepsilon_{l+1}$, by Lemma 2.3,

$$\begin{aligned}
\operatorname{tr} w_l &= (z - x^2) \operatorname{tr} w_{l+1} - \operatorname{tr} w_{l+2} + x \operatorname{tr} u_{l+1} + x \operatorname{tr} v_{l+1} \\
&= (z - x^2) T_{d-l}(z) - T_{d-l-1}(z) \\
&\quad + 2x^2 (T_{d-1-l}(z) - T_{d-2-l}(z) + \cdots + (-1)^{d-l-2} T_1(z) + (-1)^{d-l-1}).
\end{aligned}$$

Hence $\Phi_w(x, z)$ is equal to

$$\begin{aligned}
&\operatorname{tr} w_1 - \operatorname{tr} w_2 + \cdots + (-1)^{l-1} \operatorname{tr} w_l + (-1)^l \operatorname{tr} w_{l+1} + \cdots + (-1)^{d-1} \operatorname{tr} w_d + (-1)^d \\
&= (S_{l-1}(z) - S_{l-2}(z) + \cdots + (-1)^{l-2} S_1(z) + (-1)^{l-1} S_0(z)) \operatorname{tr} w_l \\
&\quad - (S_{l-2}(z) - S_{l-3}(z) + \cdots + (-1)^{l-2} S_0(z) + (-1)^{l-1} S_{-1}(z)) \operatorname{tr} w_{l+1} \\
&\quad + (-1)^l \operatorname{tr} w_{l+1} + \cdots + (-1)^{d-1} \operatorname{tr} w_d + (-1)^d \\
&= P(z) + x^2 Q(z) R(z),
\end{aligned}$$

where

$$\begin{aligned}
P(z) &= T_{d-l+1}(z)(S_{l-1}(z) - S_{l-2}(z) + \cdots + (-1)^{l-2} S_1(z) + (-1)^{l-1} S_0(z)) \\
&\quad - (S_{l-2}(z) - S_{l-3}(z) + \cdots + (-1)^{l-2} S_0(z) + (-1)^{l-1} S_{-1}(z)) T_{d-l}(z) \\
&\quad + (-1)^l T_{d-l}(z) + (-1)^{l+1} T_{d-l-1}(z) + \cdots + (-1)^{d-1} T_1(z) + (-1)^d, \\
Q(z) &= S_{l-1}(z) - S_{l-2}(z) + \cdots + (-1)^{l-2} S_1(z) + (-1)^{l-1} S_0(z), \\
R(z) &= -T_{d-l}(z) + 2(T_{d-1-l}(z) - T_{d-2-l}(z) + \cdots + (-1)^{d-l-2} T_1(z) + (-1)^{d-l-1}).
\end{aligned}$$

The following lemma gives us nice descriptions for $P(z)$, $Q(z)$ and $R(z)$.

Lemma 2.4. *The followings hold.*
(1) $P(z) = S_d(z) - S_{d-1}(z)$,
(2) $Q(z) = S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor - 1}(z))$,
(3) $R(z) = (2 - z) S_{d-l-1}(z)$.

Proof. (1) follows from [10] (see also [11, Proposition A.2]) as $P(z) = \Phi_w(0, z) = S_d(z) - S_{d-1}(z)$. This can be checked directly by $T_j(z) = S_j(z) - S_{j-2}(z)$ and Lemma 4.3 in [13] saying that for any non-negative integers r and s,

$$S_r(u) S_{r+s}(u) = S_{2r+s}(u) + S_{2r+s-2}(u) + \cdots + S_s(u).$$

To show (2), let

$$\alpha_n = S_n(z) - S_{n-1}(z) + \cdots + (-1)^{n-1} S_1(z) + (-1)^n S_0(z).$$

Then $Q(z) = \alpha_{l-1}$. If $n = 2k$ then

$$\alpha_n = (S_{2k}(z) + \cdots + S_0(z)) - (S_{2k-1}(z) + \cdots + S_1(z))$$
$$= S_k(z)^2 - S_k(z)S_{k-1}(z) = S_k(z)(S_k(z) - S_{k-1}(z)).$$

If $n = 2k + 1$ then

$$\alpha_n = (S_{2k+1}(z) + \cdots + S_1(z)) - (S_{2k}(z) + \cdots + S_0(z))$$
$$= S_{k+1}(z)S_k(z) - S_k(z)^2 = S_k(z)(S_{k+1}(z) - S_k(z)).$$

In both cases $\alpha_n = S_{n-\lfloor (n+1)/2 \rfloor}(z)(S_{\lfloor (n+1)/2 \rfloor}(z) - S_{\lfloor (n-1)/2 \rfloor}(z))$. Hence

$$Q(z) = \alpha_{l-1} = S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor -1}(z)).$$

To show (3), let

$$\beta_n = -T_{n+1}(z) + 2(T_n(z) - T_{n-1}(z) + \cdots + (-1)^{n-1} T_1(z) + (-1)^n).$$

Then $R(z) = \beta_{d-l-1}$. Note that $T_j(z) = S_j(z) - S_{j-2}(z)$. If $n = 2k$ then

$$\beta_n = -T_{2k+1}(z) + 2 + 2((T_{2k}(z) + \cdots + T_2(z)) - (T_{2k-1}(z) + \cdots + T_1(z)))$$
$$= -(S_{2k+1}(z) - S_{2k-1}(z)) + 2 + 2((S_{2k}(z) - S_0(z)) - (S_{2k-1}(z) - S_{-1}(z)))$$
$$= -(S_{2k+1}(z) + S_{2k-1}(z)) + 2S_{2k}(z)$$
$$= (2 - z)S_{2k}(z) = (2 - z)S_n(z).$$

If $n = 2k + 1$ then

$$\beta_n = -T_{2k+2}(z) - 2 + 2((T_{2k+1}(z) + \cdots + T_1(z)) - (T_{2k}(z) + \cdots + T_2(z)))$$
$$= -(S_{2k+2}(z) - S_{2k}(z)) - 2 + 2((S_{2k+1}(z) - S_{-1}(z)) - (S_{2k}(z) - S_0(z)))$$
$$= -(S_{2k+2}(z) + S_{2k}(z)) + 2S_{2k+1}(z)$$
$$= (2 - z)S_{2k+1}(z) = (2 - z)S_n(z).$$

In both cases $\beta_n = (2 - z)S_n(z)$. Hence $R(z) = \beta_{d-l-1} = (2 - z)S_{d-l-1}(z)$. $\qquad\square$

From Lemma 2.4, we get

$$\Phi_w(x, z) = P(z) + x^2 Q(z)R(z)$$
$$= S_d(z) - S_{d-1}(z) + x^2(2 - z)S_{d-l-1}(z)S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor -1}(z)).$$

This completes the proof of Proposition 2.1.

2.2. Proof of Theorem 2.2. By Proposition 2.1, we have $\Phi_w(x, z) = P(z) + x^2 Q(z)R(z)$, where

$$P(z) = S_d(z) - S_{d-1}(z),$$

$$Q(z) = S_{l-1-\lfloor l/2 \rfloor}(z)(S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor - 1}(z)),$$

$$R(z) = (2 - z)S_{d-l-1}(z).$$

Since $\deg P - \deg QR = d - ((l-1) + (d-l)) = 1$ is an odd number, by Proposition 1.3, $\Phi_w(x, z) \in \mathbb{C}[x, z]$ is irreducible if $\gcd(P(z), Q(z)R(z)) = 1$.

The following lemma is standard, see e.g. [14].

Lemma 2.5. *For $n \geq 1$, the followings hold*:
(1) $S_n(z)$ *is a monic polynomial of degree n whose n roots are exactly $2\cos(j\pi/(n+1))$, $1 \leq j \leq n$.*
(2) $S_n(z) - S_{n-1}(z)$ *is a monic polynomial of degree n whose n roots are exactly $2\cos((2j+1)\pi/(2n+1))$, $0 \leq j \leq n-1$.*

Note that Lemma 2.5 (2) shows $\gcd(P(z), 2 - z) = 1$.

Lemma 2.6. $\gcd(S_d(z) - S_{d-1}(z), S_{\lfloor l/2 \rfloor}(z) - S_{\lfloor l/2 \rfloor - 1}(z)) = 1$.

Proof. By Lemma 2.5 (2), it suffices to show that

$$(2.6) \qquad \frac{2j+1}{2d+1} \neq \frac{2j'+1}{2\lfloor l/2 \rfloor + 1}$$

where $0 \leq j \leq d-1$ and $0 \leq j' \leq \lfloor l/2 \rfloor - 1$. It is easy to see that Equation (2.6) holds true if $\gcd(2d+1, 2\lfloor l/2 \rfloor + 1) = 1$. Recall that $d = (p-1)/2$ and $l = \lfloor p/3 \rfloor$. Since $3(2\lfloor l/2 \rfloor + 1) - (2d+1)$ is equal to either $3l - p$ or $3(l+1) - p$, and $3l - p = 3\lfloor p/3 \rfloor - p$ is equal to either -1 or -2 (note that $\gcd(p, 3) = 1$), $3(2\lfloor l/2 \rfloor + 1) - (2d+1)$ is equal to either ± 1 or ± 2. It follows that $\gcd(2d+1, 2\lfloor l/2 \rfloor + 1)$ is a divisor of 2. Since $2d+1$ is odd, we must have $\gcd(2d+1, 2\lfloor l/2 \rfloor + 1) = 1$. $\square$

Lemma 2.7. $\gcd(S_d(z) - S_{d-1}(z), S_{l-1-\lfloor l/2 \rfloor}(z)) = \gcd(S_d(z) - S_{d-1}(z), S_{d-l-1}(z)) = 1$.

Proof. By Lemma 2.5 (1), it suffices to show that

$$\frac{2j+1}{2d+1} \neq \frac{j'}{l - \lfloor l/2 \rfloor}, \qquad \frac{2k+1}{2d+1} \neq \frac{k'}{d-l}$$

where $0 \leq j \leq d-1$, $0 \leq j' \leq l - \lfloor l/2 \rfloor - 1$, $0 \leq k \leq d-1$ and $0 \leq k' \leq d - l - 1$. These hold true if $\gcd(2d+1, l - \lfloor l/2 \rfloor) = 1$ and $\gcd(2d+1, d-l) = 1$. Since the proof is similar to that of Lemma 2.6, we omit the details. $\square$

We now finish the proof of Theorem 2.2. From Lemmas 2.5, 2.6 and 2.7, we have $\gcd(P(z), Q(z)R(z)) = 1$. Hence Proposition 1.3 implies that $\Phi_w(x, z)$ is irreducible in $\mathbb{C}[x, z]$ for the 2-bridge knot $\mathfrak{b}(p, 3)$ and this completes the proof of Theorem 2.2.

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