Grindstaff, G.R. and Nelson, S.
Osaka J. Math.
53 (2016), 1015–1027

LIE IDEAL ENHANCEMENTS OF COUNTING INVARIANTS

GILLIAN ROXANNE GRINDSTAFF and SAM NELSON

(Received May 19, 2014, revised October 19, 2015)

Abstract

We define enhancements of the quandle counting invariant for knots and links with a finite labeling quandle Q embedded in the quandle of units of a Lie algebra $\mathfrak{a}$ using Lie ideals. We provide examples demonstrating that the enhancement is stronger than the associated unenhanced counting invariant and image enhancement invariant.

1. Introduction

In the early 1980s, Joyce [4] and Matveev [6] introduced an algebraic structure called *quandles* or *distributive groupoids* with connections to knot theory. In particular the quandle axioms can be understood as the conditions required by the Reidemeister moves for labelings of the arcs in a knot diagram by elements of a quandle to correspond one to one before and after the move.

Associated to an oriented knot or link L is a *fundamental quandle* $Q(L)$ which determines the knot group as well as the peripheral subgroup and hence is a complete invariant of knots up to ambient homeomorphism. Given a finite quandle Q, the set $\mathrm{Hom}(Q(L), Q)$ is a finite set (provided L is tame) and its cardinality is a computable knot invariant known as the *quandle counting invariant*. In [1] the first *enhancement* of the quandle counting invariant, known as the *quandle* 2*-cocycle* invariant, was introduced. An enhancement is a stronger invariant which determines the counting invariant but distinguishes some knots or links which have the same counting invariant value. Since then, numerous enhancements of the quandle counting invariant and its generalizations have been explored. Each homomorphism $f \in \mathrm{Hom}(Q(L), Q)$ can be identified with a labeling of the arcs in a diagram of L with elements of Q; in particular, enhancements of the counting invariant associated to a finite quandle Q can be understood as invariants of Q-labeled knots.

In this paper we introduce a new enhancement of the quandle counting invariant defied via Lie ideals in a Lie algebra, analogous to the symplectic quandle enhancement defined in [7]. Our construction involves embedding a finite quandle Q in the quandle of units $Q(\mathfrak{a})$ of the universal enveloping algebra A of a Lie algebra $\mathfrak{a}$, then using the Lie algebra structure to obtain an invariant signature for each quandle coloring of our

---

2010 Mathematics Subject Classification. 55M25, 55M27.
Partially supported by Simons Foundation Collaboration Award 316709.

knot or link K. Different embeddings of the same finite quandle into Lie algebras in general yield different enhancements. We can think of the embedding $Q \to Q(\mathfrak{a})$ as a kind of knotting analogous to embedding copies of S^1 into S^3; then these invariants are defined via pairs of "knotted quandles" philosophically similar to the approach to finite type invariants in [3], where a knot is compared to another "knot" via evaluation of a bilinear form.

The paper is organized as follows. In Section 2 we review the basics of quandles and the quandle counting invariant, as well as some previously studied enhancements. In Section 3 we introduce the Lie Ideal Enhanced polynomial invariant or LIE polynomial associated to Lie algebra with finite quandle of units. In Section 4 we collect computations and examples of the new invariants, and in Section 5 we conclude with some questions for future research.

2. Quandles

We begin with a definition (see [2, 4, 6]).

DEFINITION 1. A *quandle* is a set Q with binary operations $\triangleright, \triangleright^{-1}\colon Q \times Q \to Q$ satisfying for all $x, y, z \in Q$

(i) $x \triangleright x = x$,

(ii) $(x \triangleright y) \triangleright^{-1} y = x = (x \triangleright^{-1} y) \triangleright y$, and

(iii) $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

If we have only (ii) and (iii), Q is a *rack*.

EXAMPLE 1. Let G be a group. Then for each $n \in \mathbb{Z}$, G is a quandle under n-fold conjugation

$$x \triangleright y = y^{-n} x y^n, \quad x \triangleright^{-1} y = y^n x y^{-n}.$$

We usually denote this quandle structure as $\mathrm{Conj}_n(G)$.

EXAMPLE 2. Let $\Lambda = \mathbb{Z}[t^{\pm 1}]$ and let A be any Λ-module. Then, A is a quandle under the operations

$$x \triangleright y = tx + (1-t)y, \quad x \triangleright^{-1} y = t^{-1}x + (1-t^{-1})y.$$

Quandles of this type are known as *Alexander quandles*.

EXAMPLE 3. Let $\mathbb{F}$ be a field and V an $\mathbb{F}$-vector space with symplectic form $\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{F}$. Then, V is a quandle under the operations

$$\vec{x} \triangleright \vec{y} = \vec{x} + \langle \vec{x}, \vec{y} \rangle \vec{y}, \quad \vec{x} \triangleright^{-1} \vec{y} = \vec{x} - \langle \vec{x}, \vec{y} \rangle \vec{y}.$$

Quandles of this type are known as *symplectic quandles*.

More generally, if $X = \{1, 2, \ldots, n\}$ then we can define a quandle structure on X by $i \triangleright j = M_{ij}$ with an $n \times n$ matrix M_X with the properties that $M_{ii} = i$, each column is a permutation of X, and $M_{M_{ij}k} = M_{M_{ik}M_{jk}}$ for $i, j, k \in X$. We call M_X the *quandle matrix* of X.

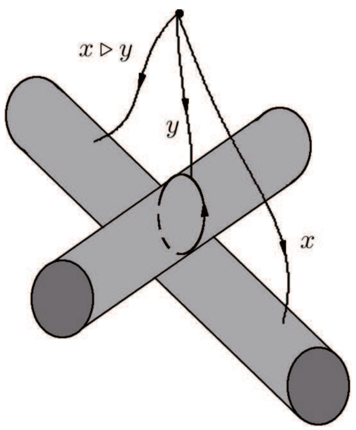
EXAMPLE 4. The Alexander quandle $X = \Lambda/(3, t-2) = \mathbb{Z}_3[t]/(t-2)$ has quandle matrix

$$M_X = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

EXAMPLE 5. The *fundamental quandle* of a link L with diagram D has a generator for each arc in D with relation $z = x \triangleright y$ at each crossing. Elements of $Q(L)$ are then equivalence classes of quandle words in these generators modulo the equivalence relation generated by the quandle axioms and the crossing relations.

$$Q(L) = \langle x, y, x : x \triangleright y = z,\ y \triangleright z = x,\ z \triangleright x = y \rangle.$$

EXAMPLE 6. Let L be an oriented link in S^3 and $N(L)$ a regular neighborhood of L. Geometrically, the fundamental quandle of L is the set of homotopy classes of paths in the link complement $S^3 \setminus N(L)$ from a base point $* \in S^3 \setminus N(L)$ to $N(L)$ where the endpoint is allowed to wander along $N(L)$ during the homotopy. For each such path y there is is a canonical meridian $m(y)$ based at the terminal point of y and linking the core of the component of $N(L)$ once. The quandle operation is then given by the path along y, around $m(y)$, backwards along y, then along x



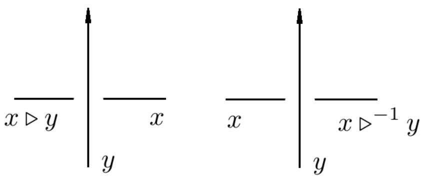
$$x \triangleright y = xy^{-1}m(y)^{-1}y$$

as depicted. See [2, 4, 6] for more.

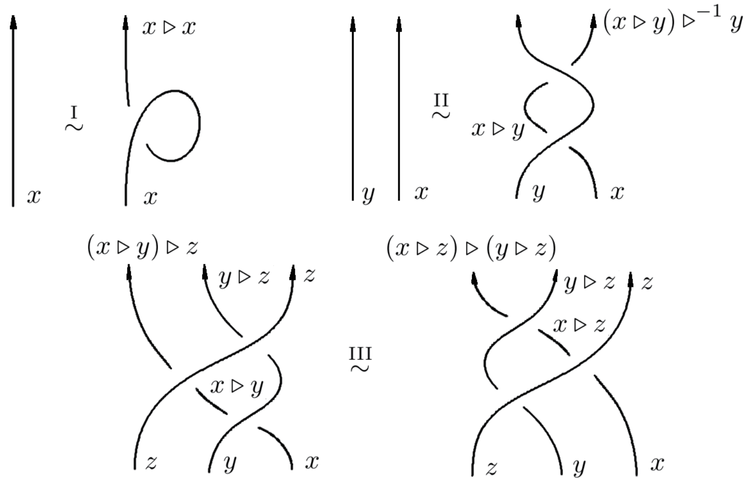
For each $y \in Q$, define f_y: $Q \to Q$ by $f_y(x) = x \triangleright y$. Then quandle axiom (ii) says that f_y is invertible for each y; in particular, the operation $\triangleright$ determines the operation $\triangleright^{-1}$. A subset $S \subset Q$ of a quandle Q which is closed under $\triangleright$ and $\triangleright^{-1}$ is a *subquandle* of Q; if Q is finite, then closure under $\triangleright$ implies closure under $\triangleright^{-1}$. A map $f : Q \to Q'$ between quandles Q and Q' with operations $\triangleright$ and $\triangleright'$ is a *homomorphism of quandles* if for all $x, y \in Q$ we have

$$f(x \triangleright y) = f(x) \triangleright' f(y).$$

The quandle axioms can be understood as algebraically encoding the Reidemeister moves. More precisely, a *quandle coloring* of an oriented knot diagram K by a quandle Q is an assignment of an element of Q to each arc in K such that at each crossing, the result of an arc labeled x crossing under and arc labeled y from right to left is an arc labeled $x \triangleright y$.



In particular, the quandle axioms are the conditions required to ensure that for each quandle coloring of a knot diagram before a move, there is a unique corresponding quandle coloring of the diagram after the move.



Then we have:

Theorem 1 ([4, 6]). *Let Q be a finite quandle and L an oriented knot or link. Then the number $|\mathrm{Hom}(Q(K), Q)|$ of quandle colorings of a diagram of L is invariant* *under Reidemeister moves.*

A quandle coloring of a knot or link L by a quandle Q determines a unique homomorphism of quandles $f\colon Q(L) \to Q$ by assigning an image in Q to each generator of the fundamental quandle $Q(L)$; then the homomorphism condition requires that $f(x \triangleright y) = f(x) \triangleright f(y)$ for each $x, y \in Q(L)$. In particular, an assignment of elements of Q to arcs in a diagram of L determines a quandle homomorphism $f\colon Q(L) \to Q$ if and only if the crossing relations are satisfied at every crossing, and the homomorphism so determined is unique. The number of quandle colorings $|\mathrm{Hom}(Q(L), Q)|$ of a knot or link L is called the *quandle counting invariant* of K with coloring quandle Q. If Q is finite, then $|\mathrm{Hom}(Q(L), Q)|$ is a positive integer greater than or equal to $|Q|$, since monochromatic colorings are always valid.

Now let ϕ be an invariant of Q-colored knot diagrams. Instead of counting "1" for each element of $\mathrm{Hom}(Q(L), Q)$, we can collect $\phi(f)$ values for each $f \in \mathrm{Hom}(Q(L), Q)$ to obtain a multiset

$$\Phi_Q^{\phi}(K) = \{\phi(f)\colon\ f \in \mathrm{Hom}(Q(L), Q)\}$$

whose cardinality is the quandle counting invariant. We call such an invariant an *enhancement* of the counting invariant. Especially for integer-valued enhancements, we often replace the multiset with its generating function to get a polynomial invariant for ease of comparison. For instance, the multiset $\{0, 0, 1, 2, 2, 2\}$ corresponds to the polynomial $2 + u + 3u^2$.

EXAMPLE 7. Let Q be a finite quandle and L an oriented knot or link. For each $f \in \mathrm{Hom}(Q(L), Q)$, let $\phi(f)$ be the cardinality of the image subquandle $\phi(f) = |\mathrm{Im}(f)|$. Then the *image enhancement* is the multiset

$$\Phi_Q^{\mathrm{Im},M}(L) = \{|\mathrm{Im}(f)|\colon\ f \in \mathrm{Hom}(Q(L), Q)\}$$

or the polynomial

$$\Phi_Q^{\mathrm{Im}}(L) = \sum_{f \in \mathrm{Hom}(Q(L), Q)} u^{|\mathrm{Im}(f)|}.$$

For instance, the trefoil knot 3_1 has three monochromatic colorings by the quandle $Q = \Lambda_3/(t-2)$ and six surjective colorings, so we have quandle counting invariant $|\mathrm{Hom}(Q(K3_1), Q)| = 9$ and image enhancement $\Phi_Q^{\mathrm{Im}}(3_1) = 3u + 6u^3$. See [8] for more.

3. Lie ideal enhancements

Let R be a commutative ring, A be an associative unital R-algebra and let $\mathfrak{a}$ be the associated Lie algebra (that is, $\mathfrak{a}$ is A endowed with the Lie bracket $[u, v] = uv - vu$). Then for any integer $n \in \mathbb{Z}$, the group of units of A, $A^* = \{u \in A\colon \exists v \in A \text{s.t. } uv = vu = 1\}$, is a quandle under the n-fold conjugation operation

$$u \triangleright v = v^{-n} u v^n = u + v^{-n}[u, v^n]$$

which we call the *n-th quandle of units* of $\mathfrak{a}$, denoted $Q_n(\mathfrak{a})$. If $\mathfrak{a}$ is finite, then so is $Q_n(\mathfrak{a})$, though the converse is not necessarily true. If $n = 1$, we will write $Q(\mathfrak{a})$ instead of $Q_1(\mathfrak{a})$.

EXAMPLE 8. Let $\mathbb{F}$ be a field. Then for any positive integer m, the matrix algebra $M_m(\mathbb{F})$ of square $m \times m$ matrices with entries in F has quandle of units $GL_m(\mathbb{F})$ consisting of invertible matrices. In particular, if $m = 1$ then the quandle of units is the conjugation quandle of the abelian group $\mathbb{F} \setminus 0$ and hence is a trivial quandle.

EXAMPLE 9. Let G be a group and $\mathbb{F}$ a field. Then the group $\mathbb{F}$-algebra

$$\mathfrak{a} = \mathbb{F}[G] = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F},\ g \in G \right\}$$

has quandle of units $Q(\mathfrak{a})$ containing a subquandle isomorphic to $\mathrm{Conj}(G)$.

In some cases this quandle of units coincides with symplectic quandles as defined in [7]. More precisely, we have the following:

Theorem 2. *Let A be an associative unital $\mathbb{F}$-algebra with symplectic form $\langle\cdot,\cdot\rangle : A \times A \to \mathbb{F}$. If the symplectic quandle structure on A^* agrees with the quandle of units $Q(\mathfrak{a})$, then $[\vec{v}^{-1}, \vec{u}] = [\vec{v}, \vec{u}^{-1}]$ for all $\vec{u}, \vec{v} \in A^*$.*

Proof. For the symplectic quandle structure on A^* we have $\vec{u} \triangleright \vec{v} = \vec{u} + \langle \vec{u}, \vec{v}\rangle \vec{v}$, where $\langle\cdot,\cdot\rangle$ is antisymmetric. If $\vec{u} \triangleright \vec{v}$ is the same as the quandle action defining $Q(\mathfrak{a})$ above, then we have

$$\langle \vec{u}, \vec{v}\rangle \vec{v} = \vec{v}^{-1}[\vec{u}, \vec{v}]$$

and thus

$$\langle \vec{u}, \vec{v}\rangle \vec{1} = \langle \vec{u}, \vec{v}\rangle \vec{v}\vec{v}^{-1} = \vec{v}^{-1}[\vec{u}, \vec{v}]\vec{v}^{-1} = \vec{v}^{-1}(\vec{u}\vec{v} - \vec{v}\vec{u})\vec{v}^{-1} = \vec{v}^{-1}u - \vec{u}\vec{v}^{-1} = [\vec{v}^{-1}, \vec{u}].$$

Now, we also have

$$\langle \vec{u}, \vec{v}\rangle \vec{1} = -\langle \vec{v}, \vec{u}\rangle \vec{1} = -[\vec{u}^{-1}, \vec{v}] = [\vec{v}, \vec{u}^{-1}]$$

and thus $[\vec{v}^{-1}, \vec{u}] = [\vec{v}, \vec{u}^{-1}]$. □

Proposition 3. *The quandle of units in a Lie algebra A over a field $\mathbb{F}$ is involutory if and only if either* (1) *$\mathbb{F}$ has characteristic* 2 *or* (2) *the group of units A^* is abelian.*

Proof. Recall that a quandle Q is involutory if we have $\vec{u} \triangleright \vec{v} = \vec{u} \triangleright^{-1} \vec{v}$ for all $\vec{u}, \vec{v} \in Q$. In the quandle of units case, we have

$$\vec{u} \triangleright \vec{v} = \vec{u} + \vec{v}^{-1}[\vec{u}, \vec{v}]$$

and

$$\vec{u} \triangleright^{-1} \vec{v} = \vec{u} - \vec{v}^{-1}[\vec{u}, \vec{v}].$$

Then $Q(\mathfrak{a})$ is involutory iff

$$[\vec{u}, \vec{v}] = -[\vec{u}, \vec{v}] = [\vec{v}, \vec{u}],$$

that is, iff

$$\vec{u}\vec{v} - \vec{v}\vec{u} = \vec{v}\vec{u} - \vec{u}\vec{v}$$

i.e., iff

$$2\vec{u}\vec{v} = 2\vec{v}\vec{u}$$

for all $\vec{u}, \vec{v} \in A^*$. If $\mathbb{F}$ has characteristic 2, this is automatic; otherwise, $Q(\mathfrak{a})$ is involutory iff $\vec{u}\vec{v} = \vec{v}\vec{u}$ for all $\vec{u}, \vec{v} \in A^*$. □

Now, if L is a knot or link and $Q \subset Q(\mathfrak{a})$ is a finite subquandle of the quandle of units of a Lie algebra $\mathfrak{a}$, then the set of quandle homomorphisms $\mathrm{Hom}(Q(L), Q)$ is a finite set which is unchanged by Reidemeister moves. In particular, we can use the structure of $\mathfrak{a}$ to obtain invariant signatures of the quandle labelings of L by Q to enhance the quandle counting invariant.

DEFINITION 2. Let L be a link, $\mathfrak{a}$ a finite dimensional Lie algebra and $Q \subset Q(\mathfrak{a})$ a finite subquandle of the quandle of units $Q(\mathfrak{a})$. Then the multiset

$$\Phi^M_{Q,\mathfrak{a}}(L) = \{I(\mathrm{Im}(f)) : f \in \mathrm{Hom}(Q(K), Q)\}$$

of Lie ideals $I(\mathrm{Im}(f))$ generated by the image subquandles $\mathrm{Im}(f)$ for $f \in \mathrm{Hom}(Q(K), Q)$ is the *Lie Ideal Enhancement* of the quandle counting invariant. The polynomial

$$\Phi_{Q,\mathfrak{a}}(L) = \begin{cases} \displaystyle\sum_{f \in \mathrm{Hom}(Q(L),Q)} u^{|I(\mathrm{Im}(f))|} & \mathfrak{a} \text{ finite,} \\ \displaystyle\sum_{f \in \mathrm{Hom}(Q(L),Q)} u^{\mathrm{rank}(I(\mathrm{Im}(f)))} & \mathfrak{a} \text{ infinite} \end{cases}$$

is the *Lie Ideal Enhancement polynomial* of L with respect to the the subquandle $Q \subset Q(\mathfrak{a})$ of the quandle of units of the Lie algebra $\mathfrak{a}$. If $Q = Q(\mathfrak{a})$, we will denote $\Phi_{Q,\mathfrak{a}}$ simply as $\Phi_{\mathfrak{a}}$.

By construction, we have the following theorem:

Theorem 4. *For any Lie algebra* $\mathfrak{a}$, *the multiset* $\Phi^I_{Q,\mathfrak{a}}(L)$ *and polynomial* $\Phi_{Q,\mathfrak{a}}(L)$ *are link invariants.*

We can also define enhancements using the associative ideals $AI(\mathrm{Im}(f))$ in the enveloping algebra A

$$\Phi^M_{Q,A}(L) = \{AI(\mathrm{Im}(f))\colon\ f \in \mathrm{Hom}(Q(K), Q)\}$$

and

$$\Phi_{Q,A}(L) = \sum_{f\in\mathrm{Hom}(Q(L),Q)} u^{|AI(\mathrm{Im}(f))|}$$

and combine these to get a two-variable polynomial

$$\Phi_{Q,A,\mathfrak{a}}(L) = \sum_{f\in\mathrm{Hom}(Q(L),Q)} u^{|I(\mathrm{Im}(f))|}v^{|AI(\mathrm{Im}(f))|}.$$

REMARK 1. These invariants are defined for virtual knots as well as classical knots via the usual method of ignoring virtual crossings; see [5] for more about virtual knot invariants defined from quandles.

4. Computations and Examples

In this section we collect a few examples of LIE invariants and their computation. Our computations use custom `Python` code available for download from the second author's website at `www.esotericka.org`.

EXAMPLE 10. If $A = \mathbb{Z}[G]$ where G is an abelian group, then $Q(\mathfrak{a})$ is a trivial quandle, i.e. $x \triangleright y = x$ for all $x, y \in Q(\mathfrak{a})$. If L is a link of c components, then each component is monochromatic in any quandle coloring, and every coloring assigning the same fixed colors to the arcs comprising a component is valid. Hence, there are $|Q(\mathfrak{a})|^c$ colorings of L by such a quandle. Since the Lie bracket is zero for G abelian, the Lie ideals are just the subspaces generated by the quandle elements. Such a coloring using k distinct colors will then generate a Lie ideal of dimension k. Let $n = |Q(\mathfrak{a})|$; then for each $k \in \{1, 2, \dots, c\}$ there are $(n)_k S(c, k)$ such colorings where $(n)_k = n(n-1)\cdots(n-k+1)$ is the kth falling factorial of n and $S(c, k)$ is the Stirling number of the second kind, i.e. the number of ways of partitioning a set of cardinality c into k nonempty subsets. For instance, if $\mathfrak{a} = \mathbb{Z}_3[C_4]$ where $C_4 = \{1, x, x^2, x^3\colon x^4 = 1\}$ then $Q(\mathfrak{a})$ is the trivial quandle of $n = 4$ elements; if L is a link of $c = 3$ components, then $S(3, 1) = 1$, $S(3, 2) = 3$ and $S(3, 3) = 1$, and we have

$$\Phi_{\mathfrak{a}}(L) = (4)_1S(3, 1)u^3 + (4)_2S(3, 2)u^9 + (4)_3S(3, 3)u^{27} = 4u^3 + 36u^9 + 24u^{27}.$$

In particular, the previous example implies the following:

Theorem 5. *Let L be a link of c components and $\mathfrak{a}$ an abelian Lie algebra over a field $\mathbb{F}$ with trivial quandle of units $Q(\mathfrak{a})$ of cardinality n. Then the LIE polynomial* *is given by*

$$\Phi_{\mathfrak{a}}(L) = \sum_{k=1}^{c} (|Q(\mathfrak{a})|)_k S(c,k) u^{p^k}$$

if $|\mathbb{F}| = p$ *or*

$$\Phi_{\mathfrak{a}}(L) = \sum_{k=1}^{c} (|Q(\mathfrak{a})|)_k S(c,k) u^{k}$$

if $|\mathbb{F}|$ *is not finite.*

Thus, to get nontrivial invariant values, we must (unsurprisingly) focus on non-abelian Lie algebras.

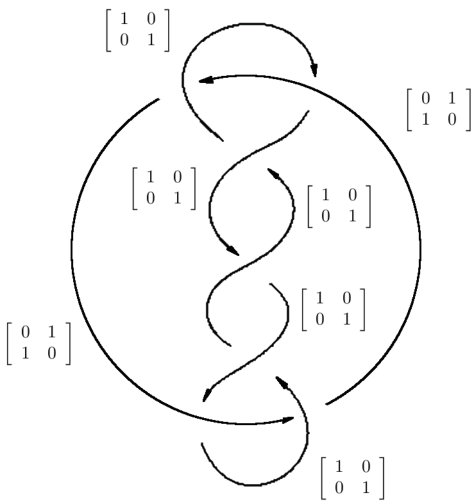
EXAMPLE 11. Let $\mathfrak{a} = M_2(\mathbb{Z}_2)$, the Lie algebra of 2×2 matrices with entries in $\mathbb{Z}_2$. Then the quandle of units of $\mathfrak{a}$ is isomorphic to the conjugation quandle of $S_3 = \langle \alpha, \beta : \alpha^2 = \beta^3 = 1, (\alpha\beta)^2 = 1 \rangle$ with

$$\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\beta = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The link $L7a4$ has 30 quandle labellings by $Q(\mathfrak{a})$ including for example



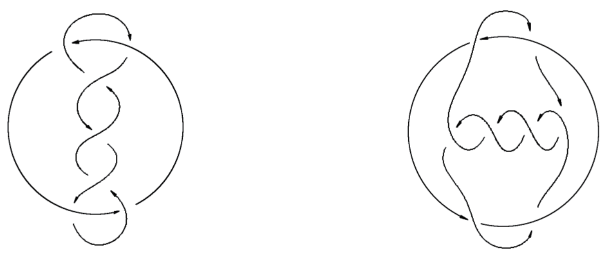
The subquandle of $Q(\mathfrak{a})$ generated by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a trivial quandle of just those two elements; the Lie ideal they generate consists of the four matrices

$$\left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}$$

so this quandle coloring contributes u^4 to $\Phi_{\mathfrak{a}}(L7a4)$.

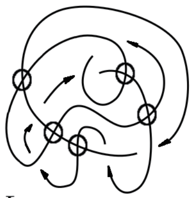
REMARK 2. In the previous example, the constant coloring of the link by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ generates a four-element Lie ideal, but spans only a 2 element subspace of $\mathfrak{a}$ considered as a symplectic vector space. Thus, this constant coloring contributes u^4 to the LIE invariant and u^2 to the symplectic quandle enhancement, so we expect the symplectic enhancement and LIE invariant to contain different information. Indeed, the following examples show that the LIE invariant is not determined by the quandle counting invariant, the image enhancement invariant or the symplectic quandle enhancement (in case the quandle of units is symplectic).

EXAMPLE 12. The links $L7a4$ and $L7n1$ both have counting invariant value 30 with respect to the quandle of units $Q(\mathfrak{a})$ from the previous example but have distinct Lie ideal enhanced polynomials of $\Phi_{\mathfrak{a}}(L7a4) = 20u^{16} + 9u^4 + u^2$ and $\Phi_{\mathfrak{a}}(L7n1) = 14u^{16} + 6u^8 + 9u^4 + u^2$. In particular, this example shows that $\Phi_{\mathfrak{a}}$ is a proper enhancement of the counting invariant.

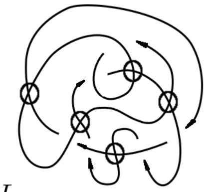


$\Phi_{\mathfrak{a}}(L7a4) = 20u^{16} + 9u^4 + u^2$. $\quad \Phi_{\mathfrak{a}}(L7n1) = 14u^{16} + 6u^8 + 9u^4 + u^2$.

EXAMPLE 13. Example 12 shows that the LIE invariant is not determined by the quandle counting invariant; however, the two links listed are also distinguished by the image enhancement invariant, with $\Phi_{\mathfrak{a}}^{\mathrm{Im}}(L7a4) = 12u^5 + 12u^2 + 6u$ and $\Phi_{\mathfrak{a}}^{\mathrm{Im}}(L7n1) = 6u^5 + 6u^4 + 12u^2 + 6u$. The virtual links L_1 and L_2 below have the same quandle counting invariant and image enhancement values for the two-fold conjugation quandle of S^3 but are distinguished by the LIE invariant for $\mathfrak{a} = \mathbb{Z}_2[S_3]$ and $Q = Q_2(\mathfrak{a})$:



L_1



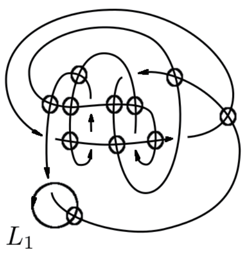
L_2

$$\begin{aligned}\Phi^{\mathbb{Z}}_{Q_2(\mathfrak{a})}(L_1) &= 126,\\ \Phi^{\mathrm{Im}}_{Q_2(\mathfrak{a})}(L_1) &= 24u^5 + 12u^4 + 30u^3 + 54u^2 + 6u,\\ \Phi_{Q_2(\mathfrak{a}),\mathfrak{a}}(L_1) &= 12u^{64} + 60u^{32} + 50u^{16} + 3u^8 + u^2.\end{aligned}$$

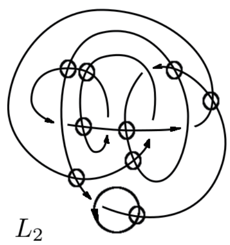
$$\begin{aligned}\Phi^{\mathbb{Z}}_{Q_2(\mathfrak{a})}(L_2) &= 126,\\ \Phi^{\mathrm{Im}}_{Q_2(\mathfrak{a})}(L_2) &= 24u^5 + 12u^4 + 30u^3 + 54u^2 + 6u,\\ \Phi_{Q_2(\mathfrak{a}),\mathfrak{a}}(L_2) &= 18u^{64} + 54u^{32} + 50u^{16} + 3u^8 + u^2.\end{aligned}$$

Hence, the LIE invariant is not determined by the image enhancement or the quandle counting invariant.

EXAMPLE 14. Let $\mathfrak{a} = M_2(\mathbb{Z}_2)$. Then the virtual links below are not distinguished by the quandle counting invariant, image enhancement or symplectic quandle enhancement invariant, but are distinguished by the LIE invariant:



L_1



L_2

$$\begin{aligned}\Phi^{\mathbb{Z}}_{Q(\mathfrak{a})}(L_1) &= 108,\\ \Phi^{\mathrm{Im}}_{Q(\mathfrak{a})}(L_1) &= 12u^6 + 30u^5 + 12u^4 + 12u^3 + 36u^2 + 6u,\\ \Phi^{\mathrm{Symp}}_{Q(\mathfrak{a})}(L_1) &= 30u^{16} + 24u^8 + 48u^4 + 6u^2,\\ \Phi_{Q(\mathfrak{a}),\mathfrak{a}}(L_1) &= 74u^{16} + 12u^8 + 21u^4 + u^2.\end{aligned}$$

$$\begin{aligned}\Phi^{\mathbb{Z}}_{Q(\mathfrak{a})}(L_2) &= 108,\\ \Phi^{\mathrm{Im}}_{Q(\mathfrak{a})}(L_2) &= 12u^6 + 30u^5 + 12u^4 + 12u^3 + 36u^2 + 6u,\\ \Phi^{\mathrm{Symp}}_{Q(\mathfrak{a})}(L_2) &= 30u^{16} + 24u^8 + 48u^4 + 6u^2,\\ \Phi_{Q(\mathfrak{a}),\mathfrak{a}}(L_2) &= 68u^{16} + 18u^8 + 21u^4 + u^2.\end{aligned}$$

EXAMPLE 15. For our final example we chose a quandle embedded in a relatively small Lie algebra for speed of computation and computed the LIE polynomial invariant for all prime classical links with up to seven crossings. The results are collected in the table.

$$M_Q = \begin{bmatrix} 1 & 3 & 1 & 3 & 3 \\ 4 & 2 & 5 & 5 & 4 \\ 3 & 1 & 3 & 1 & 1 \\ 5 & 5 & 2 & 4 & 2 \\ 2 & 4 & 4 & 2 & 5 \end{bmatrix}, \quad \mathfrak{a} = \mathbb{Z}_2[S_3].$$

$\Phi_{Q,\mathfrak{a}}(L)$	L
$4u^{16} + 3u^8$	$L2a1$, $L6a2$, $L7a6$
$8u^{16} + 3u^8$	$L6a4$
$10u^{16} + 3u^8$	$L6a3$, $L7a5$
$6u^{32} + 4u^{16} + 3u^8$	$L7a2$, $L7a3$, $L7n1$, $L7n2$
$12u^{32} + 4u^{16} + 3u^8$	$L4a1$, $L5a1$, $L7a4$
$12u^{32} + 10u^{16} + 3u^8$	$L6a1$, $L7a1$
$18u^{32} + 8u^{16} + 3u^8$	$L6n1$, $L7a7$
$18u^{32} + 14u^{16} + 3u^8$	$L6a5$

5. Questions for future work

We close with a few questions and direction for future research.

In [2], it is noted that the exponential map satisfies the augmented rack identity, giving a Lie algebra the structure of an augmented rack with augmentation group the associated Lie group. What are sufficient conditions for the resulting rack to be a finite quandle? In the case of infinite Lie quandles, can Lie ideals be used to enhance the topological version of the counting invariant defined in [9] or to define power series-valued counting invariants and enhancements? What are some other quandle structures on Lie algebras?

References

[1] J.S. Carter, D. Jelsovsky, S. Kamada, L. Langford, and M. Saito: *Quandle cohomology and state-sum invariants of knotted curves and surfaces*, Trans. Amer. Math. Soc. **355** (2003), 3947–3989.

[2] R. Fenn and C. Rourke: *Racks and links in codimension two*, J. Knot Theory Ramifications **1** (1992), 343–406.

[3] M. Goussarov, M. Polyak and O. Viro: *Finite-type invariants of classical and virtual knots*, Topology **39** (2000), 1045–1068.

[4] D. Joyce: *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra **23** (1982), 37–65.

[5] L.H. Kauffman: *Virtual knot theory*, European J. Combin. **20** (1999), 663–690.

[6] S.V. Matveev: *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) **119** (**161**) (1982), 78–88, 160.

[7] E.A. Navas and S. Nelson: *On symplectic quandles*, Osaka J. Math. **45** (2008), 973–985.

[8] S. Nelson: *The combinatorial revolution in knot theory*, Notices Amer. Math. Soc. **58** (2011), 1553–1561.

[9] R.L. Rubinsztein: *Topological quandles and invariants of links*, J. Knot Theory Ramifications **16** (2007), 789–808.

Gillian Roxanne Grindstaff
e-mail: grg02010@mymail.pomona.edu

Sam Nelson
Department of Mathematical Sciences
Claremont McKenna College
850 Columbia Ave.
Claremont, CA 91767
U.S.A.
e-mail: Sam.Nelson@cmc.edu