Inaba, K.
Osaka J. Math.
53 (2016), 813–842

ON DEFORMATIONS OF ISOLATED SINGULARITIES OF POLAR WEIGHTED HOMOGENEOUS MIXED POLYNOMIALS

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(Received September 30, 2014, revised August 20, 2015)

Abstract

In the present paper, we deform isolated singularities of $f\bar{g}$, where f and g are 2-variable weighted homogeneous complex polynomials, and show that there exists a deformation of $f\bar{g}$ which has only indefinite fold singularities and mixed Morse singularities.

1. Introduction

Let $f(\mathbf{z})$ be a complex polynomial of variables $\mathbf{z}=(z_1,\dots,z_n)$. A *deformation of* $f(\mathbf{z})$ is a polynomial mapping $F_t\colon \mathbb{C}^n\times\mathbb{R}\to\mathbb{C}$, $(\mathbf{z},t)\mapsto F_t(\mathbf{z})$, with $F_0(\mathbf{z})=f(\mathbf{z})$. Assume that the origin $\mathbf{o}$ is an isolated singularity of $f(\mathbf{z})$. For complex singularities, it is known that there exist a neighborhood U of the origin and a deformation F_t of $f(\mathbf{z})$ such that $F_t(\mathbf{z})$ is a complex polynomial and any singularity of $F_t(\mathbf{z})$ is a Morse singularity in U for any $0<t\ll 1$ [9, Chapter 4]. Here *a Morse singularity* is the singularity of the polynomial map $f(\mathbf{z})=z_1^2+\cdots+z_n^2$ at the origin. Let $\rho_1(\mathbf{x},\mathbf{y})$ and $\rho_2(\mathbf{x},\mathbf{y})$ be real polynomial maps from $\mathbb{R}^{2n}$ to $\mathbb{R}$ of variables $\mathbf{x}=(x_1,\dots,x_n)$ and $\mathbf{y}=(y_1,\dots,y_n)$. Then these real polynomials define a polynomial of variables $\mathbf{z}=(z_1,\dots,z_n)$ and $\bar{\mathbf{z}}=(\bar{z}_1,\dots,\bar{z}_n)$ as

$$P(\mathbf{z},\bar{\mathbf{z}}):=\rho_1\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2},\frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right)+i\rho_2\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2},\frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right),$$

where $z_j=x_j+iy_j$ $(j=1,\dots,n)$. The polynomial $P(\mathbf{z},\bar{\mathbf{z}})$ is called *a mixed polynomial*. M. Oka introduced the terminology of mixed polynomials and proposed a wide class of mixed polynomials which admit Milnor fibrations, see for instance [11, 12].

Let $\mathbf{w}$ be an isolated singularity of a mixed polynomial $P(\mathbf{z},\bar{\mathbf{z}})$, $c=P(\mathbf{w},\bar{\mathbf{w}})$ and $S_{\mathbf{w}}^{2n-1}$ be the $(2n-1)$-dimensional sphere centered at $\mathbf{w}$. If the link $P^{-1}(c)\cap S_{\mathbf{w}}^{2n-1}$ is isotopic to the link defined by a complex Morse singularity as an oriented link, we say that $\mathbf{w}$ is *a mixed Morse singularity*. In [6, Theorem 1], [7, Corollary 1, 2], there exist isolated singularities of mixed polynomials whose homotopy types of the vector fields introduced in [10] are different from those of complex polynomials. Thus there

2010 Mathematics Subject Classification. Primary 58K60; Secondary 58K15, 37P05.

exist isolated singularities of real polynomial maps which cannot deform mixed Morse singularities.

Let $C^\infty(X, Y)$ be the set of smooth maps from X to Y, where X is a $2n$-dimensional manifold and Y is a 2-dimensional manifold. It is known that the subset of smooth maps from X to Y which have only definite fold singularities, indefinite fold singularities or cusps is open and dense in $C^\infty(X, Y)$ topologized with the C^∞-topology. Moreover definite fold singularities and cusps can be eliminated by homotopy under some conditions [8, Theorem 1, 2], [15, Theorem 2.6]. In $\dim X = 4$, the fibration with only indefinite fold singularities and Morse singularities is called *a broken Lefschetz fibration*, which is recently studied in several papers, see for instance [2, Theorem 1.1], [5, Theorem 1.1], (cf. [1, Theorem 1]). We are interested in making deformations of singularities with only indefinite fold singularities and mixed Morse singularities. This deformed map can be topologically regarded as a broken Lefschetz fibration. If any singularity of a C^∞-map $f\colon X \to Y$ is an indefinite fold singularity or a mixed Morse singularity, we call f *a mixed broken Lefschetz fibration*.

To know if a smooth map $f\colon X \to Y$ has only fold singularities or cusps, we observe f in the bundle of r-jets. We introduce the bundle $J^r(X, Y)$ of r-jets and its submanifolds $S_k(X, Y)$ and $S_1^2(X, Y)$ for $k = 1, 2$. Let $j^r f(p)$ be the r-jet of f at p and set

$$J^r(X, Y) := \bigcup_{(p,q)\in X\times Y} J^r(X, Y, p, q),$$

where $J^r(X, Y, p, q) = \{j^r f(p) \mid f(p) = q\}$. The set $J^r(X, Y)$ is called *the bundle of r-jets of maps from X into Y*. It is known that $J^r(X, Y)$ is a smooth manifold. The *r-extension* $j^r f\colon X \to J^r(X, Y)$ of f is defined by $p \mapsto j^r f(p)$ where $p \in X$. The 1-jet space is the $(6n + 2)$-dimensional smooth manifold and the 1-extension $j^1 f$ of f is a smooth map. We define a codimension $(2n - 2 + k)k$-submanifold of $J^1(X, Y)$ for $k = 1, 2$ as follows:

$$S_k(X, Y) = \{j^1 f(p) \in J^1(X, Y) \mid \operatorname{rank} df_p = 2 - k\}.$$

A smooth map $f\colon X \to Y$ is said to be *generic* if f satisfies the following conditions:
(1) $j^1 f$ is transversal to $S_1(X, Y)$ and $S_2(X, Y)$,
(2) $j^2 f$ is transversal to $S_1^2(X, Y)$,
where $S_1(f) = \{p \in X \mid \operatorname{rank} df_p = 1\}$, $S_1^2(f) = S_1(f \mid S_1(f))$ and $S_1^2(X, Y)$ is defined as follows:

$$S_1^2(X, Y) = \left\{ j^2 f(p) \in J^2(X, Y) \;\middle|\; \begin{array}{l} j^1 f(p) \in S_1(X, Y), \\ j^1 f \text{ is transversal to } S_1(X, Y) \text{ at } p, \\ \operatorname{rank} d(f \mid S_1(f))(p) = 0 \end{array} \right\}.$$

It is well-known that a smooth map $f\colon X \to Y$ is generic if and only if each singularity of f is either a fold singularity or a cusp. Here *a fold singularity* is the singularity of $(x_1, \ldots, x_{2n}) \mapsto \bigl(x_1, \sum_{j=2}^{2n} \pm x_j^2\bigr)$ and *a cusp* is the singularity of $(x_1, \ldots, x_{2n}) \mapsto \bigl(x_1, \sum_{j=3}^{2n} \pm x_j^2 + x_1x_2 + x_2^3\bigr)$, where $(x_1, \ldots, x_{2n})$ are the coordinates centered at the singularity. If the coefficients of x_j for $j = 2,\ldots,2n$ is either all positive or all negative, we say that x is a *definite fold* singularity, otherwise it is an *indefinite fold* singularity.

Now we state the main theorems. Let $f(\mathbf{z})$ and $g(\mathbf{z})$ be weighted homogeneous complex polynomials. Assume $f(\mathbf{z})$ and $g(\mathbf{z})$ have same weights. Then $f(\mathbf{z})\overline{g(\mathbf{z})}$ satisfies

$$f(c \circ \mathbf{z})\overline{g(c \circ \mathbf{z})} = c^{pq(m-n)} f(\mathbf{z})\overline{g(\mathbf{z})},$$

where $c \circ \mathbf{z} = (c^q z_1, c^p z_2)$, $c \in \mathbb{C}^*$ and pqm and pqn are the degrees of the $\mathbb{C}^*$-action of $f(\mathbf{z})$ and $g(\mathbf{z})$ respectively. Then we have the Euler equality:

$$(pqm)f(\mathbf{z}) = qz_1\frac{\partial f}{\partial z_1} + pz_2\frac{\partial f}{\partial z_2}, \quad (pqn)g(\mathbf{z}) = qz_1\frac{\partial g}{\partial z_1} + pz_2\frac{\partial g}{\partial z_2}.$$

The mixed polynomial $f(\mathbf{z})\bar{g}(\mathbf{z})$ is *a polar and radial weighted homogeneous mixed polynomial*, see Section 2.2 for the definitions. Polynomials of this type admit Milnor fibrations [14, 3, 13, 11, 12].

We study singularities appearing in a deformation $\{F_t\}$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ for any $0 < t \ll 1$. The main theorem is the following.

Theorem 1. *Let $f(\mathbf{z})$ and $g(\mathbf{z})$ be 2-variable convenient weighted homogeneous complex polynomials such that $f(\mathbf{z})\bar{g}(\mathbf{z})$ has an isolated singularity at $\mathbf{o}$ and U be a sufficiently small neighborhood of $\mathbf{o}$. Assume that $f(\mathbf{z})$ and $g(\mathbf{z})$ have same weights and the degree of the $\mathbb{C}^*$-action of $f(\mathbf{z})$ is greater than that of $g(\mathbf{z})$. Then there exists a deformation $F_t(\mathbf{z})$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ such that any singularity of $F_t(\mathbf{z})$ in $U \setminus \{\mathbf{o}\}$ is an indefinite fold singularity, $F_t(\mathbf{o}) = 0$ and the link $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ is a $(p(m-n), q(m-n))$-torus link, where $S^3_{\varepsilon_t} \subset U$ is a sufficiently small 3-sphere centered at $\mathbf{o}$ for any $0 < t \ll 1$.*

As an application of Theorem 1, we show that there exists a deformation into mixed broken Lefschetz fibrations.

Theorem 2. *Let $F_t(\mathbf{z})$ be a deformation of $f(\mathbf{z})\bar{g}(\mathbf{z})$ in* Theorem 1. *Then there exists a deformation $F_{t,s}(\mathbf{z})$ of $F_t(\mathbf{z})$ such that $F_{t,s}(\mathbf{z})$ is a mixed broken Lefschetz fibration on U where $0 < s \ll t \ll 1$.*

This paper is organized as follows. In Section 2 we introduce the definition of higher differentials of smooth maps, define mixed Hessian $H(P)$ of a mixed polynomial $P(\mathbf{z}, \bar{\mathbf{z}})$ and show properties of mixed Hessians to study singularities of mixed polynomials. In Sections 3 and 4 we prove Theorems 1 and 2 respectively.

2. Preliminaries

2.1. Higher differentials. In this subsection, we assume that X is an n-dimensional manifold and Y is a 2-dimensional manifold. Let $f\colon X \to Y$ be a smooth map and $df : T(X) \to T(Y)$ be the induced map of f, where $T(X)$ and $T(Y)$ are the tangent bundles of X and Y respectively. If $\tilde{X}$ is a bundle over X and $\Upsilon : \tilde{X} \to W$ is a map from $\tilde{X}$ to a space W, we denote by X_x and $\Upsilon_x = \Upsilon|_{X_x}$ the fiber over $x \in X$ and the restriction map of Υ to X_x respectively. We set the subset $S_k(f)$ of X as

$$S_k(f) = \{x \in X \mid \text{the rank of } df_x = 2 - k\} \quad (k = 0, 1, 2).$$

Note that $S_0(f)$ is the set of regular points of f and $S(f) = S_1(f) \cup S_2(f)$ is the set of singularities of f.

The following notations are introduced in [8, Section 2]. Let U and V be small neighborhoods of $x \in X$ and $f(x) \in Y$ such that $f(U) \subset V$. Since $T(X)|U$ and $T(Y)|V$ are trivial bundles, we can choose bases $\{u_i\}$ and $\{v_j\}$ of the sections of these restricted bundles such that

$$\langle u_i(x), u_k^*(x)\rangle = \delta_{i,k} \quad \text{for all} \quad x \in U,$$
$$\langle v_i(y), v_k^*(y)\rangle = \delta_{i,k} \quad \text{for all} \quad y \in V,$$

where $\langle\ ,\ \rangle$ denotes the pairing of a vector space with its dual, $\{u_i^*\}$ and $\{v_j^*\}$ are dual bases of $\{u_i\}$ and $\{v_j\}$ respectively.

Choose coordinates $\{\xi_i\}$ in U and $\{\eta_j\}$ in V such that $\partial/\partial\xi_i = u_i$, $d\xi_i = u_i^*$, $(\partial/\partial\eta_j) = v_j$ and $d\eta_j = v_j^*$. Then df can be represented by

$$df = \sum_{i,j} \frac{\partial(\eta_j \circ f)}{\partial\xi_i}\, d\xi_i \otimes v_j.$$

Set $E = T(X)|S_1(f)$ and $F = T(Y)|f(S_1(f))$. Then we can define the following exact sequence

$$0 \to L \to E \xrightarrow{df} F \xrightarrow{\pi_1} G \to 0,$$

where $L = \ker df$, $G = \operatorname{coker} df$ and π is the linear map such that $\operatorname{Im}\pi = \operatorname{coker} df$.

Let $k \in X_x$, $t \in L_x$ and $a_{i,j} = \partial(\eta_j \circ f)/\partial\xi_i$. We define the map $\varphi^1\colon E \to L^* \otimes F$ by

$$\begin{aligned}\varphi_x^1(k, t) &= \sum_{i,j} (\langle k, da_{i,j}(x)\rangle \langle t, u_i^*(x)\rangle) v_j(x)\\ &= \sum_{i,j,m} \left(\langle k, d\xi_m(x)\rangle \frac{\partial^2(\eta_j \circ f)}{\partial\xi_i\partial\xi_m} \langle t, d\xi_i(x)\rangle \right) v_j(x)\end{aligned}$$

and then define the map $d^2 f : E \to L^* \otimes G$ by

$$d^2 f_x(k)(t) = \pi_1(\varphi_x^1(k)(t)).$$

By choosing bases of L_x, X_x and G_x, the map $d^2 f$ determines a $n\times(n-1)$ matrix ϕ. $j^1 f$ is transversal to $S_1(X,Y)$ at $S_1(f)$ if and only if the rank of ϕ is equal to $n-1$. Moreover the singularity $x \in S_1(f)$ is a fold singularity if and only if the rank of ϕ is equal to $n-1$ and the dimension of the kernel of $d^2 f_x$ restricted to L_x is equal to 0 [8].

2.2. Polar weighted homogeneous mixed polynomials. Let $P(\mathbf{z}, \bar{\mathbf{z}})$ be a polynomial of variables $\mathbf{z} = (z_1, \dots, z_n)$ and $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$ given as

$$P(\mathbf{z}, \bar{\mathbf{z}}) := \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu,$$

where $\mathbf{z}^\nu = z_1^{\nu_1}\cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1}\cdots\bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$). $\bar{z}_j$ represents the complex conjugate of z_j. A polynomial $P(\mathbf{z},\bar{\mathbf{z}})$ of this form is called a *mixed polynomial* [11, 12]. If $P((0,\dots,0,z_j,0,\dots,0),(0,\dots,0,\bar{z}_j,0,\dots,0))$ is non-zero for each $j = 1, \dots, n$, then we say that $P(\mathbf{z}, \bar{\mathbf{z}})$ is *convenient*. A point $\mathbf{w} \in \mathbb{C}^n$ is *a singularity of* $P(\mathbf{z}, \bar{\mathbf{z}})$ if the gradient vectors of $\Re P$ and $\Im P$ are linearly dependent at $\mathbf{w}$. A singularity $\mathbf{w}$ of $P(\mathbf{z}, \bar{\mathbf{z}})$ has the following property.

Proposition 1 ([11] Proposition 1)**.** *The following conditions are equivalent*:
(1) $\mathbf{w}$ *is a singularity of* $P(\mathbf{z}, \bar{\mathbf{z}})$.
(2) *There exists a complex number* α *with* $|\alpha| = 1$ *such that*

$$\left(\overline{\frac{\partial P}{\partial z_1}}(\mathbf{w}), \dots, \overline{\frac{\partial P}{\partial z_n}}(\mathbf{w})\right) = \alpha\left(\frac{\partial P}{\partial \bar{z}_1}(\mathbf{w}), \dots, \frac{\partial P}{\partial \bar{z}_n}(\mathbf{w})\right).$$

Let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be integers such that $\gcd(p_1, \dots, p_n) = 1$. We define the S^1-action and the $\mathbb{R}^*$-action on $\mathbb{C}^n$ as follows:

$$s \circ \mathbf{z} = (s^{p_1}z_1, \dots, s^{p_n}z_n), \quad s \in S^1,$$
$$r \circ \mathbf{z} = (r^{q_1}z_1, \dots, r^{q_n}z_n), \quad r \in \mathbb{R}^*.$$

If there exist positive integers d_p and d_r such that the mixed polynomial $P(\mathbf{z},\bar{\mathbf{z}})$ satisfies

$$P(s^{p_1}z_1, \dots, s^{p_n}z_n, \bar{s}^{p_1}\bar{z_1}, \dots, \bar{s}^{p_1}\bar{z_n}) = s^{d_p}P(\mathbf{z}, \bar{\mathbf{z}}), \quad s \in S^1,$$
$$P(r^{q_1}z_1, \dots, r^{q_n}z_n, r^{q_1}\bar{z_1}, \dots, r^{q_n}\bar{z_n}) = r^{d_r}P(\mathbf{z}, \bar{\mathbf{z}}), \quad r \in \mathbb{R}^*,$$

we say that $P(\mathbf{z}, \bar{\mathbf{z}})$ is *a polar and radial weighted homogeneous mixed polynomial*. If a polar and radial weighted homogeneous mixed polynomial $P(\mathbf{z}, \bar{\mathbf{z}})$ is a complex polynomial, we call $P(\mathbf{z}, \bar{\mathbf{z}})$ *a weighted homogeneous complex polynomial*. Polar and radial weighted homogeneous mixed polynomials admit Milnor fibrations, see for instance [14, 3, 11, 12]. Suppose that $P(\mathbf{z},\bar{\mathbf{z}})$ is a polar and radial weighted homogeneous mixed polynomial. Then we have

$$d_p P(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{n} p_j \left(\frac{\partial P}{\partial z_j} z_j - \frac{\partial P}{\partial \bar{z}_j} \bar{z}_j \right),$$
$$d_r P(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{n} q_j \left(\frac{\partial P}{\partial z_j} z_j + \frac{\partial P}{\partial \bar{z}_j} \bar{z}_j \right).$$

If $p_j = q_j$ for $j = 1, \dots, n$, the above equations give:

$$\sum_{j=1}^{n} p_j \frac{\partial P}{\partial z_j} z_j = \frac{d_p + d_r}{2} P(\mathbf{z}, \bar{\mathbf{z}}). \tag{1}$$

The following claim says that the singularities of $P(\mathbf{z},\bar{\mathbf{z}})$ are orbits of the S^1-action.

Proposition 2. *Let $P(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous mixed polynomial. If $\mathbf{w}$ is a singularity of $P(\mathbf{z}, \bar{\mathbf{z}})$, $s \circ \mathbf{w}$ is also a singularity of $P(\mathbf{z}, \bar{\mathbf{z}})$, where $s \in S^1$.*

Proof. Let $\mathbf{w}$ be a singularity of a polar weighted homogeneous mixed polynomial $P(\mathbf{z}, \bar{\mathbf{z}})$. Then there exists $\alpha \in S^1$ such that

$$\left(\overline{\frac{\partial P}{\partial z_1}}(\mathbf{w}), \dots, \overline{\frac{\partial P}{\partial z_n}}(\mathbf{w}) \right) = \alpha \left(\frac{\partial P}{\partial \bar{z}_1}(\mathbf{w}), \dots, \frac{\partial P}{\partial \bar{z}_n}(\mathbf{w}) \right).$$

Since $P(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous mixed polynomial, $\partial P/\partial z_j$ and $\partial P/\partial \bar{z}_j$ are also. Then we have

$$\frac{\partial P}{\partial z_j}(s \circ \mathbf{w}) = s^{d_p - p_j} \frac{\partial P}{\partial z_j}(\mathbf{w}), \quad \frac{\partial P}{\partial \bar{z}_j}(s \circ \mathbf{w}) = s^{d_p + p_j} \frac{\partial P}{\partial \bar{z}_j}(\mathbf{w}),$$

where $j = 1, \dots, n$ and $s \in S^1$. So the above equations lead to the following equation:

$$\left(\overline{\frac{\partial P}{\partial z_1}}(s \circ \mathbf{w}), \dots, \overline{\frac{\partial P}{\partial z_n}}(s \circ \mathbf{w}) \right) = (s^{-2d_p}\alpha) \left(\frac{\partial P}{\partial \bar{z}_1}(s \circ \mathbf{w}), \dots, \frac{\partial P}{\partial \bar{z}_n}(s \circ \mathbf{w}) \right).$$

Since $|s^{-2d_p}\alpha| = 1$, by Proposition 1, $s \circ \mathbf{w}$ is also a singularity of $P(\mathbf{z}, \bar{\mathbf{z}})$. □

2.3. Mixed Hessians. To study a necessary condition for $P(\mathbf{z}, \bar{\mathbf{z}})$ so that the rank of the representation matrix of $d^2 P$ is equal to $n-1$, we define the matrix $H(P)$ as follows:

$$H(P) := \begin{pmatrix} \left(\dfrac{\partial^2 P}{\partial z_j \partial z_k} \right) & \left(\dfrac{\partial^2 P}{\partial z_j \partial \bar{z}_k} \right) \\ \left(\dfrac{\partial^2 P}{\partial \bar{z}_j \partial z_k} \right) & \left(\dfrac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k} \right) \end{pmatrix},$$

where $P(\mathbf{z}, \bar{\mathbf{z}})$ is a mixed polynomial. We call the matrix $H(P)$ the *mixed Hessian* of $P(\mathbf{z}, \bar{\mathbf{z}})$ and show some properties of $H(P)$ to study singularities of $P(\mathbf{z}, \bar{\mathbf{z}})$.

The next lemma is useful to understand the mixed Hessian of $P(\mathbf{z}, \bar{\mathbf{z}})$.

Lemma 1. *Let A and B be $n \times n$ real matrices such that* $\det(A + iB) \neq 0$. *Then there exists a real number u_0 such that* $\det(A + u_0 B) \neq 0$.

Proof. Let u be a complex variable. If B is the zero matrix, then $\det(A + uB) = \det(A + iB) \neq 0$. Suppose that B is not the zero matrix. By the assumption, $\det(A + uB)$ is not identically zero. Since $\det(A + uB)$ is a polynomial of degree at most n, the equation $\det(A + uB) = 0$ has finitely many roots. Thus there exists a real number u_0 which is not a root of $\det(A + uB) = 0$. $\square$

Let $H_{\mathbb{R}}(\eta)$ denote the Hessian of a smooth function $\eta\colon \mathbb{R}^n \to \mathbb{R}$.

Lemma 2. *Suppose that the rank of $H(P)$ is $2n$. By changing the coordinates of $\mathbb{R}^2$ if necessary, the rank of $H_{\mathbb{R}}(\Im P)$ is $2n$. By the same argument, we can also say that by changing the coordinates of $\mathbb{R}^2$ if necessary, the rank of $H_{\mathbb{R}}(\Re P)$ is $2n$.*

Proof. Recall that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

The second differentials of complex variables can be represented as follows:

$$\begin{aligned}
\frac{\partial^2}{\partial z_j \partial z_k} &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial x_k} - \frac{\partial^2}{\partial y_j \partial y_k}\right) - \frac{i}{4}\left(\frac{\partial^2}{\partial y_j \partial x_k} + \frac{\partial^2}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2}{\partial z_j \partial \bar{z}_k} &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k}\right) - \frac{i}{4}\left(\frac{\partial^2}{\partial y_j \partial x_k} - \frac{\partial^2}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2}{\partial \bar{z}_j \partial z_k} &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_j \partial y_k}\right) + \frac{i}{4}\left(\frac{\partial^2}{\partial y_j \partial x_k} - \frac{\partial^2}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2}{\partial \bar{z}_j \partial \bar{z}_k} &= \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial x_k} - \frac{\partial^2}{\partial y_j \partial y_k}\right) + \frac{i}{4}\left(\frac{\partial^2}{\partial y_j \partial x_k} + \frac{\partial^2}{\partial x_j \partial y_k}\right).
\end{aligned}$$

So the second differentials of a mixed polynomial $P(\mathbf{z},\bar{\mathbf{z}})$ satisfy the following equations:

$$\begin{aligned}
\frac{\partial^2 P}{\partial z_j \partial z_k} &= \frac{1}{4}\left(\frac{\partial^2 \Re P}{\partial x_j \partial x_k} + \frac{\partial^2 \Im P}{\partial x_j \partial y_k} + \frac{\partial^2 \Im P}{\partial y_j \partial x_k} - \frac{\partial^2 \Re P}{\partial y_j \partial y_k}\right)\\
&\quad + \frac{i}{4}\left(-\frac{\partial^2 \Re P}{\partial y_j \partial x_k} - \frac{\partial^2 \Im P}{\partial y_j \partial y_k} + \frac{\partial^2 \Im P}{\partial x_j \partial x_k} - \frac{\partial^2 \Re P}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2 P}{\partial z_j \partial \bar{z}_k} &= \frac{1}{4}\left(\frac{\partial^2 \Re P}{\partial x_j \partial x_k} - \frac{\partial^2 \Im P}{\partial x_j \partial y_k} + \frac{\partial^2 \Im P}{\partial y_j \partial x_k} + \frac{\partial^2 \Re P}{\partial y_j \partial y_k}\right)\\
&\quad + \frac{i}{4}\left(-\frac{\partial^2 \Re P}{\partial y_j \partial x_k} + \frac{\partial^2 \Im P}{\partial y_j \partial y_k} + \frac{\partial^2 \Im P}{\partial x_j \partial x_k} + \frac{\partial^2 \Re P}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2 P}{\partial \bar{z}_j \partial z_k} &= \frac{1}{4}\left(\frac{\partial^2 \Re P}{\partial x_j \partial x_k} + \frac{\partial^2 \Im P}{\partial x_j \partial y_k} - \frac{\partial^2 \Im P}{\partial y_j \partial x_k} + \frac{\partial^2 \Re P}{\partial y_j \partial y_k}\right)\\
&\quad + \frac{i}{4}\left(\frac{\partial^2 \Re P}{\partial y_j \partial x_k} + \frac{\partial^2 \Im P}{\partial y_j \partial y_k} + \frac{\partial^2 \Im P}{\partial x_j \partial x_k} - \frac{\partial^2 \Re P}{\partial x_j \partial y_k}\right),\\
\frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k} &= \frac{1}{4}\left(\frac{\partial^2 \Re P}{\partial x_j \partial x_k} - \frac{\partial^2 \Im P}{\partial x_j \partial y_k} - \frac{\partial^2 \Im P}{\partial y_j \partial x_k} - \frac{\partial^2 \Re P}{\partial y_j \partial y_k}\right)\\
&\quad + \frac{i}{4}\left(\frac{\partial^2 \Re f}{\partial y_j \partial x_k} - \frac{\partial^2 \Im P}{\partial y_j \partial y_k} + \frac{\partial^2 \Im P}{\partial x_j \partial x_k} + \frac{\partial^2 \Re P}{\partial x_j \partial y_k}\right).
\end{aligned}$$

The above equations show that the matrix $H_{\mathbb{R}}(\Re P) + i H_{\mathbb{R}}(\Im P)$ has the form:

$$\begin{aligned}
&H_{\mathbb{R}}(\Re P) + i H_{\mathbb{R}}(\Im P)\\
&= \begin{pmatrix} \left(\frac{\partial^2 P}{\partial z_j \partial z_k} + \frac{\partial^2 P}{\partial \bar{z}_j \partial z_k} + \frac{\partial^2 P}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k}\right) & i\left(\frac{\partial^2 P}{\partial z_j \partial z_k} + \frac{\partial^2 P}{\partial \bar{z}_j \partial z_k} - \frac{\partial^2 P}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k}\right)\\ i\left(\frac{\partial^2 P}{\partial z_j \partial z_k} - \frac{\partial^2 P}{\partial \bar{z}_j \partial z_k} + \frac{\partial^2 P}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k}\right) & \left(-\frac{\partial^2 P}{\partial z_j \partial z_k} + \frac{\partial^2 P}{\partial \bar{z}_j \partial z_k} + \frac{\partial^2 P}{\partial z_j \partial \bar{z}_k} - \frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k}\right) \end{pmatrix}.
\end{aligned}$$

We see that $H_{\mathbb{R}}(\Re P) + i H_{\mathbb{R}}(\Im P)$ is congruent to the Hessian $H(P)$. Therefore the rank of $H(P)$ is equal to the rank of $H_{\mathbb{R}}(\Re P) + i H_{\mathbb{R}}(\Im P)$. We assume that the rank of $H(P)$ is equal to $2n$. By Lemma 1, we can change the coordinates (w_1, w_2) of $\mathbb{R}^2$ as

$$(w_1, w_2) \mapsto (w_1, w_1 + u_0 w_2)$$

such that u_0 satisfies $\det(H_{\mathbb{R}}(\Re P) + u_0 H_{\mathbb{R}}(\Im P)) \neq 0$. With these new coordinates, $P(\mathbf{z}, \bar{\mathbf{z}})$ satisfies $\det H_{\mathbb{R}}(\Im P) \neq 0$. Thus the rank of $H_{\mathbb{R}}(\Im P))$ is $2n$. □

We show a necessary condition of $P(\mathbf{z}, \bar{\mathbf{z}})$ so that the rank of the representation matrix of $d^2 P$ is equal to $2n - 1$.

Lemma 3. *Let* $\mathbf{w}$ *belong to* $S_1(P)$*. Suppose the rank of* $H(P)$ *is* $2n$*. The rank of the representation matrix of* d^2P *is equal to* $2n-1$*.*

Proof. Since $\mathbf{w} \in S_1(P)$, one of $\mathrm{grad}(\Re P)(\mathbf{w})$ and $\mathrm{grad}(\Im P)(\mathbf{w})$ is non-zero. We may assume that $\mathrm{grad}(\Re P)(\mathbf{w})$ is non-zero. By a change of coordinates of $\mathbb{R}^2$ as in the proof of Lemma 2, we assume that the rank of $H_{\mathbb{R}}(\Im P)$ is $2n$. By change of coordinates of $\mathbb{R}^{2n}$, we may further assume that $\partial \Re P/\partial x_1(\mathbf{w}) \neq 0$ and write $\mathrm{grad}\,\Im P(\mathbf{w}) = s\,\mathrm{grad}\,\Re P(\mathbf{w})$ for some $s \in \mathbb{R}$.

We then change the coordinates of $\mathbb{R}^{2n}$ as follows:

$$\tilde{x}_1 = \sum_{l=1}^{2n} \frac{\partial \Re P}{\partial x_l}(\mathbf{w}) x_l, \quad \tilde{x}_j = x_j \quad \text{for} \quad j \geq 2.$$

By an easy calculus, the gradient of $\Re P$ at $\mathbf{w}$ is equal to $(1, 0, \ldots, 0)$.

We define the map $\psi \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$(\tilde{x}_1, \ldots, \tilde{x}_{2n}) \mapsto (\Re P, \tilde{x}_2, \ldots, \tilde{x}_{2n}).$$

Since the Jacobi matrix of ψ at $\mathbf{w}$ is the identity matrix, there exists the inverse function ψ^{-1} on a neighborhood of $\mathbf{w}$. Then the map $(\Re P, \Im P)$ can be represented as follows:

$$\begin{aligned}(\Re P, \Im P) &= P(\tilde{x}_1, \ldots, \tilde{x}_{2n}) \\ &= (P \circ \psi^{-1}) \circ \psi(\tilde{x}_1, \ldots, \tilde{x}_{2n}) \\ &= (P \circ \psi^{-1})(\Re P, \tilde{x}_2, \ldots, \tilde{x}_{2n}).\end{aligned}$$

Let $(x'_1, \ldots, x'_{2n})$ be the coordinates of $\mathbb{R}^{2n}$ given by

$$(x'_1, \ldots, x'_{2n}) = (\Re P, \tilde{x}_2, \ldots, \tilde{x}_{2n}).$$

Then there exists a map $Q \colon \mathbb{R}^{2n} \to \mathbb{R}$ such that $P \circ \psi^{-1}(x'_1, \ldots, y'_n) = (x'_1, Q(x'_1, \ldots, x'_{2n}))$. Since the singularity $\mathbf{w}$ belongs to $S_1(P)$, the gradient of Q at $\mathbf{w}$ can be represented by $(s, 0, \ldots, 0)$. Let (w_1, w_2) be the coordinates of $\mathbb{R}^2$. Set $\sum_{j=1}^{2n} a_j(\partial/\partial x'_j) \in X_{\mathbf{w}}$, then we have

$$\begin{aligned}&dP\left(\sum_{j=1}^{2n}\left(a_j \frac{\partial}{\partial x'_j}\right)\right) \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix} \frac{\partial}{\partial w_1} + \begin{pmatrix} s & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_{2n} \end{pmatrix} \frac{\partial}{\partial w_2} \\ &= a_1\left(\frac{\partial}{\partial w_1} + s\frac{\partial}{\partial w_2}\right).\end{aligned}$$

So the kernel $L_{\mathbf{w}}$ of dP is $\left\{\sum_{j=2}^{2n} a_j(\partial/\partial x'_j) \;\middle|\; a_j \in \mathbb{R}\right\}$ and the cokernel $G_{\mathbf{w}}$ of dP is generated by $\partial/\partial w_2$. By the definition of d^2P, we see that the representation matrix of d^2P is the Hessian $H_{\mathbb{R}}(Q)$ of Q taking away the first column with these basis. Thus the rank of the representation matrix of d^2P is equal to $2n-1$ if and only if the rank of the Hessian $H_{\mathbb{R}}(Q)$ of Q taking away the first column is $2n-1$.

By the definition of $Q(x'_1, \dots, x'_{2n})$, $Q(x'_1, \dots, x'_{2n}) = \Im P(x'_1, \dots, x'_{2n})$. Therefore the rank of the representation matrix of d^2P is equal to $H_{\mathbb{R}}(\Im P)$ taking away the first column. Since $\operatorname{rank} H_{\mathbb{R}}(\Im P) = 2n$ by the assumption, the rank of the representation matrix of d^2P is equal to $2n-1$. □

3. Proof of Theorem 1

Let $f(\mathbf{z})$ and $g(\mathbf{z})$ be complex polynomials such that $f(\mathbf{z})\bar{g}(\mathbf{z})$ has an isolated singularity at the origin. We define the $\mathbb{C}^*$-action on $\mathbb{C}^2$:

$$c \circ (z_1, z_2) := (c^q z_1, c^p z_2), \quad c \in \mathbb{C}^*.$$

Assume that $f(\mathbf{z})$ and $g(\mathbf{z})$ are convenient weighted homogeneous complex polynomials, i.e., $f(c \circ \mathbf{z}) = c^{pqm} f(\mathbf{z})$ and $g(c \circ \mathbf{z}) = c^{pqn} g(\mathbf{z})$. Assume that $m > n$ and $q \geq p$. We prepare two lemmas.

Lemma 4. *Let $g(\mathbf{z})$ be a convenient weighted homogeneous complex polynomial which has an isolated singularity at the origin. Then*

$$\det H_{\mathbb{C}}(g)(\mathbf{z}) := \left(\frac{\partial^2 g}{\partial z_1 \partial z_1}\right)(\mathbf{z})\left(\frac{\partial^2 g}{\partial z_2 \partial z_2}\right)(\mathbf{z}) - \left(\frac{\partial^2 g}{\partial z_1 \partial z_2}\right)(\mathbf{z})\left(\frac{\partial^2 g}{\partial z_2 \partial z_1}\right)(\mathbf{z})$$

is not identically equal to 0.

Proof. Put $g(\mathbf{z}) = \sum_j c_j z_1^{l_j} z_2^{k_j}$, where $l_1 \geq 2$, $k_1 = 0$ and $l_j > l_{j'}$ for $j < j'$. We calculate the degrees

$$\deg_{z_1}\left(\frac{\partial^2 g}{\partial z_1 \partial z_1}\right)(\mathbf{z})\left(\frac{\partial^2 g}{\partial z_2 \partial z_2}\right)(\mathbf{z})$$

and

$$\deg_{z_1}\left(\frac{\partial^2 g}{\partial z_1 \partial z_2}\right)(\mathbf{z})\left(\frac{\partial^2 g}{\partial z_2 \partial z_1}\right)(\mathbf{z})$$

of z_1. If $k_2 \geq 2$, two degrees are $l_1 + l_2 - 2$ and $2(l_2 - 1)$ respectively. Since l_1 is greater than l_2, two degrees are not equal. If $k_2 = 1$, by using equation (1), $l_1 = l_2 + (p/q)$. If q is greater than p, l_1 and l_2 does not satisfy $l_1 = l_2 + (p/q)$.

So we may assume that $p = q$, $k_2 = 1$. Then $g(\mathbf{z})$ has the form:

$$g(\mathbf{z}) = (z_1 - \tilde{c} z_2)\tilde{g}(\mathbf{z}),$$

where $\tilde{g}(\mathbf{z})$ is a weighted homogeneous polynomial such that $\tilde{g}(\mathbf{z})$ and $z_1 - \tilde{c}z_2$ have no common branches. On $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 - \tilde{c}z_2 = 0\}$, $\det H_{\mathbb{C}}(g)(\mathbf{z})$ is equal to $-(\tilde{c}\,\partial\tilde{g}/\partial z_1 + \partial\tilde{g}/\partial z_2)^2$. If $\det H_{\mathbb{C}}(g)(\mathbf{z})$ is identically equal to 0, the differentials of $\tilde{g}(\mathbf{z})$ satisfy

$$\tilde{c}\frac{\partial\tilde{g}}{\partial z_1} + \frac{\partial\tilde{g}}{\partial z_2} = 0.$$

Since $\tilde{g}(\mathbf{z})$ is a weighted homogeneous polynomial, by using equation (1), $\tilde{g}(\mathbf{z})$ is equal to

$$\frac{1}{n-1}\left(z_1\frac{\partial\tilde{g}}{\partial z_1} + z_2\frac{\partial\tilde{g}}{\partial z_2}\right).$$

So $\tilde{g}(\mathbf{z})$ vanishes on $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 - \tilde{c}z_2 = 0\}$. Since $\tilde{g}(\mathbf{z})$ and $z_1 - \tilde{c}z_2$ have no common branches, this is a contradiction. Thus $\det H_{\mathbb{C}}(g)(\mathbf{z})$ is not identically equal to 0 for $l_1 \geq 2$. $\square$

We define the following matrix:

$$A=\begin{pmatrix} \dfrac{\partial^2 f}{\partial z_1\partial z_1}\bar{g}-\dfrac{p(m-n)-1}{z_1}\dfrac{\partial f}{\partial z_1}\bar{g} & \dfrac{\partial^2 f}{\partial z_2\partial z_1}\bar{g} & \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial^2 f}{\partial z_1\partial z_2}\bar{g} & \dfrac{\partial^2 f}{\partial z_2\partial z_2}\bar{g}-\dfrac{q(m-n)-1}{z_2}\dfrac{\partial f}{\partial z_2}\bar{g} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & f\overline{\dfrac{\partial^2 g}{\partial z_1\partial z_1}} & f\overline{\dfrac{\partial^2 g}{\partial z_2\partial z_1}} \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_2}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} & f\overline{\dfrac{\partial^2 g}{\partial z_1\partial z_2}} & f\overline{\dfrac{\partial^2 g}{\partial z_2\partial z_2}} \end{pmatrix},$$

where (q, p) are weights of $f(\mathbf{z})$ and $g(\mathbf{z})$. Suppose that $g(\mathbf{z})$ does not have an isolated singularity at the origin. By changing coordinates of $\mathbb{C}^2$, we may assume that $g(\mathbf{z})$ has the following form:

$$g(\mathbf{z}) = \beta_1 z_1 + \beta_2 z_2^k.$$

Lemma 5. *Let $g(\mathbf{z}) = \beta_1 z_1 + \beta_2 z_2^k$ with $k \geq 2$. Then the determinant of A is not identically equal to* 0.

Proof. The determinant of A is equal to

$$f\overline{\frac{\partial^2 g}{\partial z_2\partial z_2}}\bar{\beta}_1^2\bar{g}\left(2\frac{\partial^2 f}{\partial z_1\partial z_2}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2} - \left(\frac{\partial^2 f}{\partial z_1\partial z_1} - \frac{m-2}{z_1}\frac{\partial f}{\partial z_1}\right)\left(\frac{\partial f}{\partial z_2}\right)^2 - \left(\frac{\partial^2 f}{\partial z_2\partial z_2} - \frac{k(m-1)-1}{z_2}\frac{\partial f}{\partial z_2}\right)\left(\frac{\partial f}{\partial z_1}\right)^2\right).$$

By the assumption, $\partial^2 g/\partial z_2\partial z_2 \not\equiv 0$. By using equation (1),

$$\frac{\partial^2 f}{\partial z_1\partial z_1} = \frac{m-1}{z_1}\frac{\partial f}{\partial z_1} - \frac{z_2}{kz_1}\frac{\partial^2 f}{\partial z_1\partial z_2},$$
$$\frac{\partial^2 f}{\partial z_2\partial z_2} = \frac{(km-1)}{z_2}\frac{\partial f}{\partial z_2} - \frac{kz_1}{z_2}\frac{\partial^2 f}{\partial z_1\partial z_2}.$$

Then we have

$$\begin{aligned}
&2\frac{\partial^2 f}{\partial z_1\partial z_2}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2} - \left(\frac{\partial^2 f}{\partial z_1\partial z_1} - \frac{m-2}{z_1}\frac{\partial f}{\partial z_1}\right)\left(\frac{\partial f}{\partial z_2}\right)^2 \\
&- \left(\frac{\partial^2 f}{\partial z_2\partial z_2} - \frac{k(m-1)-1}{z_2}\frac{\partial f}{\partial z_2}\right)\left(\frac{\partial f}{\partial z_1}\right)^2 \\
&= 2\frac{\partial^2 f}{\partial z_1\partial z_2}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2} - \frac{1}{z_1}\left(\frac{\partial f}{\partial z_1} - \frac{z_2}{k}\frac{\partial^2 f}{\partial z_1\partial z_2}\right)\left(\frac{\partial f}{\partial z_2}\right)^2 - \frac{k}{z_2}\left(\frac{\partial f}{\partial z_2} - z_1\frac{\partial^2 f}{\partial z_1\partial z_2}\right)\left(\frac{\partial f}{\partial z_1}\right)^2 \\
&= \frac{1}{z_1z_2}\left(\frac{z_2}{\sqrt{k}}\frac{\partial f}{\partial z_2} + \sqrt{k}z_1\frac{\partial f}{\partial z_1}\right)\left(\frac{\partial^2 f}{\partial z_1\partial z_2}\left(\frac{z_2}{\sqrt{k}}\frac{\partial f}{\partial z_2} + \sqrt{k}z_1\frac{\partial f}{\partial z_1}\right) - \sqrt{k}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2}\right).
\end{aligned}$$

Set $f(\mathbf{z}) = \sum_{j=0}^{m} \delta_j z_1^j z_2^{k(m-j)}$ and $l = \min\{j \mid j \neq 0, \delta_j \neq 0\}$. Then

$$\frac{\partial^2 f}{\partial z_1\partial z_2}\left(\frac{z_2}{\sqrt{k}}\frac{\partial f}{\partial z_2} + \sqrt{k}z_1\frac{\partial f}{\partial z_1}\right) - \sqrt{k}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2}$$

has the following form:

$$\begin{aligned}
&\frac{\partial^2 f}{\partial z_1\partial z_2}\left(\frac{z_2}{\sqrt{k}}\frac{\partial f}{\partial z_2} + \sqrt{k}z_1\frac{\partial f}{\partial z_1}\right) - \sqrt{k}\frac{\partial f}{\partial z_1}\frac{\partial f}{\partial z_2} \\
&= \sqrt{k}klm(m-l-1)\delta_0\delta_l z_1^{l-1}z_2^{2km-kl-1} \\
&\quad + \sqrt{k}kl(m-l)(m-1)\delta_l^2 z_1^{2l-1}z_2^{2k(m-l)-1} + \cdots.
\end{aligned}$$

If $m-l-1 \neq 0$, $\sqrt{k}klm(m-l-1)\delta_0\delta_l \neq 0$. Thus the determinant of A is not identically equal to 0. Suppose that $m-l-1 = 0$. Since m is greater than 1, $\det A \not\equiv 0$. □

To prove Theorem 1, we choose $h(\mathbf{z})$ such that the determinant of $H(f\bar{g} + th)$ is not identically equal to 0. We divide the proof of Theorem 1 into two cases:
(1) $g(\mathbf{z})$ is not a linear function,
(2) $g(\mathbf{z}) = \beta_1 z_1 + \beta_2 z_2$.

3.1. Case (1). We define a deformation of $f(\mathbf{z})\bar{g}(\mathbf{z})$ as follows:

$$F_t(\mathbf{z}) = f(\mathbf{z})\overline{g(\mathbf{z})} + th(\mathbf{z}),$$

where $h(\mathbf{z}) = \gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)}$ and $0 < t \ll 1$. Let s be a complex number such that $|s| = 1$. Then $F_t(s \circ \mathbf{z})$ satisfies

$$F_t(s \circ \mathbf{z}) = f(s \circ \mathbf{z})\bar{g}(s \circ \mathbf{z}) + th(s \circ \mathbf{z}) = s^{pq(m-n)} F_t(\mathbf{z}).$$

So $F_t(\mathbf{z})$ is also a polar weighted homogeneous mixed polynomial. Suppose that m is greater than n. Assume that $f(\mathbf{z})\bar{g}(\mathbf{z})$ and $h(\mathbf{z})$ have no common branches.

Lemma 6. *Let $F_t(\mathbf{z})$ be the above deformation of $f(\mathbf{z})\bar{g}(\mathbf{z})$. If $S_2(F_t) \neq \emptyset$, $S_2(F_t)$ is only the origin. If $S_2(F_t) = \emptyset$, the origin is a regular point of $F_t(\mathbf{z})$.*

Proof. If $\mathbf{w}$ belongs to $S_2(F_t)$, by Proposition 1, the singularity $\mathbf{w}$ satisfies $(\partial f/\partial z_j)(\mathbf{w})\bar{g}(\mathbf{w}) + (t\partial h/\partial z_j)(\mathbf{w}) = 0$ and $f(\mathbf{w})(\overline{\partial g/\partial z_j})(\mathbf{w}) = 0$ for $j = 1, 2$. By using equation (1), $f(\mathbf{w})\bar{g}(\mathbf{w}) = 0$ and $th(\mathbf{w}) = 0$. By the assumption of $h(\mathbf{w})$, $\mathbf{w}$ is equal to the origin. Since the origin $\mathbf{o}$ is an isolated singularity of $f(\mathbf{z})\bar{g}(\mathbf{z})$, $f(\mathbf{o})(\overline{\partial g/\partial z_j})(\mathbf{o}) = 0$ for $j = 1, 2$. If $S_2(F_t) = \emptyset$, there exists j such that $(\partial f/\partial z_j)(\mathbf{o})\bar{g}(\mathbf{o}) + (t\partial h/\partial z_j)(\mathbf{o}) \neq 0$. Thus the origin is not a singularity of $F_t(\mathbf{z})$. □

Set $f(\mathbf{z}) = a_1 z_1^{pm} + a_2 z_2^{qm} + z_1^p z_2^q f'(\mathbf{z})$ and $g(\mathbf{z}) = b_1 z_1^{pn} + b_2 z_2^{qn} + z_1^p z_2^q g'(\mathbf{z})$, where $f'(\mathbf{z})$ and $g'(\mathbf{z})$ are weighted homogeneous complex polynomials.

Lemma 7. *Suppose that γ_j is a coefficient of $h(\mathbf{z})$ which satisfies $\Re(\overline{a_j} b_j/\overline{\gamma_j}) > 0$ for $j = 1, 2$. Then z_1 and z_2 are non-zero for any $\mathbf{w} = (z_1, z_2) \in S_1(F_t)$ where $0 < t \ll 1$.*

Proof. Assume that $\mathbf{w} = (0, z_2) \in S_1(F_t)$. By Proposition 1 and Lemma 6, $z_2 \neq 0$ and

$$qm\overline{a_2} b_2 z_2^{qn} \bar{z}_2^{qm-1} + tq(m-n)\overline{\gamma_2}\bar{z}_2^{q(m-n)-1} = \alpha qn a_2 \overline{b_2} z_2^{qm} \bar{z}_2^{qn-1},$$

where $\alpha \in S^1$. Then we have

$$m\frac{\overline{a_2} b_2}{\overline{\gamma_2}} z_2^{qn} \bar{z}_2^{qn} + t(m-n) = \alpha n \frac{a_2 \overline{b_2}}{\overline{\gamma_2}} z_2^{qm} \bar{z}_2^{-qm+2qn}. \tag{2}$$

Since m is greater than n and $\alpha \in S^1$, the absolute value of $m(\overline{a_2} b_2/\overline{\gamma_2}) z_2^{qn} \bar{z}_2^{qn}$ is greater than that of $\alpha n(a_2 \overline{b_2}/\overline{\gamma_2}) z_2^{qm} \bar{z}_2^{-qm+2qn}$. We take $\gamma_2 \in \mathbb{C}$ which satisfies $\Re(\overline{a_2} b_2/\overline{\gamma_2}) > 0$. Then z_2 does not satisfy equation (2). This is a contradiction. Suppose that $\mathbf{w} = (z_1, 0) \in S_1(F_t)$. If we take $\gamma_1 \in \mathbb{C}$ which satisfies $\Re(\overline{a_1} b_1/\overline{\gamma_1}) > 0$, the proof is analogous in case $\mathbf{w} = (0, z_2)$. Thus we show that coefficients γ_1 and γ_2 of $h(\mathbf{z})$ such that z_1 and z_2 are non-zero for any $\mathbf{w} = (z_1, z_2) \in S_1(F_t)$. □

We now consider $h(\mathbf{z})$ satisfying the following condition:

$$\{\det H(F_t) = 0\} \cap S_1(F_t) = \emptyset. \tag{3}$$

Note that if $h(\mathbf{z})$ satisfies the condition (3), the rank of the representation matrix of d^2F_t is equal to 3 by Lemma 3.

Lemma 8. *There exist coefficients* γ_1 *and* γ_2 *of* $h(\mathbf{z})$ *such that* $\Re(\overline{a_j}b_j/\overline{\gamma_j}) > 0$, $h(\mathbf{z})$ *satisfies the condition* (3) *and, on* $S_1(F_t)$,

$$F_t(\mathbf{z}) = f(\mathbf{z})\bar{g}(\mathbf{z}) + th(\mathbf{z}) \neq 0,$$

where $j = 1, 2$ *and* $0 < t \ll 1$.

Proof. We define the mixed polynomial $\Phi(\mathbf{z}, \alpha)$ as follows:

$$\begin{aligned}\Phi(\mathbf{z}, \alpha) = &\left(\overline{\frac{\partial f}{\partial z_1}}(\mathbf{z})g(\mathbf{z}) - \alpha f(\mathbf{z})\overline{\frac{\partial g}{\partial z_1}}(\mathbf{z})\right)\overline{\frac{\partial h}{\partial z_2}}(\mathbf{z})\\ &- \left(\overline{\frac{\partial f}{\partial z_2}}(\mathbf{z})g(\mathbf{z}) - \alpha f(\mathbf{z})\overline{\frac{\partial g}{\partial z_2}}(\mathbf{z})\right)\overline{\frac{\partial h}{\partial z_1}}(\mathbf{z}),\end{aligned}$$

where $\alpha \in S^1$. Since f and g are convenient, $(\overline{\partial f/\partial z_j})(\mathbf{w})g(\mathbf{w}) - \alpha f(\mathbf{w})(\overline{\partial g/\partial z_j})(\mathbf{w}) \not\equiv 0$ for $j = 1, 2$. So there exist γ_1 and γ_2 such that $\Phi(\mathbf{z}, \alpha)$ is not identically equal to 0. If $\mathbf{w}$ is a singularity of $F_t(\mathbf{z})$, there exists $\tilde{\alpha} \in S^1$ such that

$$\begin{aligned}\overline{\frac{\partial f}{\partial z_1}}(\mathbf{w})g(\mathbf{w}) + t\overline{\frac{\partial h}{\partial z_1}}(\mathbf{w}) = \tilde{\alpha} f(\mathbf{w})\overline{\frac{\partial g}{\partial z_1}}(\mathbf{w}),\\ \overline{\frac{\partial f}{\partial z_2}}(\mathbf{w})g(\mathbf{w}) + t\overline{\frac{\partial h}{\partial z_2}}(\mathbf{w}) = \tilde{\alpha} f(\mathbf{w})\overline{\frac{\partial g}{\partial z_2}}(\mathbf{w}).\end{aligned}$$

So $\Phi(\mathbf{z}, \alpha)$ vanishes on $S_1(F_t)$.

We take a coefficient γ_j of $h(\mathbf{z})$ which satisfies $\Re(\overline{a_j}b_j/\overline{\gamma_j}) > 0$ for $j = 1, 2$. By using equation (1), Proposition 1 and Lemma 7, for any $\mathbf{w} \in S_1(F_t)$ there exists $\alpha \in S^1$ which satisfies the following equalities:

$$pqm\,\bar{f}(\mathbf{w})g(\mathbf{w}) + pq(m-n)t\overline{h}(\mathbf{w}) = \alpha\, pqn f(\mathbf{w})\bar{g}(\mathbf{w}), \tag{4}$$

$$\begin{aligned}t\frac{\partial^2 h}{\partial z_j \partial z_j} &= t\frac{pq(m-n) - p_j}{p_j z_j}\frac{\partial h}{\partial z_j}\\ &= \frac{pq(m-n) - p_j}{p_j z_j}\left(\bar{\alpha}\bar{f}\frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j}\bar{g}\right),\end{aligned} \tag{5}$$

where $p_1 = q$, $p_2 = p$ and $j = 1, 2$. By equation (4), $F_t(\mathbf{z})$ satisfies

$$\begin{aligned} F_t(\mathbf{z}) &= f(\mathbf{z})\bar{g}(\mathbf{z}) + th(\mathbf{z}) \\ &= f(\mathbf{z})\bar{g}(\mathbf{z}) + \frac{1}{pq(m-n)}(-pqm f(\mathbf{z})\bar{g}(\mathbf{z}) + \bar{\alpha} pqn \bar{f}(\mathbf{z})g(\mathbf{z})) \\ &= f(\mathbf{z})\bar{g}(\mathbf{z}) - \frac{m}{m-n} f(\mathbf{z})\bar{g}(\mathbf{z}) + \frac{n}{m-n}\bar{\alpha} \bar{f}(\mathbf{z})g(\mathbf{z}) \\ &= \frac{-n}{m-n}(f(\mathbf{z})\bar{g}(\mathbf{z}) - \bar{\alpha} \bar{f}(\mathbf{z})g(\mathbf{z})). \end{aligned}$$

So for $\mathbf{w} \in S_1(F_t)$, $F_t(\mathbf{w})$ is equal to 0 if and only if $f(\mathbf{w})\bar{g}(\mathbf{w}) - \bar{\alpha} \bar{f}(\mathbf{w})g(\mathbf{w}) = 0$ for some $\alpha \in S^1$. By equation (5), the Hessian $H(F_t)$ of $F_t(\mathbf{z})$ is equal to

$$\begin{pmatrix} \frac{\partial^2 f}{\partial z_1 \partial z_1}\bar{g} + \frac{p(m-n)-1}{z_1}\left(\bar{\alpha}\bar{f}\frac{\partial g}{\partial z_1} - \frac{\partial f}{\partial z_1}\bar{g}\right) & \frac{\partial^2 f}{\partial z_2 \partial z_1}\bar{g} & \frac{\partial f}{\partial z_1}\overline{\frac{\partial g}{\partial z_1}} & \frac{\partial f}{\partial z_1}\overline{\frac{\partial g}{\partial z_2}} \\ \frac{\partial^2 f}{\partial z_1 \partial z_2}\bar{g} & \frac{\partial^2 f}{\partial z_2 \partial z_2}\bar{g} + \frac{q(m-n)-1}{z_2}\left(\bar{\alpha}\bar{f}\frac{\partial g}{\partial z_2} - \frac{\partial f}{\partial z_2}\bar{g}\right) & \frac{\partial f}{\partial z_2}\overline{\frac{\partial g}{\partial z_1}} & \frac{\partial f}{\partial z_2}\overline{\frac{\partial g}{\partial z_2}} \\ \frac{\partial f}{\partial z_1}\overline{\frac{\partial g}{\partial z_1}} & \frac{\partial f}{\partial z_2}\overline{\frac{\partial g}{\partial z_1}} & f\overline{\frac{\partial^2 g}{\partial z_1 \partial z_1}} & f\overline{\frac{\partial^2 g}{\partial z_2 \partial z_1}} \\ \frac{\partial f}{\partial z_1}\overline{\frac{\partial g}{\partial z_2}} & \frac{\partial f}{\partial z_2}\overline{\frac{\partial g}{\partial z_2}} & f\overline{\frac{\partial^2 g}{\partial z_1 \partial z_2}} & f\overline{\frac{\partial^2 g}{\partial z_2 \partial z_2}} \end{pmatrix}$$

at $\mathbf{w} \in S_1(F_t)$. Let $\Psi(\mathbf{z}, \alpha)$ be the determinant of the above matrix. Set $\Psi(\mathbf{z}, \alpha) = (1/z_1 z_2)(\Psi_2(\mathbf{z})\bar{\alpha}^2 + \Psi_1(\mathbf{z})\bar{\alpha} + \Psi_0(\mathbf{z}))$ where $\Psi_j(\mathbf{z})$ is a mixed polynomial of $\mathbf{z}$ and $\bar{\mathbf{z}}$ for $j = 0, 1, 2$. Then we have

$$\begin{aligned} \Psi_2(\mathbf{z}) &= \{p(m-n)-1\}\{q(m-n)-1\}(f(\mathbf{z}))^2\overline{(f(\mathbf{z}))^2}\frac{\partial g}{\partial z_1}(\mathbf{z})\frac{\partial g}{\partial z_2}(\mathbf{z})\overline{\det H_{\mathbb{C}}(g)(\mathbf{z})}, \\ \Psi_0(\mathbf{z}) &= z_1 z_2 \det A. \end{aligned}$$

By Lemmas 4 and 5, either $\Psi_2(\mathbf{z})$ or $\Psi_0(\mathbf{z})$ is not identically equal to 0. So $\Psi(\mathbf{z}, \alpha)$ is not identically equal to 0. We define the S^1-action and the $\mathbb{R}^*$-action on $\mathbb{C}^2 \times S^1$ as follows:

$$\begin{aligned} s \circ (z_1, z_2, \alpha) &:= (s^q z_1, s^p z_2, s^{-2d_f + 2d_g}\alpha), \\ r \circ (z_1, z_2, \alpha) &:= (r^q z_1, r^p z_2, \alpha), \end{aligned}$$

where $s \in S^1$, $r \in \mathbb{R}^*$. Then $f(\mathbf{z})\bar{g}(\mathbf{z}) - \bar{\alpha} \bar{f}(\mathbf{z})g(\mathbf{z})$, $\Phi(\mathbf{z}, \alpha)$ and $\Psi(\mathbf{z}, \alpha)$ are polar and radial weighted homogeneous mixed polynomials. Set $V_1 = \{(z_1, z_2, \alpha) \in \mathbb{C}^2 \times S^1 \mid f(\mathbf{z})\bar{g}(\mathbf{z}) - \bar{\alpha} \bar{f}(\mathbf{z})g(\mathbf{z}) = 0\}$ and $V_2 = \{(z_1, z_2, \alpha) \in \mathbb{C}^2 \times S^1 \mid z_1 z_2 \Psi(\mathbf{z}, \alpha) = 0\}$. Since z_1 and z_2 are non-zero on $S_1(F_t)$ by Lemma 7, we may assume that $z_2 \neq 0$. Let (z_1, z_2, α) be a point of V_j, where $j = 1, 2$. Set $z_1 = r^q s^q z_1'$, $z_2 = r^p s^p$ and $\alpha = s^{-2d_f + 2d_g}\alpha'$, where $s \in S^1$, $r \in \mathbb{R}^*$. Then $(z_1', 1, \alpha')$ also belongs to V_j. So we may assume that $z_2 = 1$. The dimension of $V_j \cap (\mathbb{C} \times \{1\} \times S^1)$ is 1 for $j = 1, 2$. Then the curves V_1

and V_2 have finitely many branches which depend only $f(\mathbf{z})$ and $g(\mathbf{z})$. On $\mathbb{C}\times\{1\}\times S^1$, each branch of $(V_1 \cup V_2)$ is given by a convergent power series

$$\xi_k(u) = \left(\sum_l a_l u^l,\, 1,\, \sum_l b_l u^l\right) \in \mathbb{C}\times\{1\}\times S^1,$$

where $0 \le u \le 1$ for $k = 1, \ldots, d$.

Since the curves V_1 and V_2 have finitely many branches, we can choose coefficients γ_1 and γ_2 of $h(\mathbf{z})$ such that, on $\mathbb{C}\times\{1\}\times S^1$, the intersection of $(V_1 \cup V_2)$ and $\{\Phi(\mathbf{z},\alpha) = 0\}$ is empty, i.e., $\Phi(\xi_k(u)) \neq 0$ and $\Re(\overline{a_j} b_j/\overline{\gamma_j}) > 0$ for $0 \le u \le 1$, $k = 1, \ldots, d$ and $j = 1, 2$. Thus a deformation $F_t(\mathbf{z})$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ satisfies the condition (3) and $F_t(\mathbf{z}) \neq 0$ on $S_1(F_t)$. □

Lemma 9. *Let $F_t(\mathbf{z})$ be a deformation of $f(\mathbf{z})\bar{g}(\mathbf{z})$ in* Lemma 8. *$S_1(F_t)$ are indefinite fold singularities.*

Proof. By Proposition 2, $S_1(F_t)$ is a union of the orbits of the S^1-action. So a connected component of $S_1(F_t)$ can be represented by

$$(e^{iq\theta} z_1,\, e^{ip\theta} z_2), \quad \theta \in [0, 2\pi].$$

We first show that the differential of $F_t|_{S_1(F_t)} \colon S_1(F_t) \to \mathbb{R}^2$ is non-zero. On a connected component of $S_1(F_t)$, the map F_t has the following form:

$$F_t(e^{iq\theta} z_1,\, e^{ip\theta} z_2) = e^{ipq(m-n)\theta} F_t(z_1, z_2).$$

Thus the differential of F_t satisfies

$$\frac{dF_t}{d\theta}(e^{iq\theta} z_1,\, e^{ip\theta} z_2) = ipq(m-n) e^{ipq(m-n)\theta} F_t(z_1, z_2).$$

Since $F_t(\mathbf{z})$ does not vanish on $S_1(F_t)$, the differential does not vanish on $S_1(F_t)$. Thus any point of $S_1(F_t)$ is a fold singularity.

Next we calculate the differential of $|F_t|^2$. Let S be a connected component of $S_1(F_t)$. Set the coordinates of $\mathbb{C}^2$ as follows:

$$z_1 = r_1 e^{iq\theta},\, z_2 = r_2 e^{i(p\theta+\tau)},$$

where $S = \{(r_1' e^{iq\theta},\, r_2' e^{i(p\theta+\tau')}) \mid 0 \le \theta \le 2\pi\}$. Since $|F_t|^2$ is constant on $S_1(F_t)$, $\partial|F_t|^2/\partial\theta \equiv 0$. Then we have

$$\frac{\partial|F_t|^2}{\partial z_1} = \frac{1}{2}\frac{\partial|F_t|^2}{\partial r_1} e^{-iq\theta}, \quad \frac{\partial|F_t|^2}{\partial \bar{z}_1} = \frac{1}{2}\frac{\partial|F_t|^2}{\partial r_1} e^{iq\theta}.$$

On S, second differentials of $|F_t|^2$ satisfy

$$\frac{\partial^2|F_t|^2}{\partial z_1\partial\bar z_2}=\frac{1}{4}\left(\frac{\partial^2|F_t|^2}{\partial x_2\partial r_1}+i\frac{\partial^2|F_t|^2}{\partial y_2\partial r_1}\right)e^{-iq\theta},\quad \frac{\partial^2|F_t|^2}{\partial\bar z_1\partial\bar z_2}=\frac{1}{4}\left(\frac{\partial^2|F_t|^2}{\partial x_2\partial r_1}+i\frac{\partial^2|F_t|^2}{\partial y_2\partial r_1}\right)e^{iq\theta}.$$

So $z_1\,\partial|F_t|^2/\partial z_1\bar z_2-\bar z_1\,\partial|F_t|^2/\partial\bar z_1\bar z_2$ is equal to 0 on S. Since $\partial|F_t|^2/\partial\bar z_2$ is a polar weighted homogeneous mixed polynomial, on S, $\partial|F_t|^2/\partial\bar z_2$ satisfies

$$p\frac{\partial|F_t|^2}{\partial\bar z_2}=q\left(z_1\frac{\partial|F_t|^2}{\partial z_1\bar z_2}-\bar z_1\frac{\partial|F_t|^2}{\partial\bar z_1\bar z_2}\right)+p\left(z_2\frac{\partial|F_t|^2}{\partial z_2\bar z_2}-\bar z_2\frac{\partial|F_t|^2}{\partial\bar z_2\bar z_2}\right)=0.$$

Thus $z_2\,\partial|F_t|^2/\partial z_2\bar z_2-\bar z_2\,\partial|F_t|^2/\partial\bar z_2\bar z_2$ is equal to 0 on S. Since mixed polynomial $z_2\,\partial|F_t|^2/\partial z_2\bar z_2-\bar z_2\,\partial|F_t|^2/\partial\bar z_2\bar z_2$ is equal to

$$\begin{aligned}&\frac{r_2}{2}\left(\cos(p\theta+\tau)\frac{\partial|F_t|^2}{\partial y_2\,\partial y_2}-\sin(p\theta+\tau)\frac{\partial|F_t|^2}{\partial x_2\,\partial y_2}\right)\\&+i\frac{r_2}{2}\left(\sin(p\theta+\tau)\frac{\partial|F_t|^2}{\partial x_2\,\partial x_2}-\cos(p\theta+\tau)\frac{\partial|F_t|^2}{\partial x_2\,\partial y_2}\right),\end{aligned}$$

$(\partial|F_t|^2/\partial x_2x_2)(\partial|F_t|^2/\partial y_2y_2)-(\partial|F_t|^2/\partial x_2y_2)^2$ is equal to 0 on S. Set a curve $z(u)=(w_1,w_2+su)$, where $s\in\mathbb{C}$, $0\le u\ll 1$ and $(w_1,w_2)\in S_1(F_t)$. Then we have

$$\begin{aligned}\frac{\partial^2|F_t|^2}{\partial u\partial u}(z(0))&=\frac{\partial^2|F_t|^2}{\partial z_2\partial z_2}(z(0))\left(\frac{dz}{du}\right)^2+2\frac{\partial^2|F_t|^2}{\partial z_2\,\partial\bar z_2}(z(0))\frac{dz}{du}\frac{d\bar z}{du}+\frac{\partial^2|F_t|^2}{\partial\bar z_2\,\partial\bar z_2}(z(0))\left(\frac{d\bar z}{du}\right)^2\\&=\frac{s^2}{4}\left(\frac{\partial^2|F_t|^2}{\partial x_2\,\partial x_2}-\frac{\partial^2|F_t|^2}{\partial y_2\,\partial y_2}-2i\frac{\partial^2|F_t|^2}{\partial x_2\,\partial y_2}\right)+\frac{s\bar s}{2}\left(\frac{\partial^2|F_t|^2}{\partial x_2\,\partial x_2}+\frac{\partial^2|F_t|^2}{\partial y_2\,\partial y_2}\right)\\&\quad+\frac{\bar s^2}{4}\left(\frac{\partial^2|F_t|^2}{\partial x_2\,\partial x_2}-\frac{\partial^2|F_t|^2}{\partial y_2\,\partial y_2}+2i\frac{\partial^2|F_t|^2}{\partial x_2\,\partial y_2}\right).\end{aligned}$$

Since

$$\frac{\partial|F_t|^2}{\partial x_2x_2}\frac{\partial|F_t|^2}{\partial y_2y_2}-\frac{\partial|F_t|^2}{\partial x_2y_2)^2}$$

is equal to 0 on S, $(\partial^2|F_t|^2/\partial u\,\partial u)(z(0))$ is equal to

$$\begin{aligned}&\frac{s^2}{4}\left(\sqrt{\frac{\partial^2|F_t|^2}{\partial x_2\partial x_2}}-i\sqrt{\frac{\partial^2|F_t|^2}{\partial y_2\partial y_2}}\right)^2+\frac{s\bar s}{2}\left(\frac{\partial^2|F_t|^2}{\partial x_2\partial x_2}+\frac{\partial^2|F_t|^2}{\partial y_2\partial y_2}\right)\\&+\frac{\bar s^2}{4}\left(\sqrt{\frac{\partial^2|F_t|^2}{\partial x_2\partial x_2}}+i\sqrt{\frac{\partial^2|F_t|^2}{\partial y_2\partial y_2}}\right)^2\\&=\frac{1}{4}\left\{s\left(\sqrt{\frac{\partial^2|F_t|^2}{\partial x_2\partial x_2}}-i\sqrt{\frac{\partial^2|F_t|^2}{\partial y_2\partial y_2}}\right)+\bar s\left(\sqrt{\frac{\partial^2|F_t|^2}{\partial x_2\partial x_2}}+i\sqrt{\frac{\partial^2|F_t|^2}{\partial y_2\partial y_2}}\right)\right\}^2\ge 0.\end{aligned}$$

We consider the Hessian of $|F_t|^2\colon \mathbb{R}^3 \to \mathbb{R}$, $(r_1, r_2, \tau) \mapsto |F_t|^2(r_1, r_2, \tau)$. By changing coordinates of (r_2, τ), $H_{\mathbb{R}}(|F_t|^2)$ is congruent to

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & 0 \\ a_3 & 0 & 0 \end{pmatrix},$$

where $a_3 \neq 0$ and $a_4 > 0$. Thus $H_{\mathbb{R}}(|F_t|^2)$ is congruent to

$$H' = \begin{pmatrix} a_4 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & -a_3 \end{pmatrix}.$$

Since S is the set of the fold singularities of F_t, a_3 and a_4 are non-zero. So the matrix H' has two positive eigenvalues and a negative eigenvalue. Thus we show that $S_1(F_t)$ is the set of indefinite fold singularities. □

3.2. Case (2). In this subsection, $g(\mathbf{z})$ is equal to $\beta_1 z_1 + \beta_2 z_2$. Since we study $f(\mathbf{z})\bar{g}(\mathbf{z})$, we may assume that $g(\mathbf{z}) = z_1 + \beta z_2$. We study the following deformation $F_t(\mathbf{z})$ of $f(\mathbf{z})\overline{(z_1 + \beta z_2)}$:

$$F_t(\mathbf{z}) := f(\mathbf{z})\overline{(z_1 + \beta z_2)} + th(\mathbf{z}),$$

where $h(\mathbf{z}) = z_1^m \bar{z}_1 + z_1^{m-1} + \gamma z_2^{m-1}$. We study the rank of $H(F_t)$ and the differential of $F_t|_{S_1(F_t)}$.

Lemma 10. *There exists a coefficient γ such that the singularities of $F_t(\mathbf{z})$ in a sufficiently small neighborhood U of $\mathbf{o}$ are indefinite fold singularities except for the origin.*

Proof. Set

$$\begin{aligned}\Phi'(\mathbf{z}, \alpha) := &\left(\overline{\frac{\partial f}{\partial z_1}}(\mathbf{z})g(\mathbf{z}) - \alpha f(\mathbf{z})\right)\overline{\frac{\partial h}{\partial z_2}}(\mathbf{z}) \\ &+ \left(\alpha\bar{\beta} f(\mathbf{z}) - \overline{\frac{\partial f}{\partial z_2}}(\mathbf{z})g(\mathbf{z})\right)\left(\overline{\frac{\partial h}{\partial z_1}}(\mathbf{z}) - \alpha z_1^m\right).\end{aligned}$$

On $S_1(F_t)$, there exists $\alpha \in S^1$ such that α depends on $\mathbf{w} \in S_1(F_t)$ and $\Phi'(\mathbf{w}, \alpha) = 0$. Since $\det H(F_0) \equiv 0$, the determinant of $H(F_t)$ has the form:

$$t^2 m^2 \overline{\beta^2} z_1^{2m-2}\left(\frac{\partial f}{\partial z_2}\right)^2.$$

Suppose that γ is a coefficient of $h(\mathbf{z})$ which satisfies $\Re(\bar{a}_2\beta/\bar{\gamma}) > 0$, where $f(\mathbf{z}) = a_1z_1^m + a_2z_2^m + z_1z_2f'(\mathbf{z})$. By the same argument in Lemma 7, z_1 is non-zero for any $S_1(F_t)$. Assume that $\partial f/\partial z_2 = 0$. Then we have

$$\begin{aligned}\Phi'(\mathbf{z},\alpha) &= \left(\overline{\frac{\partial f}{\partial z_1}}(\mathbf{z})g(\mathbf{z}) - \alpha\frac{z_1}{m}\frac{\partial f}{\partial z_1}(\mathbf{z})\right)\overline{\frac{\partial h}{\partial z_2}}(\mathbf{z}) + \alpha\bar{\beta}\frac{z_1}{m}\frac{\partial f}{\partial z_1}(\mathbf{z})\left(\overline{\frac{\partial h}{\partial z_1}}(\mathbf{z}) - \alpha z_1^m\right)\\ &= (m-1)\bar{\gamma}\left(\overline{\frac{\partial f}{\partial z_1}}(\mathbf{z})g(\mathbf{z}) - \alpha\frac{z_1}{m}\frac{\partial f}{\partial z_1}(\mathbf{z})\right)\bar{z}_2^{m-2}\\ &\quad + \alpha\bar{\beta}\frac{z_1}{m}\frac{\partial f}{\partial z_1}(\mathbf{z})(mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m).\end{aligned}$$

Let U be a sufficiently small neighborhood of the origin $\mathbf{o}$. Since $f(\mathbf{z})$ has an isolated singularity at the origin $\mathbf{o}$ and m is greater than 1, the intersection of $\{(z_1,z_2,\alpha) \in U \times S^1 \mid \partial f/\partial z_2 = 0\}$ and $\{(z_1,z_2,\alpha) \in U \times S^1 \mid (z_1/m)(\partial f/\partial z_1)(\mathbf{z})(mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m) = 0\}$ is included in $\{\mathbf{o}\} \times S^1$. So $(z_1/m)(\partial f/\partial z_1)(\mathbf{z})(mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m)$ is non-zero on $\{(z_1,z_2,\alpha) \in U \times S^1 \mid \partial f/\partial z_2 = 0\} \setminus \{\mathbf{o}\} \times S^1$.

Since $\partial f/\partial z_2$ is also a weighted homogeneous polynomial, we may assume that $\partial f/\partial z_2$ has the following form:

$$\frac{\partial f}{\partial z_2} = z_2^{m_1}\prod_{j=1}^{m_2}(z_1 + \tau_j z_2).$$

We take a coefficient γ as follows:

$$\begin{aligned}\frac{1}{\bar{\gamma}} &\neq \frac{-(m-1)((\overline{\partial f/\partial z_1})(\mathbf{z})g(\mathbf{z}) - \alpha(z_1/m)(\partial f/\partial z_1)(\mathbf{z}))\bar{z}_2^{m-2}}{\alpha\bar{\beta}(z_1/m)(\partial f/\partial z_1)(\mathbf{z})(mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m)}\\ &= \frac{-(m-1)((\overline{\partial f/\partial z_1})(-\tau_j,1)g(-\tau_j,1) - \alpha'(-\tau_j/m)(\partial f/\partial z_1)(-\tau_j,1))}{(-1)^{m-2}\alpha'\bar{\beta}(-\tau_j/m)(\partial f/\partial z_1)(-\tau_j,1)(m\tau_j\bar{\tau}_j^{m-1}r^2 + (m-1)\bar{\tau}_j^{m-2} - \alpha'\tau_j^m r^2)}\end{aligned}$$

where $\alpha, \alpha' \in S^1$, $z_1 = -\tau_j re^{i\theta}$, $z_2 = re^{i\theta}$ for $0 \le r \ll 1$ and $j = 1, \ldots, m_3$. Hence we can choose $h(\mathbf{z})$ such that the intersection of $\{\det H(F_t) = 0\}$ and $\{\Phi'(\mathbf{z},\alpha) = 0\}$ is included in $\{\mathbf{o}\} \times S^1$. The origin $\mathbf{o}$ is a regular point of $F_t(\mathbf{z})$ or belongs to $S_2(F_t)$. Thus $\det H(F_t)$ is non-zero on $S_1(F_t)$.

On $S_1(F_t)$, $(m-1)F_t$ is equal to $-(f\bar{g} + tz_1^m\bar{z}_1) + \bar{\alpha}(\bar{f}g + tz_1\bar{z}_1^m)$. By the same argument in Lemma 8, the intersection of $\{-f\bar{g} + \bar{\alpha}\bar{f}g = 0\}$ and $\{\Phi'(\mathbf{z},\alpha) = 0\}$ is included in $\{\mathbf{o}\} \times S^1$. Since t is sufficiently small, the intersection of $\{-(f\bar{g} + tz_1^m\bar{z}_1) + \bar{\alpha}(\bar{f}g + tz_1\bar{z}_1^m) = 0\}$ and $\{\Phi'(\mathbf{z},\alpha) = 0\}$ is also included in $\{\mathbf{o}\} \times S^1$. Since $F_t(\mathbf{z})$ is also a polar weighted homogeneous mixed polynomial, we can show that the singularities of $F_t(\mathbf{z})$ are indefinite fold singularities, by using the same way as Lemma 9. □

If the origin of $\mathbb{C}^2$ is a singularity of $F_t(\mathbf{z})$, $F_t^{-1}(0) \cap S_{\varepsilon_t}^3$ is an oriented link in $S_{\varepsilon_t}^3$ for a sufficiently small ε_t. We study the topology of $F_t^{-1}(0) \cap S_{\varepsilon_t}^3$.

Lemma 11. *The link $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ is a $(p(m-n), q(m-n))$-torus link.*

Proof. The deformation $F_t(\mathbf{z})$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ is a convenient non-degenerate mixed polynomial in sense of [12]. Let Δ be the compact face of the Newton boundary of $F_t(\mathbf{z})$. Then the face function $F_{t\Delta}(\mathbf{z})$ is $t(\gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)})$. In [12, Theorem 43], the number of the connected components of $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ is equal to that of $F_{t\Delta}^{-1}(0) \cap S^3_{\varepsilon_t}$ where $0 < \varepsilon_t \ll 1$. Since $F_{t\Delta}(\mathbf{z}) = t(\gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)})$ has $m-n$ irreducible components, the number of link components of $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ is equal to $m-n$. By the choice of $h(\mathbf{z})$, $F_{t\Delta}$ is a polar weighted homogeneous polynomial. So $F_{t\Delta}^{-1}(0)$ is an invariant set of the S^1-action. In [4], the connected component of $F_{t\Delta}^{-1}(0) \cap S^3_{\varepsilon_t}$ is isotopic to a (p, q)-torus knot whose orientation coincides with that of the S^1-action and the linking numbers of components of $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ are equal to pq. $\square$

Proof of Theorem 1. The singularities of the deformation $F_t(\mathbf{z})$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ except for the origin are indefinite fold singularities by Lemma 8, Lemma 9 and Lemma 10. The link $F_t^{-1}(0) \cap S^3_{\varepsilon_t}$ is a $(p(m-n), q(m-n))$-torus link by Lemma 11. $\square$

3.3. Examples. Set $f(\mathbf{z}) = z_1^m + z_2^m$ and $g(\mathbf{z}) = z_1 + 2z_2$ where $m \geq 3$. Two polynomials $f(\mathbf{z})$ and $g(\mathbf{z})$ are convenient weighted homogeneous and $f(\mathbf{z})\bar{g}(\mathbf{z})$ has an isolated singularity at the origin $\mathbf{o}$. We consider a deformation $F_t = f(\mathbf{z})\bar{g}(\mathbf{z}) + t(z_1^m \bar{z}_1 + z_1^{m-1} + \gamma z_2^{m-1})$ of $f(\mathbf{z})\bar{g}(\mathbf{z})$ where $\gamma \neq 0$. Then we have

$$\begin{aligned}\Phi'(\mathbf{z}, \alpha) &= (m-1)\bar{\gamma}(m\bar{z}_1^{m-1} g(\mathbf{z}) - \alpha f(\mathbf{z}))\bar{z}_2^{m-2} \\ &\quad + (2\alpha f(\mathbf{z}) - m\bar{z}_2^{m-1} g(\mathbf{z}))(mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m), \\ \det H(F_t) &= 4t^2 m^4 z_1^{2m-2} z_2^{2m-2}.\end{aligned}$$

So $\det H(F_t)$ is equal to 0 if and only if $z_1 = 0$ or $z_2 = 0$. Since m is greater than 2, $\Phi'(\mathbf{z}, \alpha)|_{z_1=0} = -(m-1)\alpha\bar{\gamma} z_2^m \bar{z}_2^{m-2}$ and $\Phi'(\mathbf{z}, \alpha)|_{z_2=0} = \alpha\bar{\beta} z_1^m (mz_1\bar{z}_1^{m-1} + (m-1)\bar{z}_1^{m-2} - \alpha z_1^m)$. Hence (z_1, z_2, α) satisfies $\Phi'(\mathbf{z}, \alpha) = 0$ and $H(F_t) = 0$ in U if and only if $z_1 = z_2 = 0$. Since the origin $\mathbf{o}$ does not belong to $S_1(F_t)$, $\det H(F_t)$ is non-zero on $S_1(F_t)$.

If (z_1, z_2, α) satisfies $-f\bar{g} + \alpha \bar{f} g = 0$, (z_1, z_2, α) can be represented by

$$z_1 = zre^{i\theta}, \quad z_2 = re^{i\theta}, \quad \alpha = \alpha' e^{(-2m+2)i\theta},$$

where $(z^m + 1)\overline{(z+2)} = \alpha'\overline{(z^m+1)}(z+2)$. We take a coefficient γ of $h(\mathbf{z})$ such that

$$\bar{\gamma} \neq \frac{-(2\alpha' f(z,1) - mg(z,1))(mz\bar{z}^{m-1}r^2 + (m-1)\bar{z}^{m-2} - \alpha' z^m r^2)}{(m-1)(m\bar{z}^{m-1} g(z,1) - \alpha' f(z,1))}.$$

Then $F_t(\mathbf{z})$ is non-zero on $S_1(F_t)$. Thus $S_1(F_t)$ is the set of fold singularities and the link $S^3_\varepsilon \cap F_t^{-1}(0)$ is a $(m-1, m-1)$-torus link, where $S^3_\varepsilon = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = \varepsilon\}$, $\varepsilon \ll 1$.

4. Proof of Theorem 2

Let $F_t(\mathbf{z})$ be a deformation of $f(\mathbf{z})\bar{g}(\mathbf{z})$ in Theorem 1. In this section, we study the deformation of $F_t(\mathbf{z})$:

$$F_{t,s}(\mathbf{z}) := f(\mathbf{z})\bar{g}(\mathbf{z}) + th(\mathbf{z}) + sl(\mathbf{z}),$$

where $l(\mathbf{z}) = c_1 z_1 + c_2 z_2$, $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $0 < s \ll t \ll 1$. Suppose that c_1 and c_2 satisfy

(i) $\{td_h h(\mathbf{z}) + s(qc_1 z_1 + pc_2 z_2) = 0\}$ and $\{f(\mathbf{z})\bar{g}(\mathbf{z}) = 0\}$ have no common branches,

(ii) $\{(d_h - q)qc_1 z_1 + (d_h - p)pc_2 z_2 = 0\}$ and $\{f(\mathbf{z})\bar{g}(\mathbf{z}) = 0\}$ have no common branches,

where $d_h = pq(m-n)$ and $0 < s \ll t \ll 1$. The mixed Hessian $H(F_{t,s})$ of $F_{t,s}$ is equal to $H(F_{t,0})$. To prove Theorem 2, we first show that non-isolated singularities of $F_{t,s}(\mathbf{z})$ are indefinite fold singularities.

Lemma 12. *There exist c_1 and c_2 such that any point of $S_1(F_{t,s})$ is an indefinite fold singularity, where $0 < s \ll t \ll 1$.*

Proof. In the proof of Theorem 1, we proved that $\{\det H(F_{t,0}) = 0\} \cap S_1(F_{t,0}) = \emptyset$ and the differential of $F_{t,0}|_{S_1(F_{t,0})}$ is non-zero. So there exists a neighborhood $U_{F_{t,0}}$ of $S_1(F_{t,0})$ such that

$$\{\det H(F_{t,0}) = 0\} \cap U_{F_{t,0}} = \emptyset,$$
$$\frac{d}{dx_1^0} F_{t,0}(\mathbf{w}) \neq 0,$$

where x_1^0 is a coordinate of $S_1(F_{t,0})$ in $\mathbb{R}^4$ and $\mathbf{w} \in S_1(F_{t,0})$. We take non-zero complex numbers c_1, c_2 and sufficiently small positive real number s_0 such that $S_1(F_{t,s}) \subset U_{F_{t,0}}$ for any $0 < s \leq s_0$. Then the intersection of $S_1(F_{t,s})$ and $\{\det H(F_{t,s}) = 0\}$ is empty. Thus $j^1 F_{t,s}$ is transversal to $S_1(\mathbb{R}^4, \mathbb{R}^2)$ at $S_1(F_{t,s})$. We check the differential of $F_{t,s} \colon S_1(F_{t,s}) \to \mathbb{R}^2$. Let $(x_1^s, \ldots, x_4^s)$ be a family of coordinates of $\mathbb{R}^4$, smoothly parametrized by s, such that x_1^s is the coordinate of $S_1(F_{t,s})$. Then we have

$$\frac{dF_{t,s}}{dx_1^s} = \frac{\partial F_{t,0}}{\partial x_1^0}\frac{\partial x_1^0}{\partial x_1^s} + \cdots + \frac{\partial F_{t,0}}{\partial x_4^0}\frac{\partial x_4^0}{\partial x_1^s} + s\left(\frac{\partial l}{\partial x_1^0}\frac{\partial x_1^0}{\partial x_1^s} + \cdots + \frac{\partial l}{\partial x_4^0}\frac{\partial x_4^0}{\partial x_1^s}\right).$$

Since $\partial F_{t,0}/\partial x_1^0$ is non-zero on $U_{F_{t,0}}$, $dF_{t,s}/dx_1^s$ is non-zero for $0 < s \ll 1$. Thus any point of $S_1(F_{t,s})$ is a fold singularity.

By changing coordinates of $\mathbb{R}^4$ and $\mathbb{R}^2$, on $U_{F_{t,0}}$, we may assume that a point of $S_1(F_{t,s})$ is equal to $(x_1^s, 0, 0, 0)$ and $F_{t,s} \colon \mathbb{R}^4 \to \mathbb{R}^2$ has the following form:

$$F_{t,s} = (x_1^s, I_{t,s}(x_1^s, x_2^s, x_3^s, x_4^s)),$$

where grad $I_{t,s}(\mathbf{w}) = (0, 0, 0, 0)$ for any $\mathbf{w} \in S_1(F_{t,s})$. Set $I_{t,s}(x_1^0, \dots, x_4^0) = I_{t,0}(x_1^0, x_2^0, \dots, x_4^0) + sI'_{t,s}(x_1^0, \dots, x_4^0)$. Since $S_1(F_{t,0})$ are indefinite fold singularities, by choosing suitable coordinates $(x_1^0, x_2^0, x_3^0, x_4^0)$, we may assume that

$$I_{t,s} = -(x_2^0)^2 + (x_3^0)^2 + (x_4^0)^2 + sI'_{t,s}(x_1^0, \dots, x_4^0).$$

Let $(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s)$ be a point of $S_1(F_{t,s})$. Then we have

$$\frac{\partial I_{t,s}}{\partial x_2^0}(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s) = -2\iota_2^s + s\frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s) = 0.$$

We first fix x_1^0, x_3^0 and x_4^0, i.e., $x_1^0 = \iota_1^s$, $x_3^0 = \iota_3^s$ and $x_4^0 = \iota_4^s$. Since s is sufficiently small, $\partial I_{t,s}/\partial x_2^0$ satisfies

$$\begin{aligned}
\frac{\partial I_{t,s}}{\partial x_2^0} &= -2x_2^0 + s\frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, x_2^0, \iota_3^s, \iota_4^s) \\
&= -2x_2^0 + s\frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, x_2^0, \iota_3^s, \iota_4^s) + 2\iota_2^0 - s\frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s) \\
&= -2(x_2^0 - \iota_2^s) + s\left(\frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, x_2^0, \iota_3^s, \iota_4^s) - \frac{\partial I'_{t,s}}{\partial x_2^0}(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s)\right) \\
&= (x_2^0 - \iota_2^s)\left(-2 + s\frac{(\partial I'_{t,s}/\partial x_2^0)(\iota_1^s, x_2^0, \iota_3^s, \iota_4^s) - (\partial I'_{t,s}/\partial x_2^0)(\iota_1^s, \iota_2^s, \iota_3^s, \iota_4^s)}{x_2^0 - \iota_2^s}\right) < 0
\end{aligned}$$

for $x_2^0 > \iota_2^s$. Thus there exists a curve $\mathbf{z}_s(u) = (\iota_1^s, x_{2,s}(u), \iota_3^s, \iota_4^s)$ on $U_{F_{t,0}}$ such that $\mathbf{z}_s(0) = \iota_2^s$ and $I_{t,s}$ is monotone decreasing on $\mathbf{z}_s(u)$, i.e., $x_{2,s}(u) \geq \iota_2^s$. Next we fix x_1^0, x_2^0 and x_4^0. Then we can show that there exists a curve $\mathbf{z}'_s(u)$ on $U_{F_{t,0}}$ such that $\mathbf{z}'_s(0) \in S_1(F_{t,s})$ and $I_{t,s}$ is monotone increasing on $\mathbf{z}'_s(u)$. So $S_1(F_{t,s})$ is the set of indefinite fold singularities. □

Next we consider isolated singularities of $F_{t,s}(\mathbf{z})$. Then these singularities belong to $S_2(F_{t,s})$. We study the topological types of the links at each point of $S_2(F_{t,s})$.

Lemma 13. *Let $F_{t,s}(\mathbf{z})$ be a deformation of $F_t(\mathbf{z})$ in* Lemma 12. *Then $S_2(F_{t,s})$ is the set of finite mixed Morse singularities.*

Proof. Let $\mathbf{w} = (w_1, w_2)$ be a point of $S_2(F_{t,s})$. If $g(\mathbf{z})$ is not a linear function, $f(\mathbf{w})\overline{(\partial g/\partial z_j)}(\mathbf{w}) = 0$ for $j = 1, 2$ by Proposition 1. Since $g(\mathbf{z})$ has an isolated singularity at $\mathbf{o}$ and $f(\mathbf{o}) = 0$, f vanishes on $S_2(F_{t,s})$. By equation (1), $\mathbf{w}$ satisfies $f(\mathbf{w})\bar{g}(\mathbf{w}) = 0$ and $td_h h(\mathbf{w}) + s(qc_1z_1 + pc_2z_2) = 0$. Since c_1 and c_2 satisfy the condition (i), the number of $S_2(F_{t,s})$ is finite. If $g(\mathbf{z}) = z_1 + \beta z_2$, $\mathbf{w}$ satisfies

$$f(\mathbf{w}) + w_1^m = 0, \quad f(\mathbf{w})\bar{\beta} = 0.$$

So f and w_1 vanish on $S_2(F_{t,s})$. By Proposition 1 and equation (1), we have

$$m\bar{f}(\mathbf{w})g(\mathbf{w}) + t(m-1)(\overline{w_1^{m-1} + \gamma w_2^{m-1}}) + m\overline{w_1^m}w_1 + s(\overline{c_1w_1 + c_2w_2}) = 0,$$
$$t(m-1)\gamma w_2^{m-1} + sc_2w_2 = 0.$$

Thus $\{(0, z_2) \mid t(m-1)\gamma z_2^{m-1} + sc_2z_2 = 0\}$ includes $S_2(F_{t,s})$. The number of $S_2(F_{t,s})$ is finite.

Since f vanishes on $S_2(F_{t,s})$, $H(F_{t,s})$ is equal to

$$H(F_{t,s}) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial z_1 \partial z_1}\bar{g} + t\dfrac{\partial^2 h}{\partial z_1 \partial z_1} & \dfrac{\partial^2 f}{\partial z_2 \partial z_1}\bar{g} & \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial^2 f}{\partial z_1 \partial z_2}\bar{g} & \dfrac{\partial^2 f}{\partial z_2 \partial z_2}\bar{g} + t\dfrac{\partial^2 h}{\partial z_2 \partial z_2} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 & 0 \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_2}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} & 0 & 0 \end{pmatrix}.$$

Assume that $(\partial f/\partial z_j)(\mathbf{w})(\overline{\partial g/\partial z_k})(\mathbf{w}) = 0$ for any $j, k \in \{1, 2\}$. By using equation (1), $f(\mathbf{w})(\overline{\partial g/\partial z_j})(\mathbf{w}) = (\partial f/\partial z_j)(\mathbf{w})\bar{g}(\mathbf{w}) = 0$. Hence $\mathbf{w}$ is a singularity of $f(\mathbf{z})\bar{g}(\mathbf{z})$ by Proposition 1. Since $f(\mathbf{z})\bar{g}(\mathbf{z})$ has an isolated singularity at the origin, $\mathbf{w}$ is the origin. Then we have

$$\begin{aligned} \frac{\partial F_{t,s}}{\partial z_j}(\mathbf{w}) &= \frac{\partial F_{t,s}}{\partial z_j}(\mathbf{o}) \\ &= \frac{\partial f}{\partial z_j}(\mathbf{o})\bar{g}(\mathbf{o}) + t\frac{\partial h}{\partial z_j}(\mathbf{o}) + sc_j \\ &= t\frac{\partial h}{\partial z_j}(\mathbf{o}) + sc_j \neq 0, \end{aligned}$$

where $0 < s \ll t \ll 1$. The origin $\mathbf{o}$ does not belong to $S_2(F_{t,s})$. This is a contradiction. So there exist $j, k \in \{1, 2\}$ such that $(\partial f/\partial z_j)(\mathbf{w})(\overline{\partial g/\partial z_k})(\mathbf{w})$ is non-zero at $\mathbf{w} \in S_2(F_{t,s})$. Assume $(\partial f/\partial z_2)(\mathbf{w})(\overline{\partial g/\partial z_1})(\mathbf{w})$ is non-zero. If $g(\mathbf{z})$ is not a linear function, we calculate $H(F_{t,s})$ by using equation (1).

$$H(F_{t,s}) \cong \begin{pmatrix} h_{1,1} & h_{1,2} & pqmf\overline{\dfrac{\partial g}{\partial z_1}} & pqmf\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial^2 f}{\partial z_1 \partial z_2}\bar{g} & \dfrac{\partial^2 f}{\partial z_2 \partial z_2}\bar{g} + t\dfrac{\partial^2 h}{\partial z_2 \partial z_2} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 & 0 \\ \dfrac{\partial f}{\partial z_1}\overline{\dfrac{\partial g}{\partial z_2}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} & 0 & 0 \end{pmatrix}$$

$$\cong \begin{pmatrix} h'_{1,1} & h_{1,2} & pqmf\overline{\dfrac{\partial g}{\partial z_1}} & pqmf\overline{\dfrac{\partial g}{\partial z_2}} \\ h_{1,2} & \dfrac{\partial^2 f}{\partial z_2 \partial z_2}\bar{g} + t\dfrac{\partial^2 h}{\partial z_2 \partial z_2} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} \\ pqmf\overline{\dfrac{\partial g}{\partial z_1}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 & 0 \\ pqmf\overline{\dfrac{\partial g}{\partial z_2}} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_2}} & 0 & 0 \end{pmatrix},$$

where

$$h_{1,1} = q(pm-1)\frac{\partial f}{\partial z_1}\bar{g} + tq(p(m-n)-1)\frac{\partial h}{\partial z_1},$$
$$h_{1,2} = p(qm-1)\frac{\partial f}{\partial z_2}\bar{g} + tp(q(m-n)-1)\frac{\partial h}{\partial z_2}$$

and

$$h'_{1,1} = ((pm-1)q^2 z_1\frac{\partial f}{\partial z_1} + (qm-1)p^2 z_2\frac{\partial f}{\partial z_2}\bar{g} \\ + t\left((p(m-n)-1)q^2 z_1\frac{\partial h}{\partial z_1} + (q(m-n)-1)p^2 z_2\frac{\partial h}{\partial z_2}\right).$$

Since f vanishes on $S_2(F_{t,s})$, $h'_{1,1}$ is equal to

$$\left((pm-1)q^2 z_1\frac{\partial f}{\partial z_1} + (qm-1)p^2 z_2\frac{\partial f}{\partial z_2}\right)\bar{g} \\ + t\left((p(m-n)-1)q^2 z_1\frac{\partial h}{\partial z_1} + (q(m-n)-1)p^2 z_2\frac{\partial h}{\partial z_2}\right)$$

$$\begin{aligned}
&= \left((p(m-n)-1)q^2 z_1 \frac{\partial f}{\partial z_1} + (q(m-n)-1)p^2 z_2 \frac{\partial f}{\partial z_2}\right)\bar{g} \\
&\quad + t\left((p(m-n)-1)q^2 z_1 \frac{\partial h}{\partial z_1} + (q(m-n)-1)p^2 z_2 \frac{\partial h}{\partial z_2}\right) \\
&\quad + \left(pq^2 n z_1 \frac{\partial f}{\partial z_1} + p^2 q n z_2 \frac{\partial f}{\partial z_2}\right)\bar{g} \\
&= \left((p(m-n)-1)q^2 z_1 \frac{\partial f}{\partial z_1} + (q(m-n)-1)p^2 z_2 \frac{\partial f}{\partial z_2}\right)\bar{g} \\
&\quad + t\left((p(m-n)-1)q^2 z_1 \frac{\partial h}{\partial z_1} + (q(m-n)-1)p^2 z_2 \frac{\partial h}{\partial z_2}\right) \\
&\quad + pqn\left(q z_1 \frac{\partial f}{\partial z_1} + p z_2 \frac{\partial f}{\partial z_2}\right)\bar{g} \\
&= \{pq(m-n)-q\} q z_1 \left(\frac{\partial f}{\partial z_1}\bar{g} + t\frac{\partial h}{\partial z_1}\right) \\
&\quad + \{pq(m-n)-p\} p z_2 \left(\frac{\partial f}{\partial z_2}\bar{g} + t\frac{\partial h}{\partial z_2}\right) + p^2 q^2 m n f \bar{g} \\
&= -s\{(d_h - q) q c_1 z_1 + (d_h - p) p c_2 z_2\},
\end{aligned}$$

where $d_h = pq(m-n)$. Hence we have

$$H(F_{t,s}) \cong \begin{pmatrix} -s\{(d_h-q)qc_1z_1+(d_h-p)pc_2z_2\} & h_{1,2} & 0 & 0 \\ h_{1,2} & \dfrac{\partial^2 f}{\partial z_2 \partial z_2}\bar{g}(\mathbf{w})+t\dfrac{\partial^2 h}{\partial z_2 \partial z_2} & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 \\ 0 & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since c_1 and c_2 satisfy the condition (ii), the Hessian $H(F_{t,s})$ is congruent to

$$\begin{aligned}
H(F_{t,s}) &\cong \begin{pmatrix} -s\{(d_h-q)qc_1z_1+(d_h-p)pc_2z_2\} & 0 & 0 & 0 \\ 0 & 0 & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 \\ 0 & \dfrac{\partial f}{\partial z_2}\overline{\dfrac{\partial g}{\partial z_1}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\cong \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

at $\mathbf{w} \in S_2(F_{t,s})$. If $g(\mathbf{z})$ is a linear function, we use the same method of non-linear cases. So $H(F_{t,s})$ is congruent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By the same argument, we can check that if either

$$\frac{\partial f}{\partial z_1}(\mathbf{w})\overline{\frac{\partial g}{\partial z_1}}(\mathbf{w}), \quad \frac{\partial f}{\partial z_1}(\mathbf{w})\overline{\frac{\partial g}{\partial z_2}}(\mathbf{w})$$

or

$$\frac{\partial f}{\partial z_2}(\mathbf{w})\overline{\frac{\partial g}{\partial z_2}}(\mathbf{w})$$

is non-zero, $H(F_{t,s})$ is congruent to the above right-hand matrix. We change the coordinates of $\mathbb{C}^2$ such that, at the singularity of $F_{t,s}(\mathbf{z})$, the mixed Hessian $H(F_{t,s})$ is equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We identify $\mathbb{C}^2$ with $\mathbb{R}^4$. Then $H_{\mathbb{R}}(\Re F_{t,s}) + i H_{\mathbb{R}}(\Im F_{t,s})$ has the following form:

$$\begin{aligned} H_{\mathbb{R}}(\Re F_{t,s}) + i H_{\mathbb{R}}(\Im F_{t,s}) &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ i & 0 & -i & 0 \\ 0 & i & 0 & -i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & i & i \\ 1 & 0 & -i & 0 \\ i & -i & -1 & 1 \\ i & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Since $H_{\mathbb{R}}(\Re F_{t,s})$ and $H_{\mathbb{R}}(\Im F_{t,s})$ are regular matrices, $\Re F_{t,s}$ and $\Im F_{t,s}$ are Morse functions. Put $R = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned} {}^t R H_{\mathbb{R}}(\Re F_{t,s}) R &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence there exist the coordinates of $\mathbb{R}^4$ such that $\Re F_{t,s}$ has the following form:

$$\Re F_{t,s} = x_1^2 - x_2^2 - y_1^2 + y_2^2.$$

On the other hand, the Hessian of $\Im F_{t,s}$ is congruent to

$$\begin{aligned} {}^t R H_{\mathbb{R}}(\Im F_{t,s}) R &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \\ 1 & -2 & 0 & 0 \\ 2 & -3 & 0 & 0 \end{pmatrix}. \end{aligned}$$

So the 2-jet $j^2 \Im F_{t,s}(\mathbf{w})$ of $\Im F_{t,s}$ at $\mathbf{w}$ is equal to $j^2 \Im \tilde{F}_{t,s}(\mathbf{w})$, where $\Im \tilde{F}_{t,s} = 2(x_1 y_1 + 2x_1 y_2 - 2x_2 y_1 - 3x_2 y_2)$.

Put $z_j = x_j + i y_j$ for $j = 1, 2$. We calculate $\tilde{F}_{t,s}(\mathbf{z}) := \Re F_{t,s} + i \Im \tilde{F}_{t,s}$ on a neighborhood of $\mathbf{w}$:

$$\begin{aligned} \Re F_{t,s} + i \Im \tilde{F}_{t,s} &= x_1^2 - x_2^2 - y_1^2 + y_2^2 + 2i(x_1 y_1 + 2x_1 y_2 - 2x_2 y_1 - 3x_2 y_2) \\ &= (x_1 + i y_1)^2 - (x_2 + i y_2)^2 + 4i(x_1 y_2 - x_2 y_1 - x_2 y_2) \\ &= z_1^2 - z_2^2 + 4i\left(\frac{z_1 + \bar{z}_1}{2} \frac{z_2 - \bar{z}_2}{2i} - \frac{z_2 + \bar{z}_2}{2} \frac{z_1 - \bar{z}_1}{2i} - \frac{z_2 + \bar{z}_2}{2} \frac{z_2 - \bar{z}_2}{2i} \right) \\ &= z_1^2 - 2z_2^2 - 2z_1 \bar{z}_2 + 2z_2 \bar{z}_1 + \bar{z}_2^2. \end{aligned}$$

We define a family of mixed functions:

$$F_{t,s,\tau} := \tilde{F}_{t,s} + \tau f_{t,s},$$

where $f_{t,s} = F_{t,s} - \tilde{F}_{t,s}$ and $0 \leq \tau \leq 1$. Since $F_{t,s,\tau}$ is true non-degenerate for any $0 \leq \tau \leq 1$ in sense of [12], there exist a positive real number r_0 such that the sphere S_r^3 with $0 < r \leq r_0$ intersects $F_{t,s,\tau}^{-1}(0)$ transversely. Set

$$K_\tau = \{(z_1, z_2) \in S_r^3 \mid F_{t,s,\tau}(z_1, z_2) = 0\},$$
$$\tilde{K} = \{(z_1, z_2, \tau) \in S_r^3 \times [0, 1] \mid F_{t,s,\tau}(z_1, z_2) = 0\}.$$

Then the projection $\pi\colon \tilde{K} \to [0,1]$ is a fiber bundle by the Ehresmann fibering theorem [16]. Thus K_0 is isotopic to K_1.

We change coordinates of $\mathbb{C}^2$:

$$v_1 = z_1 - \bar{z}_2, \quad v_2 = z_2.$$

Then $\tilde{F}_{t,s}$ is equal to $v_1^2 + 2\overline{v}_1 v_2$. So the algebraic set $\{(v_1, v_2) \mid v_1^2 + 2\overline{v}_1 v_2 = 0\}$ has two components:

$$\{v_1 = 0\}, \quad \{(v_1, v_2) = (2\tilde{r}e^{i\theta'}, -\tilde{r}e^{3i\theta'}) \mid 0 < \tilde{r},\ 0 \leq \theta' \leq 2\pi\}.$$

We define the S^1-action on $\mathbb{C}^2$:

$$(v_1, v_2) \mapsto (\tilde{s}v_1, \tilde{s}^3 v_2), \quad \tilde{s} \in S^1.$$

Then the set of the zero points of $\tilde{F}_{t,s}(\mathbf{z})$ is an invariant set of the S^1-action. So the link of $\tilde{F}_{t,s}(\mathbf{z})$ is the Seifert link in [4]. Since two components of the link of $\tilde{F}_{t,s}$ are trivial knots and the absolute value of the linking number is 1, $\mathbf{w}$ defines a Hopf link as an unoriented link.

Let B_δ^4 be the 4-dimensional ball such that $F_{t,0}^{-1}(0) \cap \partial B_\delta^4$ is isotopic to $F_{t\Delta}^{-1}(0) \cap \partial B_\delta^4$ and the intersection of B_δ^4 and the singularities of $F_{t,s}(\mathbf{z})$ is equal to $S_2(F_{t,s})$, where $F_{t\Delta}(\mathbf{z})$ is the face function of $F_t(\mathbf{z})$. The restricted map $F_{t,s}\colon B_\delta^4 \to D^2$ is an unfolding of $F_{t\Delta}^{-1}(0) \cap \partial B_\delta^4$ in the sense of [10]. By Lemma 11, $F_{t\Delta}^{-1}(0) \cap \partial B_\delta^4$ is isotopic to the $(p(m-n), q(m-n))$-torus link whose orientations coincide with that of links of holomorphic functions. Then there exists an unfolding which has only positive Hopf links and the enhancement to the Milnor number is equal to 0 [10, Theorem 5.6]. Note that the enhancement to the Milnor number is a homotopy invariant of fibered links. Assume that there exists singularities of $F_{t,s}(\mathbf{z})$ such that they define negative Hopf links. Then the enhancement to the Milnor number is positive [10, Theorem 5.4]. The homotopy type of $F_{t,0}^{-1}(0) \cap \partial B_\delta^4$ is different from that of links of holomorphic functions. By Lemma 11, this is a contradiction. Any point of $S_2(F_{t,s})$ defines a positive Hopf link. Thus $\mathbf{w}$ is a mixed Morse singularity. □

Proof of Theorem 2. Let $l(\mathbf{z}) = c_1 z_1 + c_2 z_2$ be a linear function in Lemma 12. Any point of $S_1(F_{t,s})$ is an indefinite fold singularity. By Lemma 13, isolated singularities of $F_{t,s}(\mathbf{z})$ are mixed Morse singularities. Thus $F_{t,s}(\mathbf{z})$ is a mixed broken Lefschetz fibration. □

4.1. Examples. Let F_t be a deformation of $f\bar{g}$ in Section 3.3. We consider a deformation $F_{t,s} = F_t + s(c_1 z_1 + c_2 z_2)$ of F_t, where $0 \leq s \ll t \ll 1$. Suppose that c_1 and c_2 satisfy $c_2/c_1 \neq 2$ and $(-c_2/c_1)^m \neq -1$. Then $S_1(F_{t,s})$ is the set of indefinite fold singularities and $S_2(F_{t,s})$ is the set of mixed Morse singularities.

ACKNOWLEDGMENTS. The author would like to thank Masaharu Ishikawa, Masayuki Kawashima and Nguyen Tat Thang for precious comments. He also thank to the referee for careful reading of the manuscript and fruitful suggestions.

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