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A PAIRWISE INDEPENDENT RANDOM SAMPLING METHOD IN THE RING OF *p*-ADIC INTEGERS

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Abstract

For the ring of p-adic integers, p being a fixed prime, any sequence which plays a similar role to Weyl's irrational rotation has not been proposed yet. We will see that a modified p-adic van der Corput sequence provides us with a reasonable counterpart of Weyl's irrational rotation in the ring. We will present a similar random Weyl sampling on the ring to the one proposed by Sugita and Takanobu. In the process of establishing the counterpart, a sampling method based on a function with naturally extended domain to the field of p-adic numbers in terms of the additive characters will be mentioned.

1. Introduction

For the ring $\mathbb{Z}_p$ of p-adic integers, p being a fixed prime, any sequence which plays a similar role to Weyl's irrational rotation has not been proposed yet. In mainframe of the article, we are going to investigate how a sequence of points in the ring can be generated relying on algorithmic procedure, aiming at an approximation to the integral of function with respect to the Haar measure on the ring $\mathbb{Z}_p$ without losing advantages in use of the sequence given by purely random choice of points. To achieve this objective on $\mathbb{Z}_p$ in the present article, we will introduce a sequence on $\mathbb{Z}_p$ hinted by the p-adic van der Corput sequence, similarly to the sequence with randomness proposed by Sugita and Takanobu on the multidimensional torus.

Numerical approximation methods with an empirical average of function at algorithmically generated points could result unsatisfactory rate of convergence to the integral, if the function takes exceptional values at those sampling points. To avoid such a problem, we can shift our focus onto so called i.i.d.-sampling, which is a core idea supporting the Monte Carlo method. Sugita and Takanobu mentioned in [8] two facts on the i.i.d.-sampling, one of which says that sampling with large sample size provides us with a secure approximation for square integrable functions and the other says that the i.i.d.-sampling with large sample size responds the quality of the generated pseudorandom numbers, i.e., the statistical bias of them may largely be amplified and diminish the quality of the sampling method. We can find some advantage in a sequence of randomly generated points on the state space. However, it may create a problem arising

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from statistical bias.

For improving this dichotomous situation, Sugita and Takanobu focused on a sequence with a hybrid effect of random and algorithmic choice of sampling points and proposed a sampling method with the sequence $\{\{x + n\alpha\}\}_{n=0}^{\infty}$ each term of which is given as the fractional part $\{x + n\alpha\}$ of $x + n\alpha$ with a random initial value x and a random common difference α in the k-dimensional torus T^k.

As for the algorithmically generated points on $\mathbb{Z}_p$, it has been revealed in [3] that such non-random sequence of numbers as the sequence of non-negative integers in $\mathbb{Z}_p$ plays a similar role to the p-adic van der Corput sequence in the unit interval as traditionally studied in [5]. However, non-random sequence of sampling points could again result unsatisfactory rate of convergence, if the integrand has exceptional values at those sampling points.

One might imagine that the sequence $\{x + n\alpha\}_{n=0}^{\infty}$ with randomly taken initial value x and common difference α from $\mathbb{Z}_p$ gives us some hints. However, one fails to achieve this by simply using the sequence. In fact, when x and α is taken from a ball centered at zero with a small radius, the non-archimedean inequality shows that $\|x + n\alpha\|_p \le \max\{\|x\|_p, \|\alpha\|_p\}$. This fact results that the empirical average can not cover the value of the function at any points outside the ball centered at zero with radius $\max\{\|x\|_p, \|\alpha\|_p\}$.

In this article, instead of non-negative integers, we will use the p-adic van der Corput sequence for approximating the integral of functions on $\mathbb{Z}_p$. Let

$$D_p = \{0, 1, \dots, p-1\} \tag{1}$$

and

$$L = \left\{ \frac{a_{-1}}{p} + \cdots + \frac{a_{-M}}{p^M} \;\middle|\; M \in \{1, 2, \dots\}, a_{-1}, \dots, a_{-M} \in D_p \right\}. \tag{2}$$

We define the fractional part $\{x\}_p$ of $x \in \mathbb{Q}_p$ as a unique element $y \in L$ which satisfies $x - y \in \mathbb{Z}_p$. Accordingly, the integer part $[x]_p$ of $x \in \mathbb{Q}_p$ is defined by $[x]_p := x - \{x\}_p$.

In Section 2, we will see that, for any $\alpha \in \mathbb{Z}_p$, the subset $\{[n\alpha/p^m]_p \mid n \in \{0, 1, \dots, p^m - 1\}, m \in \{1, 2, \dots\}\}$ in the ring $\mathbb{Z}_p$ of p-adic integers is dense in the ring if and only if $\alpha \notin \mathbb{Q}$. This suggests that the p-adic van der Corput sequence provides us with a counterpart of Weyl's irrational rotation in $\mathbb{Z}_p$.

In the one-dimensional case in [8], the method with the Fourier series is employed based on the complete orthogonal system $\{e^{2\pi\sqrt{-1}kt}\}_{k\in\mathbb{Z}}$ in $L^2([0, 1))$, which is viewed as a sequence of periodic functions on the real line with period 1. The fundamental system of functions is used for extending domain of functions to the real line without removing the integer part of the variable of functions in the description. In accordance with the procedure, we will take a complete orthogonal system described by the additive characters on $\mathbb{Q}_p$ for extending the original domain $\mathbb{Z}_p$ of functions to the whole space $\mathbb{Q}_p$ without removing the fractional part of variable of functions in their description. For square integrable function f with respect to the Haar measure on $\mathbb{Z}_p$, we will examine the behavior of the sequence $\{(1/N)\sum_{k=0}^{N-1} f(x + x_k\alpha) - \int_{\mathbb{Z}_p} f(y)\,dy\}$ involving the k-th term x_k of the p-adic van der Corput sequence and f with the extended domain, under independently random choice of $x, \alpha \in \mathbb{Z}_p$ for achieving a similar result to random sampling method established by Sugita and Takanobu in [8]. We will finally be in a position to regard this approximation for $\int_{\mathbb{Z}_p} f(y)\,dy$ as the one based on a modified random Weyl sampling.

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2. Fundamental property of p-adic van der Corput sequence in $\mathbb{Z}_p$

For the random Weyl sampling in the unit interval in the real line based on Weyl's irrational rotation, one takes the sequence each term of which is given as the fractional part of the product of non-negative integer and a fixed irrational real number. As pointed out in Introduction, it is required to find some sequence in $\mathbb{Z}_p$ other than the one involving an irrational number as common difference for creating a similar effect to Weyl's irrational rotation. We will make an attempt of taking the integer part of the terms in the sequence in $\mathbb{Q}_p$ obtained as the product of a fixed number in $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$ and the p-adic van der Corput sequence.

Proposition 2.1. *Let $\alpha \in \mathbb{Z}_p$. The set*

$$U(\alpha) = \left\{ \left[\frac{n\alpha}{p^m} \right]_p \;\middle|\; n \in \{0, 1, \ldots, p^m - 1\},\ m \in \{1, 2, \ldots\} \right\}$$

is dense in $\mathbb{Z}_p$, if and only if $\alpha \notin \mathbb{Q}$.

Proof. Let $\alpha \notin \mathbb{Q}$. To prove of the denseness of $U(\alpha)$, it suffices to show $[n\alpha/p^m]_p$ can be an element of any given ball in $\mathbb{Z}_p$ with radius p^{-k} for some positive integer m and $n \in \{0, 1, \ldots, p^m - 1\}$. For that purpose, we employ the canonical representation $\alpha = \cdots a_n a_{n-1} \cdots a_1 a_0$ in [9]. If $U(-\alpha)$ is dense in $\mathbb{Z}_p$, so is $U(\alpha)$. Consequently, if it is necessary, we may prove the assertion by replacing α with $-\alpha$. This means that if $n > -\log_p|\alpha|_p$, the n-th digit a_n of $\alpha \in \mathbb{Z}_p \setminus \mathbb{Q}$ in its canonical representation is replaced with $p - 1 - a_n$, if $n = -\log_p|\alpha|_p$, a_n is replaced with $p - a_n$, and if $n < -\log_p|\alpha|_p$, a_n remains the same, i.e., $a_n = 0$, where $|\alpha|_p$ is the p-adic norm of α.

First we consider the case the canonical representation admits $k+1$ consecutive digits $a_{l+k+1} \neq 0$, $a_{l+k} = \cdots = a_{l+1} = 0$. Since the canonical representation for α/p^{k+l+1} gives $\alpha/p^{k+l+1} = \cdots a_{l+k+1}.\underbrace{0\cdots0}_{k} a_l a_{l-1} \cdots$ with $a_{l+k+1} \neq 0$, the set $\{[n\alpha/p^{k+l+1}]_p \mid n = 1, \ldots, p^k\}$ has an element in each of p^k balls with radius p^{-k} in $\mathbb{Z}_p$.

Second, in the case that a sequence of consecutive k digits $b_{l+k},\dots,b_{l+1}$ appears as $\alpha=\cdots a_{l+k+1}b_{l+k}\cdots b_{l+1}a_l\cdots$ and as $\alpha=\cdots a_{l'+k+1}b_{l+k}\cdots b_{l+1}a_{l'}\cdots$ with $a_{l+k+1}\neq a_{l'+k+1}$, $a_l>a_{l'}$ for some integers $l'>l>0$, it turns out that

$$\frac{\alpha}{p^{k+l+1}}-\frac{\alpha}{p^{k+l'+1}}=\cdots c_2c_1c_0.\underbrace{0\cdots0}_{k}c_{-k-1}c_{-k-2}\cdots$$

with some digits $\dots,c_2,c_1,c_0$ and $c_{-k-1},c_{-k-2},c_{-k-3},\dots$. Accordingly, the set $\{[n\alpha/p^m]_p\mid n\in\{0,1,\dots,p^m-1\},\ m\in\{1,2,\dots\}\}$ has at least one element in each of p^k balls with radius p^{-k} in $\mathbb{Z}_p$.

We can establish an algorithmic method for finding a sequence of consecutive k' digits $b_{l+k'},\dots,b_{l+1}$ which appears as $\alpha=\cdots a_{l+k'+1}b_{l+k'}\cdots b_{l+1}a_l\cdots$, and as $\alpha=\cdots a_{l'+k'+1}b_{l+k'}\cdots b_{l+1}a_{l'}\cdots$ in the canonical representation of α with $a_{l+k'+1}\neq a_{l'+k'+1}$ and $a_l>a_{l'}$ for some positive integer k' with $k'\geq k$ and some positive integers l', l with $l'>l$. For that purpose, we define a set of integers

$$I_k(b_{k+1},\dots,b_1)=\{l\in\{1,2,\dots\}\mid(a_{(l+1)k+1},\dots,a_{lk+1})=(b_{k+1},\dots,b_1)\}$$

which is determined by an integer $b_{k+1}p^{k+1}+\cdots+b_1p\in\mathbb{Z}_p$. We note that $I_k(b_{k+1},\dots,b_1)$ consists of infinitely many non-negative integers for some $b_{k+1}p^{k+1}+\cdots+b_1p\in\mathbb{Z}_p$. Therefore, we can take the increasing sequence $\{l_i\}_{i=0}^{\infty}$ consisting of all elements in $I_k(b_{k+1},\dots,b_1)$.

By using the increasing sequence, we can find a sequence of consecutive k digits $b_{k+1},\dots,b_1$ which appears as $\alpha=\cdots a_{(l_{i+1}+1)k+2}b_{k+1}\cdots b_1a_{l_{i+1}k}\cdots$ and as $\alpha=\cdots a_{(l_i+1)k+2}b_{k+1}\cdots b_1a_{l_ik}\cdots$ with some non-negative integer j satisfying $a_{l_ik-j}\neq a_{l_{i+1}k-j}$. In fact, if $a_{l_ik}=a_{l_{i+1}k}$, $a_{l_ik-1}=a_{l_{i+1}k-1}$, $a_{l_ik-2}=a_{l_{i+1}k-2},\dots,a_0=a_{l_{i+1}k-l_ik}$ for each i, it turns out that the sequence $\{a_n\}$ in the canonical representation $\alpha=\cdots a_na_{n-1}\cdots a_1a_0$ is eventually periodic. This contradicts the assumption $\alpha\notin\mathbb{Q}$ (see [4]). Consequently, we can find that $a_{l_ik-j}\neq a_{l_{i+1}k-j}$ with $a_{l_ik+1}=a_{l_{i+1}k+1}$, $a_{l_ik}=a_{l_{i+1}k}$, $a_{l_ik-1}=a_{l_{i+1}k-1},\dots,a_{l_ik-j+1}=a_{l_{i+1}k-j+1}$ for some sufficiently large l_ik-j. If it is necessary, by replacing α with $-\alpha$ and applying the canonical representation for $-\alpha$, we see that $\alpha=\cdots a_{(l_i+1)k+2}b_{k+1}\cdots b_1a_{l_ik}\cdots a_{l_ik-j+1}a_{l_ik-j}$ and as $\alpha=\cdots a_{(l_{i+1}+1)k+2}b_{k+1}\cdots b_1a_{l_ik}\cdots a_{l_ik-j+1}a_{l_{i+1}k-j}$ with $a_{l_ik-j}>a_{l_{i+1}k-j}$ in the canonical representation of α. Since α is not a rational number, we see $\{0,1,\dots\}\setminus I_k(b_{k+1},\dots,b_1)$ contains infinity many positive integers. Therefore, we see $a_{(l_i+1)k+j'+1}\neq a_{(l_{i+1}+1)k+j'+1}$ for some positive integer j'. This shows that the set $\{[n\alpha/p^m]_p\mid n\in\{0,1,\dots,p^m-1\},\ m\in\{1,2,\dots\}\}$ has at least one element in each of p^k balls with radius p^{-k} in $\mathbb{Z}_p$.

Conversely, we assume that $U(\alpha)$ is dense in $\mathbb{Z}_p$. Let $\alpha\in\mathbb{Q}\setminus\{0\}$. Then α is uniquely represented by $\alpha=a/b$ with mutually prime non-zero rational integers a and b, where we may assume $b>0$. Since $\alpha\in\mathbb{Z}_p$, b is not divisible by p and $1/b\in\mathbb{Z}_p$. The assumption implies that there exist some integer $n_k\in\{1,\dots,p^{m_k}-1\}$ and positive integer m_k such that $[n_k\alpha/p^{m_k}]_p$ is some element in the ball centered at p^{k-1}/b with radius p^k for any positive integer k. Therefore, by taking some $\nu_k\in\{0,\dots,p^{m_k}-1\}$ and

p-adic integer N_k, we obtain $(n_k/p^{m_k})\alpha - \nu_k/p^{m_k} = p^{k-1}/b + N_k p^k$. This implies that bN_k is a rational integer and $a/b = \nu_k/n_k + ((1+bN_k p)/bn_k)p^{k-1}p^{m_k}$. If $\#\{k \mid bN_k \geq 0\} = \infty$, then $a/b \geq (1/bn_k)p^{k-1}p^{m_k}$ for infinitely many k. Otherwise, $a/b \leq p^{m_k}/n_k - p^{k-1}p^{m_k}/bn_k$ for infinitely many k. In either case, we have a contradiction. □

3. A modified random Weyl sampling on $\mathbb{Z}_p$

In this section, we will present some results for establishing a reasonable modified random Weyl sampling on $\mathbb{Z}_p$ by taking the results of the previous section into account. In what follows, the Haar measure on $\mathbb{Z}_p$ will be denoted by μ and assumed to be normalized as $\mu(\mathbb{Z}_p) = 1$. The integral of complex valued integrable function f on $\mathbb{Z}_p$ with respect to the Haar measure will be denoted by $\int_{\mathbb{Z}_p} f(y)\,dy$.

DEFINITION 3.1. A random element y in $\mathbb{Z}_p$ is said to be uniformly distributed if $P(y \in E) = \mu(E)$ for any topological Borel subset E in $\mathbb{Z}_p$. For any complex-valued square integrable function f on $\mathbb{Z}_p$, the variance $\mathrm{Var}(f)$ of the function is defined by $\mathrm{Var}(f) = \int_{\mathbb{Z}_p} \left| f(x) - \int_{\mathbb{Z}_p} f(y)\,dy \right|^2 dx$.

We introduce the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{Z}_p} f(x) e^{2\sqrt{-1}\pi\{\xi x\}_p}\,dx, \quad \xi \in \mathbb{Q}_p,$$

for any complex valued square integrable function f. Then, the original function f is restored as

$$f(x) = \int_{\mathbb{Q}_p} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p}\,d\xi, \quad x \in \mathbb{Z}_p,$$

by performing the inverse Fourier transform (see Chapter 1, VIII in [9]). Since f can be regarded as a function on $\mathbb{Q}_p$ whose support is contained in $\mathbb{Z}_p$, i.e., the ball $B(0,1)$ centered at the origin and with radius 1, $\hat{f}$ takes a constant on every ball with radius 1 as seen in Chapter 1, VII in [9]. Accordingly, another representation of the function is given as

$$\begin{aligned} f(x) &= \sum_{\xi \in L} \hat{f}(\xi) \int_{B(0,1)} e^{-2\sqrt{-1}\pi\{(\xi+\eta)x\}_p}\,d\eta \\ &= \sum_{\xi \in L} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p}, \quad x \in \mathbb{Z}_p, \end{aligned}$$

where L has been defined by (2).

As for the additive character $\chi(\xi t) = e^{2\sqrt{-1}\pi\{\xi t\}_p}$, we easily observe that $\int_{\mathbb{Z}_p} |\chi(\xi t)|^2\,dt = \int_{\mathbb{Z}_p} \chi(\xi t)\chi(-\xi t)\,dt = \int_{\mathbb{Z}_p} \chi((\xi - \xi)t)\,dt = 1$ for any $\xi \in L$ and

$\int_{\mathbb{Z}_p} \chi(\xi t)\bar{\chi}(\eta t)\,dt = \int_{\mathbb{Z}_p} \chi((\xi - \eta)t)\,dt = 0$ for any pair of distinct $\xi, \eta \in L$. From these identities, we can derive $\int_{\mathbb{Z}_p} |f(t)|^2\,dt = \sum_{\xi \in L} |\hat{f}(\xi)|^2$. Consequently, we see that the additive characters provide us with the complete orthonormal system $\{\chi(\xi t)\}_{\xi \in L}$ in $L^2(\mathbb{Z}_p, \mu)$.

A natural extension of the domain of f to $\mathbb{Q}_p$ is performed by

$$f(x) = \sum_{\xi \in L} \hat{f}(\xi) e^{-2\sqrt{-1}\pi\{\xi x\}_p} = \sum_{\xi \in L} \hat{f}(\xi)\chi(-\xi x), \quad x \in \mathbb{Q}_p.$$

For any positive integer M, we take function f_M on $\mathbb{Q}_p$ defined by

$$\begin{aligned} f_M(x) &= \sum_{\xi \in L \cap B(0,p^M)} \hat{f}(\xi) \int_{B(0,1)} e^{-2\sqrt{-1}\pi\{(\xi+\eta)x\}_p}\,d\eta \\ &= \sum_{\xi \in L \cap B(0,p^M)} \hat{f}(\xi)\chi(-\xi x), \quad x \in \mathbb{Q}_p, \end{aligned} \tag{3}$$

where $B(0, p^M)$ stands for the ball centered at the origin with radius p^M.

We recall that the sequence

$$x_k = \frac{d_0}{p} + \cdots + \frac{d_l}{p^{l+1}} \quad (k = 0, 1, 2, \ldots)$$

determined by the p-adic expansion $k = d_0 + \cdots + d_l p^l$ of non-negative integer k constitutes the p-adic van der Corput sequence in [6]. Hereafter, we will take the p-adic van der Corput sequence $\{x_k\}_{k=0}^{\infty}$. We will focus only on $f(x + x_k\alpha)$ and $f_M(x + x_k\alpha)$ instead of $f(x + [x_k\alpha]_p)$ and $f_M(x + [x_k\alpha]_p)$ respectively without removing fractional part of variables, similarly to the method of extending domain of functions in [8] without removing integer part of the variables.

In the first theorem in this section, we will consider the sequence $\bigl\{(1/N)\sum_{k=0}^{N-1} f(x+x_k\alpha)\bigr\}_{N=1}^{\infty}$ with uniformly distributed independent random variables α and x on $\mathbb{Z}_p$. Our main interest is under what condition the sequence $\bigl\{(1/\sqrt{N})\sum_{k=0}^{N-1}\bigl(f(x + x_k\alpha) - \int_{\mathbb{Z}_p} f(y)\,dy\bigr)\bigr\}_{N=1}^{\infty}$ with a slower growth rate in the denominator can be expected to converge to zero as $N \to \infty$ for any square integrable function f on $\mathbb{Z}_p$. In the present article, the method focusing on the sequence for fast approximation to the integral $\int_{\mathbb{Z}_p} f(y)\,dy$ by the empirical average $(1/N)\sum_{k=0}^{N-1} f(x + x_k\alpha)$ is called modified random Weyl sampling on $\mathbb{Z}_p$.

We see that the modified random Weyl sampling on $\mathbb{Z}_p$ has the robustness in the sense in [7] as the method proposed by Sugita and Takanobu.

Lemma 3.2. *Let f be a complex valued function in $L^2(\mathbb{Z}_p, \mu)$ and $\xi \in L \setminus \{0\}$. Then,*

(i) $f(x+\alpha\xi) \in L^2(\mathbb{Z}_p \times \mathbb{Z}_p, \mu \times \mu)$,
(ii) $\lim_{M\to\infty} \| f(x+\alpha\xi) - f_M(x+\alpha\xi)\|_{L^2(\mathbb{Z}_p\times\mathbb{Z}_p,\mu\times\mu)} = 0$,
(iii) $\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} f(x+\alpha\xi)\,dx\,d\alpha = \int_{\mathbb{Z}_p} f(y)\,dy$, $\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} | f(x+\alpha\xi) - \int_{\mathbb{Z}_p} f(y)\,dy|^2\,dx\,d\alpha =$ Var(f),
(iv) $\xi' \in L \setminus \{0\}$ *and* $\xi' \neq \xi$ *imply*

$$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left(f(x+\alpha\xi) - \int_{\mathbb{Z}_p} f(y)\,dy \right) \overline{\left(g(x+\alpha\xi') - \int_{\mathbb{Z}_p} g(y)\,dy \right)}\,dx\,d\alpha = 0,$$

for any complex valued function g in $L^2(\mathbb{Z}_p, \mu)$.

Proof. For any $f \in L^2(\mathbb{Z}_p, \mu)$, we can take a sequence $\{M_j\}_{j=1}^{\infty}$ of positive integers such that

$$f(x) = \lim_{j\to\infty} f_{M_j}(x) = \lim_{j\to\infty} \sum_{\eta\in L\cap B(0,p^{M_j})} \hat{f}(\eta)\chi(-\eta x) \quad \mu\text{-a.e.} \quad x \in \mathbb{Q}_p.$$

This implies that

$$\begin{aligned} f(x+\alpha\xi) &= \lim_{j\to\infty} \sum_{\eta\in L\cap B(0,p^{M_j})} \hat{f}(\eta)\chi(-\eta(x+\alpha\xi)) \\ &= \lim_{j\to\infty} \sum_{\eta\in L\cap B(0,p^{M_j})} \hat{f}(\eta)\chi(-\eta\alpha\xi)\chi(-\eta x) \quad \mu\times\mu\text{-a.e. } (x,\alpha) \end{aligned}$$

from which we can derive (i) and (ii). In fact, for any $\xi \in L \setminus \{0\}$, (ii) is obtained by replacing $| f(x+\alpha\xi)|^2$ with $| f(x+\alpha\xi) - f_M(x+\alpha\xi)|^2$ as follows:

$$\begin{aligned} &\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} | f(x+\alpha\xi) - f_M(x+\alpha\xi)|^2\,dx\,d\alpha \\ &= \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \lim_{j\to\infty} \Bigg| \sum_{\eta\in L\cap B(0,p^{M_j})} \hat{f}(\eta)\chi(-\eta\alpha\xi)\chi(-\eta x) \\ &\qquad\qquad - \sum_{\eta\in L\cap B(0,p^{M})} \hat{f}(\eta)\chi(-\eta\alpha\xi)\chi(-\eta x)\Bigg|^2\,dx\,d\alpha \\ &\le \liminf_{j\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \Bigg| \sum_{\eta\in L\cap(B(0,p^{M_j})\setminus B(0,p^M))} \hat{f}(\eta)\chi(-\eta\alpha\xi)\chi(-\eta x)\Bigg|^2\,dx\,d\alpha \\ &= \liminf_{j\to\infty} \int_{\mathbb{Z}_p} d\alpha \int_{\mathbb{Z}_p} \sum_{\eta\in L\cap(B(0,p^{M_j})\setminus B(0,p^M))} | \hat{f}(\eta)\chi(-\eta\alpha\xi)\chi(-\eta x)|^2\,dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{j\to\infty} \sum_{\eta\in L\cap(B(0,p^{M_j})\setminus B(0,p^M))} |\hat{f}(\eta)|^2 \\
&\leq \sum_{\eta\in L\setminus B(0,p^M)} |\hat{f}(\eta)|^2 \to 0 \quad \text{as } M\to\infty.
\end{aligned}
$$

The square integrability (i) is shown by the following finiteness which is given by replacing $B(0, p^M)$ with the empty set in the preceding estimates:

$$
\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} |f(x+\alpha\xi)|^2\,dx\,d\alpha \leq \sum_{\eta\in L} |\hat{f}(\eta)|^2 < \infty.
$$

(iii) Thanks to (ii), we see that

$$
\begin{aligned}
&\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} f(x+\alpha\xi)\,dx\,d\alpha \\
&= \lim_{M\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} f_M(x+\alpha\xi)\,dx\,d\alpha \\
&= \lim_{M\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \sum_{\eta\in L\cap B(0,p^M)} \hat{f}(\eta)\chi(-\eta(x+\alpha\xi))\,dx\,d\alpha \\
&= \lim_{M\to\infty} \sum_{\eta\in L\cap B(0,p^M)} \hat{f}(\eta) \int_{\mathbb{Z}_p} \chi(-\eta x)\,dx \int_{\mathbb{Z}_p} \chi(-\eta\alpha\xi)\,d\alpha \\
&= \lim_{M\to\infty} \sum_{\eta\in L\cap B(0,p^M)} \hat{f}(\eta)\,\delta_{\eta,0}\,\delta_{\eta\xi,0} \\
&= \hat{f}(0) \\
&= \int_{\mathbb{Z}_p} f(y)\,dy.
\end{aligned}
$$

The first identity of (iii) is proved. The second identity of (iii) and the one in (iv) follow from this identity. In fact, we observe that

$$
\begin{aligned}
&\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left(f(x+\alpha\xi) - \int_{\mathbb{Z}_p} f(y)\,dy\right)\overline{\left(g(x+\alpha\xi') - \int_{\mathbb{Z}_p} g(y)\,dy\right)}\,dx\,d\alpha \\
&= \lim_{M\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} (f_M(x+\alpha\xi) - \hat{f}(0))\overline{(g_M(x+\alpha\xi') - \hat{g}(0))}\,dx\,d\alpha \\
&= \lim_{M\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left(\sum_{\eta\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\chi(-\eta(x+\alpha\xi))\right) \\
&\qquad\qquad \times \overline{\left(\sum_{\eta'\in(L\setminus\{0\})\cap B(0,p^M)} \hat{g}(\eta')\chi(-\eta'(x+\alpha\xi'))\right)}\,dx\,d\alpha
\end{aligned}
$$

$$= \lim_{M\to\infty} \sum_{\eta,\eta'\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\overline{\hat{g}(\eta')} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \chi(-\eta x)\chi(-\eta\alpha\xi) \times \overline{\chi(-\eta' x)\chi(-\eta'\alpha\xi')}\, dx\, d\alpha$$

$$= \lim_{M\to\infty} \sum_{\eta,\eta'\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\overline{\hat{g}(\eta')} \int_{\mathbb{Z}_p} \chi(-\eta x)\overline{\chi(-\eta' x)}\, dx \times \int_{\mathbb{Z}_p} \chi(-\eta\alpha\xi)\overline{\chi(-\eta'\alpha\xi')}\, d\alpha$$

$$= \lim_{M\to\infty} \sum_{\eta,\eta'\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\overline{\hat{g}(\eta')}\, \delta_{\eta,\eta'}\, \delta_{\eta\xi,\eta'\xi'}$$

$$= \lim_{M\to\infty} \sum_{\eta\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\overline{\hat{g}(\eta)}\, \delta_{\eta\xi,\eta\xi'}$$

$$= \lim_{M\to\infty} \left(\sum_{\eta\in(L\setminus\{0\})\cap B(0,p^M)} \hat{f}(\eta)\overline{\hat{g}(\eta)} \right) \delta_{\xi,\xi'}$$

$$= \begin{cases} \int_{\mathbb{Z}_p} (f(y) - \hat{f}(0))\overline{(g(y)-\hat{g}(0))}\, dy & \text{if } \xi = \xi', \\ 0 & \text{if } \xi \neq 0\xi'. \end{cases} \qquad \square$$

Theorem 3.3. *For any complex valued function* $f \in L^2(\mathbb{Z}_p, \mu)$,

$$\left\{ f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right\}_{n=0}^{\infty}$$

constitute an orthonormal family in $L^2(\mathbb{Z}_p \times \mathbb{Z}_p, \mu\times\mu)$ *satisfying*

$$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right|^2 dx\, d\alpha = \mathrm{Var}(f).$$

In particular,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x+\alpha x_n) = \int_{\mathbb{Z}_p} f(y)\, dy \quad \mu\times\mu\text{-a.e. } (x,\alpha),$$

and for any positive integer N,

$$\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| \frac{1}{N} \sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right|^2 dx\, d\alpha = \frac{1}{N}\mathrm{Var}(f).$$

Proof. The assertions on the family $\left\{ f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right\}$ of functions with two variables x and α follow from the previous lemma. As seen in [1] and [2], by applying general theory, we see the validity of $\lim_{N\to\infty}(1/N)\sum_{n=0}^{N-1} f(x+\alpha x_n) = \int_{\mathbb{Z}_p} f(y)\,dy$ at $\mu\times\mu$-almost every (x,α). The final identity in the assertion is proved by the following observation:

$$
\begin{aligned}
&\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\frac{1}{N}\sum_{n=0}^{N-1} f(x+\alpha x_n)-\int_{\mathbb{Z}_p} f(y)\,dy\right|^2 dx\,d\alpha\\
&=\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\frac{1}{N}\sum_{n=0}^{N-1}\left(f(x+\alpha x_n)-\int_{\mathbb{Z}_p} f(y)\,dy\right)\right|^2 dx\,d\alpha\\
&=\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\frac{1}{N^2}\sum_{n,n'=0}^{N-1}\left(f(x+\alpha x_n)-\int_{\mathbb{Z}_p} f(y)\,dy\right)\\
&\qquad\qquad\times\overline{\left(f(x+\alpha x_{n'})-\int_{\mathbb{Z}_p} f(y)\,dy\right)}\,dx\,d\alpha\\
&=\frac{1}{N^2}\sum_{n,n'=0}^{N-1}\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left(f(x+\alpha x_n)-\int_{\mathbb{Z}_p} f(y)\,dy\right)\\
&\qquad\qquad\times\overline{\left(f(x+\alpha x_{n'})-\int_{\mathbb{Z}_p} f(y)\,dy\right)}\,dx\,d\alpha\\
&=\frac{1}{N^2}\sum_{n,n'=0}^{N-1}\delta_{n,n'}\operatorname{Var}(f)\\
&=\frac{1}{N}\operatorname{Var}(f). \qquad \square
\end{aligned}
$$

Let us recall $D_p=\{0,1,2,\dots,p-1\}$ (cf. (1)). To each integer N with $N-1\ge p^2$, there corresponds a unique pair of sequence of integers $h_1>h_2>\cdots>h_s\ge 0$ and $r_1,r_2,\dots,r_s\in D_p\setminus\{0\}$ such that

$$N-1=r_1p^{h_1}+r_2p^{h_2}+\cdots+r_sp^{h_s}, \tag{4}$$

where $s=\max\{l\in\{1,2,\dots\}\mid N-1\ge p^l\}$. We let (ν_1,ν_2) denote the greatest common divisor of integers ν_1 and ν_2.

Lemma 3.4. *For any q with $1<q<2$, any $\xi\in L\setminus\{0\}$, and any $N\in\{1,2,\dots\}$ with $N-1\ge p^2$,*

(i) *under the correspondence* (4)

$$\int_{\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\chi(\xi\alpha x_n)\right|^q d\alpha$$
$$\le \frac{1}{\sqrt{N}}+\frac{1}{\sqrt{N}}\sum_{j=1}^{s}\left(\int_{\mathbb{Z}_p}\left|\sum_{a=0}^{r_j-1}\sum_{n=0}^{p^{h_j}-1} e^{2\sqrt{-1}\pi a\{\xi\alpha/p^{h_j+1}\}_p}e^{2\sqrt{-1}\pi n\{\xi\alpha/p^{h_j}\}_p}\right|^q d\alpha\right)^{1/q},$$

where $s=\max\{l \mid l$ *is positive integer satisfying* $N-1\ge p^l\}$,
(ii) *for any* $j\in\{1,\dots,s\}$ *and* $\xi\in L\setminus\{0\}$ *represented as* $\xi=\nu/p^M$ *with* $\nu\in\{0,\dots,p^M-1\}$ *and* $(\nu,\,p)=1$,

$$\int_{\mathbb{Z}_p}\left|\sum_{a=0}^{r_j-1}\sum_{n=0}^{p^{h_j}-1} e^{2\sqrt{-1}\pi a\{\xi\alpha/p^{h_j+1}\}_p}e^{2\sqrt{-1}\pi n\{\xi\alpha/p^{h_j}\}_p}\right|^q d\alpha$$
$$=\frac{1}{p^M}\frac{1}{p^{h_j+1}}\left((r_jp^{h_j})^q+\sum_{c_0\in D_p\setminus\{0\}}\left|\frac{\sin(\pi r_jc_0/p)}{\sin(\pi c_0/p)}p^{h_j}\right|^q\right.$$
$$+\sum_{l=h_j+1}^{M+h_j}\sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}}\left|\frac{\sin\pi r_j(c_0+c_1p+\cdots+c_lp^l)/p^{l+1}}{\sin\pi(c_0+c_1p+\cdots+c_lp^l)/p^{l+1}}\right.$$
$$\left.\left.\times\frac{\sin\pi p^{h_j}(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l}{\sin\pi(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l}\right|^q\right),$$

(iii) *for any positive integer* M, *non-negative integer* h *and* $r\in D_p$,

$$\sum_{l=h+1}^{M+h}\sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}}\left|\frac{\sin\pi r(c_0+c_1p+\cdots+c_lp^l)/p^{l+1}}{\sin\pi(c_0+c_1p+\cdots+c_lp^l)/p^{l+1}}\right.$$
$$\left.\times\frac{\sin\pi p^h(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l}{\sin\pi(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l}\right|^q$$
$$\le p|r|^q\sum_{l=h+1}^{M+h}\sum_{a=1}^{p^l-1}\frac{1}{|\sin\pi a/p^l|^q}$$
$$\le (p-1)^q\frac{Mp^{Mq}\zeta(q)}{2^{q-1}}p^{hq+1},$$

where ζ *stands for the Riemann zeta function, i.e.,* $\zeta(z)=\sum_{n=1}^{\infty}1/n^z$.

Proof. We can divide the set $\{0, 1, \dots, N-1\}$ of consecutive integers into the disjoint subsets $I_1, \dots, I_{s+1}$ of consecutive integers defined by

$$\begin{aligned}
I_1 &= \{0, 1, \dots, r_1 p^{h_1} - 1\},\\
I_2 &= \{r_1 p^{h_1}, \dots, r_1 p^{h_1} + r_2 p^{h_2} - 1\},\\
&\dots\\
I_j &= \{r_1 p^{h_1} + \dots + r_{j-1} p^{h_{j-1}}, \dots, r_1 p^{h_1} + \dots + r_j p^{h_j} - 1\},\\
&\dots\\
I_s &= \{r_1 p^{h_1} + \dots + r_{s-1} p^{h_{s-1}}, \dots, r_1 p^{h_1} + \dots + r_s p^{h_s} - 1\}
\end{aligned}$$

and

$$I_{s+1} = \{r_1 p^{h_1} + \dots + r_s p^{h_s}\} = \{N-1\}.$$

To each integer $m_j \in \{0, 1, \dots, p^{h_j} - 1\}$, there corresponds a sequence of integers $a_0, a_1, \dots, a_{h_j - 1} \in D_p$ such that

$$m_j = a_0 p^{h_j - 1} + a_1 p^{h_j - 2} + \dots + a_{h_j - 1} p^0. \tag{5}$$

(i) If $n \in I_1$, then $n = a_0 + a_1 p^1 + \dots + a_{h_1 - 1} p^{h_1 - 1} + a_{h_1} p^{h_1}$ for some $a_0, a_1, \dots, a_{h_1 - 1} \in D_p$ and $a_{h_1} \in \{0, \dots, r_1 - 1\}$. Accordingly, the integer n assigns a fractional number

$$\begin{aligned}
x_n &= \frac{a_0}{p} + \frac{a_1}{p^2} + \dots + \frac{a_{h_1 - 1}}{p^{h_1}} + \frac{a_{h_1}}{p^{h_1 + 1}}\\
&= \frac{m_1}{p^{h_1}} + \frac{a_{h_1}}{p^{h_1 + 1}} \quad (0 \le m_1 \le p^{h_1} - 1,\ 0 \le a_{h_1} \le r_1 - 1).
\end{aligned}$$

If $n \in I_j$ with some $2 \le j \le s$, we see

$$r_1 p^{h_1} + \dots + r_{j-1} p^{h_{j-1}} \le n < r_1 p^{h_1} + \dots + r_j p^{h_j},$$

equivalently, $0 \le n - (r_1 p^{h_1} + \dots + r_{j-1} p^{h_{j-1}}) < r_j p^{h_j}$. Consequently, by taking $a_0, a_1, \dots, a_{h_j - 1} \in D_p$ and $a_{h_j} \in \{0, \dots, r_j - 1\}$, we see

$$n - (r_1 p^{h_1} + \dots + r_{j-1} p^{h_{j-1}}) = a_0 p^0 + a_1 p^1 + \dots + a_{h_j - 1} p^{h_j - 1} + a_{h_j} p^{h_j}.$$

Such expression of n as

$$n = a_0 + a_1 p^1 + \dots + a_{h_j - 1} p^{h_j - 1} + a_{h_j} p^{h_j} + (r_{j-1} p^{h_{j-1}} + \dots + r_1 p^{h_1})$$

provides us with a representation of the fractional number x_n written as

$$\begin{aligned} x_n &= \frac{a_0}{p} + \frac{a_1}{p^2} + \cdots + \frac{a_{h_{j-1}}}{p^{h_j}} + \frac{a_{h_j}}{p^{h_j+1}} + \frac{r_{j-1}}{p^{h_{j-1}+1}} + \cdots + \frac{r_1}{p^{h_1+1}} \\ &= \frac{m_j}{p^{h_j}} + \frac{a_{h_j}}{p^{h_j+1}} + R_j \end{aligned}$$

under the correspondence (5), where $a_{h_j} \in \{0, 1, \ldots, r_j - 1\}$ and

$$R_j = \frac{r_{j-1}}{p^{h_{j-1}+1}} + \cdots + \frac{r_1}{p^{h_1+1}}.$$

If $n \in I_{s+1}$, then

$$n = r_s p^{h_s} + \cdots + r_1 p^{h_1} = N - 1.$$

This implies that

$$x_n = \frac{r_s}{p^{h_s+1}} + \cdots + \frac{r_1}{p^{h_1+1}} = R_{s+1}.$$

From these observations, we can derive that, for any $\alpha \in \mathbb{Z}_p$,

$$\begin{aligned} &\sum_{n=0}^{N-1} \chi(\xi \alpha x_n) \\ &= \sum_{n=0}^{N-1} e^{2\sqrt{-1}\pi\{\xi\alpha x_n\}_p} \\ &= \sum_{n\in I_1} e^{2\sqrt{-1}\pi\{\xi\alpha x_n\}_p} + \sum_{j=2}^{s}\sum_{n\in I_j} e^{2\sqrt{-1}\pi\{\xi\alpha x_n\}_p} + \sum_{n\in I_{s+1}} e^{2\sqrt{-1}\pi\{\xi\alpha x_n\}_p} \\ &= \sum_{m_1=0}^{p^{h_1-1}} \sum_{a_{h_1}=0}^{r_1-1} e^{2\sqrt{-1}\pi\{\xi\alpha(m_1/p^{h_1}+a_{h_1}/p^{h_1+1})\}_p} \\ &\quad + \sum_{j=2}^{s} \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi\{\xi\alpha(m_j/p^{h_j}+a_{h_j}/p^{h_j+1}+R_j)\}_p} + e^{2\sqrt{-1}\pi\{\xi\alpha R_{s+1}\}_p}. \end{aligned}$$

Therefore, by applying Minkowski's inequality, we can conclude that

$$\begin{aligned}
&\left(\int_{\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}\sum_{n=1}^{N-1}\chi(\xi\alpha x_n)\right|^q d\alpha\right)^{1/q}\\
&=\Bigg(\int_{\mathbb{Z}_p}\Bigg|\frac{1}{\sqrt{N}}\Bigg(\sum_{m_1=0}^{p^{h_1-1}}\sum_{a_{h_1}=0}^{r_1-1} e^{2\sqrt{-1}\pi m_1\{\xi\alpha/p^{h_1}\}_p}e^{2\sqrt{-1}\pi a_{h_1}\{\xi\alpha/p^{h_1+1}\}_p}\\
&\qquad+\sum_{j=2}^{s}\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j\{\xi\alpha/p^{h_j}\}_p}e^{2\sqrt{-1}\pi a_{h_j}\{\xi\alpha/p^{h_j+1}\}_p}e^{2\sqrt{-1}\pi\{\xi\alpha R_j\}_p}\\
&\qquad+e^{2\sqrt{-1}\pi\{\xi\alpha R_{s+1}\}_p}\Bigg)\Bigg|^q d\alpha\Bigg)^{1/q}\\
&\le\left(\int_{\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}(e^{2\sqrt{-1}\pi\{\xi\alpha R_{s+1}\}_p})\right|^q d\alpha\right)^{1/q}\\
&\quad+\left(\int_{\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}\sum_{m_1=0}^{p^{h_1-1}}\sum_{a_{h_1}=0}^{r_1-1} e^{2\sqrt{-1}\pi m_1\{\xi\alpha/p^{h_1}\}_p}e^{2\sqrt{-1}\pi a_{h_1}\{\xi\alpha/p^{h_1+1}\}_p}\right|^q d\alpha\right)^{1/q}\\
&\quad+\sum_{j=2}^{s}\Bigg(\int_{\mathbb{Z}_p}\Bigg|\frac{1}{\sqrt{N}}\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j\{\xi\alpha/p^{h_j}\}_p}\\
&\qquad\qquad\times e^{2\sqrt{-1}\pi a_{h_j}\{\xi\alpha/p^{h_j+1}\}_p}e^{2\sqrt{-1}\pi\{\xi\alpha R_j\}_p}\Bigg|^q d\alpha\Bigg)^{1/q}\\
&\le\frac{1}{\sqrt{N}}+\frac{1}{\sqrt{N}}\sum_{j=1}^{s}\left(\int_{\mathbb{Z}_p}\left|\sum_{n=0}^{p^{h_j-1}}\sum_{a=0}^{r_j-1}e^{2\sqrt{-1}\pi a\{\xi\alpha/p^{h_j+1}\}_p}e^{2\sqrt{-1}\pi n\{\xi\alpha/p^{h_j}\}_p}\right|^q d\alpha\right)^{1/q}.
\end{aligned}$$

(ii) For any $j\in\{1,\ldots,s\}$, we first note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left|\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j\{\xi\alpha/p^{h_j}\}_p}e^{2\sqrt{-1}\pi a_{h_j}\{\xi\alpha/p^{h_j+1}\}_p}\right|^q d\alpha\\
&=\int_{\mathbb{Q}_p}\mathbf{1}_{\mathbb{Z}_p}(\alpha)\left|\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j\{\xi\alpha/p^{h_j}\}_p}e^{2\sqrt{-1}\pi a_{h_j}\{\xi\alpha/p^{h_j+1}\}_p}\right|^q d\alpha.
\end{aligned}$$

Since ξ is represented by $\xi=\nu/p^M$ with some $\nu\in\{1,\ldots,p^M-1\}$ satisfying $(\nu,p)=1$, by taking variable β given by $\beta=\xi\alpha$, we can perform the integral with respect to the variable β on $B(0, p^M)$. Accordingly, it turns out that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\xi\alpha/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\xi\alpha/p^{h_j+1}\}_p} \right|^q d\alpha \\
&= \int_{\mathbb{Q}_p} \mathbf{1}_{\mathbb{Z}_p}\left(\frac{\beta}{\xi}\right) \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\beta/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\beta/p^{h_j+1}\}_p} \right|^q \frac{d\beta}{|\xi|_p} \\
&= \frac{1}{p^M} \int_{B(0,p^M)} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\beta/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\beta/p^{h_j+1}\}_p} \right|^q d\beta \\
&= \frac{1}{p^M} \Bigg(\int_{B(0,p^{-h_j-1})} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\beta/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\beta/p^{h_j+1}\}_p} \right|^q d\beta \\
&\qquad + \sum_{k=-h_j-1}^{M-1} \int_{S(0,p^{k+1})} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\beta/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\beta/p^{h_j+1}\}_p} \right|^q d\beta \Bigg).
\end{aligned}$$

Here and in the sequel, $S(0, p^{k+1}) = B(0, p^{k+1}) \setminus B(0, p^k)$. We can perform change of variables given by $\gamma = p^{-h_j-1}\beta$ in the first integral and change of variables given by $\gamma = p^{k+1}\beta$ in the second integral to obtain

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\xi\alpha/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\xi\alpha/p^{h_j+1}\}_p} \right|^q d\alpha \\
&= \frac{1}{p^M} \Bigg(\int_{\mathbb{Z}_p} (r_j p^{h_j})^q \frac{d\gamma}{p^{h_j+1}} \\
&\qquad + \sum_{k=-h_j-1}^{M-1} \int_{S(0,1)} \Bigg| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\gamma/p^{k+1+h_j}\}_p} \\
&\qquad\qquad \times e^{2\sqrt{-1}\pi a_{h_j} \{\gamma/p^{k+1+h_j+1}\}_p} \Bigg|^q p^{k+1}\, d\gamma \Bigg)
\end{aligned}$$

$$= \frac{1}{p^M}\Bigg(\frac{(r_j p^{h_j})^q}{p^{h_j+1}} + \sum_{k=-h_j-1}^{M-1} p^{k+1} \int_{S(0,1)} \Bigg| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\gamma/p^{k+1+h_j}\}_p} \times e^{2\sqrt{-1}\pi a_{h_j}\{\gamma/p^{k+1+h_j+1}\}_p}\Bigg|^q d\gamma\Bigg)$$

$$= \frac{1}{p^M}\Bigg(\frac{(r_j p^{h_j})^q}{p^{h_j+1}} + \sum_{l=0}^{M+h_j} p^{l-h_j} \int_{S(0,1)} \Bigg| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\gamma/p^{l}\}_p} e^{2\sqrt{-1}\pi a_{h_j}\{\gamma/p^{l+1}\}_p}\Bigg|^q d\gamma\Bigg)$$

$$= \frac{1}{p^M}\Bigg(\frac{(r_j p^{h_j})^q}{p^{h_j+1}} + \sum_{l=0}^{M+h_j} p^{l-h_j} \sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}} \int_{B(c_0+c_1p+\cdots+c_lp^l,\,p^{l+1})} \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} |e^{2\sqrt{-1}\pi m_j (c_0/p^l+c_1/p^{l-1}+\cdots+c_{l-1}/p)} \times e^{2\sqrt{-1}\pi a_{h_j}(c_0/p^{l+1}+c_1/p^l+\cdots+c_l/p)}|^q d\gamma\Bigg)$$

$$= \frac{1}{p^M}\Bigg(\frac{(r_j p^{h_j})^q}{p^{h_j+1}} + \sum_{l=0}^{M+h_j} \frac{1}{p^{h_j+1}} \sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}} \Bigg| \sum_{m_j=0}^{p^{h_j-1}} e^{2\sqrt{-1}\pi m_j((c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l)} \times \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi a_{h_j}((c_0+c_1p+\cdots+c_lp^l)/p^{l+1})}\Bigg|^q\Bigg)$$

$$
\begin{aligned}
&= \frac{1}{p^M}\frac{1}{p^{h_j+1}}\Bigg((r_j p^{h_j})^q + \sum_{c_0\in D_p\setminus\{0\}}\Bigg|\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j c_0} e^{2\sqrt{-1}\pi a_{h_j} c_0/p}\Bigg|^q \\
&\qquad + \sum_{l=1}^{M+h_j}\sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}}\Bigg|\sum_{m_j=0}^{p^{h_j-1}} e^{2\sqrt{-1}\pi m_j((c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l)} \\
&\qquad\qquad \times \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi a_{h_j}((c_0+c_1p+\cdots+c_lp^l)/p^{l+1})}\Bigg|^q\Bigg) \\
&= \frac{1}{p^M}\frac{1}{p^{h_j+1}}\Bigg((r_j p^{h_j})^q + \sum_{c_0\in D_p\setminus\{0\}}\Bigg|\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi a_{h_j} c_0/p} p^{h_j}\Bigg|^q \\
&\qquad + \sum_{l=1}^{M+h_j}\sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}}\Bigg|\sum_{m_j=0}^{p^{h_j-1}} e^{2\sqrt{-1}\pi m_j((c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l)} \\
&\qquad\qquad \times \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi a_{h_j}((c_0+c_1p+\cdots+c_lp^l)/p^{l+1})}\Bigg|^q\Bigg) \\
&= \frac{1}{p^M}\frac{1}{p^{h_j+1}}\Bigg((r_j p^{h_j})^q + \sum_{c_0\in D_p\setminus\{0\}}\Bigg|\frac{\sin(\pi r_j c_0/p)}{\sin(\pi c_0/p)} p^{h_j}\Bigg|^q \\
&\qquad + \sum_{l=1}^{M+h_j}\sum_{\substack{c_0\in D_p\setminus\{0\}\\ c_1,\dots,c_l\in D_p}}\Bigg|\frac{\sin(\pi p^{h_j}(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l)}{\sin(\pi(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l)} \\
&\qquad\qquad \times \frac{\sin(\pi r_j(c_0+c_1p+\cdots+c_lp^l)/p^{l+1})}{\sin(\pi(c_0+c_1p+\cdots+c_lp^l)/p^{l+1})}\Bigg|^q\Bigg).
\end{aligned}
$$

Thanks to the fact that $l \in \{1,\dots,h_j\}$ implies $\sin(\pi p^{h_j}(c_0+c_1p+\cdots+c_{l-1}p^{l-1})/p^l) = 0$, we can conclude that

$$\int_{\mathbb{Z}_p} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\xi\alpha/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\xi\alpha/p^{h_j+1}\}_p} \right|^q d\alpha$$

$$= \frac{1}{p^M} \frac{1}{p^{h_j+1}} \left((r_j p^{h_j})^q + \sum_{c_0 \in D_p \setminus \{0\}} \left| \frac{\sin(\pi r_j c_0/p)}{\sin(\pi c_0/p)} p^{h_j} \right|^q \right.$$

$$+ \sum_{l=h_j+1}^{M+h_j} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1,\dots,c_l \in D_p}} \left| \frac{\sin(\pi r_j (c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})}{\sin(\pi (c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})} \right.$$

$$\left. \left. \times \frac{\sin(\pi p^{h_j} (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)}{\sin(\pi (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)} \right|^q \right).$$

(iii) From the estimate

$$\left| \frac{\sin(n+1)\pi x}{\sin \pi x} \right| = \left| \frac{\sin n\pi x \cos \pi x + \cos n\pi x \sin \pi x}{\sin \pi x} \right| \le \left| \frac{\sin n\pi x}{\sin \pi x} \right| + 1,$$

we can inductively derive $|\sin(n\pi x)/\sin(\pi x)| \le n$, and so we have $|\sin(\pi r(c_0 + c_1 p + \cdots + c_l p^l/p^{l+1}))/\sin(\pi(c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})| \le r \le p-1$. Accordingly, we easily see that

$$\sum_{l=h+1}^{M+h} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1,\dots,c_l \in D_p}} \left| \frac{\sin(\pi p^h (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)}{\sin(\pi (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)} \right.$$

$$\left. \times \frac{\sin(\pi r (c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})}{\sin(\pi (c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})} \right|^q$$

$$\le \sum_{l=h+1}^{M+h} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1,\dots,c_l \in D_p}} \frac{1}{|\sin(\pi (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)|^q} \cdot |r|^q$$

$$= \sum_{l=h+1}^{M+h} \sum_{a=1}^{p^l-1} \frac{p}{|\sin(\pi a/p^l)|^q} \cdot |r|^q$$

$$= p|r|^q \sum_{l=h+1}^{M+h} \sum_{a=1}^{p^l-1} \frac{1}{|\sin(\pi a/p^l)|^q}.$$

On the other hand, If $(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l \leq 1/2$, we see $c_0 + c_1 p + \cdots + c_{l-1} p^{l-1} \in \{1, 2, \ldots, \lfloor p^l/2 \rfloor\}$. Since $0 \leq x \leq 1/2$ implies $\sin \pi x \geq 2x$,

$$\frac{1}{|\sin(\pi(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)|} \leq \frac{p^l}{2(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})}.$$

Otherwise, namely in the case that $1/2 < (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l < 1$, we see $c_0 + c_1 p + \cdots + c_{l-1} p^{l-1} \in \{\lfloor p^l/2 \rfloor + 1, \ldots, p^l - 1\}$, which implies $p^l - (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}) \in \{1, 2, \ldots, \lfloor p^l/2 \rfloor - 1\}$. Consequently, we have

$$\begin{aligned}
&\frac{1}{|\sin(\pi(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)|} \\
&= \frac{1}{|\sin((\pi - \pi(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l))|} \\
&\leq \frac{1}{|2 \cdot (p^l - (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}))/p^l|} \\
&\leq \frac{p^l}{2(p^l - (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1}))}.
\end{aligned}$$

By combining these observations, we conclude

$$\begin{aligned}
&\sum_{l=h+1}^{M+h} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1, \ldots, c_l \in D_p}} \left| \frac{\sin(\pi p^h (c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)}{\sin(\pi(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)} \right. \\
&\qquad \left. \times \frac{\sin(\pi r(c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})}{\sin(\pi(c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})} \right|^q \\
&\leq p|r|^q \sum_{l=h+1}^{M+h} \left(\sum_{a=1}^{\lfloor p^l/2 \rfloor} \frac{1}{|\sin(\pi a/p^l)|^q} + \sum_{a=\lfloor p^l/2 \rfloor + 1}^{p^l - 1} \frac{1}{|\sin(\pi a/p^l)|^q} \right) \\
&\leq p|r|^q \sum_{l=h+1}^{M+h} \left(\sum_{a=1}^{\lfloor p^l/2 \rfloor} \left| \frac{p^l}{2a} \right|^q + \sum_{a=\lfloor p^l/2 \rfloor + 1}^{p^l - 1} \left| \frac{p^l}{2(p^l - a)} \right|^q \right) \\
&= p|r|^q \sum_{l=h+1}^{M+h} \left(\sum_{a=1}^{\lfloor p^l/2 \rfloor} \left| \frac{p^l}{2a} \right|^q + \sum_{b=1}^{\lfloor p^l/2 \rfloor - 1} \left| \frac{p^l}{2b} \right|^q \right) \\
&= |r|^q \sum_{l=h+1}^{M+h} \frac{p^{lq+1}}{2^q} \left(\sum_{a=1}^{\lfloor p^l/2 \rfloor} \frac{1}{a^q} + \sum_{b=1}^{\lfloor p^l/2 \rfloor - 1} \frac{1}{b^q} \right) \\
&= |r|^q \frac{p^{hq+1}}{2^q} \sum_{l=h+1}^{M+h} p^{(l-h)q} \left(\sum_{a=1}^{\lfloor p^l/2 \rfloor} \frac{1}{a^q} + \sum_{b=1}^{\lfloor p^l/2 \rfloor - 1} \frac{1}{b^q} \right)
\end{aligned}$$

$$\begin{aligned}
&\leq |r|^q \frac{p^{hq+1}}{2^q} \sum_{l=h+1}^{M+h} p^{(l-h)q} \left(\sum_{a=1}^{\infty} \frac{1}{a^q} + \sum_{b=1}^{\infty} \frac{1}{b^q} \right) \\
&\leq (p-1)^q \frac{p^{hq+1}}{2^q} \sum_{l=h+1}^{M+h} p^{Mq} \cdot 2\zeta(q) \\
&\leq (p-1)^q \frac{p^{hq+1}}{2^q} M p^{Mq} \cdot 2\zeta(q) \\
&= (p-1)^q \frac{M p^{Mq} \zeta(q)}{2^{q-1}} p^{hq+1}. \qquad \square
\end{aligned}$$

Proposition 3.5. *For any* $\xi \in L \setminus \{0\}$,

$$\lim_{N\to\infty} \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi \alpha x_n) \right|^q d\alpha \right)^{1/q} = 0.$$

Proof. We can deduce from (ii) and (iii) in Lemma 3.4 that

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \left| \sum_{m_j=0}^{p^{h_j-1}} \sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j \{\xi\alpha/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} \{\xi\alpha/p^{h_j+1}\}_p} \right|^q d\alpha \\
&\leq \frac{1}{p^M} \frac{1}{p^{h_j+1}} \left((r_j p^{h_j})^q + \sum_{c_0 \in D_p \setminus \{0\}} \left| \frac{\sin(\pi r_j c_0/p)}{\sin(\pi c_0/p)} p^{h_j} \right|^q \right. \\
&\qquad + \sum_{l=h_j+1}^{M+h_j} \sum_{\substack{c_0 \in D_p \setminus \{0\} \\ c_1,\ldots,c_l \in D_p}} \left| \frac{\sin(\pi p^{h_j}(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)}{\sin(\pi(c_0 + c_1 p + \cdots + c_{l-1} p^{l-1})/p^l)} \right. \\
&\qquad\qquad \left. \left. \times \frac{\sin(\pi r_j (c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})}{\sin(\pi(c_0 + c_1 p + \cdots + c_l p^l)/p^{l+1})} \right|^q \right) \\
&\leq \frac{1}{p^M} \frac{1}{p^{h_j+1}} \left((p-1)^q p^{h_j q} + (p-1)^{q+1} p^{h_j q} + (p-1)^q \frac{M p^{Mq} \zeta(q)}{2^{q-1}} p^{h_j q+1} \right) \\
&= \frac{(p-1)^q}{p^{M+1}} \left(p + \frac{M p^{Mq+1} \zeta(q)}{2^{q-1}} \right) p^{h_j(q-1)} \\
&= C p^{h_j(q-1)} \quad (1 \leq j \leq s),
\end{aligned}$$

where $C = ((p-1)^q/p^{M+1})(p + M p^{Mq+1} \zeta(q)/2^{q-1})$.

On the other hand, we will show that

$$\frac{1}{\sqrt{N}}\sum_{j=1}^{s}(p^{h_j})^{(q-1)/q} \le \left(\frac{1}{N-1}\right)^{(2-q)/(2q)}\left(\frac{\log((p-1)(N-1)+1)}{\log p}\right)^{1/q}.$$

The inequalities $h_1 > h_2 > \cdots > h_s \ge 0$ provide us with the estimate on h_j given by

$$\begin{aligned} h_j &= (h_j - h_{j+1}) + (h_{j+1} - h_{j+2}) + \cdots + (h_{s-1} - h_s) + h_s \\ &\ge (s-1) - j + 1 + 0 \\ &= s - j \quad (1 \le j \le s), \end{aligned}$$

which implies

$$N - 1 = \sum_{j=1}^{s} r_j p^{h_j} \ge \sum_{j=1}^{s} p^{h_j} \ge \sum_{j=1}^{s} p^{s-j} = \sum_{k=0}^{s-1} p^k = \frac{p^s - 1}{p-1}.$$

Since this shows $s \le \log((p-1)(N-1)+1)/\log p$, by applying Hölder's inequality we have

$$\begin{aligned} \frac{1}{\sqrt{N}}\sum_{j=1}^{s}(p^{h_j})^{(q-1)/q} &\le \frac{1}{\sqrt{N}}\left(\sum_{j=1}^{s}(p^{h_j})^{(q-1)/q\cdot q/(q-1)}\right)^{(q-1)/q}\left(\sum_{j=1}^{s}1^q\right)^{1/q} \\ &= \frac{1}{\sqrt{N}}\left(\sum_{j=1}^{s}p^{h_j}\right)^{(q-1)/q} s^{1/q} \\ &\le \frac{1}{\sqrt{N}}(N-1)^{(q-1)/q}\left(\frac{\log((p-1)(N-1)+1)}{\log p}\right)^{1/q} \\ &< \left(\frac{1}{N-1}\right)^{(2-q)/(2q)}\left(\frac{\log((p-1)(N-1)+1)}{\log p}\right)^{1/q}, \end{aligned}$$

which yields

$$\begin{aligned} &\frac{1}{\sqrt{N}}\sum_{j=1}^{s}\left(\int_{\mathbb{Z}_p}\left|\sum_{m_j=0}^{p^{h_j-1}}\sum_{a_{h_j}=0}^{r_j-1} e^{2\sqrt{-1}\pi m_j\{\xi\alpha/p^{h_j}\}_p} e^{2\sqrt{-1}\pi a_{h_j} a\{\xi\alpha/p^{h_j+1}\}_p}\right|^q d\alpha\right)^{1/q} \\ &\le C^{1/q}\frac{1}{\sqrt{N}}\sum_{j=1}^{s}(p^{h_j})^{(q-1)/q} \\ &< C^{1/q}\left(\frac{1}{N-1}\right)^{(2-q)/(2q)}\left(\frac{\log((p-1)(N-1)+1)}{\log p}\right)^{1/q} \\ &\to 0 \quad \text{as } N \to \infty. \end{aligned}$$

Finally, by combining this with (i) in the Lemma 3.4, it turns out that

$$\lim_{N\to\infty}\left(\int_{\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\chi(\xi\alpha x_n)\right|^q d\alpha\right)^{1/q} = 0. \qquad \square$$

Theorem 3.6. *For any q satisfying $1 < q < 2$ and any complex-valued function $f \in L^2(\mathbb{Z}_p, \mu)$,*

$$\lim_{N\to\infty}\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\sqrt{N}\left(\frac{1}{N}\sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\,dy\right)\right|^q dx\,d\alpha = 0.$$

Proof. We use f_M defined by (3) to approximate f;

$$\begin{aligned}
&\frac{1}{N}\sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\,dy \\
&= \frac{1}{N}\sum_{n=0}^{N-1}(f_M(x+\alpha x_n) - \hat{f}(0)) + \frac{1}{N}\sum_{n=0}^{N-1}(f-f_M)(x+\alpha x_n) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)}\hat{f}(\xi)\chi(-\xi(x+\alpha x_n)) + \frac{1}{N}\sum_{n=0}^{N-1}(f-f_M)(x+\alpha x_n) \\
&= \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)}\hat{f}(\xi)\chi(-\xi x)\frac{1}{N}\sum_{n=0}^{N-1}\chi(-\xi\alpha x_n) + \frac{1}{N}\sum_{n=0}^{N-1}(f-f_M)(x+\alpha x_n).
\end{aligned}$$

By applying Minkowski's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\sqrt{N}\left(\frac{1}{N}\sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\,dy\right)\right|^q dx\,d\alpha\right) \\
&= \left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)}\hat{f}(\xi)\chi(-\xi x)\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\chi(-\xi\alpha x_n)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}(f-f_M)(x+\alpha x_n)\right|^q dx\,d\alpha\right)^{1/q} \\
&\le \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)}|\hat{f}(\xi)|\left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}|\overline{\chi(\xi x)}|^q\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\overline{\chi(\xi\alpha x_n)}\right|^q dx\,d\alpha\right)^{1/q} \\
&\quad + \left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p}\left|\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}(f-f_M)(x+\alpha x_n)\right|^q dx\,d\alpha\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)} |\hat{f}(\xi)| \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi\alpha x_n) \right|^q d\alpha \right)^{1/q} \\
&\quad + \left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (f - f_M)(x+\alpha x_n) \right|^q dx\, d\alpha \right)^{1/q} \\
&\le \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)} |\hat{f}(\xi)| \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi\alpha x_n) \right|^q d\alpha \right)^{1/q} \\
&\quad + \left(\left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (f - f_M)(x+\alpha x_n) \right|^{q\cdot 2/q} dx\, d\alpha \right)^{q/2} \right. \\
&\qquad \left. \times \left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} 1^{q/(2-q)}\, dx\, d\alpha \right)^{(2-q)/2} \right)^{1/q} \\
&= \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)} |\hat{f}(\xi)| \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi\alpha x_n) \right|^q d\alpha \right)^{1/q} \\
&\quad + \left(\iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} (f - f_M)(x+\alpha x_n) \right|^2 dx\, d\alpha \right)^{1/2} \\
&= \sum_{\xi\in(L\setminus\{0\})\cap B(0,p^M)} |\hat{f}(\xi)| \left(\int_{\mathbb{Z}_p} \left| \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \chi(\xi\alpha x_n) \right|^q d\alpha \right)^{1/q} + \| f - f_M \|_2.
\end{aligned}
$$

In the final identity, Theorem 3.3 is applied. By passing the limit as $N \to \infty$, we can derive that the first term tends to zero from Proposition 3.5. Subsequently by passing the limit as $M \to \infty$, we can conclude that

$$
\lim_{N\to\infty} \iint_{\mathbb{Z}_p\times\mathbb{Z}_p} \left| \sqrt{N} \left(\frac{1}{N} \sum_{n=0}^{N-1} f(x+\alpha x_n) - \int_{\mathbb{Z}_p} f(y)\, dy \right) \right|^q dx\, d\alpha = 0. \qquad \square
$$

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