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RIGHT-ANGLED ARTIN GROUPS AND FINITE SUBGRAPHS OF CURVE GRAPHS

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Abstract

We show that for a sufficiently simple surface S, if a right-angled Artin group $A(\Gamma)$ embeds into $\mathrm{Mod}(S)$ then Γ embeds into the curve graph $\mathcal{C}(S)$ as an induced subgraph. When S is sufficiently complicated, there exists an embedding $A(\Gamma) \to \mathrm{Mod}(S)$ such that Γ is not contained in $\mathcal{C}(S)$ as an induced subgraph.

1. Introduction

1.1. Statement of the main results. Let $S = S_{g,n}$ be a connected orientable surface of genus g and n punctures, and let $\mathrm{Mod}(S)$ denote its mapping class group. As is standard, we will write

$$\xi(S) = \max(3g - 3 + n, 0)$$

for the *complexity* of S. It is clear that $\xi(S)$ is the number of components of a maximal multicurve on S. A celebrated result of Birman, Lubotzky and McCarthy is the following.

Theorem 1 ([1, Theorem A]). *The torsion free rank of an abelian group in* $\mathrm{Mod}(S)$ *is at most* $\xi(S)$.

In this article, we study a generalization of Theorem 1 for right-angled Artin subgroups of $\mathrm{Mod}(S)$. Let Γ be a finite simplicial graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We will write $A(\Gamma)$ for the *right-angled Artin group* on Γ, which is defined by

$$A(\Gamma) = \langle V(\Gamma) \mid [u, v] = 1 \text{ if and only if } \{u, v\} \in E(\Gamma)\rangle.$$

We will use $\mathcal{C}(S)$ to denote the *curve graph* of S, which is the 1-skeleton of the *curve complex* of S. The vertices of $\mathcal{C}(S)$ are isotopy classes of essential, non-peripheral, simple closed curves on S. Two vertices are adjacent if the corresponding isotopy classes admit disjoint representatives. Let us denote a complete graph on n vertices as K_n.

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Equipped with this language, we can rephrase Theorem 1 as follows:

> If $A(K_n)$ embeds into $\mathrm{Mod}(S)$, then K_n is an induced subgraph of $\mathcal{C}(S)$.

Our main results are the following:

Theorem 2. *Let S be a surface with $\xi(S) < 3$. If $A(\Gamma)$ embeds into* $\mathrm{Mod}(S)$, *then Γ is an induced subgraph of $\mathcal{C}(S)$.*

Theorem 3. *Let S be a surface with $\xi(S) > 3$. Then there exists a finite graph Γ such that $A(\Gamma)$ embeds into* $\mathrm{Mod}(S)$ *but Γ is not an induced subgraph of $\mathcal{C}(S)$.*

Note that the converse of Theorem 1 is easily seen to be true. More generally, the second author proved the following theorem:

Theorem 4 ([12], Theorem 1.1 and Proposition 7.16)**.** *If Γ is an induced subgraph of $\mathcal{C}(S)$ then $A(\Gamma)$ embeds into* $\mathrm{Mod}(S)$.

Theorems 2 and 3 characterizes the surfaces for which the converse of Theorem 4 is true, except for the case $\xi(S) = 3$. In this latter case, the methods developed in this paper are ineffective. There are exactly three surfaces with $\xi(S) = 3$, though there are only two different cases to consider among them (see Section 5).

QUESTION 1. Let S be a surface of complexity 3. Do there exist subgroups $A(\Gamma) \leq \mathrm{Mod}(S)$ such that Γ is not an induced subgraph of $\mathcal{C}(S)$?

A *clique* is a subset of the vertex set which spans a complete subgraph. A *facet* of a triangulation of a manifold means a top-dimensional simplex. For a positive integer N, we will say that Γ has *N-thick stars* if each vertex v of Γ is contained in two cliques $K_1 \cong K_2$ on N vertices of Γ whose intersection is exactly v. Equivalently, the link $\mathrm{Lk}(v)$ of v in Γ contains two disjoint copies of complete graphs on $N-1$ vertices. For example, a *proper* (namely, no two facets share more than one faces) triangulation of a compact surface with no triangular links has 3-thick stars. The following generalization is immediate.

Proposition 5. *A proper triangulation of a compact $(N-1)$-manifold has N-thick stars if and only if the link of each vertex has at least $N+1$ facets.*

Having N-thick stars forces the converse of Theorem 4 to be true.

Theorem 6. *Suppose S is a surface with $\xi(S) = N$ and Γ is a finite graph with N-thick stars. If $A(\Gamma)$ embeds into* $\mathrm{Mod}(S)$, *then Γ is an induced subgraph of $\mathcal{C}(S)$.*

1.2. Notes and references. Throughout this article, a *graph* will always mean a simplicial 1-complex with vertex set V and edge set E. Let X be a graph. A subgraph Λ of X is called an *induced subgraph* if Λ is the subgraph of X spanned by the vertices $V(\Lambda) \subset V(X)$. Thus, a pair of vertices in Λ are adjacent if and only if they are adjacent in X. We write $\Lambda \leq X$ if Λ is an induced subgraph of X. If X is a simplicial graph, the *clique graph* X_k of X is the graph whose vertices $\{v_K\}$ are nonempty complete subgraphs $K \leq X$, and two vertices $\{v_{K_1}, v_{K_2}\}$ of X_k are adjacent if and only if the corresponding complete subgraphs K_1 and K_2 span a complete subgraph of X. The *extension graph* Γ^e of a finite simplicial graph Γ is the graph whose vertices are given by $\{gvg^{-1} \mid g \in A(\Gamma),\ v \in V(\Gamma)\}$, and whose edges are given by pairs of vertices which commute as elements of $A(\Gamma)$. The *complement* $\binom{V(X)}{2} \setminus X$ of a simplicial graph X is the graph with the same vertex set as X, but where a pair of vertices spans an edge in $\binom{V(X)}{2} \setminus X$ if and only if it does not span an edge in X. The join $X * Y$ of two graphs X and Y is the graph whose vertex set is $V(X) \cup V(Y)$, and where a pair of vertices $\{v, w\}$ is adjacent if and only if the vertices span an edge in X, an edge in Y, or if one vertex lies in $V(X)$ and the other lies in $V(Y)$.

For background on mapping class groups, we refer the reader to [7]. We briefly recall that every mapping class $1 \neq \psi \in \mathrm{Mod}(S)$ is either *finite order*, *infinite order reducible*, or *pseudo-Anosov*, according to whether it has finite order in $\mathrm{Mod}(S)$, fixes the homotopy class of a multicurve on S, or neither. This is called the *Nielsen–Thurston classification* of surface diffeomorphisms.

The relationship between right-angled Artin groups and mapping class groups of surfaces has been studied by many authors from various perspectives (see [5], [6], [4], [12], [10] and the references therein, for instance). Our perspective stems from the following theorem, which can be obtained by combining a result of the authors with a result of the second author (see [12] and [9] or [10]):

Theorem 7 (See [11])**.** *Let Γ be a finite graph and let S be a surface.*
(1) *Let i be an embedding of Γ into $\mathcal{C}(S)$ as an induced subgraph. Then for all sufficiently large N, the map*

$$i_{*,N} \colon A(\Gamma) \to \mathrm{Mod}(S)$$

given by sending v to the N^{th} power of a Dehn twist $T_{i(v)}^N$ is injective.
(2) *If $A(\Gamma)$ embeds into* $\mathrm{Mod}(S)$*, then Γ is an induced subgraph of $\mathcal{C}(S)_k$.*

Observe that the first part of Theorem 7 is a more precise version of Theorem 4. As defined above, the graph $\mathcal{C}(S)_k$ denotes the clique graph of $\mathcal{C}(S)$. From a topological perspective, $\mathcal{C}(S)_k$ can be defined as the graph whose vertices are isotopy classes of essential, non-peripheral multicurves on S, and where two vertices are adjacent if the corresponding multicurves are component-wise parallel or disjoint. Theorem 3 shows that $\mathcal{C}(S)_k$ in Theorem 7 cannot be replaced by $\mathcal{C}(S)$ for a general surface.

In [10], in [9] and in [11], the authors develop an analogous theory of curve graphs for right-angled Artin groups. In particular, a verbatim analogue of Theorem 7 holds with $\mathrm{Mod}(S)$ replaced by a right-angled Artin group $A(\Lambda)$ and the curve graph $\mathcal{C}(S)$ replaced by the extension graph Λ^e of Λ. For many classes of graphs, it is known that $A(\Gamma)$ embeds into $A(\Lambda)$ if and only if Γ is an induced subgraph of Λ^e; for instance, this statement holds when Λ is triangle-free [9], or when Λ is C_4- and P_3-free [2]. However Casals–Ruiz, Duncan and Kazachkov proved that this is not always the case:

Theorem 8 ([2]). *There exist finite graphs Γ and Λ such that $A(\Gamma)$ embeds into $A(\Lambda)$ but Γ is not an induced subgraph of Λ^e.*

Thus, Theorem 3 can be viewed as an analogue of Theorem 8 for mapping class groups. We note briefly that Theorem 8 does not imply Theorem 3, for even if a particular graph Γ embeds in $\mathcal{C}(S)_k$, the graph $\mathcal{C}(S)_k$ is vastly more complicated than Γ_k^e. However, our example in Section 3 gives another example of $A(\Lambda)$ embedded in $A(\Gamma)$ such that Λ is not an induced subgraph of Γ^e; see the remark following Lemma 14.

The concept of N-thick stars used in Theorem 6 is related to the well-studied graph-theoretic notion of a *quasi-line* (see [3], for instance). A graph is a quasi-line if the star of each vertex is the union of two complete graphs.

2. Proof of Theorem 2

Let S be a surface with punctures. A mapping class $\phi \in \mathrm{Mod}(S)$ is called a *multi-twist* if ϕ can be represented by a multiplication of powers of Dehn twists along disjoint pairwise-non-isotopic simple closed curves. We call a regular neighborhood of the union of those simple closed curves as the *support* of ϕ.

For two groups G and H, we will write $G \le H$ if there is an embedding from G into H. As defined above, we will write $\Lambda \le \Gamma$ for two graphs Λ and Γ if Λ is isomorphic to an induced subgraph of Γ. The following is a refinement of [10, Lemma 2.3].

Lemma 9. *Let X be a finite graph. If $A(X) \le \mathrm{Mod}(S)$ then there exists an embedding $f\colon A(X) \to \mathrm{Mod}(S)$ satisfying the following*:

(i) *The map f maps each vertex of X to a multi-twist*;

(ii) *For two distinct vertices u and v of X, the support of $f(u)$ is not contained in the support of $f(v)$.*

Proof. Let f_0 be an embedding of $A(X)$ into $\mathrm{Mod}(S)$. By raising the generators to powers if necessary, we may assume that the image of each vertex v is written as $\phi_1^v\phi_2^v\cdots\phi_{n_v}^v$ where each ϕ_i^v is either a Dehn twist or a pseudo-Anosov on a connected subsurface and ϕ_i^v's have disjoint supports. Choose a minimal collection $\{\psi_1,\ldots,\psi_m\} \subseteq \mathrm{Mod}(S)$ such that for every i and v, the mapping class ϕ_i^v is a power of some ψ_j. By

[12] (see also [4]), there exists $N > 0$ and a graph Y with $V(Y) = \{v_1, \ldots, v_m\}$ such that the map $g_0\colon A(Y) \to \mathrm{Mod}(S)$ defined by $g_0(v_j) = \psi_j^N$ is an embedding. Moreover, we can find simple closed curves $\gamma_1, \ldots, \gamma_m$ such that $\gamma_i \subseteq \operatorname{supp}\psi_i$ and $\operatorname{supp}\psi_i \cap \operatorname{supp}\psi_j = \varnothing$ if and only if $\gamma_i \cap \gamma_j = \varnothing$ for every i and j. By raising N further if necessary, we have an embedding $g\colon A(Y) \to \mathrm{Mod}(S)$ defined by $v_j \mapsto T_{\gamma_j}^N$. We may assume that $f = g \circ h$ for some $h\colon A(X) \to A(Y)$, by further raising the image of each $f(v)$ for $v \in X$ to some power. Then $g \circ h$ is an embedding from $A(X)$ to $\mathrm{Mod}(S)$ such that each vertex maps to a multi-twist. Note that if u and v are adjacent vertices in X then the multi-curves corresponding to $f(u)$ and $f(v)$ also form a multi-curve.

Now among the embeddings $f\colon A(X) \to \mathrm{Mod}(S)$ that map each vertex to a multi-twist, we choose f so that

$$\sum_w \#\operatorname{supp} f(w)$$

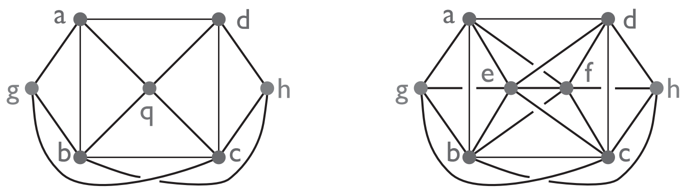
is minimal. Here, # of a support of a multi-twist denotes the number of components. Suppose that $\operatorname{supp} f(u) \subseteq \operatorname{supp} f(v)$ for two distinct vertices u, v of X. Since $[f(u), f(v)] = 1$, we have $[u, v] = 1$. If $w \in \mathrm{Lk}_X(v)$, then each curve in $\operatorname{supp} f(w)$ is equal to or disjoint from each curve in $\operatorname{supp} f(v)$. This implies that $[w, u] = 1$ for each $w \in \mathrm{Lk}_X(v)$ and hence, $\mathrm{Lk}_X(v) \subseteq \mathrm{St}_X(u)$. For each non-zero P, Q, we have a map $\tau\colon A(X) \to A(X)$ defined $\tau(w) = w$ for $w \neq v$ and $\tau(v) = u^P v^Q$. If $Q = 1$, such a map is called a *transvection* automorphism; see [14]. For a general Q, the map τ is a monomorphism, since it is obtained from a transvection by pre-composing with the monomorphism $v \mapsto v^Q$ and $w \mapsto w$ for $w \neq v$. We claim that there exist P, Q such that $\#\operatorname{supp} f(u^P v^Q) < \#\operatorname{supp} f(v)$. Once the claim is proved, we have that $\sum_w \#\operatorname{supp} f \circ \tau(w) < \sum_w \#\operatorname{supp} f(w)$ and a contradiction to the minimality.

The argument for the claim is similar to [10, Lemma 2.3], and we recall the details for the convenience of the reader. Write $f(u) = T_\alpha^Q g_1$ and $f(v) = T_\alpha^{-P} g_2$ so that g_1, g_2 are multi-twists whose supports are disjoint from α and $\operatorname{supp} g_1 \subseteq \operatorname{supp} g_2$. Then $\operatorname{supp} f(u^P v^Q) \subseteq \operatorname{supp} g_2 = \operatorname{supp} f(v) \setminus \{\alpha\}$ and this proves the claim. □

REMARK. In the above lemma, if $\operatorname{supp} f(v)$ is a maximal clique in $\mathcal{C}(S)$ then the condition (ii) implies that v is an isolated vertex.

DEFINITION 10. An embedding of a right-angled Artin group into a mapping class group is called *standard* if conditions (i) and (ii) in Lemma 9 are satisfied.

The lowest complexity surfaces with nontrivial mapping class groups are $S_{0,4}$ and $S_{1,1}$ (so that $\xi(S) = 1$). Both of these surfaces admit simple closed curves, but neither admits a pair of disjoint isotopy classes of simple closed curves. Because of this fact, most authors define edges in $\mathcal{C}(S)$ to lie between curves with minimal intersection (two or one intersection point, respectively). This definition is not suitable for our purposes and we will keep the standard definition of curve graphs, so that $\mathcal{C}(S)$ is an infinite union of isolated vertices in both of these cases.



(a) Γ_0 (b) Γ_1

Fig. 1. Two graphs Γ_0 and Γ_1.

Proof of Theorem 2. Suppose first that $\xi(S) = 1$. We have that $\mathcal{C}(S)$ is discrete, since there are no pairs of disjoint simple closed curves on S. The conclusion follows from that $\mathrm{Mod}(S)$ is virtually free; see [7], Sections 2.2.4 and 2.2.5. Now let us assume $\xi(S) = 2$, so that $S = S_{1,2}$ or $S = S_{0,5}$. We note that $\mathcal{C}(S)$ contains no triangles.

The conclusion of the theorem holds for Γ if and only if it holds for each component of Γ. This is an easy consequence of the fact that $\mathcal{C}(S)$ has infinite diameter and that a pseudo-Anosov mapping class on S exists. So, we may suppose that Γ is connected and contains at least one edge. By Lemma 9, we can further assume to have a standard embedding $f : A(\Gamma) \to \mathrm{Mod}(S)$. Since Γ has no isolated vertices and $\mathcal{C}(S)$ is triangle-free, the remark following Lemma 9 implies that each vertex maps to a power of a single Dehn twist. This gives a desired embedding $\Gamma \to \mathcal{C}(S)$. $\square$

3. High complexity surfaces

The strategy for dealing with high complexity surfaces (surfaces S for which $\xi(S) > 3$) is to build an example which works for surfaces with $\xi(S) = 4$ and then bootstrapping to obtain examples in all higher complexities. In particular, we will take the three surfaces with $\xi(S) = 4$ and build graphs Γ_0 and Γ_1 such that $A(\Gamma_0) < A(\Gamma_1) < \mathrm{Mod}(S)$ but such that $\Gamma_0 \not\leq \mathcal{C}(S)$. We will then use Γ_0 and Γ_1 to build corresponding graphs for surfaces of complexity greater than four.

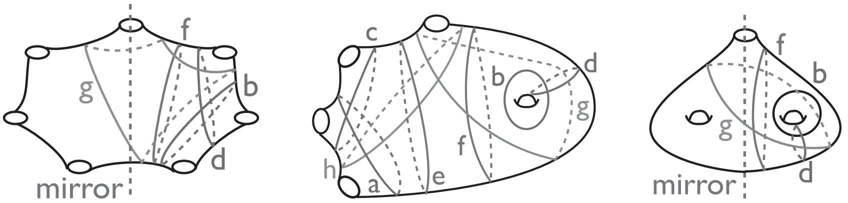
The source of our examples in this section will be the graphs Γ_0 and Γ_1 shown in Fig. 1. Observe that the graph Γ_0 is obtained from the graph Γ_1 by collapsing e and f to a single vertex q and retaining all common adjacency relations. We will denote by C_4 the 4-cycle spanned by $\{a, b, c, d\}$.

3.1. An algebraic lemma.

Let us consider the map $\phi : A(\Gamma_0) \to A(\Gamma_1)$ defined by $\phi : q \mapsto ef$ and which is the identity on the remaining vertices.

Lemma 11. *The map $\phi : A(\Gamma_0) \to A(\Gamma_1)$ is injective.*

Proof. We first claim that the restriction $\psi : \langle C_4, q, h\rangle \to \langle C_4, ef, h\rangle$ of ϕ is an isomorphism. Here, we mean $\langle C_4, q, h\rangle = \langle a, b, c, d, q, h\rangle \leq A(\Gamma_0)$ and $\langle C_4, ef, h\rangle =$



(a) $S_{0,7}$ (b) $S_{1,4}$ (c) $S_{2,1}$

Fig. 2. Complexity four surfaces.

$\langle a,b,c,d,ef,h\rangle \le A(\Gamma_1)$. To see this, consider the projection p: $\langle C_4,e,f,h\rangle \to \langle C_4,e,h\rangle$ defined by $p(f)=1$. The claim follows from that $p\circ\psi$ is an isomorphism.

Now suppose w is a reduced word in $\ker\phi \setminus \{1\}$. Since ψ is an isomorphism, g or g^{-1} appears in w. From the assumption that $\phi(w)=1$, the occurrences of g or g^{-1} in $\phi(w)$ can be paired, so that each pair consists of g and g^{-1} and that g commutes with the subword of $\phi(w)$ between the pair. This is due to the solution to the word problem in right-angled Artin groups; see [6, 8] for more details. There must exist a pair of g and g^{-1} in $\phi(w)$ so that there does not exist any more g or g^{-1} between the pair; such a pair is called an *innermost* $\{g, g^{-1}\}$*-pair in the cancellation diagram* [6, 8]. In other words, we can write $w = w_0 g^{\pm 1} w_1 g^{\mp 1} w_2$ so that $w_1 \in \langle V(\Gamma_0)\setminus\{g\}\rangle = \langle C_4, q, h\rangle$ and $\phi(w_1) \in Z(g) \cap \langle V(\Gamma_1)\setminus\{g\}\rangle = \langle a, b, c, e\rangle$.

It follows that

$$\phi(w_1) \in \phi\langle C_4, q, h\rangle \cap \langle a, b, c, e\rangle = \langle C_4, ef, h\rangle \cap (\langle a, c\rangle \times \langle b\rangle \times \langle e\rangle).$$

Since $\phi(w_1) \in \langle a, b, c, e\rangle$, the exponent sum of f in $\phi(w_1)$ is zero. From $\phi(w_1) \in \langle C_4, ef, h\rangle$, it follows that the exponent sum of e in $\phi(w_1)$ is also zero. Since $\langle e\rangle$ is a direct factor of $\langle a, b, c, e\rangle$, we see $\phi(w_1) \in \langle a, b, c\rangle$. Combined with the claim in the first paragraph, we have

$$w_1 \in \phi^{-1}\langle a, b, c\rangle \cap \langle C_4, q, h\rangle = \psi^{-1}\langle a, b, c\rangle = \langle a, b, c\rangle.$$

This contradicts the fact that w is reduced. □

3.2. The case $\xi(S) = 4$. Let S be a connected surface with complexity four. This means S is one of $S_{0,7}$, $S_{1,4}$ and $S_{2,1}$.

Lemma 12. *The graph Γ_1 embeds into $\mathcal{C}(S)$ as an induced subgraph.*

Proof. The corresponding surfaces are shown in Fig. 2. In (a) and (c), the curves for the vertices a, c, e and h are given by the mirror images of those for b, d, f and g, respectively. One can verify that the curves with the shown configuration have the minimal intersections by observing that the intersection numbers are either 0, 1 or 2. □

Now suppose $\{a, b, c, d\}$ are simple closed curves on S (here we still have $\xi(S) = 4$) which form a four-cycle in $\mathcal{C}(S)$ with this cyclic order. Let S_1 be a closed regular neighborhood of the curves a and c along with disks glued to null-homotopic boundary components. Similarly we define S_2 for b and d so that $S_1 \cap S_2 = \varnothing$. Define S_0 as the closure of $S \setminus (S_1 \cup S_2)$. By isotopically enlarging S_1 and S_2 if necessary, we may assume that whenever A is an annulus component of S_0 then both components of ∂A intersect $S_1 \cup S_2$ (i.e. no component of S_0 is a punctured disk). Note that $\xi(S_1), \xi(S_2) \geq 1$, since they both contain a pair of non-isotopic simple closed curves. Since S is connected and at least one component of S_0 intersects each of S_1 and S_2, we have that S_0 has at least two boundary components.

Lemma 13. *The triple (S_0, S_1, S_2) satisfies exactly one of the following conditions, possibly after switching the roles of S_1 and S_2.*

(i) $S_1 \in \{S_{1,2}, S_{0,5}\}$, $S_2 \in \{S_{1,1}, S_{0,4}\}$, $S_0 \approx S_{0,2}$, *and S_0 intersects both S_1 and S_2.*

(ii) $S_1, S_2 \in \{S_{0,4}, S_{1,1}\}$, $S_0 \approx S_{0,3}$, *and S_0 intersects each of S_1 and S_2 at only one boundary component.*

(iii) $S_1, S_2 \in \{S_{0,4}\}$, $S_0 \approx S_{0,2} \coprod S_{0,2}$, *and each component of S_0 intersects both S_1 and S_2.*

(iv) $(S_1, S_2) \in \{(S_{0,4}, S_{0,4}), (S_{0,4}, S_{1,1})\}$, S_0 $approx S_{0,2} \coprod S_{0,2}$, *and one component of S_0 intersects each of S_1 and S_2 at only one boundary component, while the other component of S_0 intersects S_1 at two boundary components.*

(v) $(S_1, S_2) \in \{(S_{0,4}, S_{0,4}), (S_{0,4}, S_{1,1})\}$ *and* $S_0 \approx S_{0,2} \coprod S_{0,3}$ *such that the $S_{0,2}$ component intersects both S_1 and S_2 and the $S_{0,3}$ component is disjoint from S_2, and moreover,* $S_{0,3} \cap S_1 \approx S^1$.

Proof. Let α be the number of free isotopy classes of boundary components of S_0 that are contained in $S_1 \cup S_2$. We have $\alpha > 0$ since S is connected and $S_1 \cap S_2 = \varnothing$. Then $\xi(S) = \xi(S_1) + \xi(S_2) + \xi(S_0) + \alpha$; here, $\xi(S_0)$ is defined as the sum of the complexities of the components of S_0 [1]. It follows that $2 \leq \xi(S_1) + \xi(S_2) \leq 3$.

Let us first assume $\xi(S_1) + \xi(S_2) = 3$. From $\xi(S_0) + \alpha = 1$, we see that S_0 is an annulus joining S_1 and S_2. Case (i) is immediate.

Now we assume $\xi(S_1) = \xi(S_2) = 1$. If $\alpha = 1$, then S_0 is forced to be an annulus and we have a contradiction of the fact that $\xi(S_0) + \alpha = 2$. So we have $\xi(S_0) = 0$ and $\alpha = 2$. If S_0 is connected, then $\alpha = 2$ implies that S_0 cannot be an annulus, and hence, Case (ii) follows. So we may assume S_0 is disconnected.

Suppose $S_0 \approx S_{0,2} \coprod S_{0,2}$. If each component of S_0 intersects both of S_1 and S_2, then each S_i has at least two boundary components for $i = 1, 2$. In particular, $S_i \neq S_{1,1}$ and Case (iii) follows. Without loss of generality, let us assume that one component of S_0 intersects only S_1. Then $S_1 \neq S_{1,1}$ and we have Case (iv).

Let us finally assume $S_0 \approx S_{0,2} \coprod S_{0,3}$. This is the only remaining case, for $\alpha = 2$. The subsurface $S_{0,3}$ can contribute only one to α. In particular, $S_{0,3}$ is glued to say, S_1 but not S_2. The annulus component of S_0 joins S_1 and S_2 and therefore, $S_1 \neq S_{1,1}$ and Case (v) follows. □

The following special case of Theorem 3 will be central to our discussion of surfaces with $\xi(S) \geq 4$:

Lemma 14. *Let S be a surface with $\xi(S) = 4$. There exists an embedding from $A(\Gamma_0)$ into $\mathrm{Mod}(S)$, but Γ_0 does not embed into $\mathcal{C}(S)$ as an induced subgraph.*

Proof. The first half of the conclusion follows from Lemmas 11 and 12, combined with Theorem 7 (1). For the second half, let us assume $\Gamma_0 \leq \mathcal{C}(S)$ and regard the vertices $a, b, c, \ldots$ as simple closed curves on S. From $C_4 \leq \Gamma_0$, we have one of the five cases in Lemma 13. From the adjacency relations in Γ_0, we observe that $q \cap g$, $q \cap h$, $g \cap S_2$, $h \cap S_1$ and $g \cap h$ are all non-empty, and also that $q \subseteq S_0$ and $g \cap S_1 = h \cap S_2 = \varnothing$.

In Case (i), the annulus S_0 connects S_1 and S_2. This implies that $g \subseteq S_2$, $h \subseteq S_1$ and so, $g \cap h = \varnothing$. This is a contradiction. In Case (iii) and (iv), we similarly obtain a contradiction from $g \cap h = \varnothing$.

In Case (ii), the curve q must be boundary parallel in S_0. Hence, it must be either $S_0 \cap S_1$ or $S_0 \cap S_2$. By symmetry, we may assume $q = S_0 \cap S_1$. Then q separates S_1 from S, and so, $g \cap q \subseteq g \cap S_0 = \varnothing$. This is a contradiction. The proof for Case (v) is similar and goes as follows. The subsurface S_1 separates S_2 and $S_{0,3} \subseteq S_0$. This forces $g \subseteq S_2$, so that $g \cap q \subseteq g \cap S_0 = \varnothing$. □

Remark. Since $\Gamma_1 \leq \mathcal{C}(S)$, we have $\Gamma_1^e \leq \mathcal{C}(S)$ by [10]. Hence we have another example of graphs $\Gamma_0 \not\leq \Gamma_1^e$ but $A(\Gamma_0) \leq A(\Gamma_1)$; see [9, 2].

3.3. Surfaces with complexity larger than four. For a graph X, let us define $\eta(X)$ to be the minimum of $\xi(S)$ among connected surfaces S satisfying $X \leq \mathcal{C}(S)$. Note that $\eta(X)$ is at least the size of a maximal clique in X. Lemma 14 implies $\eta(\Gamma_0) > 4$, and we see from Fig. 3 that $\eta(\Gamma_0) = 5$.

A graph is *anti-connected* if its complement graph $\binom{V(X)}{2} \setminus X$ is connected. Note that the graphs Γ_0 and Γ_1 are both anti-connected.

Lemma 15. *If X is a finite anti-connected graph and $n \geq 0$, then $\eta(X * K_n) \geq \eta(X) + n$.*

Proof. Choose a surface S such that $\xi(S) = \eta(X * K_n)$ and $X * K_n \leq \mathcal{C}(S)$. Let N denote a regular neighborhood of curves in K_n. Since the graph X is anti-connected, the curves in $V(X)$ must fill a connected subsurface of S. Indeed, otherwise there

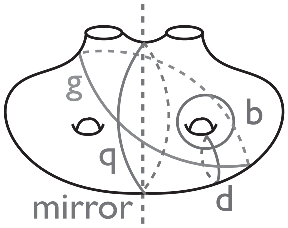


Fig. 3. Realizing Γ_0 in $\mathcal{C}(S_{2,2})$. The curves a, c, h are the mirror images of d, b, g.

would be a nontrivial partition $V(X) = J_1 \cup J_2$, where every vertex of J_1 is adjacent to every vertex of J_2, which violates the assumption that X is anti-connected. It follows that the curves in $V(X)$ are contained in a component S_1 of $S \setminus N$. Since $X \le \mathcal{C}(S_1)$, we have $\xi(S) = \xi(S \setminus N) + n \ge \xi(S_1) + n \ge \eta(X) + n$. □

Put $\Lambda_n = \Gamma_0 * K_{n-4}$ for $n \ge 4$. Theorem 3 is an immediate consequence of the following.

Proposition 16. *If S is a surface with $\xi(S) = n$, then $A(\Lambda_n)$ embeds into* Mod(S) *but Λ_n is not an induced subgraph of $\mathcal{C}(S)$.*

Proof. Choose a multicurve X on S with $n-4$ components such that $S \setminus X$ has a connected component S_0 of complexity at least four. We have that $\mathcal{C}(S_0)$ contains a copy of Γ_1, so that $A(\Gamma_1) \times \mathbb{Z}^{n-4}$ embeds in Mod(S). It follows that $A(\Gamma_0) \times \mathbb{Z}^{n-4} \cong A(\Lambda_n)$ embeds in Mod(S). On the other hand, Lemma 15 implies that $\eta(\Lambda_n) \ge \eta(\Gamma_0) + n - 4 > n$. □

4. Proof of Theorem 6

In this section, we give a proof of Theorem 6. For a multi-curve A on a surface S, we denote by $\langle A \rangle$ the subgroup of Mod(S) generated by the Dehn twist about the curves in A.

Proof of Theorem 6. By Lemma 9, there exists a standard embedding $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$. Let v be an arbitrary vertex of Γ. Write K and L for two disjoint cliques of Γ such that $K \coprod \{v\}$ and $L \coprod \{v\}$ are cliques on N vertices. The support of $\phi\langle K \rangle$ is a multi-curve, say A. Similarly we write $B = \operatorname{supp} \phi\langle L \rangle$ and $C = \operatorname{supp} \phi\langle v \rangle$. Since $\xi(S) = N$, the multi-curves $A \cup C$ and $B \cup C$ are maximal. Note that $\langle C \rangle$ is a subgroup of $\langle A \cup C \rangle \cap \langle B \cup C \rangle$. In the diagram below, we see that $\phi\langle v \rangle$ is of finite-index in $\langle C \rangle \cong \mathbb{Z}^{|C|}$ and hence, $|C| = 1$. It follows that the support of $\phi\langle v \rangle$ consists of exactly one curve on S. Thus, the map $\Gamma \to \mathcal{C}(S)$ given by sending a vertex v to the unique curve in the support of $\phi\langle v \rangle$ is a well-defined map of graphs. This map realizes Γ as an induced subgraph of $\mathcal{C}(S)$, since ϕ is an injective map of groups and must therefore send nonadjacent vertices to Dehn twists which do not commute in $\mathrm{Mod}(S)$. $\square$

$$\begin{tikzcd}
 & & \phi(A(\Gamma)) \leq \mathrm{Mod}(S) \arrow[dll, dash] \arrow[drr, dash] & & \\
\langle A \cup C \rangle \cong \mathbb{Z}^N \arrow[d, dash, "\text{finite-index}"] \arrow[drr, dash] & & & & \langle B \cup C \rangle \cong \mathbb{Z}^N \arrow[d, dash, "\text{finite-index}"] \arrow[dll, dash] \\
\phi\langle K, v \rangle \cong \mathbb{Z}^N \arrow[drr, dash] & & \langle C \rangle \arrow[d, dash] & & \phi\langle L, v \rangle \cong \mathbb{Z}^N \arrow[dll, dash] \\
 & & \phi\langle K, v \rangle \cap \phi\langle L, v \rangle = \phi\langle v \rangle \cong \mathbb{Z} & &
\end{tikzcd}$$

5. Remarks on intermediate complexity surfaces

There are only three surfaces of complexity three: $S_{2,0}$, $S_{1,3}$ and $S_{0,6}$. From the perspective of Theorem 3, these three surfaces collapse into at most two cases:

Lemma 17. *Either conclusion of* Theorem 2 *or* 3 *holds for* $S \cong S_{0,6}$ *if and only if it holds for* $S \cong S_{2,0}$.

Proof. It is well-known that $\mathrm{Mod}(S_{2,0})$ and $\mathrm{Mod}(S_{0,6})$ are commensurable (see [7], Theorem 9.2, for instance). It follows that $A(\Gamma) < \mathrm{Mod}(S_{2,0})$ if and only if $A(\Gamma) < \mathrm{Mod}(S_{0,6})$. It is also well-known (see [13], for instance) that the curve complexes $\mathcal{C}(S_{2,0})$ and $\mathcal{C}(S_{0,6})$ are isomorphic (in fact, the fact that the mapping class groups are commensurable implies that the curve graphs are isomorphic; see [11], Lemma 3 and Proposition 4). In particular, the two curve graphs have the same finite subgraphs. The lemma follows immediately. $\square$

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