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COTTON TENSOR AND CONFORMAL DEFORMATIONS OF THREE-DIMENSIONAL RICCI FLOW

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Abstract

In this paper, we study the deformation of the three-dimensional conformal structures by the Ricci flow. We drive the evolution equation of the Cotton–York tensor and the L^1-norm of it under the Ricci flow. In particular, we investigate the behavior of the L^1-norm of the Cotton–York tensor under the Ricci flow on three-dimensional simply-connected Riemannian homogeneous spaces which admit compact quotients. For a non-homogeneous case, we also investigate the behavior of the L^1-norm for the product metric of the Rosenau solution for the Ricci flow on S^2 and the standard metric of S^1.

1. Introduction

Let M^n be a C^∞ manifold. A one-parameter family of Riemannian metrics $g(t)$ is called the *Ricci flow* if it satisfies

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}_g\ .$$

We are interested in the properties of the Ricci flow from the viewpoint of three-dimensional conformal geometry. More precisely, we study the deformation of the three-dimensional conformal structures by the Ricci flow.

It is well known that the conformal flatness in dimension $n \geq 4$ is equivalent to the vanishing of the Weyl tensor. In dimension $n = 3$, the Weyl tensor vanishes identically, and hence the conformal flatness cannot be detected by the Weyl tensor. However, there is a conformally invariant tensor which in $n = 3$ plays a role analogous to that of the Weyl tensor in $n \geq 4$. This tensor is called the *Cotton tensor* and defined by

$$C_3 = C_{ijk} := \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{4}(\nabla_i R g_{jk} - \nabla_j R g_{ik}),$$

where R_{ij} is the Ricci tensor, R is the scalar curvature and ∇ is the Levi-Civita connection. It can be shown that C_3 is conformally invariant and the conformal flatness is equivalent to $C_3 = 0$. By a direct computation, we can see that the following properties hold:

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1. $C_{ijk} + C_{jik} = 0$,
2. $C_{ijk} + C_{jki} + C_{kij} = 0$,
3. $g^{ij}C_{ijk} = g^{ik}C_{ijk} = g^{jk}C_{ijk} = 0$.

We can write the Cotton tensor in an algebraically equivalent form which is called the *Cotton–York tensor* [17]

$$C_2 = C_{ij} := g_{ik}\varepsilon^{klm}\left(\nabla_l R_{mj} - \frac{1}{4}\nabla_l R g_{mj}\right) = \frac{1}{2}g_{ik}\varepsilon^{klm}C_{lmj},$$

where ε^{ijk} is a tensor constructed by using the completely anti-symmetric tensor density η^{klm} of weight $+1$ with $\eta^{123} = 1$ and the determinant of the metric tensor g for the given coordinate system:

$$\varepsilon^{klm} := \frac{\eta^{klm}}{\sqrt{\det g}}.$$

The tensor ε satisfies the following:

1. $\varepsilon_{ijk}\varepsilon^{ilm} = \delta^l_j\delta^m_k - \delta^m_j\delta^l_k$,
2. $\varepsilon_{ijk}\varepsilon^{ijl} = 2\delta^l_k$,
3. $\varepsilon_{ijk}\varepsilon^{ijk} = 6$,
4. $\nabla_i\varepsilon^{jkl} = \nabla_i\varepsilon_{jkl} = 0$.

From the relation $C_{ijk} = \varepsilon_{ijl}g^{lm}C_{mk}$, we can see that the conformal flatness is equivalent to $C_2 = 0$. The (2, 0)-tensor C_2 has the following properties:

1. $C_{ij} = C_{ji}$ (symmetric),
2. $g^{ij}C_{ij} = 0$ (trace-free),
3. $\nabla^i C_{ij} = 0$ (divergence-free/transverse),
4. $|C_3|_g = \sqrt{2}|C_2|_g$,

where $|C_3|^2_g(x) = (g^{ip}g^{jq}g^{kr}C_{ijk}C_{pqr})(x)$ and $|C_2|^2_g(x) = (g^{ip}g^{jq}C_{ij}C_{pq})(x)$. We consider the L^1-norm of the Cotton–York tensor on a closed Riemannian manifold (M^3, g)

$$C(g) := \int_{M^3} |C_2|_g \, d\mu_g,$$

where $d\mu_g$ is the volume element of g. Note that the L^1-norm of the Cotton tensor differs from that of the Cotton–York tensor by $\sqrt{2}$ multiple. By the property of the Cotton–York tensor, it is easy to see the conformal invariance of the L^1-norm and the equivalence between the vanishing of the L^1-norm and the conformal flatness. If the manifold is non-compact, we consider the L^1-norm on an arbitrary compact set K

$$C_K(g) := \int_K |C_2|_g \, d\mu_g.$$

We are interested in the behavior of the L^1-norm under the Ricci flow.

As a fundamental result of the Ricci flow by R. Hamilton [6], it is known that the Ricci flow starting at an initial metric with positive Ricci curvature on a three-dimensional closed manifold converges to a constant curvature metric up to scaling. Since a constant curvature metric is conformally flat, we can regard this result as a convergence to a conformally flat metric and a vanishing of the L^1-norm of the Cotton–York tensor. In general, the Ricci flow develops singularities, but it is not clear whether or not the conformal structure degenerates in the sense that the L^1-norm of the Cotton–York tensor blows up. This observations have motivated us to look into the properties of the Cotton–York tensor under the Ricci flow.

C. Mantegazza, S. Mongodi, and M. Rimoldi [10] described the evolution of the Cotton tensor C_3 under the Ricci flow. By using this evolution equation, we derive the evolution equation of the Cotton–York tensor C_2 and the L^1-norm of C_2 (Proposition 2.3 and Theorem 2.1). In particular, we investigate the behavior of the L^1-norm of the Cotton–York tensor on two separate contexts. The first is evolution of the L^1-norm under the Ricci flow on simply-connected three-dimensional Riemannian homogeneous spaces $M = G/H$ which admit compact quotients. Here G is a transitive group of diffeomorphisms of M and H is the compact isotropy subgroup. We assume that G is minimal, i.e. no proper subgroup of G acts transitively on M. The second context is for the product metric of the Rosenau solution [15] for the Ricci flow on S^2 (which is ancient and shrinks to a round point as $t \nearrow 0$) and the standard metric of S^1. Note that the product metric on $S^2 \times S^1$ is also a solution to the Ricci flow. Recall the complete list of the Riemannian homogeneous spaces M ([11], [16]): $\mathbb{R}^3$, SU(2), $\widetilde{\mathrm{Isom}(\mathbb{R}^2)}$, $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, the Heisenberg group, $\widetilde{\mathrm{Isom}(\mathbb{R}^1_1)}$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, and $S^2 \times \mathbb{R}$, where the tildes denote the universal covering spaces, $\mathbb{R}^1_1$ is the two-dimensional Minkowski space, $\mathbb{H}^3$ is three-dimensional hyperbolic space, and $\mathbb{H}^2$ is two-dimensional hyperbolic space. Since homogeneous geometries on $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, and $S^2 \times \mathbb{R}$ are conformally flat, the L^1-norm of Cotton–York tensor for these geometries trivial. In the first six homogeneous spaces, for an arbitrary left invariant metric g_0, J. Milnor [11] provided a left invariant frame field $\{F_i\}_i^3$ (called the *Milnor frame* for g_0) such that

$$g_0 = A_0 \omega^1 \otimes \omega^1 + B_0 \omega^2 \otimes \omega^2 + C_0 \omega^3 \otimes \omega^3$$

where A_0, B_0, C_0 are positive constants and

$$[F_2, F_3] = 2\lambda F_1, \quad [F_3, F_1] = 2\mu F_2, \quad [F_1, F_2] = 2\nu F_3,$$

where $\lambda, \mu, \nu \in \{-1, 0, 1\}$ and $\lambda \le \mu \le \nu$ are satisfied. Recall that the value of the triplet λ, μ, ν completely determines the corresponding Lie group for the six homogeneous spaces. With respect to the Milnor frame, not only g_0 but also Ric_{g_0} are diagonalized. As g_0 and Ric_{g_0} remain diagonalized under the Ricci flow, it follows that the metric $g(t)$ evolves as

$$g(t) = A(t)\omega^1 \otimes \omega^1 + B(t)\omega^2 \otimes \omega^2 + C(t)\omega^3 \otimes \omega^3$$

Lie group	Behavior of the L^1-norm $C(g)$, $C_K(g)$
SU(2)	$C(g) \to 0$ and it has a unique local extremum if $A_0/B_0 < 1/2$. $C(g) \searrow 0$ if $1/2 \leq A_0/B_0 < 1$ or $1 < A_0/B_0$. $C(g) = 0$ if $A_0 = B_0$.
$\widetilde{\mathrm{Isom}(\mathbb{R}^2)}$	$C_K(g) \searrow 0$ if $A_0 \neq B_0$. $C_K(g) = 0$ if $A_0 = B_0$.
$\widetilde{\mathrm{SL}(2, \mathbb{R})}$	$C_K(g) \searrow 0$.
Heisenberg	$C_K(g) \searrow 0$.
$\widetilde{\mathrm{Isom}(\mathbb{R}^1_1)}$	$C_K(g) \searrow 0$.
$\mathbb{R}^3$	$C_K(g) = 0$.

Table 1. The behavior of the L^1-norm of the Cotton–York tensor.

and that the Ricci flow equation becomes a system of three ODE's [7] for $A(t)$, $B(t)$, and $C(t)$. J. Isenberg and M. Jackson [7] studied the behavior of the normalized Ricci flow on all the homogeneous spaces. The behavior of the (unnormalized) Ricci flow on those spaces was studied by D. Knopf and K. McLeod [8] (see also [3]). The Ricci flow on $\mathbb{R}^3$ is trivial. It becomes asymptotically round as $t \nearrow T < \infty$ on SU(2). It converges to the flat space as $t \nearrow \infty$ on $\widetilde{\mathrm{Isom}(\mathbb{R}^2)}$. For the other Lie groups, the each solution to the Ricci flow approaches a flat degenerate geometry of either two or one dimensions as $t \nearrow \infty$. We follow the calculations as done in these previous works.

We suppose that in the case of SU(2) and $\widetilde{\mathrm{SL}(2, \mathbb{R})}$, the initial metric g_0 satisfies $B_0 = C_0$. The results for the six homogeneous spaces are summarized in the Table 1 (Theorems 3.2, 3.4, 3.6, 3.8 and 3.10). The main conclusions are the following:

1. In all cases, the each L^1-norm of the Cotton–York tensor converges to zero.
2. If the initial metric g_0 on SU(2) satisfies $B_0 = C_0$ and $A_0/B_0 < 1/2$, the L^1-norm has a unique local extremum at t_0 with $A(t_0)/B(t_0) = 1/2$.
3. In other cases, the each L^1-norm is strictly decreasing or identically zero.

The L^1-norm of the Cotton–York tensor for the product metric of the Rosenau solution and the standard metric of S^1 is strictly decreasing and converges to zero as $t \nearrow 0$ (Theorem 3.11).

It is interesting that in these examples the L^1-norm of the solution to the Ricci flow starting at the initial metric with non-positive scalar curvature is strictly decreasing. The following are topics for further investigation:

- The monotonicity of the L^1-norm of the Cotton–York tensor (or the lack of it).
- The characterization of the Riemannian manifold at the time which the L^1-norm takes a local extremum.

2. The evolution equation of the L^1-norm of the Cotton–York tensor

For any tensor T, S such as T_{ij}, S_{ij}, we define $\langle T, S\rangle_g := g^{ip}g^{jq}T_{ij}S_{pq}$, $T^2 := T_{ik}g^{kl}T_{lj}$, $\mathrm{div}_g T := \nabla^i T_{ij}$, and $\Delta_g T := g^{ij}\nabla_i\nabla_j T_{ij}$. Our goal in this section is to derive the following evolution equation.

Theorem 2.1. *Let $(M^3, g(t))$, $0 \le t < T$ be a solution of the Ricci flow on a closed manifold. Suppose the norm of the Cotton–York tensor C_2 does not vanish in $M \times [0, T)$. Then the L^1-norm of C_2 satisfies the following evolution equation*

$$\begin{aligned}
&\frac{d}{dt}\int_M |C_2|_g d\mu_g \\
&= \int_M \frac{1}{2|C_2|_g}(\Delta_g |C_2|_g^2 - 2|\nabla C_2|_g^2 - 16\langle \mathrm{Ric}, C_2^2\rangle_g + 6R|C_2|_g^2 \\
&\qquad - 4\langle \mathrm{Ric}, \mathrm{div}_g\, D\rangle_g + 4\langle \mathrm{Ric}^2, \mathrm{div}_g\, C_3\rangle_g - 2\langle \nabla R, \mathrm{div}_g(\mathrm{div}_g\, C_3)\rangle_g)\, d\mu_g,
\end{aligned}$$

where $D = D_{ijk} := C_{ijp} g^{pq} R_{qk}$.

The evolution equation of the Cotton tensor C_3 under the Ricci flow is obtained by Mantegazza, Mongodi, and Rimoldi.

Proposition 2.2 ([10]). *Let $(M^3, g(t))$ be a solution of the Ricci flow. Then the Cotton tensor C_3 satisfies the following evolution equation*

$$\begin{aligned}
\frac{\partial}{\partial t} C_{ijk} &= \Delta_g C_{ijk} + g^{pq} R_{pj}(C_{kqi} + C_{kiq}) + 5 g^{pq} R_{kp} C_{jiq} + g^{pq} R_{pi}(C_{qkj} + C_{jkq}) \\
&\quad + 2RC_{ijk} + 2g^{pq} g^{rs} R_{pr} C_{sjq} g_{ki} - 2g^{pq} g^{rs} R_{pr} C_{siq} g_{kj} \\
&\quad + \frac{1}{2}(\nabla_i |\mathrm{Ric}|_g^2) g_{kj} - \frac{1}{2}(\nabla_j |\mathrm{Ric}|_g^2) g_{ki} + \frac{R}{2}(\nabla_j R) g_{ki} - \frac{R}{2}(\nabla_i R) g_{kj} \\
&\quad + 2g^{pq} R_{pi} \nabla_j R_{qk} - 2g^{pq} R_{pj} \nabla_i R_{qk} + R_{kj} \nabla_i R - R_{ki} \nabla_j R.
\end{aligned}$$

By using Proposition 2.2, we obtain the evolution equation of the Cotton–York tensor C_2 under the Ricci flow.

Proposition 2.3. *Let $(M^3, g(t))$ be a solution of the Ricci flow. Then the Cotton–York tensor C_2 satisfies the following evolution equation*

$$\begin{aligned}
\frac{\partial}{\partial t} C_{ij} &= \Delta_g C_{ij} - 5g^{pq} R_{ip} C_{qj} - 5g^{pq} C_{iq} R_{pj} + 2\langle C_2, \mathrm{Ric}\rangle_g g_{ij} + 4RC_{ij} \\
&\quad + \frac{1}{2} g_{ik} g_{jm} \varepsilon^{klm} \nabla_l |\mathrm{Ric}|_g^2 + \frac{R}{2} g_{ik} g_{jl} \varepsilon^{klm} \nabla_m R + 2 g_{ik} g^{pq} \varepsilon^{klm} R_{pl} \nabla_m R_{qj} \\
&\quad + g_{ik} \varepsilon^{klm} R_{jm} \nabla_l R.
\end{aligned}$$

Corollary 2.4. *Let $(M^3, g(t))$ be a solution of the Ricci flow. Then the squared norm of the Cotton–York tensor C_2 satisfies the following evolution equation*

$$\frac{\partial}{\partial t}|C_2|_g^2 = \Delta_g|C_2|_g^2 - 2|\nabla C_2|_g^2 - 16\langle \mathrm{Ric}, C_2^2\rangle_g + 8R|C_2|_g^2$$
$$- 4\langle \mathrm{Ric}, \mathrm{div}_g\, D\rangle_g + 4\langle \mathrm{Ric}^2, \mathrm{div}_g\, C_3\rangle_g - 2\langle \nabla R, \mathrm{div}_g(\mathrm{div}_g\, C_3)\rangle_g.$$

Proof of Proposition 2.3. Note that

$$\frac{\partial}{\partial t}\varepsilon^{klm} = R\varepsilon^{klm}.$$

Indeed,

$$\frac{\partial}{\partial t}\varepsilon^{klm} = \frac{\partial}{\partial t}\left(\frac{\eta^{klm}}{\sqrt{\det(g_{ij})}}\right) = -\frac{\eta^{klm}}{\det(g_{ij})}\cdot(-R\sqrt{\det(g_{ij})}) = R\varepsilon^{klm}.$$

By this equation and Proposition 2.2,

$$\frac{\partial}{\partial t}C_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial t}g_{ik}\right)\varepsilon^{klm}C_{lmj} + \frac{1}{2}g_{ik}\left(\frac{\partial}{\partial t}\varepsilon^{klm}\right)C_{lmj} + \frac{1}{2}g_{ik}\varepsilon^{klm}\left(\frac{\partial}{\partial t}C_{lmj}\right)$$
$$= -R_{ik}\varepsilon^{klm}C_{lmj} + \frac{1}{2}g_{ik}R\varepsilon^{klm}C_{lmj} + \frac{1}{2}g_{ik}\varepsilon^{klm}$$
$$\times \Big\{\Delta_g C_{lmj} + g^{pq}R_{pm}(C_{jql} + C_{jlq}) + 5g^{pq}R_{jp}C_{mlq} + g^{pq}R_{pl}(C_{qjm} + C_{mjq})$$
$$+ 2RC_{lmj} + 2g^{pq}g^{rs}R_{pr}C_{smq}g_{jl} - 2g^{pq}g^{rs}R_{pr}C_{slq}g_{jm}$$
$$+ \frac{1}{2}(\nabla_l|\mathrm{Ric}|_g^2)g_{jm} - \frac{1}{2}(\nabla_m|\mathrm{Ric}|_g^2)g_{jl} + \frac{R}{2}(\nabla_m R)g_{jl} - \frac{R}{2}(\nabla_l R)g_{jm}$$
$$+ 2g^{pq}R_{pl}\nabla_m R_{qj} - 2g^{pq}R_{pm}\nabla_l R_{qj} + R_{jm}\nabla_l R - R_{jl}\nabla_m R\Big\}.$$

We compute each term by using the identities $C_{ij} = (1/2)g_{ik}\varepsilon^{klm}C_{lmj}$, $C_{ijk} = \varepsilon_{ijl}g^{lm}C_{mk}$, and the properties of C_3, C_2, ε.

$$(\text{1st term of RHS}) = -R_{ik}\varepsilon^{klm}\varepsilon_{lmp}g^{pq}C_{qj} = -R_{ik}\cdot 2\delta_p^k g^{pq}C_{qj} = -2R_{ip}g^{pq}C_{qj},$$
$$(\text{2nd}) = RC_{ij},$$
$$(\text{3rd}) = \frac{1}{2}g_{ik}\varepsilon^{klm}\Delta_g C_{lmj} = \Delta_g\left(\frac{1}{2}g_{ik}\varepsilon^{klm}C_{lmj}\right) = \Delta_g C_{ij},$$
$$(\text{4th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}C_{jql} = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}(-C_{qlj} - C_{ljq})$$
$$= -\frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}\varepsilon_{qlr}g^{rs}C_{sj} - \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}\varepsilon_{ljr}g^{rs}C_{sq}$$
$$= -\frac{1}{2}g_{ik}g^{pq}g^{rs}(\delta_q^k\delta_r^m - \delta_r^k\delta_q^m)R_{pm}C_{sj} + \frac{1}{2}g_{ik}g^{pq}g^{rs}(\delta_j^k\delta_r^m - \delta_r^k\delta_j^m)R_{pm}C_{sq}$$

$$= -\frac{1}{2}g^{rs}R_{ir}C_{sj} + \frac{1}{2}RC_{ij} + \frac{1}{2}\langle C_2, \mathrm{Ric}\rangle_g g_{ij} - \frac{1}{2}g^{pq}C_{iq}R_{pj},$$

$$(5\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}C_{jlq} = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pm}\varepsilon_{jlr}g^{rs}C_{sq}$$

$$= \frac{1}{2}g_{ik}g^{pq}g^{rs}(\delta^k_j\delta^m_r - \delta^k_r\delta^m_j)R_{pm}C_{sq} = \frac{1}{2}g_{ij}g^{pq}g^{rs}R_{pr}C_{sq} - \frac{1}{2}g_{ir}g^{pq}g^{rs}R_{pj}C_{sq}$$

$$= \frac{1}{2}\langle C_2, \mathrm{Ric}\rangle_g g_{ij} - \frac{1}{2}g^{pq}C_{iq}R_{pj},$$

$$(6\text{th}) = \frac{5}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{jp}\varepsilon_{mlr}g^{rs}C_{sq} = -\frac{5}{2}g_{ik}g^{pq}g^{rs}\cdot 2\delta^k_r R_{jp}C_{sq} = -5g^{pq}C_{iq}R_{pj},$$

$$(7\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pl}C_{qjm} = -\frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pl}C_{jqm} = (4\text{th}),$$

$$(8\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pl}C_{mjq} = -\frac{1}{2}g_{ik}\varepsilon^{klm}g^{pq}R_{pl}C_{jmq} = (5\text{th}),$$

$$(9\text{th}) = 2RC_{ij},$$

$$(10\text{th}) = g_{ik}\varepsilon^{klm}g^{pq}g^{rs}R_{pr}\varepsilon_{sma}g^{ab}C_{bq}g_{jl} = -g_{ik}g^{pq}g^{rs}g^{ab}g_{jl}(\delta^k_s\delta^l_a - \delta^k_a\delta^l_s)R_{pr}C_{bq}$$

$$= -g^{pq}R_{pi}C_{jq} + g^{pq}C_{iq}R_{pj},$$

$$(11\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{kml}\cdot(-2)g^{pq}g^{rs}R_{pr}C_{smq}g_{jl} = \frac{1}{2}g_{ik}\varepsilon^{klm}\cdot 2g^{pq}g^{rs}R_{pr}C_{smq}g_{jl} = (10\text{th}),$$

$$(12\text{th}) = \frac{1}{4}g_{ik}g_{jm}\varepsilon^{klm}\nabla_l|\mathrm{Ric}|^2_g,$$

$$(13\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}\cdot\left(-\frac{1}{2}\nabla_m|\mathrm{Ric}|^2_g\right)g_{jl} = \frac{1}{2}g_{ik}\varepsilon^{kml}\cdot\frac{1}{2}\left(\nabla_m|\mathrm{Ric}|^2_g\right)g_{jl} = (12\text{th}),$$

$$(14\text{th}) = \frac{R}{4}g_{ik}g_{jl}\varepsilon^{klm}\nabla_m R,$$

$$(15\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}\cdot\left(-\frac{R}{2}\nabla_l R\right)g_{jm} = \frac{1}{2}g_{ik}\varepsilon^{kml}\cdot\frac{R}{2}(\nabla_l R)g_{jm} = (14\text{th}),$$

$$(16\text{th}) = g_{ik}g^{pq}\varepsilon^{klm}R_{pl}\nabla_m R_{qj},$$

$$(17\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}\cdot(-2)g^{pq}R_{pm}\nabla_l R_{qj} = \frac{1}{2}g_{ik}\varepsilon^{kml}\cdot 2g^{pq}R_{pm}\nabla_l R_{qj} = (16\text{th}),$$

$$(18\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}R_{jm}\nabla_l R,$$

$$(19\text{th}) = \frac{1}{2}g_{ik}\varepsilon^{klm}\cdot(-1)R_{jl}\nabla_m R = \frac{1}{2}g_{ik}\varepsilon^{kml}R_{jl}\nabla_m R = (18\text{th}).$$

Hence, we obtain the result. □

Proof of Corollary 2.4. By Proposition 2.3,

$$
\begin{aligned}
&\frac{\partial}{\partial t}|C_2|_g^2 \\
&= 2\left(\frac{\partial}{\partial t}g^{i_1 i_2}\right)g^{j_1 j_2}C_{i_1 j_1}C_{i_2 j_2} + 2g^{i_1 i_2}g^{j_1 j_2}\left(\frac{\partial}{\partial t}C_{i_1 j_1}\right)C_{i_2 j_2} \\
&= 4R^{i_1 i_2}C^2_{i_1 i_2} + 2g^{i_1 i_2}g^{j_1 j_2} \\
&\quad \times \Big\{\Delta_g C_{i_1 j_1} - 5g^{pq}R_{i_1 p}C_{qj_1} - 5g^{pq}C_{i_1 q}R_{pj_1} + 2\langle C_2, \mathrm{Ric}\rangle_g g_{i_1 j_1} + 4RC_{i_1 j_1} \\
&\qquad + \frac{1}{2}g_{i_1 k}g_{j_1 m}\varepsilon^{klm}\nabla_l|\mathrm{Ric}|_g^2 + \frac{R}{2}g_{i_1 k}g_{j_1 l}\varepsilon^{klm}\nabla_m R + 2g_{i_1 k}g^{pq}\varepsilon^{klm}R_{pl}\nabla_m R_{qj_1} \\
&\qquad + g_{i_1 k}\varepsilon^{klm}R_{j_1 m}\nabla_l R\Big\} \times C_{i_2 j_2}.
\end{aligned}
$$

We compute each term by using $(1/2)g_{ik}\varepsilon^{klm}C_{lmj}$, $C_{ijk} = \varepsilon_{ijl}g^{lm}C_{mk}$, and the properties of C_3, C_2, ε.

$$
\begin{aligned}
&\text{(1st term of RHS)} = 4\langle \mathrm{Ric}, C_2^2\rangle_g, \\
&\text{(2nd)} = 2g^{i_1 i_2}g^{j_1 j_2}(\Delta_g C_{i_1 j_1})C_{i_2 j_2} = 2\langle \Delta C_2, C_2\rangle_g = \Delta_g|C_2|_g^2 - 2|\nabla C_2|_g^2, \\
&\text{(3rd)} = -10g^{i_1 i_2}g^{pq}R_{i_1 p}C^2_{i_2 q} = -10\langle \mathrm{Ric}, C_2^2\rangle_g, \\
&\text{(4th)} = -10g^{j_1 j_2}g^{pq}C^2_{qj_2}R_{pj_1} = -10\langle C_2^2, \mathrm{Ric}\rangle_g, \\
&\text{(5th)} = 4\langle C_2, \mathrm{Ric}\rangle_g g^{i_2 j_2}C_{i_2 j_2} = 4\langle C_2, \mathrm{Ric}\rangle_g \mathrm{tr}_g C_2 = 0, \\
&\text{(6th)} = 8RC_{i_1 j_1}g^{i_1 i_2}g^{j_1 j_2}C_{i_2 j_2} = 8R|C_2|_g^2, \\
&\text{(7th)} = \delta_k^{i_2}\delta_m^{j_2}\varepsilon^{klm}\nabla_l|\mathrm{Ric}|_g^2 C_{i_2 j_2} = \varepsilon^{klm}(\nabla_l|\mathrm{Ric}|_g^2)C_{km} = 0, \\
&\text{(8th)} = \delta_k^{i_2}\delta_l^{j_2}R\varepsilon^{klm}\nabla_m RC_{i_2 j_2} = R\varepsilon^{klm}(\nabla_m R)C_{kl} = 0, \\
&\text{(9th)} = 4g^{i_1 i_2}g^{j_1 j_2}g_{i_1 k}g^{pq}\varepsilon^{klm}R_{pl}(\nabla_m R_{qj_1})\cdot\frac{1}{2}g_{i_2 a}\varepsilon^{ars}C_{rsj_2} \\
&\qquad = 2g^{pq}\varepsilon^{i_2 lm}\varepsilon_{i_2 rs}R_{pl}(\nabla_m R_{qj_1})C^{rsj_1} = 2g^{pq}(\delta_r^l\delta_s^m - \delta_s^l\delta_r^m)R_{pl}(\nabla_m R_{qj_1})C^{rsj_1} \\
&\qquad = -4g^{pq}R_{pr}C^{srj_1}\nabla_s R_{j_1 q} = -4g^{pq}R_{pr}\nabla_s(C^{srj_1}R_{j_1 q}) + 4g^{pq}R_{pr}(\nabla_s C^{srj_1})R_{j_1 q} \\
&\qquad = -4\langle \mathrm{Ric}, \mathrm{div}_g\, D\rangle_g + 4\langle \mathrm{Ric}^2, \mathrm{div}_g\, C_3\rangle_g, \\
&\text{(10th)} = 2g^{i_1 i_2}g^{j_1 j_2}g_{i_1 k}\varepsilon^{klm}R_{j_1 m}(\nabla_l R)\cdot\frac{1}{2}g_{i_2 a}\varepsilon^{ars}C_{rsj_2} \\
&\qquad = \varepsilon^{i_2 lm}\varepsilon_{i_2 rs}R_{j_1 m}(\nabla_l R)C^{rsj_1} = (\delta_r^l\delta_s^m - \delta_s^l\delta_r^m)R_{j_1 m}(\nabla_l R)C^{rsj_1} \\
&\qquad = 2(\nabla^r R)C_{rsj_1}R^{sj_1} = 2(\nabla_r R)(-\nabla^q\nabla^p C_{pqr}) \\
&\qquad = -2\langle \nabla R, \mathrm{div}_g(\mathrm{div}_g\, C_3)\rangle_g,
\end{aligned}
$$

where we use the identity $\nabla^j\nabla^i C_{ijk} = -C_{klm}R^{lm}$ (see for example [1, p. 9]). Hence, we obtain the result. □

Theorem 2.1 follows from Corollary 2.4 and $(d/dt)\,d\mu_g = -R\,d\mu_g$.

3. Examples of behavior of the L^1-norm

3.1. The Lie group SU(2). We consider the Ricci flow $g(t)$ starting at a left invariant metric g_0 on SU(2), and fix a Milnor frame for g_0 such that $\lambda = \mu = \nu = -1$. Note that SU(2) is identified topologically with standard three-sphere of radius one embedded in $\mathbb{R}^4$.

The Ricci tensor of g is

$$\begin{aligned} R(F_1, F_1) &= 4 - 2\frac{B^2 + C^2 - A^2}{BC}, \\ R(F_2, F_2) &= 4 - 2\frac{C^2 + A^2 - B^2}{CA}, \\ R(F_3, F_3) &= 4 - 2\frac{B^2 + A^2 - C^2}{BA}. \end{aligned}$$

Then the Ricci flow equation is equivalent to the system of ODE's

$$\begin{cases} \dfrac{d}{dt}A = -8 + 4\dfrac{B^2 + C^2 - A^2}{BC}, \\ \dfrac{d}{dt}B = -8 + 4\dfrac{C^2 + A^2 - B^2}{CA}, \\ \dfrac{d}{dt}C = -8 + 4\dfrac{B^2 + A^2 - C^2}{BA}. \end{cases}$$

Proposition 3.1 ([3, Proposition 1.17]). *For any choice of initial data A_0, B_0, $C_0 > 0$, the unique solution $g(t)$ exists for a maximal finite time interval $0 \le t < T < \infty$. The metric $g(t)$ becomes asymptotically round as $t \nearrow T$.*

Now we are interested in the behavior of the L^1-norm of the Cotton–York tensor C_2. Since the L^1-norm is very complicated for general initial data, we assume that $B_0 = C_0$. Then $B(t) = C(t)$ holds from the symmetry in the Ricci flow equation.

Theorem 3.2. *For any choice of initial data A_0, $B_0 = C_0 > 0$, the behavior of the L^1-norm $C(g)$ of the Cotton–York is the following*:

1. *If $0 < A_0/B_0 < 1/2$, $C(g)$ has a unique local extremum at t_0 with $A(t_0)/B(t_0) = 1/2$ and converges to zero as $t \to T$.*
2. *If $1/2 \le A_0/B_0 < 1$ or $1 < A_0/B_0$, $C(g)$ is strictly decreasing and converges to zero as $t \to T$.*
3. *If $A_0 = B_0$, $C(g)$ is identically zero.*

Proof. In this case, the Ricci flow equation is reduced to

$$\frac{d}{dt}A = -4\left(\frac{A}{B}\right)^2, \quad \frac{d}{dt}B = -8 + 4\frac{A}{B},$$

and the scalar curvature is

$$R = \frac{2(4B - A)}{B^2}.$$

Note that $A_0/B_0 = 1$, $A_0/B_0 < 1$, and $A_0/B_0 > 1$ are preserved under the Ricci flow, and $\lim_{t\nearrow T} A = \lim_{t\nearrow T} B = 0$ in all cases.

The Cotton–York tensor is

$$C_2(F_1, F_1) = 8\frac{A^{3/2}}{B^2}\left(\frac{A}{B} - 1\right), \quad C_2(F_2, F_2) = C_2(F_3, F_3) = 4\frac{A^{1/2}}{B}\left(1 - \frac{A}{B}\right).$$

Then for an arbitrary compact set K,

$$\int_K |C_2(t)|_{g(t)}\, d\mu_{g(t)} = 4\sqrt{6}\frac{A}{B}\left|\frac{A}{B} - 1\right| \mathrm{Vol}(K, g_{S^3}),$$

where g_{S^3} is the standard metric of radius one on S^3. In particular,

$$\int_{S^3} |C_2(t)|_{g(t)}\, d\mu_{g(t)} = \begin{cases} 4\sqrt{6}A/B(1 - A/B)\,\mathrm{Vol}(S^3, g_{S^3}), & 0 < A_0/B_0 \le 1, \\ 4\sqrt{6}A/B(A/B - 1)\,\mathrm{Vol}(S^3, g_{S^3}), & 1 \le A_0/B_0. \end{cases}$$

If $A_0 = B_0$, $C(g)$ is identically zero. We assume that $A_0 \neq B_0$. We show that as $t \nearrow T$, $A/B \nearrow 1$ if $A_0/B_0 < 1$ and $A/B \searrow 1$ if $A_0/B_0 > 1$. Indeed,

$$\frac{d}{dt}\frac{A}{B} = \frac{-4(A/B)^2 B - A(-8 + 4(A/B))}{B^2} = 8\frac{A}{B^2}\left(1 - \frac{A}{B}\right),$$

hence A/B is strictly increasing if $A_0/B_0 < 1$ and strictly decreasing if $A_0/B_0 > 1$. Since A/B is bounded and monotone, it converges to some constant $\alpha > 0$. By l'Hôspital' rule,

$$\alpha = \lim_{t\nearrow T}\frac{A}{B} = \lim_{t\nearrow T}\frac{-4(A/B)^2}{-8 + 4(A/B)} = \frac{-4\alpha^2}{-8 + 4\alpha}.$$

Hence we obtain $\alpha = 1$.

We define the functions f and h on $\mathbb{R}$ respectively as

$$f(x) := x(1 - x) \quad \text{and} \quad h(x) := x(x - 1).$$

Since $A/B \nearrow 1$ if $A_0/B_0 < 1$, the function $f(A/B)$ has a maximal value at t_0 with $A(t_0)/B(t_0) = 1/2$ and $f(A/B) \to 0$ if $A_0/B_0 < 1/2$, and $f(A/B) \searrow 0$ if $1/2 \le A_0/B_0 < 1$. Since $A/B \searrow 1$ if $1 < A_0/B_0$, the function $h(A/B) \searrow 0$ if $1 < A_0/B_0$. Hence if $0 < A_0/B_0 < 1/2$, the L^1-norm $C(g)$ has a maximal value at t_0 with $A(t_0)/B(t_0) = 1/2$ and converges to zero as $t \to T$. If $1/2 \le A_0/B_0 < 1$ or $1 < A_0/B_0$, it is strictly decreasing and converges to zero as $t \to T$. □

3.2. The Lie group $\widetilde{\mathbf{Isom}(\mathbb{R}^2)}$. We consider the Ricci flow $g(t)$ starting at a left invariant metric g_0 on $\widetilde{\mathrm{Isom}(\mathbb{R}^2)}$, and fix a Milnor frame for g_0 such that $\lambda = \mu = -1$ and $\nu = 0$.

The Ricci tensor of g

$$R(F_1, F_1) = -2\frac{B^2 - A^2}{BC}, \quad R(F_2, F_2) = -2\frac{A^2 - B^2}{AC}, \quad R(F_3, F_3) = -2\frac{(A - B)^2}{AB},$$

and the scalar curvature of g is

$$R = -2\frac{(A - B)^2}{ABC}.$$

Then the Ricci flow equation is equivalent to the system of ODE's

$$\begin{cases} \dfrac{d}{dt}A = 4\dfrac{B^2 - A^2}{BC}, \\ \dfrac{d}{dt}B = 4\dfrac{A^2 - B^2}{AC}, \\ \dfrac{d}{dt}C = 4\dfrac{(A - B)^2}{AB}. \end{cases}$$

By the direct computation, we can show $(d/dt)(AB) = (d/dt)(C(A + B)) = 0$.

Proposition 3.3 ([8]). *For any choice of initial data A_0, B_0, $C_0 > 0$, the unique solution $g(t)$ exists for all positive time. For any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that*

$$|A - \sqrt{A_0 B_0}| \le \varepsilon, \quad |B - \sqrt{A_0 B_0}| \le \varepsilon, \quad \left| C - \frac{C_0}{2}\left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}}\right)\right| \le \varepsilon$$

for all $t \ge T_\varepsilon$. Moreover, as $t \nearrow \infty$, $B/A \nearrow 1$ if $B_0/A_0 < 1$, $B/A \searrow 1$ if $1 < B_0/A_0$, and $B/A = 1$ if $B_0/A_0 = 1$.

The behavior of the L^1-norm of the Cotton–York tensor C_2 is given by the next result.

Theorem 3.4. *For any choice of initial data A_0, B_0, $C_0 > 0$, the behavior of the L^1-norm $C_K(g)$ of the Cotton–York tensor on an arbitrary compact set K is the following*:

1. *If $A_0 \neq B_0$, $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$.*
2. *If $A_0 = B_0$, $C_K(g)$ is identically zero.*

Proof. The Cotton–York tensor is

$$\begin{aligned} C_2(F_1, F_1) &= \frac{4A}{(ABC)^{3/2}}(2A^3 - B^3 - A^2B), \\ C_2(F_2, F_2) &= \frac{4B}{(ABC)^{3/2}}(2B^3 - A^3 - AB^2), \\ C_2(F_3, F_3) &= -\frac{4C}{(ABC)^{3/2}}(A+B)(A-B)^2. \end{aligned}$$

Then for an arbitrary compact set K,

$$\begin{aligned} &\int_K |C_2|_g \, d\mu_g \\ &= \left(6\left(\frac{A}{B}\right)^3 - 6\left(\frac{A}{B}\right)^2 + 2\left(\frac{A}{B}\right) + 6\left(\frac{B}{A}\right)^3 - 6\left(\frac{B}{A}\right)^2 + 2\left(\frac{B}{A}\right) - 4\right)^{1/2} \\ &\quad \times \frac{4(A_0B_0)^{1/2}}{C} \operatorname{Vol}(K, h) \end{aligned}$$

where $h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

If $A_0 = B_0$, $C_K(g)$ is identically zero. We assume that $A_0 \neq B_0$. We define the function f on $\mathbb{R}$ as

$$f(x) := \left(6\left(\frac{1}{x}\right)^3 - 6\left(\frac{1}{x}\right)^2 + 2\left(\frac{1}{x}\right) + 6x^3 - 6x^2 + 2x - 4\right)^{1/2}.$$

The function f is strictly decreasing if $0 < x \leq 1$ and strictly increasing if $1 < x$. By Proposition 3.3, as $t \nearrow \infty$, $f(B/A) \searrow 0$ if $B_0/A_0 < 1$ and $f(B/A) \searrow 0$ if $1 < B_0/A_0$. Clearly $1/C$ is strictly decreasing, hence $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$. □

3.3. The Lie group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. We consider the Ricci flow $g(t)$ starting at a left invariant metric g_0 on $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and fix a Milnor frame such that $\lambda = -1$ and $\mu = \nu = 1$.

The Ricci tensor of g is

$$R(F_1, F_1) = -2\frac{(B-C)^2 - A^2}{BC},$$
$$R(F_2, F_2) = -2\frac{(A+C)^2 - B^2}{AC},$$
$$R(F_3, F_3) = -2\frac{(A+B)^2 - C^2}{AB}.$$

Then the Ricci flow equation is equivalent to the system of ODE's

$$\begin{cases} \dfrac{d}{dt}A = 4\dfrac{(B-C)^2 - A^2}{BC}, \\ \dfrac{d}{dt}B = 4\dfrac{(A+C)^2 - B^2}{AC}, \\ \dfrac{d}{dt}C = 4\dfrac{(A+B)^2 - C^2}{AB}. \end{cases}$$

Proposition 3.5 ([8]). *For any choice of initial data* $A_0, B_0, C_0 > 0$, *the unique solution* $g(t)$ *exists for all positive time. There exists* $A_\infty = A_\infty(A_0, B_0, C_0) > 0$ *such that for any* $\varepsilon > 0$, *there exists* $T_\varepsilon > 0$ *such that*

$$|A - A_\infty| \le \varepsilon, \quad \left|\frac{d}{dt}B - 8\right| \le \varepsilon, \quad \left|\frac{d}{dt}C - 8\right| \le \varepsilon$$

for all $t \ge T_\varepsilon$.

Now we are interested in the behavior of the L^1-norm of the Cotton–York tensor C_2. Since the L^1-norm is very complicated for general initial data, we assume that $B_0 = C_0$. Then $B(t) = C(t)$ holds from the symmetry in the Ricci flow equation.

Theorem 3.6. *For any choice of initial data* A_0, $B_0 = C_0 > 0$, *the* L^1*-norm* $C_K(g)$ *of the Cotton–York tensor on an arbitrary compact set* K *is strictly decreasing and converges to zero as* $t \to \infty$.

Proof. In this case, the Ricci flow equation is reduced to

$$\frac{d}{dt}A = -4\left(\frac{A}{B}\right)^2, \quad \frac{d}{dt}B = 4\frac{A}{B} + 8,$$

and the scalar curvature is

$$R = -\frac{2(A+4B)}{B^2}.$$

The Cotton–York tensor is

$$C_2(F_1, F_1) = \frac{8A^3(A+B)}{(AB^2)^{3/2}}, \quad C_2(F_2, F_2) = C_2(F_3, F_3) = -\frac{4A^2B(A+B)}{(AB^2)^{3/2}}.$$

Then for an arbitrary compact set K,

$$\int_K |C_2|_g \, d\mu_g = 4\sqrt{6}\frac{A}{B}\left(1 + \frac{A}{B}\right) \mathrm{Vol}(K, h),$$

where $h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

The function A/B is strictly decreasing and converges to zero as $t \to \infty$. Indeed,

$$\frac{d}{dt}\frac{A}{B} = \frac{-4(A/B)^2B - A\{4(A/B) + 8\}}{B^2} = -8\frac{A}{B^2}\left(\frac{A}{B} + 1\right) < 0,$$

and $\lim_{t\to\infty}(A/B) = A_\infty/\infty = 0$. Hence $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$. □

3.4. The Heisenberg group. We consider the Ricci flow $g(t)$ starting at a left invariant metric g_0 on the Heisenberg group, and fix a Milnor frame for g_0 such that $\lambda = -1$ and $\mu = \nu = 0$.

The Ricci tensor of g is

$$R(F_1, F_1) = 2\frac{A^2}{BC}, \quad R(F_2, F_2) = -2\frac{A}{C}, \quad R(F_3, F_3) = -2\frac{A}{B},$$

and the scalar curvature of g is

$$R = -2\frac{A}{BC}.$$

Then the Ricci flow equation is equivalent to the system of ODE's

$$\begin{cases} \dfrac{d}{dt}A = -4\dfrac{A^2}{BC}, \\ \dfrac{d}{dt}B = 4\dfrac{A}{C}, \\ \dfrac{d}{dt}C = 4\dfrac{A}{B}. \end{cases}$$

Proposition 3.7 ([8]). *For any choice of initial data A_0, B_0, $C_0 > 0$, the unique solution $g(t)$ exists for all positive time. Moreover, the above system of ODE's is solved explicitly*:

$$A = A_0^{2/3} B_0^{1/3} C_0^{1/3} \left(12t + \frac{B_0 C_0}{A_0}\right)^{-1/3},$$
$$B = A_0^{1/3} B_0^{2/3} C_0^{-1/3} \left(12t + \frac{B_0 C_0}{A_0}\right)^{1/3},$$
$$C = A_0^{1/3} B_0^{-1/3} C_0^{2/3} \left(12t + \frac{B_0 C_0}{A_0}\right)^{1/3}$$

for $t \in (-B_0C_0/A_0, \infty)$.

The behavior of the L^1-norm of the Cotton–York tensor C_2 is given by the following:

Theorem 3.8. *For any choice of initial data A_0, B_0, $C_0 > 0$, the L^1-norm $C_K(g)$ of the Cotton–York tensor on an arbitrary compact set K is strictly decreasing and converges to zero as $t \to \infty$.*

Proof. The Cotton–York tensor is

$$C_2(F_1, F_1) = \frac{8A^2}{BC}\sqrt{\frac{A}{BC}}, \quad C_2(F_2, F_2) = -\frac{4A^2}{C\sqrt{ABC}}, \quad C_2(F_3, F_3) = -\frac{4A^2}{B\sqrt{ABC}}.$$

Then for an arbitrary compact set K,

$$\begin{aligned}\int_K |C_2(t)|_{g(t)}\, d\mu_{g(t)} &= 2\sqrt{6}\frac{A^2}{BC}\,\mathrm{Vol}(K, h)\\ &= 2\sqrt{6} A_0^{2/3} B_0^{1/3} C_0^{1/3} \left(12t + \frac{B_0C_0}{A_0}\right)^{-4/3} \mathrm{Vol}(K, h),\end{aligned}$$

where $h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

Hence $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$. □

3.5. The Lie group $\widetilde{\mathrm{Isom}(\mathbb{R}_1^1)}$. We consider the Ricci flow $g(t)$ starting at a left invariant metric g_0 on $\widetilde{\mathrm{Isom}(\mathbb{R}_1^1)}$, and fix a Milnor frame for g_0 such that $\lambda = -1$, $\mu = 0$, and $\nu = 1$.

The Ricci tensor g is

$$R(F_1, F_1) = -2\frac{C^2 - A^2}{BC}, \quad R(F_2, F_2) = -2\frac{(A + C)^2}{AC}, \quad R(F_3, F_3) = -2\frac{A^2 - C^2}{AB}$$

and the scalar curvature of g is

$$R = -2\frac{(A+C)^2}{ABC}.$$

Then the Ricci flow equation is equivalent to the system of ODE's

$$\begin{cases} \dfrac{d}{dt}A = 4\dfrac{C^2 - A^2}{BC}, \\ \dfrac{d}{dt}B = 4\dfrac{(A+C)^2}{AC}, \\ \dfrac{d}{dt}C = 4\dfrac{A^2 - C^2}{AB}. \end{cases}$$

By the direct computation, we can show $(d/dt)(AC) = (d/dt)(B(C-A)) = 0$.

Proposition 3.9 ([8]). *For any choice of initial data $A_0, B_0, C_0 > 0$, the unique solution $g(t)$ exists for all positive time. For any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that*

$$|A - \sqrt{A_0C_0}| \le \varepsilon, \quad |C - \sqrt{A_0C_0}| \le \varepsilon, \quad \left|\frac{d}{dt}B - 16\right| \le \varepsilon$$

for all $t \ge T_\varepsilon$. Moreover, as $t \nearrow \infty$, $A/C \nearrow 1$ if $A_0/C_0 < 1$, $A/C \searrow 1$ if $1 < A_0/C_0$, and $A/C = 1$ if $A_0/C_0 = 1$.

Now we are interested in the behavior of the L^1-norm of the Cotton–York tensor C_2.

Theorem 3.10. *For any choice of initial data $A_0, B_0, C_0 > 0$, the behavior of the L^1-norm $C_K(g)$ of the Cotton–York tensor on an arbitrary compact set K is strictly decreasing and converges to zero as $t \to \infty$.*

Proof. The Cotton–York tensor is

$$C_2(F_1, F_1) = \frac{4A(A+C)}{B\sqrt{ABC}}\left(2\frac{A}{C} + \frac{C}{A} - 1\right),$$
$$C_2(F_2, F_2) = \frac{4(A+C)}{\sqrt{ABC}}\left(\frac{C}{A} - \frac{A}{C}\right),$$
$$C_2(F_3, F_3) = -\frac{4C(A+C)}{B\sqrt{ABC}}\left(2\frac{C}{A} + \frac{A}{C} - 1\right).$$

Then for an arbitrary compact set K,

$$\int_K |C_2|_g \, d\mu_g = \frac{4(A+C)}{B}\left(6\frac{A}{C}\left(\frac{A}{C} - 1\right) + 6\frac{C}{A}\left(\frac{C}{A} - 1\right) + 8\right)^{1/2} \mathrm{Vol}(K, h),$$

where $h = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$.

We show that $(A+C)/B$ is strictly decreasing and converges to zero as $t \to \infty$. Indeed,

$$\begin{aligned}\frac{d}{dt}\frac{A+C}{B} &= \frac{\{4(C^2-A^2)/(BC) - 4(A^2-C^2)/(AB)\}B - (A+C)\{(A+C)^2/(AC)\}}{B^2} \\ &= -\frac{8(A^3+A^2C+AC^2+C^3)}{AB^2C} < 0,\end{aligned}$$

and $\lim_{t\to\infty}(A+C)/B = 2\sqrt{A_0C_0}/\infty = 0$.

If $A_0 = C_0$, the L^1-norm $C_K(g)$ is reduced to

$$\int_K |C_2|_g \, d\mu_g = \frac{8\sqrt{2}(A+C)}{B}\,\mathrm{Vol}(K,h).$$

Hence $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$. We assume that $A_0 \neq C_0$. We define the function f on $\mathbb{R}$ as

$$f(x) := \left(6x(x-1) + 6\frac{1}{x}\left(\frac{1}{x}-1\right) + 8\right)^{1/2}.$$

The function f is strictly decreasing if $0 < x \le 1$ and strictly increasing if $1 < x$. By Proposition 3.9, as $t \nearrow \infty$, $f(A/C) \searrow 2\sqrt{2}$ if $A_0/C_0 < 1$ and $f(A/C) \searrow 2\sqrt{2}$ if $1 < A_0/C_0$. Hence $C_K(g)$ is strictly decreasing and converges to zero as $t \to \infty$. □

3.6. The product metric of the Rosenau solution and the standard metric of S^1. Let $(\mathbb{R}\times S^1(2), dx^2 + d\theta^2)$ denote the flat cylinder, where $\theta \in S^1(2) = \mathbb{R}/4\pi\mathbb{Z}$. We define a solution $g(x,\theta,t)$ for $t < 0$ to the Ricci flow on $\mathbb{R}\times S^1(2)$ by

$$g(x,\theta,t) = u(x,\theta,t)(dx^2+d\theta^2) = \frac{\sinh(-t)}{\cosh x + \cosh t}(dx^2 + d\theta^2).$$

It is known that the solution $g(x,\theta,t)$ extends to the complete ancient solution to the Ricci flow on S^2 (see [4, pp. 162–164], [3, pp. 31–34]). This solution on S^2 is called the *Rosenau solution*. We denote this extended solution by g as well. The scalar curvature of g on $\mathbb{R}\times S^1(2)$ is

$$R(x,\theta,t) = \frac{\cosh t \cdot \cosh x + 1}{\sinh(-t)(\cosh x + \cosh t)} > 0$$

and the scalar curvature $R(\pm\infty, t)$ at the poles $x = \pm\infty$ is

$$R(\pm\infty,t) = \lim_{|x|\to\infty} \frac{\cosh t \cdot \cosh x + 1}{\sinh(-t)(\cosh x + \cosh t)} = \coth(-t) > 0.$$

Moreover, the curvature $R(\pm\infty, t)$ at the poles is the maximum curvature of $(S^2, g(t))$ for all $t < 0$, since we have

$$\frac{\partial}{\partial x} R = \frac{\sinh x \cdot \sinh(-t)}{(\cosh x + \cosh t)^2} > 0$$

for all $x > 0$. Since $\lim_{t \nearrow 0} R(\pm\infty, t) = \infty$, the Rosenau solution is ancient but not eternal. Due to the fact that for all $(x, \theta) \in \mathbb{R} \times S^1(2)$

$$\lim_{t \nearrow 0} \frac{R(x, \theta, t)}{R(\pm\infty, t)} = \lim_{t \nearrow 0} \frac{\cosh t \cdot \cosh x + 1}{\cosh t(\cosh x + \cosh t)} = 1,$$

the solution shrinks to a round point.

Using the Rousenau solution, we define the Ricci flow on $S^2 \times S^1$ by $h(t) = g(t) + d\varphi^2$ for $t < 0$, where $d\varphi^2$ is the standard metric of radius one on S^1.

Theorem 3.11. *The L^1-norm $C(h)$ of the Cotton–York tensor C_2 for the product metric h of the Rosenau solution for the Ricci flow on S^2 and the standard metric of S^1 is strictly decreasing and converges to zero as $t \to 0$.*

Proof. On the local coordinate $(x^1, x^2, x^3) := (x, \theta, \varphi)$, the Ricci tensor is

$$R_{11} = \frac{\cosh t \cdot \cosh x + 1}{2(\cosh x + \cosh t)^2}, \quad R_{22} = \frac{\cosh t \cdot \cosh x + 1}{2(\cosh x + \cosh t)^2}, \quad R_{33} = 0,$$

and the scalar curvature is

$$R = \frac{\cosh t \cdot \cosh x + 1}{\sinh(-t)(\cosh x + \cosh t)}.$$

The Cotton–York tensor C_2 is

$$C_{23} = C_{32} = \frac{\sinh x \cdot \sinh(-t)}{4(\cosh x + \cosh t)^2}.$$

Then L^1-norm is given by the following:

$$\begin{aligned}
\int_{S^2 \times S^1} |C_2(t)|_{h(t)}\, d\mu_{h(t)} &= \int_{S^1} \left(\int_{S^2} |C_2(t)|_{h(t)}\, d\mu_{g(t)} \right) d\mu_{d\varphi^2} \\
&= 2\pi \int_{\mathbb{R} \times S^1(2)} |C_2(t)|_{h(t)} d\mu_{u(x,t)(dx^2 + d\theta^2)} \\
&= 2\pi \int_{S^1(2)} \left(\int_{\mathbb{R}} |C_2(t)|_{h(t)} u(x, t)\, d\mu_{dx^2} \right) d\mu_{d\theta^2} \\
&= 8\pi^2 \int_{\mathbb{R}} \frac{1}{2\sqrt{2}} \sqrt{\frac{\sinh^2 x \cdot \sinh(-t)}{(\cosh x + \cosh t)^3}} \cdot \frac{\sinh(-t)}{\cosh x + \cosh t}\, dx
\end{aligned}$$

$$
\begin{aligned}
&= 4\sqrt{2}\pi^2 \int_0^\infty \sqrt{\frac{\sinh^2 x \cdot \sinh(-t)}{(\cosh x + \cosh t)^3} \cdot \frac{\sinh(-t)}{\cosh x + \cosh t}}\, dx \\
&= 4\sqrt{2}\pi^2 \int_0^\infty \sqrt{\frac{\sinh^3(-t)}{(\cosh x + \cosh t)^5}} \cdot \sinh x\, dx \\
&= 4\sqrt{2}\pi^2 \int_1^\infty \sqrt{\frac{\sinh^3(-t)}{(y + \cosh t)^5}}\, dy \\
&= \frac{8\sqrt{2}\pi^2}{3}\left(\frac{\sinh(-t)}{1 + \cosh(-t)}\right)^{3/2}.
\end{aligned}
$$

Hence $C(h)$ is strictly decreasing and converges to zero as $t \to 0$. □

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