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PRIME COMPONENT-PRESERVINGLY AMPHICHEIRAL LINK WITH ODD MINIMAL CROSSING NUMBER

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Abstract

For every odd integer $c \geq 21$, we raise an example of a prime component-preservingly amphicheiral link with the minimal crossing number c. The link has two components, and consists of an unknot and a knot which is $(-)$-amphicheiral with odd minimal crossing number. We call the latter knot a *Stoimenow knot*. We also show that the Stoimenow knot is not invertible by the Alexander polynomials.

Contents

1. Introduction

Let $L = K_1 \cup \cdots \cup K_r$ be an oriented r-component link in S^3. A 1-component link is called a knot. For an oriented knot K, we denote the orientation-reversed knot by $-K$. If φ is an orientation-reversing homeomorphism of S^3 so that $\varphi(K_i) = \varepsilon_{\sigma(i)} K_{\sigma(i)}$ for all $i = 1, \dots, r$ where $\varepsilon_i = +$ or $-$, and σ is a permutation of $\{1, 2, \dots, r\}$, then L is called an $(\varepsilon_1, \dots, \varepsilon_r; \sigma)$-*amphicheiral link*. A term "amphicheiral link" is used as a general term for an $(\varepsilon_1, \dots, \varepsilon_r; \sigma)$-amphicheiral link. If φ can be taken as an involution (i.e. $\varphi^2 = \mathrm{id}$), then L is called a *strongly* amphicheiral link. If σ is the identity, then an amphicheiral link is called a *component-preservingly amphicheiral link*, and σ may be omitted from the notation. If every $\varepsilon_i = \varepsilon$ is identical for all $i = 1, \dots, r$ (including the case that σ is not the identity), then an $(\varepsilon_1, \dots, \varepsilon_r; \sigma)$-amphicheiral link is called an (ε)-amphicheiral link. We use the notations $+ = +1 = 1$ and $- = -1$. For the case of invertibility, we only replace φ with an orientation-preserving homeomorphism of S^3. We refer the reader to [19, 4, 6, 7, 8, 9].

The minimal crossing number of an alternating amphicheiral link is known to be even (cf. [8, Lemma 1.4]) from the positive answer for the *flyping conjecture* due to

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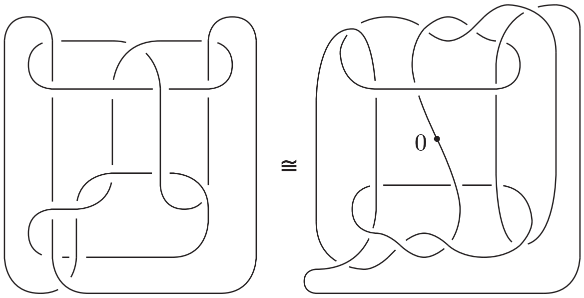

Fig. 1. 15_{224980}.

W. Menasco and M. Thistlethwaite [13]. The flyping conjecture is one of famous Tait's conjectures on alternating links, and it is also called Tait's conjecture III in [17]. The positive answer for the flyping conjecture implies those of Tait's conjecture I on the minimal crossing number (cf. [14]), and Tait's conjecture II on the writhe (cf. [15]). A. Stoimenow [17, Conjecture 2.4] sets a conjecture:

"*Amphicheiral* (*alternating*?) *knots have even crossing number.*"

as Tait's conjecture IV by guessing what Tait had in mind (i.e. Tait has not stated it explicitly). We pose the following conjecture:

Conjecture 1.1 (a generalized version of Tait's conjecture IV)**.** *The minimal crossing number of an amphicheiral link is even.*

For the case of alternating amphicheiral links, Conjecture 1.1 is affirmative as mentioned above from the answer for Tait's conjecture II. Hence it motivates to find an amphicheiral link with odd minimal crossing number. If there exists a counter-example for Conjecture 1.1, then it should be non-alternating.

A non-split link is *prime* if it is not a connected sum of non-trivial links. We assume that a prime link is non-split. There exists a prime amphicheiral knot with minimal crossing number 15 in the table of J. Hoste, M. Thistlethwaite and J. Weeks [5], which gives a negative answer for Conjecture 1.1 (the original Tait's conjecture IV). The knot is named 15_{224980} (Fig. 1). Stoimenow [18] showed that for every odd integer $c \geq 15$, there exists an example of a prime amphicheiral knot with minimal crossing number c. The case $c = 15$ corresponds to 15_{224980}. We call the sequence of knots *Stoimenow knots* (see Section 3). He also pointed out that there are no such examples for the case $c \leq 13$.

The first author and A. Kawauchi [9], and the first author [8] determined prime amphicheiral links with minimal crossing number up to 11. Then there are two prime amphicheiral links with odd minimal crossing numbers named 9^2_{61} and 11^2_{n247} (Fig. 2),

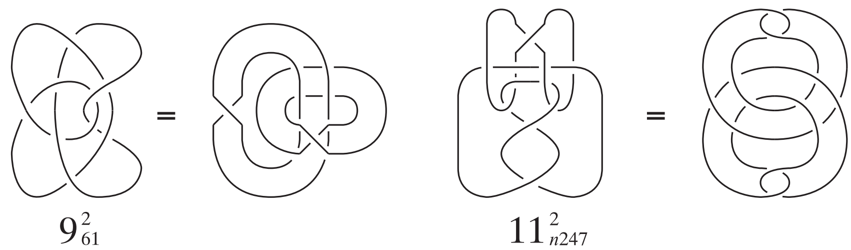

Fig. 2. 9^2_{61} and 11^2_{n247}.

where we use modified notations from Rolfsen's table [16] and Thistlethwaite's table on the web site maintained by D. Bar-Natan and S. Morrison [1]. These examples show that Conjecture 1.1 is negative for links. Since both 9^2_{61} and 11^2_{n247} are not component-preservingly amphicheiral, we ask the following question (see also Question 5.5):

QUESTION 1.2. Is there a prime component-preservingly amphicheiral link with odd minimal crossing number?

If we remove 'prime' from Question 1.2, then we can obtain nugatory examples by taking split sum of a Stoimenow knot and an unknot, or connected sum of Stoimenow knot and the Hopf link. Our main theorem is an affirmative answer for Question 1.2 which is a negative answer for Conjecture 1.1:

Theorem 1.3. *For every odd integer $c \geq 21$, there exists a prime component-preservingly amphicheiral link with minimal crossing number c* (Fig. 10).

Our example is a 2-component link with linking number 3 whose components are a Stoimenow knot and an unknot. We prove it in Section 4. The proof is divided into three parts such as to show amphicheirality, to determine the minimal crossing number, and to show primeness. We can immediately see its amphicheirality by construction. Though to find the way of linking of the two components was not so easy, to determine the minimal crossing number is easy by the help of Stoimenow's result [18] (cf. Theorem 3.1). In [18], to determine the minimal crossing number and to show primeness of his knot were very hard. Finally we show primeness by using the Kauffman bracket (cf. Subsection 2.1). This part is also eased by Stoimenow's result. In Section 5, by R. Hartley [2], R. Hartley and A. Kawauchi [3], and A. Kawauchi [10]'s necessary conditions on the Alexander polynomials of amphicheiral knots, we show that a Stoimenow knot is not invertible (Theorem 5.4).

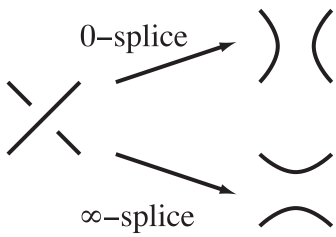

Fig. 3. Splice.

2. Link invariants

2.1. Kauffman bracket. Let L be an r-component oriented link, and D a diagram of L. Firstly we regard D as an unoriented diagram. On a crossing of D, a *splice* is a replacement from the left-hand side (the crossing) to the right-hand side as in Fig. 3. Precisely, a 0-*splice* is to the upper right-hand side, and an ∞-*splice* is to the down right-hand side, respectively. The resulting diagram is a *state*, and it is a diagram of an unlink without crossings. Let s be a state, $|s|$ the number of components of s, $t_0(s)$ the number of 0-splices to obtain s, $t_\infty(s)$ the number of ∞-splices to obtain s, $t(s) = t_0(s) - t_\infty(s)$, and $\mathcal{S}$ the set of states from D. Let A be an indeterminate, and $d = -A^2 - A^{-2}$. Then

$$\langle D \rangle = \sum_{s \in \mathcal{S}} A^{t(s)} d^{|s|-1} \in \mathbb{Z}[A, A^{-1}]$$

is the *Kauffman bracket* of D, and

$$f_L(A) = (-A^3)^{-w(D)} \langle D \rangle \tag{2.1}$$

is the f-*polynomial* of L where $w(D)$ is the writhe of D as an oriented diagram. Then $f_L(A)$ is an invariant of L, and

$$V_L(t) = f_L(t^{1/4}) \in \mathbb{Z}[t^{1/2}, t^{-1/2}] \tag{2.2}$$

is the *Jones polynomial* of L. We denote $\langle D \rangle$ as $\langle D \rangle(A)$ when we emphasis it as a function of A. We have the following facts:

Lemma 2.1. *Let L be an r-component oriented link, and D a diagram of L.*
(1) *The Kauffman bracket $\langle D \rangle$ is an invariant of L up to multiplications of* $(-A^3)$. *In particular, if we substitute a root of unity for A and take its absolute value, then it is an invariant of L, which is a non-negative real number.*
(2) *We have the following skein relation* (Fig. 4) *which can be an axiom of the Kauffman bracket*:

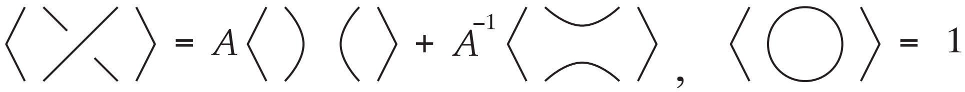

Fig. 4. Skein relation I.

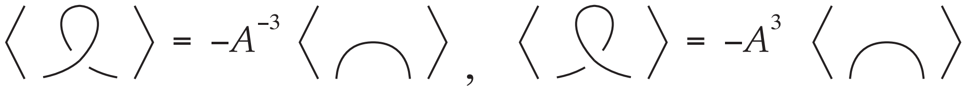

Fig. 5. Skein relation II.

(3) *Let* L_i $(i = 1, 2)$ *be a link,* D_i *a link diagram of* L_i*, and* $D_1 \amalg D_2$ $(L_1 \amalg L_2$, *respectively) the split sum of* D_1 *and* D_2 $(L_1$ *and* L_2*, respectively). Then we have*

$$\langle D_1 \amalg D_2 \rangle = d \langle D_1 \rangle \langle D_2 \rangle, \quad f_{L_1 \amalg L_2}(A) = d \cdot f_{L_1}(A) f_{L_2}(A).$$

(4) *Let* L_i $(i = 1, 2)$ *be a link,* D_i *a link diagram of* L_i*, and* $D_1 \sharp D_2$ $(L_1 \sharp L_2$, *respectively) the connected sum of* D_1 *and* D_2 $(L_1$ *and* L_2*, respectively). Then we have*

$$\langle D_1 \sharp D_2 \rangle = \langle D_1 \rangle \langle D_2 \rangle, \quad f_{L_1 \sharp L_2}(A) = f_{L_1}(A) f_{L_2}(A).$$

(5) *We have a skein relation as in* Fig. 5:
(6) *Let* D^* $(L^*$, *respectively) be the mirror image of* D $(L$, *respectively). Then we have*

$$\langle D^* \rangle (A) = \langle D \rangle (A^{-1}), \quad f_{L^*}(A) = f_L(A^{-1}).$$

(7) $f_L(A) \in A^{2(r+1)} \cdot \mathbb{Z}[A^4, A^{-4}]$.
(8) *Let* ζ *be a primitive* 8*-th root of unity (i.e.* $\zeta^4 = -1$ *and* $\zeta^8 = 1$*). Suppose that the number of the crossing number of* D *is even. Then* $\langle D \rangle (\zeta)$ *is an integer or of the form* $\sqrt{-1} \times$ *(integer), which depends on* r *and the writhe. In particular, for* $r = 1$, $\langle D \rangle (\zeta)$ *is an integer if and only if the writhe is* 0 (mod 4).
(9) *Let* ζ *be a primitive* 8*-th root of unity. Then we have* $|\langle D \rangle (\zeta)| = |V_L(-1)|$.

Lemma 2.1 (8) is obtained from (7) and (2.1), and it is a special case of (1). Lemma 2.1 (9) is obtained from (2.2).

Let T_m be an m-half twist tangle for $m \in \mathbb{Z}$, and T_∞ a tangle in Fig. 6.

By Lemma 2.1 (2), (3), (4) and (5), we have the following:

Lemma 2.2. (1) *We have*

$$\langle T_m \rangle = A^m \langle T_0 \rangle + \alpha_m(A) \langle T_\infty \rangle,$$

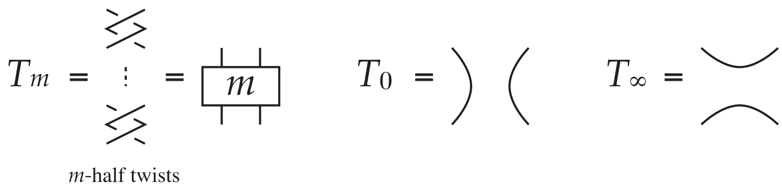

Fig. 6. m-half twists.

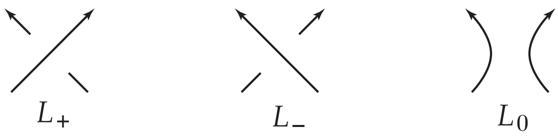

Fig. 7. Skein triple.

where

$$\alpha_m(A) = A^{m-2} \cdot \frac{1-(-A^{-4})^m}{1-(-A^{-4})}.$$

(2) $\alpha_{-m}(A) = \alpha_m(A^{-1})$.
(3) *Let* ζ *be a primitive* 8*-th root of unity. Then we have*

$$\alpha_m(\zeta) = m\zeta^{m-2}$$

and

$$\alpha_m(\zeta) \cdot \alpha_{-m}(\zeta) = m^2.$$

2.2. Alexander and Conway polynomials. Let L be an oriented link, and D a diagram of L. Pick a crossing c of D. If c is a positive crossing (a negative crossing, respectively), then we denote D by L_+ (L_-, respectively). If c is smoothed with preserving the orientation, then we denote D by L_0. We call a pair (L_+, L_-, L_0) a *skein triple* (Fig. 7).

For an oriented link L, the *Conway polynomial* of L is denoted by $\nabla_L(z)$ which is an element of $\mathbb{Z}[z]$. For a skein triple (L_+, L_-, L_0), the Conway polynomial is defined by the following skein relation:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z), \quad \nabla_O(z) = 1,$$

where O is the trivial knot.

Lemma 2.3. *Let* L *be an* r*-component oriented link, and* L^* *the mirror image of* L*. Then we have*

$$\nabla_{L^*}(z) = \nabla_L(-z).$$

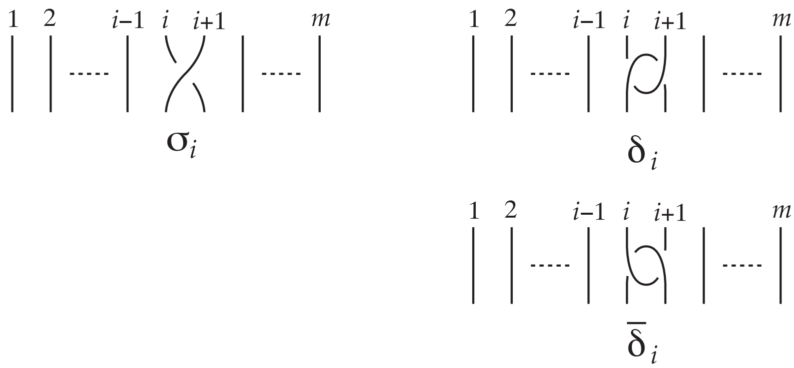

Fig. 8. Generator σ_i of braid group, and δ_i and $\bar{\delta}_i$.

More precisely, $\nabla_{L^*}(z) = \nabla_L(z)$ *if* r *is odd, and* $\nabla_{L^*}(z) = -\nabla_L(z)$ *if* r *is even.*

For an r-component oriented link L, the (*normalized one variable*) *Alexander polynomial* $\Delta_L(t)$ is defined by

$$\Delta_L(t) = \nabla_L(t^{1/2} - t^{-1/2}) \in \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

For $A, B \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$, $A \doteq B$ implies $A = \pm t^{m/2} B$ for some $m \in \mathbb{Z}$. For $f, g \in \mathbb{Z}[z]$ or $\mathbb{Z}[t^{1/2}, t^{-1/2}]$, if they are equal as elements in $(\mathbb{Z}/d\mathbb{Z})[z]$ or $(\mathbb{Z}/d\mathbb{Z})[t^{1/2}, t^{-1/2}]$, then we denote by $f =_d g$. For an oriented link L, if $\nabla_L(z)$ and $\Delta_L(t)$ are regarded as elements in $(\mathbb{Z}/d\mathbb{Z})[z]$ and $(\mathbb{Z}/d\mathbb{Z})[t^{1/2}, t^{-1/2}]$ respectively, then we call them the *mod* d *Conway polynomial* of L and the *mod* d *Alexander polynomial* of L respectively.

3. Stoimenow knots

Let σ_i $(i = 1, \dots, m-1)$ be a generator of the m-string braid group, and δ_i and $\bar{\delta}_i$ $(i = 1, \dots, m-1)$ tangles in Fig. 8. For an odd number $n \geq 15$, a *Stoimenow knot* with crossing number n, denoted by S_n, is the closure of the following composition of σ_i, δ_i and $\bar{\delta}_i$ $(i = 1, \dots, m-1)$:

$$3 \quad -1 \quad 2^2 \quad 3^{2k} \quad 4 \quad -3 \quad 2 \quad -1 \quad (-2)^{2k} \quad (-3)^2 \quad 4 \quad -2 \quad (n = 4k+11),$$

$$\delta_3 \quad -1 \quad 2^2 \quad 3^{2k} \quad 4 \quad -3 \quad 2 \quad -1 \quad (-2)^{2k} \quad (-3)^2 \quad 4 \quad \bar{\delta}_2 \quad (n = 4k+13),$$

where in the sequence above, $m = 5$, σ_i is translated into i and σ_i^{-1} is translated into $-i$, and i^l implies that i is repeated l times with $l \geq 1$. The former is *of type* I, and the latter is *of type* II, respectively. Note that $S_{15} = 15_{224980}$ in Fig. 1, and both two tangles above have $(n+1)$ crossings. We can see strong $(-)$-amphicheirality of S_n from its diagram with $(n+1)$ crossings in the righthand side of Fig. 9.

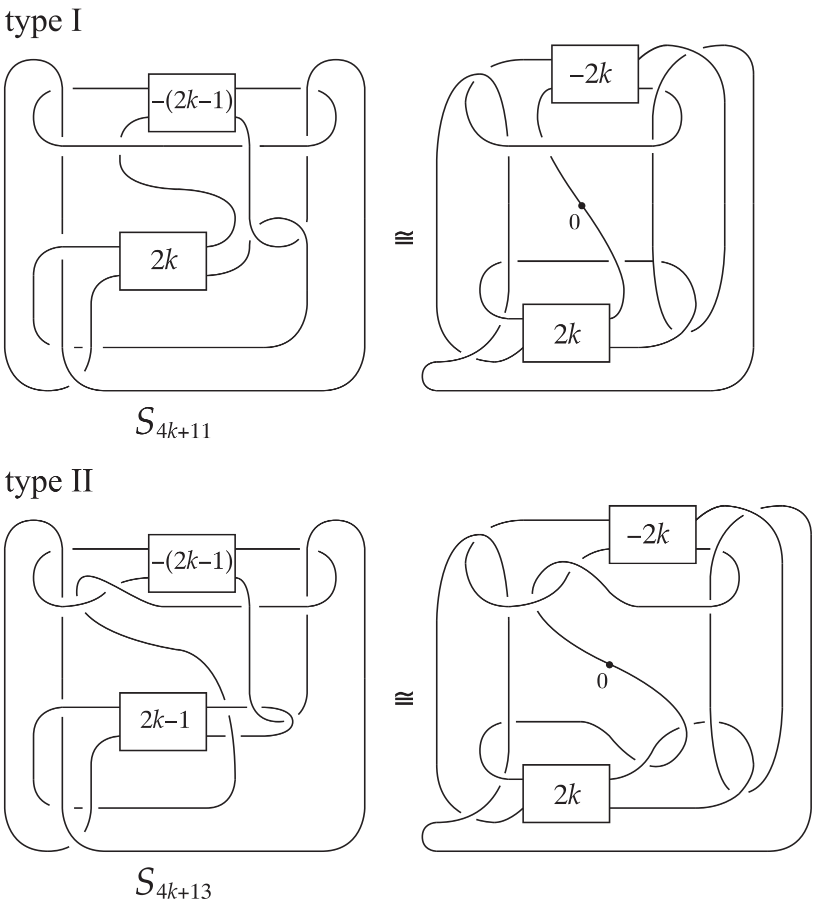

Fig. 9. Stoimenow knot S_n.

Theorem 3.1 (Stoimenow [17, 18])**.** *A Stoimenow knot S_n is a prime strongly* $(-)$*-amphicheiral knot with minimal crossing number n.*

4. Proof of Theorem 1.3

We take a 2-component link $L_n = S_n \cup U$ whose components are a Stoimenow knot S_n and an unknot U as in Fig. 10. The link L_n is *of type* I if S_n is of type I, and is *of type* II if S_n is of type II. We prove that L_n is a prime component-preservingly amphicheiral link with minimal crossing number $n+6$, where $n+6$ is odd with $n+6 \geq 21$ because n is odd with $n \geq 15$.

Proof of Theorem 1.3. By the righthand side of Fig. 10, L_n is a component-preservingly strongly $(-, +)$-amphicheiral link.

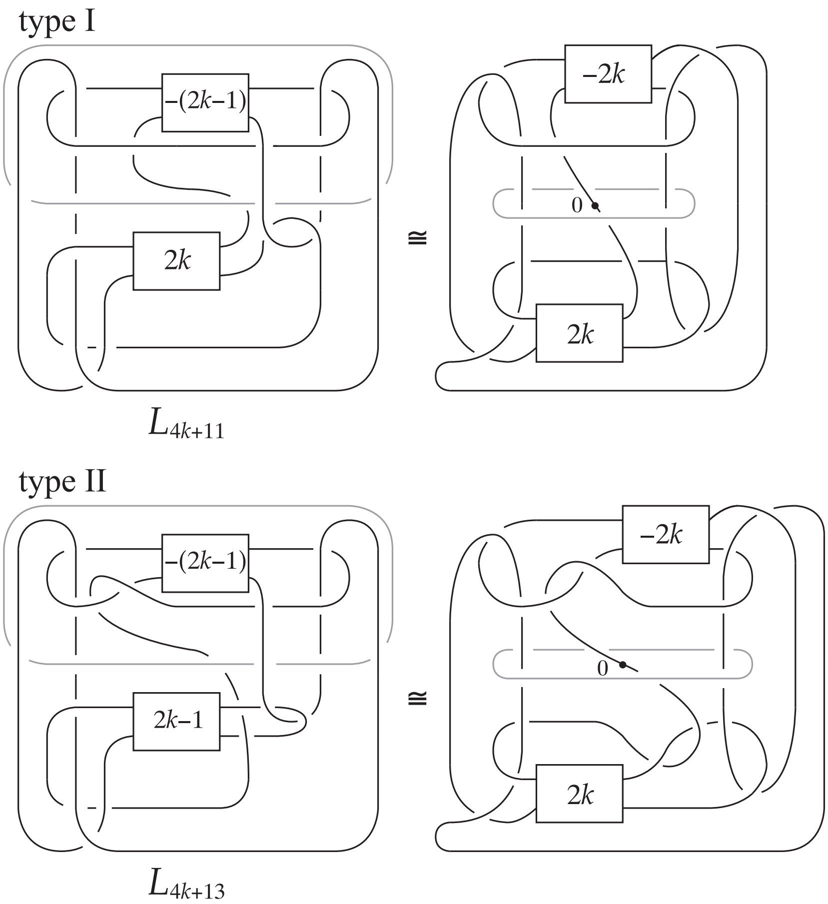

Fig. 10. Prime component-preservingly amphicheiral link L_n.

The linking number of L_n, $\mathrm{lk}(L_n)$, is 3 by a suitable orientation. Let $c(\cdot)$ denote the minimal crossing number of a link. Since

$$c(L_n) \geq c(S_n) + c(U) + 2|\mathrm{lk}(L_n)| = n + 6,$$

and the lefthand side of Fig. 10 realizes the lower bound, we have $c(L_n) = n + 6$ and it is odd.

Finally we show that L_n is prime by using the Kauffman bracket. Suppose that L_n is not prime. Then L_n is a connected sum of two links such that one is a Stoimenow knot S_n and the other is a 2-component link with unknotted components and with linking number 3 by Theorem 3.1. Hence $\langle L_n \rangle$ should be divisible by $\langle S_n \rangle$ by Lemma 2.1 (4). We compute $\langle L_n \rangle(\zeta)$ and $\langle S_n \rangle(\zeta)$, where ζ is a primitive 8-th root of unity. By Lemma 2.1 (4) and (8), $|\langle L_n \rangle(\zeta)|$ should be divisible by $|\langle S_n \rangle(\zeta)|$.

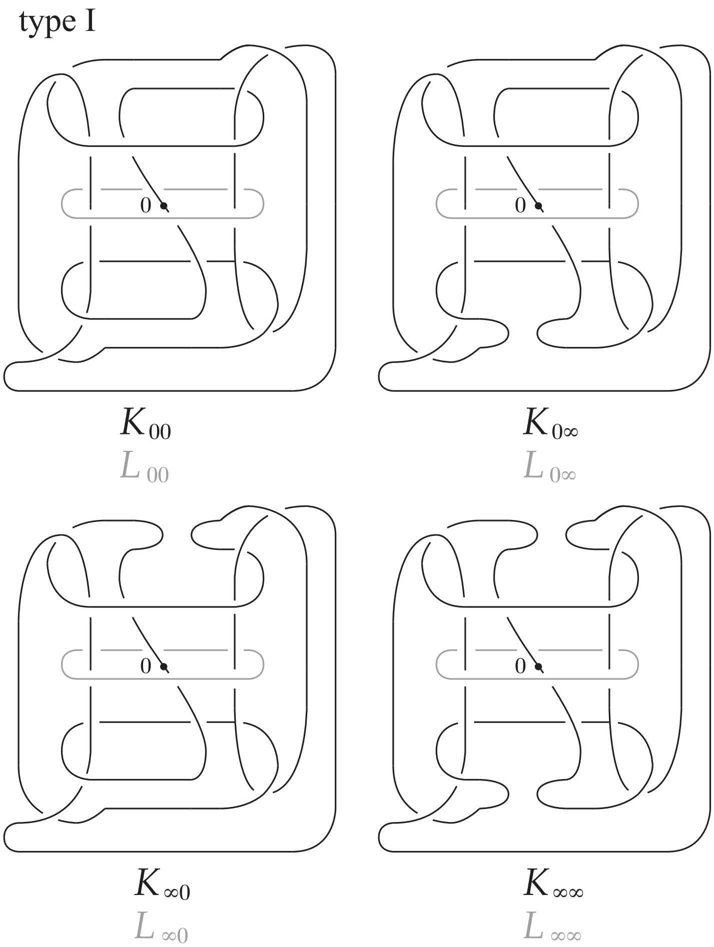

Fig. 11. Splices of L_n.

To compute $\langle S_n\rangle$ and $\langle L_n\rangle$, we set $K = S_n$ and $L = L_n$, and we denote the results of splicings by K_{00}, $K_{0\infty}$, $K_{\infty 0}$, $K_{\infty\infty}$, L_{00}, $L_{0\infty}$, $L_{\infty 0}$ and $L_{\infty\infty}$, respectively as in Fig. 11. Here we drew only the type I case. We can obtain the type II case in a similar way.

Then by Lemma 2.2 (1), we have:

$$\langle K\rangle = \langle K_{00}\rangle + A^{-2k}\alpha_{2k}(A)\langle K_{0\infty}\rangle + A^{2k}\alpha_{-2k}(A)\langle K_{\infty 0}\rangle + \alpha_{2k}(A)\alpha_{-2k}(A)\langle K_{\infty\infty}\rangle \tag{4.1}$$

and

$$\langle L\rangle = \langle L_{00}\rangle + A^{-2k}\alpha_{2k}(A)\langle L_{0\infty}\rangle + A^{2k}\alpha_{-2k}(A)\langle L_{\infty 0}\rangle + \alpha_{2k}(A)\alpha_{-2k}(A)\langle L_{\infty\infty}\rangle. \tag{4.2}$$

We can see that K_{00} and $K_{\infty\infty}$ are amphicheiral knot diagrams with writhe 0, $K_{0\infty} = (K_{\infty 0})^*$, the writhe of $K_{0\infty}$ is -10, the writhe of $K_{\infty 0}$ is 10, L_{00} and $L_{\infty\infty}$ are 2-component amphicheiral link diagrams with writhe 6, $L_{0\infty} = (L_{\infty 0})^*$, the writhe of $L_{0\infty}$ is -4, and the writhe of $L_{\infty 0}$ is 16. By Lemma 2.1 (6), we have

$$K_{00}(A) = K_{00}(A^{-1}), \quad K_{\infty\infty}(A) = K_{\infty\infty}(A^{-1}), \quad K_{\infty 0}(A) = K_{0\infty}(A^{-1}),$$
$$L_{00}(A) = L_{00}(A^{-1}), \quad L_{\infty\infty}(A) = L_{\infty\infty}(A^{-1}), \quad \text{and} \quad L_{\infty 0}(A) = L_{0\infty}(A^{-1}).$$

By Lemma 2.2 (2), $A^{2k}\alpha_{-2k}(A)$ can be obtained by replacing A with A^{-1} in $A^{-2k}\alpha_{2k}(A)$. By straight calculations using Lemma 2.1 and Lemma 2.2, we have:
(type I)

$$\begin{aligned}
\langle K_{00}\rangle &= A^{16} - 4A^{12} + 6A^{8} - 7A^{4} + 9 - 7A^{-4} + 6A^{-8} - 4A^{-12} + A^{-16},\\
\langle K_{0\infty}\rangle &= -A^{18} + 3A^{14} - 5A^{10} + 6A^{6} - 7A^{2} + 6A^{-2} - 5A^{-6} + 4A^{-10}\\
&\quad - A^{-14} + A^{-18},\\
\langle K_{\infty\infty}\rangle &= A^{16} - 3A^{12} + 5A^{8} - 6A^{4} + 7 - 6A^{-4} + 5A^{-8} - 3A^{-12} + A^{-16}.
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\langle L_{00}\rangle &= -A^{20} + 4A^{16} - 8A^{12} + 12A^{8} - 16A^{4} + 16 - 16A^{-4} + 12A^{-8}\\
&\quad - 8A^{-12} + 4A^{-16} - A^{-20},\\
\langle L_{0\infty}\rangle &= A^{22} - 3A^{18} + 6A^{14} - 9A^{10} + 12A^{6} - 12A^{2} + 11A^{-2} - 9A^{-6}\\
&\quad + 5A^{-10} - 3A^{-14} - A^{-26},\\
\langle L_{\infty\infty}\rangle &= -A^{20} + 3A^{16} - 7A^{12} + 10A^{8} - 13A^{4} + 14 - 13A^{-4} + 10A^{-8}\\
&\quad - 7A^{-12} + 3A^{-16} - A^{-20}.
\end{aligned} \tag{4.4}$$

(type II)

$$\begin{aligned}
\langle K_{00}\rangle &= -A^{20} + 4A^{16} - 9A^{12} + 14A^{8} - 17A^{4} + 19 - 17A^{-4} + 14A^{-8}\\
&\quad - 9A^{-12} + 4A^{-16} - A^{-20},\\
\langle K_{0\infty}\rangle &= A^{22} - 4A^{18} + 10A^{14} - 15A^{10} + 19A^{6} - 22A^{2} + 20A^{-2} - 18A^{-6}\\
&\quad + 12A^{-10} - 7A^{-14} + 3A^{-18} - A^{-22},\\
\langle K_{\infty\infty}\rangle &= -2A^{20} + 6A^{16} - 13A^{12} + 21A^{8} - 24A^{4} + 28 - 24A^{-4} + 21A^{-8}\\
&\quad - 13A^{-12} + 6A^{-16} - 2A^{-20}.
\end{aligned} \tag{4.5}$$

(4.6)
$$\begin{aligned}\langle L_{00}\rangle &= A^{24}-5A^{20}+13A^{16}-24A^{12}+35A^{8}-44A^{4}+46-44A^{-4}\\&\quad +35A^{-8}-24A^{-12}+13A^{-16}-5A^{-20}+A^{-24},\\ \langle L_{0\infty}\rangle &= -A^{26}+4A^{22}-11A^{18}+20A^{14}-31A^{10}+40A^{6}-42A^{2}+42A^{-2}\\&\quad -33A^{-6}+24A^{-10}-13A^{-14}+5A^{-18}-A^{-26}+A^{-30},\\ \langle L_{\infty\infty}\rangle &= A^{24}-5A^{20}+14A^{16}-27A^{12}+38A^{8}-50A^{4}+50-50A^{-4}\\&\quad +38A^{-8}-27A^{-12}+14A^{-16}-5A^{-20}+A^{-24}.\end{aligned}$$

We substitute $A=\zeta$ to (4.1) and (4.2). We set $\zeta^2=\sqrt{-1}$. By Lemma 2.2 (2) and the arguments above, we have

(4.7)
$$\langle K\rangle(\zeta)=\langle K_{00}\rangle(\zeta)-4k\sqrt{-1}\langle K_{0\infty}\rangle(\zeta)+4k^2\langle K_{\infty\infty}\rangle(\zeta)$$

and

(4.8)
$$\langle L\rangle(\zeta)=\langle L_{00}\rangle(\zeta)-4k\sqrt{-1}\langle L_{0\infty}\rangle(\zeta)+4k^2\langle L_{\infty\infty}\rangle(\zeta).$$

By (4.3), (4.4), (4.5) and (4.6), we have
(type I)

(4.9)
$$\begin{aligned}\langle K_{00}\rangle(\zeta)&=45,\\ \langle K_{0\infty}\rangle(\zeta)&=-39\sqrt{-1},\\ \langle K_{\infty\infty}\rangle(\zeta)&=37.\end{aligned}$$

(4.10)
$$\begin{aligned}\langle L_{00}\rangle(\zeta)&=98,\\ \langle L_{0\infty}\rangle(\zeta)&=-70\sqrt{-1},\\ \langle L_{\infty\infty}\rangle(\zeta)&=82.\end{aligned}$$

(type II)

(4.11)
$$\begin{aligned}\langle K_{00}\rangle(\zeta)&=109,\\ \langle K_{0\infty}\rangle(\zeta)&=-132\sqrt{-1},\\ \langle K_{\infty\infty}\rangle(\zeta)&=160.\end{aligned}$$

(4.12)
$$\begin{aligned}\langle L_{00}\rangle(\zeta)&=290,\\ \langle L_{0\infty}\rangle(\zeta)&=-264\sqrt{-1},\\ \langle L_{\infty\infty}\rangle(\zeta)&=320.\end{aligned}$$

By (4.7), (4.8), (4.9), (4.10), (4.11) and (4.12), we have
(type I)

$$\begin{aligned}\langle K\rangle(\zeta)&=148k^2-156k+45,\\ \langle L\rangle(\zeta)&=328k^2-280k+98.\end{aligned}$$

(type II)

$$\langle K\rangle(\zeta) = 640k^2 - 528k + 109,$$
$$\langle L\rangle(\zeta) = 1280k^2 - 1056k + 290.$$

Note that $148k^2 - 156k + 45$ and $640k^2 - 528k + 109$ are odd and $328k^2 - 280k + 98$ and $1280k^2 - 1056k + 290$ are of the form $2\times$(odd), and they are positive for $k \geq 1$. Hence if $148k^2 - 156k + 45$ divides $328k^2 - 280k + 98$ ($640k^2 - 528k + 109$ divides $1280k^2 - 1056k + 290$, respectively), then $148k^2 - 156k + 45$ divides $164k^2 - 140k + 49$ ($640k^2 - 528k + 109$ divides $640k^2 - 528k + 145$, respectively), and the quantity is odd.
(type I)

Suppose that $164k^2 - 140k + 49$ is divisible by $148k^2 - 156k + 45$. Since

$$(164k^2 - 140k + 49) - (148k^2 - 156k + 45) = 16k^2 + 16k + 4 > 0,$$

the quantity is not 1. Since

$$3(148k^2 - 156k + 45) - (164k^2 - 140k + 49) = 280k^2 - 328k + 86 > 0,$$

the quantity is not greater than 1. It is a contradiction.
(type II)

Suppose that $640k^2 - 528k + 145$ is divisible by $640k^2 - 528k + 109$. Since

$$(640k^2 - 528k + 145) - (640k^2 - 528k + 109) = 36 > 0,$$

the quantity is not 1. Since

$$3(640k^2 - 528k + 109) - (640k^2 - 528k + 145) = 1280k^2 - 1056k + 182 > 0,$$

the quantity is not greater than 1. It is a contradiction. □

REMARK 4.1. In [12], the second author computes the J polynomials, which are modified Jones polynomials, of S_n and L_n explicitly. The J polynomial is an invariant of unoriented links.

5. Non-invertibility of Stoimenow knots

In this section, we show that a Stoimenow knot S_n is not invertible by using the Alexander polynomials. Since S_n is $(-)$-amphicheiral, we show that it is not $(+)$-amphicheiral, which is equivalent to that it is not invertble.

Let L be a link, and $\Delta_L(t) \in \mathbb{Z}[t, t^{-1}]$ the Alexander polynomial of L. For two elements A and B in $\mathbb{Z}[t, t^{-1}]$ ($(\mathbb{Z}/d\mathbb{Z})[t, t^{-1}]$, respectively), we denote by $A \doteq B$ ($A \doteq_d B$, respectively) if they are equal up to multiplications of trivial units. A one variable Laurent polynomial $r(t) \in \mathbb{Z}[t^{\pm 1}]$ is *of type X* if there are integers $n \geq 0$ and

$\lambda \geq 3$ such that λ is odd, and $f_i(t) \in \mathbb{Z}[t, t^{-1}]$ $(i = 0, 1, \dots, n)$ such that $f_i(t) \doteq f_i(t^{-1})$, $|f_i(1)| = 1$, and for $i > 0$, $f_i(t) \doteq_2 f_0(t)^{2^i} p_\lambda(t)^{2^{i-1}}$ where $p_\lambda(t) = (t^\lambda - 1)/(t - 1)$, and

$$r(t) \doteq \begin{cases} f_0(t)^2 & (n = 0), \\ f_0(t)^2 f_1(t) \cdots f_n(t) & (n \geq 1). \end{cases} \tag{5.1}$$

R. Hartley [2], R. Hartley and A. Kawauchi [3], and A. Kawauchi [10] gave necessary conditions on the Alexander polynomials of amphicheiral knots.

Lemma 5.1 (Hartley [2]; Hartley and Kawauchi [3]; Kawauchi [10])**.** (1) *Let K be a $(-)$-amphicheiral knot. Then there exists an element $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $|f(1)| = 1$, $f(t^{-1}) \doteq f(-t)$, and*

$$\Delta_K(t^2) \doteq f(t) f(t^{-1}).$$

(2) *Let K be a $(+)$-amphicheiral knot. Then there exist $r_j(t) \in \mathbb{Z}[t, t^{-1}]$ of type X and a positive odd number α_j $(j = 1, \dots, m)$ such that*

$$\Delta_K(t) \doteq \prod_{j=1}^{m} r_j(t^{\alpha_j}).$$

In particular, if K is hyperbolic, then we can take $m = 1$ and $\alpha_1 = 1$.

We generalize Stoimenow knots as in Fig. 12. The lefthand side is called a *generalized Stoimenow link of type* I, and is denoted by $S^1_{p,q}$. The righthand side is called a *generalized Stoimenow link of type* II, and is denoted by $S^2_{r,s}$. The numbers in rectangles are the numbers of half twists. We note that $S^1_{2k,2k} = S_{4k+11}$ and $S^2_{k,k} = S_{4k+13}$. We denote the Alexander polynomials (the Conway polynomials) of $S^1_{p,q}$ and $S^2_{r,s}$ by $\Delta^{(1)}_{p,q}(t)$ and $\Delta^{(2)}_{r,s}(t)$ ($\nabla^{(1)}_{p,q}(z)$ and $\nabla^{(2)}_{r,s}(z)$), respectively. We compute $\Delta^{(1)}_{2k,2k}(t)$ and $\Delta^{(2)}_{k,k}(t)$ as the mod 2 Alexander polynomials.

Lemma 5.2. *The Alexander and the mod* 2 *Alexander polynomials of $S^1_{2k,2k}$ and $S^2_{k,k}$ are as follows*:

$$\begin{aligned}(t+1)^2 \Delta^{(1)}_{2k,2k}(t) &\doteq_2 t^{4k+6} + t^{4k+5} + t^{4k+4} + t^{4k+2} + t^{4k-1} + t^7 + t^4 + t^2 + t + 1 \\ &=_2 (t^2 + t + 1)^2 (t^{4k+2} + t^{4k+1} + t^{4k-1} + t^3 + t + 1),\end{aligned}$$

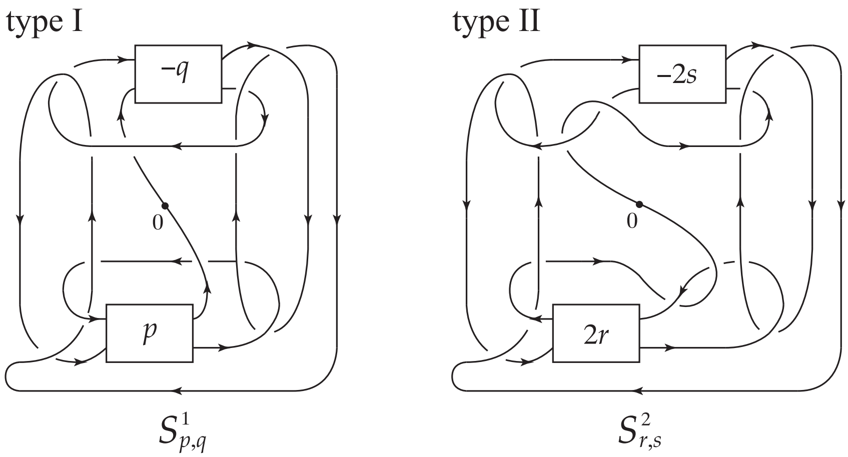

Fig. 12. Generalized Stoimenow links $S^1_{p,q}$ and $S^2_{r,s}$.

$$\begin{aligned}\Delta^{(2)}_{k,k}(t) &\doteq t^3(-t^6+9t^5-26t^4+37t^3-26t^2+9t-1)\\ &\quad -2kt^2(t-1)^2(2t^6-7t^5+15t^4-18t^3+15t^2-7t+2)\\ &\quad +k^2(t-1)^2(t^{10}-3t^9+7t^8-17t^7+32t^6-40t^5+32t^4-17t^3\\ &\qquad\qquad +7t^2-3t+1)\\ &\doteq_2 \begin{cases} t^6+t^5+t^3+t+1 & (k \textit{ is even}),\\ t^{12}+t^{11}+t^9+t^7+t^6+t^5+t^3+t+1 & (k \textit{ is odd}).\end{cases}\end{aligned}$$

Proof. We have the following relations on the Conway polynomials from the skein relation in Subsection 2.2:

$$\begin{cases}\nabla^{(1)}_{p,q}(z)-\nabla^{(1)}_{p-2,q}(z)=z\nabla^{(1)}_{p-1,q}(z),\\ \nabla^{(1)}_{p,q-2}(z)-\nabla^{(1)}_{p,q}(z)=z\nabla^{(1)}_{p,q-1}(z),\end{cases} \tag{5.2}$$

and

$$\begin{cases}\nabla^{(2)}_{r-1,s}(z)-\nabla^{(2)}_{r,s}(z)=z\nabla^{(2)}_{\infty,s}(z),\\ \nabla^{(2)}_{r,s}(z)-\nabla^{(2)}_{r,s-1}(z)=z\nabla^{(2)}_{r,\infty}(z).\end{cases} \tag{5.3}$$

For the meaning of ∞, see Fig. 6.
(type I)

From (5.2), we have:

$$\begin{cases}\Delta^{(1)}_{p,q}(t)-t^{1/2}\Delta^{(1)}_{p-1,q}(t)=(-t^{-1/2})^{p-1}(\Delta^{(1)}_{1,q}(t)-t^{1/2}\Delta^{(1)}_{0,q}(t)),\\ \Delta^{(1)}_{p,q}(t)+t^{-1/2}\Delta^{(1)}_{p-1,q}(t)=(t^{-1/2})^{p-1}(\Delta^{(1)}_{1,q}(t)+t^{-1/2}\Delta^{(1)}_{0,q}(t)),\end{cases} \tag{5.4}$$

and

$$(5.5)\qquad \begin{cases} \Delta^{(1)}_{p,q}(t) + t^{1/2}\Delta^{(1)}_{p,q-1}(t) = (t^{-1/2})^{q-1}(\Delta^{(1)}_{p,1}(t) + t^{1/2}\Delta^{(1)}_{p,0}(t)), \\ \Delta^{(1)}_{p,q}(t) - t^{-1/2}\Delta^{(1)}_{p,q-1}(t) = (-t^{-1/2})^{q-1}(\Delta^{(1)}_{p,1}(t) - t^{-1/2}\Delta^{(1)}_{p,0}(t)). \end{cases}$$

From (5.4) and (5.5), we have:

$$(5.6)\qquad \begin{cases} (t^{1/2} + t^{-1/2})\Delta^{(1)}_{p,q}(t) = (t^{p/2} - (-1)^p t^{-p/2})\Delta^{(1)}_{1,q}(t) \\ \qquad\qquad + (t^{(p-1)/2} + (-1)^p t^{-(p-1)/2})\Delta^{(1)}_{0,q}(t), \\ -(t^{1/2} + t^{-1/2})\Delta^{(1)}_{p,q}(t) = ((-1)^q t^{q/2} - t^{-q/2})\Delta^{(1)}_{p,1}(t) \\ \qquad\qquad - ((-1)^q t^{(q-1)/2} + t^{-(q-1)/2})\Delta^{(1)}_{p,0}(t). \end{cases}$$

From (5.6), if $p = q = 2k$, then we have a skein relation among the Alexander polynomials of $S^1_{2k,2k}$, $S^1_{0,0}$, $S^1_{1,0}$, $S^1_{0,1}$ and $S^1_{1,1}$ (cf. Fig. 13):

$$(5.7)\qquad \begin{aligned} (t^{1/2} + t^{-1/2})^2\Delta^{(1)}_{2k,2k}(t) = {} & (t^{k-1/2} + t^{-k+1/2})^2\Delta^{(1)}_{0,0}(t) \\ & - (t^k + t^{-k})(t^{k-1/2} + t^{-k+1/2})(\Delta^{(1)}_{1,0}(t) - \Delta^{(1)}_{0,1}(t)) \\ & - (t^k + t^{-k})^2\Delta^{(1)}_{1,1}(t). \end{aligned}$$

Since $S^1_{1,0}$ and $S^1_{0,1}$ are 2-component links with $S^1_{0,1} = -(S^1_{1,0})^*$, and (5.7), we have $\nabla^{(1)}_{0,1}(z) = -\nabla^{(1)}_{1,0}(z)$ and $\Delta^{(1)}_{0,1}(t) = -\Delta^{(1)}_{1,0}(t)$ by Lemma 2.3, and

$$(5.8)\qquad \begin{aligned} (t^{1/2} + t^{-1/2})^2\Delta^{(1)}_{2k,2k}(t) = {} & (t^{k-1/2} + t^{-k+1/2})^2\Delta^{(1)}_{0,0}(t) \\ & - 2(t^k + t^{-k})(t^{k-1/2} + t^{-k+1/2})\Delta^{(1)}_{1,0}(t) \\ & - (t^k + t^{-k})^2\Delta^{(1)}_{1,1}(t). \end{aligned}$$

Since $S^1_{0,0} = 8_{18}$,

$$\begin{aligned} \Delta^{(1)}_{0,0}(t) = \Delta_{8_{18}}(t) &= -t^3 + 5t^2 - 10t + 13 - 10t^{-1} + 5t^{-2} - t^{-3} \\ &=_2 t^3 + t^2 + 1 + t^{-2} + t^{-3}, \\ \Delta^{(1)}_{1,1}(t) = -t^{-3}(t^3 - 1)^2 &=_2 t^3 + t^{-3}, \end{aligned}$$

and (5.8), we have

$$\begin{aligned} (t+1)^2\Delta^{(1)}_{2k,2k}(t) &\doteq_2 t^{4k+6} + t^{4k+5} + t^{4k+4} + t^{4k+2} + t^{4k-1} + t^7 + t^4 + t^2 + t + 1 \\ &=_2 (t^2 + t + 1)^2(t^{4k+2} + t^{4k+1} + t^{4k-1} + t^3 + t + 1). \end{aligned}$$

(type II)

From (5.3), we have:

$$(5.9)\qquad \begin{cases} \nabla^{(2)}_{r,s}(z) = \nabla^{(2)}_{0,s}(z) - rz\nabla^{(2)}_{\infty,s}(z), \\ \nabla^{(2)}_{r,s}(z) = \nabla^{(2)}_{r,0}(z) + sz\nabla^{(2)}_{r,\infty}(z). \end{cases}$$

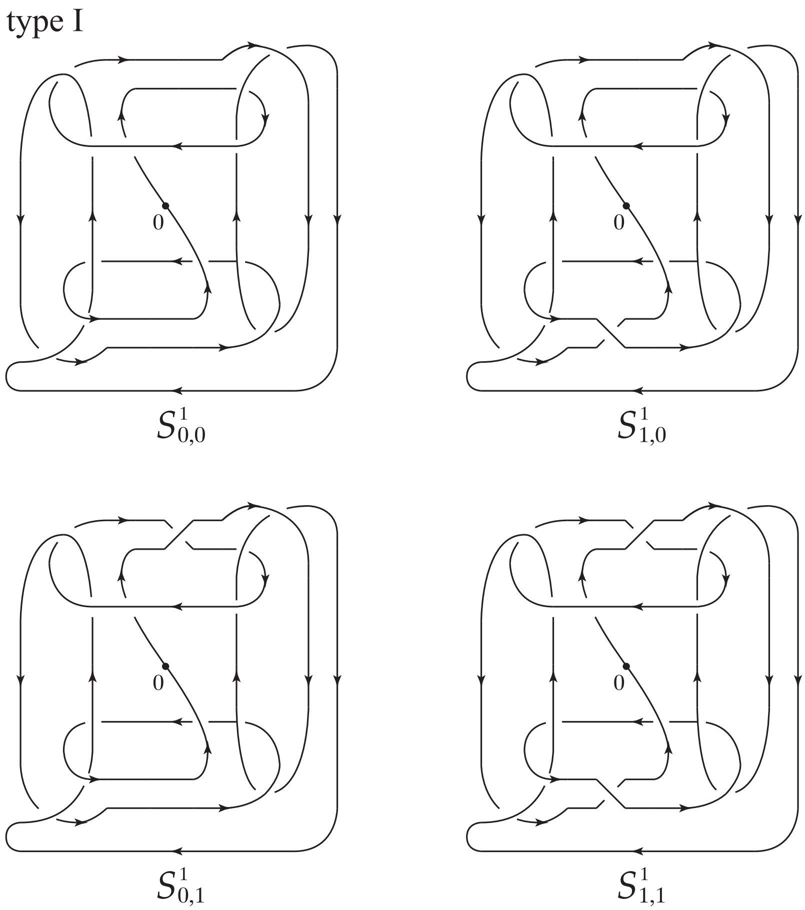

Fig. 13. $S^1_{0,0}$, $S^1_{1,0}$, $S^1_{0,1}$ and $S^1_{1,1}$.

From (5.9), we have:

$$\nabla^{(2)}_{r,s}(z) = \nabla^{(2)}_{0,0}(z) - rz\nabla^{(2)}_{\infty,0}(z) + sz\nabla^{(2)}_{0,\infty}(z) - rsz^2\nabla^{(2)}_{\infty,\infty}(z).$$

In particular, if $r = s = k$, then we have a skein relation among the Conway polynomials of $S^2_{k,k}$, $S^2_{0,0}$, $S^2_{0,\infty}$, $S^2_{\infty,0}$ and $S^2_{\infty,\infty}$ (cf. Fig. 14):

$$\nabla^{(2)}_{k,k}(z) = \nabla^{(2)}_{0,0}(z) + kz(\nabla^{(2)}_{0,\infty}(z) - \nabla^{(2)}_{\infty,0}(z)) - k^2z^2\nabla^{(2)}_{\infty,\infty}(z). \tag{5.10}$$

Since $S^2_{0,\infty}$ and $S^2_{\infty,0}$ are 2-component links with $S^2_{0,\infty} = -(S^2_{\infty,0})^*$,

$$\nabla^{(2)}_{0,0}(z) = -z^6 + 3z^4 + z^2 + 1,$$

$$\nabla^{(2)}_{0,\infty}(z) = -2z^7 - 5z^5 - 5z^3 - 2z,$$

$$\nabla^{(2)}_{\infty,\infty}(z) = -z^{10} - 7z^8 - 18z^6 - 15z^4 - 4z^2,$$

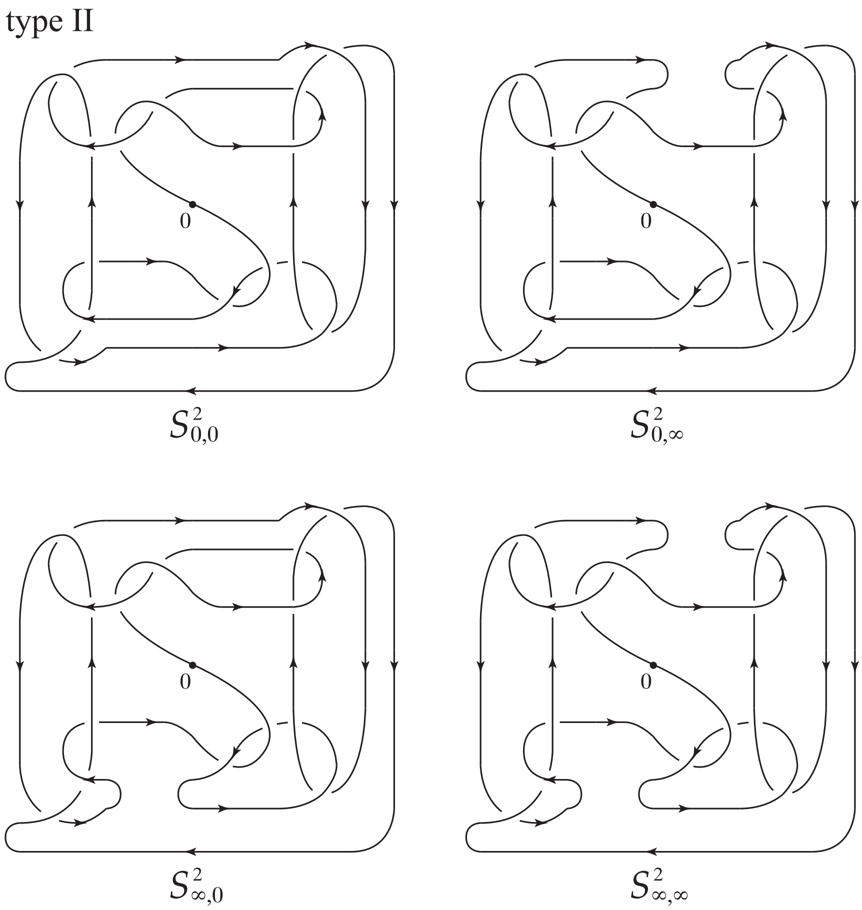

Fig. 14. $S^2_{0,0}$, $S^2_{0,\infty}$, $S^2_{\infty,0}$ and $S^2_{\infty,\infty}$.

and (5.10), we have $\nabla^{(2)}_{0,\infty}(z) = -\nabla^{(2)}_{\infty,0}(z)$ by Lemma 2.3, and

$$\begin{aligned}\Delta^{(2)}_{k,k}(t) &\doteq t^3(-t^6 + 9t^5 - 26t^4 + 37t^3 - 26t^2 + 9t - 1)\\ &\quad - 2kt^2(t-1)^2(2t^6 - 7t^5 + 15t^4 - 18t^3 + 15t^2 - 7t + 2)\\ &\quad + k^2(t-1)^2(t^{10} - 3t^9 + 7t^8 - 17t^7 + 32t^6 - 40t^5 + 32t^4 - 17t^3\\ &\qquad + 7t^2 - 3t + 1)\\ &\doteq_2 \begin{cases} t^6 + t^5 + t^3 + t + 1 & (k \text{ is even}),\\ t^{12} + t^{11} + t^9 + t^7 + t^6 + t^5 + t^3 + t + 1 & (k \text{ is odd}). \end{cases}\end{aligned}$$

□

Every element $f \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$ is of the form:

$$f = t^{k_d} + t^{k_{d-1}} + \cdots + t^{k_1} + t^{k_0}$$

where $k_0, \ldots, k_d$ are integers such that $k_0 < k_1 < \cdots < k_{d-1} < k_d$. Then we define the *mod* 2 *trace*, denoted by $\mathrm{tr}_2(f) \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, as:

$$\mathrm{tr}_2(f) = \begin{cases} 1 & (k_d - k_{d-1} = 1), \\ 0 & (k_d - k_{d-1} \geq 2). \end{cases}$$

For $f_1, f_2 \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$, $\mathrm{tr}_2(f_1 f_2) = \mathrm{tr}_2(f_1) + \mathrm{tr}_2(f_2)$. There exists an element $g \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$ such that $f = g^2$ if and only if every k_i $(i = 0, \ldots, d)$ is even. Then we call f a *square polynomial*, and we have

$$g = t^{k_d/2} + \cdots + t^{k_1/2} + t^{k_0/2}$$

and $\mathrm{tr}_2(f) = 0$.

Lemma 5.3. *Let $r(t)$ be of type X as in* (5.1), *and α a positive odd integer.*
(1) *If $n = 0$, then $r(t^\alpha)$ is a square polynomial. If $n \geq 1$, then $r(t^\alpha)$ is of the form*:

$$r(t^\alpha) = g^2 p_\lambda(t^\alpha)$$

where $g \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$ and $p_\lambda(t) = (t^\lambda - 1)/(t - 1)$.
(2) $\mathrm{tr}_2(r(t^\alpha)) = 1$ *if and only if $n \geq 1$ and $\alpha = 1$.*

Let ζ_m be a primitive m-th root of unity, and $\mathbf{\Phi}_m(t) \in \mathbb{Z}[t]$ the m-th cyclotomic polynomial defined by

$$\mathbf{\Phi}_m(t) = \prod_{\substack{1 \leq i \leq m-1 \\ \gcd(i,m)=1}} (t - \zeta_m^i).$$

The cyclotomic polynomial is a monic symmetric irreducible polynomial over $\mathbb{Z}$. For a prime q and a positive integer r,

$$\mathbf{\Phi}_{q^r}(t) = \frac{t^{q^r} - 1}{t^{q^{r-1}} - 1} = t^{q^{r-1}(q-1)} + t^{q^{r-1}(q-2)} + \cdots + t^{q^{r-1}} + 1.$$

Since

$$t^m - 1 = \prod_{d \geq 1, d | m} \mathbf{\Phi}_d(t),$$

we have

$$p_\lambda(t^\alpha) = \frac{t^{\alpha\lambda} - 1}{t^\alpha - 1} = \prod_{d | \alpha\lambda, d \nmid \alpha} \mathbf{\Phi}_d(t). \tag{5.11}$$

Theorem 5.4. *A Stoimenow knot S_n is not invertible.*

Proof. We show that both $S^1_{2k,2k}$ and $S^2_{k,k}$ with $k \geq 1$ are not $(+)$-amphicheiral.
(type I)

Suppose that $\Delta^{(1)}_{2k,2k}(t)$ satisfies the condition in Lemma 5.1 (2).

We set

$$h =_2 t^{4k+2} + t^{4k+1} + t^{4k-1} + t^3 + t + 1,$$

and $m = q^r$ with an odd prime $q \geq 3$ and $r \geq 1$. Then

$$(t+1)^2 \Delta^{(1)}_{2k,2k}(t) \doteq_2 (t^2+t+1)^2 h. \tag{5.12}$$

Claim 1. *$\mathbf{\Phi}_m(t)$ is a mod 2 divisor of h only if $m = 3, 5$ or 9.*

Proof. Take $Q(t), R(t) \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$ such that $h =_2 \mathbf{\Phi}_m(t)Q(t) + R(t)$. We can take $R(t)$ of the form:

$$R(t) =_2 t^{d+3} + t^{d+2} + t^d + t^3 + t + 1$$

where $-m/2 < d < m/2$. The span of $R(t)$ is less than $m/2 + 3$.

CASE 1 $r \geq 2$ except the case $(q, r) = (3, 2)$.

Since the degree of $\mathbf{\Phi}_m(t)$ is $q^{r-1}(q-1)$ which is greater than $q^r/2 + 3$, $R(t) = 0$ should be hold. However it does not occur.

CASE 2 $(q, r) = (3, 2)$ $(m = 9)$.

$R(t)$ is not mod 2 divisible by $\mathbf{\Phi}_9(t) = t^6 + t^3 + 1$ except the case $d = 4$.

CASE 3 $r = 1$.

We check only the cases $m = 3, 5$ and 7. The case $m = 7$ does not occur. Hence we have the result. □

Claim 2. *h is mod 2 divisible by $\mathbf{\Phi}_3(t)$ if and only if $k \equiv 0 \pmod 3$. h is mod 2 divisible by $\mathbf{\Phi}_5(t)$ if and only if $k \equiv 1 \pmod 5$. h is mod 2 divisible by $\mathbf{\Phi}_9(t)$ if and only if $k \equiv -1 \pmod 9$.*

Proof. h is mod 2 divisible by $\mathbf{\Phi}_3(t)$ if and only if $4k+1 \equiv 1 \pmod 3$ which is equivalent to $k \equiv 0 \pmod 3$.

h is mod 2 divisible by $\mathbf{\Phi}_5(t)$ if and only if $4k+1 \equiv 0 \pmod 5$ and $4k-1 \equiv 3 \pmod 5$ which is equivalent to $k \equiv 1 \pmod 5$.

h is mod 2 divisible by $\mathbf{\Phi}_9(t)$ if and only if $4k+1 \equiv 6 \pmod 9$ which is equivalent to $k \equiv -1 \pmod 9$. □

Claim 3. *$\mathbf{\Phi}_{15}(t)$ is a mod 2 divisor of h if and only if $k \equiv -5 \pmod{15}$. $\mathbf{\Phi}_{45}(t)$ is not a mod 2 divisor of h.*

Proof. For $\mathbf{\Phi}_{15}(t) = t^8 - t^7 + t^5 - t^4 + t^3 - t + 1$, we only check the cases $d = \pm 5, \pm 6$ and ± 7. For the cases, $R(t)$ is mod 2 divisible by $\mathbf{\Phi}_{15}(t)$ if and only if $4k - 1 \equiv -6 \pmod{15}$ which is equivalent to $k \equiv -5 \pmod{15}$.

For $\mathbf{\Phi}_{45}(t) = t^{24} - t^{21} + t^{15} - t^{12} + t^9 - t^3 + 1$, we only check the cases $d = \pm 21$ and ± 22. For the cases, $R(t)$ is not mod 2 divisible by $\mathbf{\Phi}_{45}(t)$. □

Claim 4. *$p_\lambda(t^\alpha)$ is a mod 2 divisor of h only if $p_3(t) = \mathbf{\Phi}_3(t) = t^2 + t + 1$, $p_5(t) = \mathbf{\Phi}_5(t) = t^4 + t^3 + t^2 + t + 1$ or $p_3(t^3) = \mathbf{\Phi}_9(t) = t^6 + t^3 + 1$.*

Proof. By Claim 1, Claim 2, Claim 3 and (5.11), we have the result. □

By Lemma 5.2, we have $\mathrm{tr}_2(\Delta^{(1)}_{2k,2k}(t)) = 1$. By Lemma 5.3, Claim 1, Claim 2, Claim 3, Claim 4 and (5.12), h is of the form:

$$h \doteq_2 g^2 p_3(t),\, g^2 p_5(t) \quad \text{or} \quad g^2 p_5(t) p_3(t^3)$$

for some $g \in (\mathbb{Z}/2\mathbb{Z})[t, t^{-1}]$. However we have

$$\frac{h}{t^2 + t + 1} =_2 t^{4k} + \cdots + t^5 + t^4 + t^2 + 1$$

for $k \equiv 0 \pmod 3$, $k \geq 3$,

$$\frac{h}{t^4 + t^3 + t^2 + t + 1} =_2 t^{4k-2} + \cdots + t^3 + t^2 + 1$$

for $k \equiv 1 \pmod 5$, $k \geq 6$, and

$$\frac{h}{(t^4 + t^3 + t^2 + t + 1)(t^6 + t^3 + 1)} =_2 t^{4k-8} + \cdots + t^5 + t^2 + 1$$

for $k \equiv 26 \pmod{45}$, $k \geq 26$ are not square polynomials. It is a contradiction.
(type II)

Suppose that $\Delta^{(2)}_{k,k}(t)$ satisfies the condition in Lemma 5.1 (2).

By Lemma 5.2, we have $\mathrm{tr}_2(\Delta^{(2)}_{k,k}(t)) = 1$. By Lemma 5.3, there exists an odd $\lambda \geq 3$ such that $p_\lambda(t)$ is a mod 2 divisor of $\Delta^{(2)}_{k,k}(t)$. If k is odd, then there is no such λ (check only the cases $\lambda = 3, 5, 7, 9, 11$). Hence we suppose that k is even. Since

$$\Delta^{(2)}_{k,k}(t) \doteq_2 (t^2 + t + 1)^3,$$

we have $\lambda = 3$. By the forms (5.1) and Lemma 5.1 (2), $\Delta^{(2)}_{k,k}(t)$ is of the form:

$$\Delta^{(2)}_{k,k}(t) \doteq r_1(t) r_2(t) r_3(t) \tag{5.13}$$

where $r_i(t) \doteq r_i(t^{-1})$, $|r_i(1)| = 1$ and $r_i(t) \doteq_2 t^2 + t + 1$ ($i = 1, 2, 3$). That is, $\Delta^{(2)}_{k,k}(t)$ is decomposed into at least three non-trivial factors in $\mathbb{Z}[t, t^{-1}]$. We set d_i as the degree (span) of $r_i(t)$ (i = 1, 2, 3), and assume $d_1 \leq d_2 \leq d_3$. There are two cases:

CASE 1 $k \equiv 0 \pmod 4$.

By Lemma 5.2, we have the mod 8 Alexander polynomial:

$$\Delta^{(2)}_{k,k}(t) \doteq_8 t^6 - t^5 + 2t^4 + 3t^3 + 2t^2 - t + 1.$$

Since $t^2 \pm t + 1$ and $t^2 \pm 3t + 1$ are not mod 8 divisors of $\Delta^{(2)}_{k,k}(t)$, the case does not occur.

CASE 2 $k \equiv 2 \pmod 4$.

By Lemma 5.2, we have the mod 8 Alexander polynomial:

$$\begin{aligned}\Delta^{(2)}_{k,k}(t) &\doteq_8 4t^{12} + 4t^{11} + 3t^9 + t^8 - 2t^7 - 3t^6 - 2t^5 + t^4 + 3t^3 + 4t + 4\\ &\doteq_8 (t^2 - t + 1)(4t^{10} + 4t^8 - t^7 + 4t^6 + 3t^5 + 4t^4 - t^3 + 4t^2 + 4).\end{aligned}$$

We set $s = 4t^{10} + 4t^8 - t^7 + 4t^6 + 3t^5 + 4t^4 - t^3 + 4t^2 + 4$. In this case, the $\mathbb{Z}$-degree of $\Delta^{(2)}_{k,k}(t)$ is 12 which is equal to the mod 8 degree of it. By the assumption, there are three cases for the triple (d_1, d_2, d_3): (d_1, d_2, d_3) = (2, 2, 8), (2, 4, 6) or (4, 4, 4). The possibilities of the degree 2 mod 8 factors are $t^2 \pm t + 1$ and $t^2 \pm 3t + 1$. Since $t^2 \pm t + 1$ and $t^2 \pm 3t + 1$ are not mod 8 divisors of s, s is decomposed into $s = s_1 s_2$ such that the degrees of s_1 and s_2 are 4 and 6 respectively, they are both irreducible, and $s_1 \doteq_2 s_2 \doteq_2 t^2 + t + 1$. By (5.13), s_1 and s_2 are of the form:

$$\begin{aligned}s_1 &\doteq_8 2t^4 + a_1 t^3 + a_2 t^2 + a_1 t + 2 \doteq_2 t^2 + t + 1,\\ s_2 &\doteq_8 2t^6 + b_1 t^5 + b_2 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + 2 \doteq_2 t^2 + t + 1\end{aligned}$$

where a_1, a_2, b_2 and b_3 are odd, and b_1 is even. Then the 9-th coefficient of $s_1 s_2$ is odd (non-zero). However it contradicts the form of s. □

At the end of the paper, we raise refined questions realted with Question 1.2:

QUESTION 5.5. (1) Is there a prime component-preservingly amphicheiral link with odd minimal crossing number less than 21?
(2) Is there a prime component-preservingly (ε)-amphicheiral link with odd minimal crossing number?

About (1), we have already known that there are no such examples for the case that the minimal crossing number ≤ 11 (cf. [8]). If we need to use an amphicheiral knot with odd minimal crossing number, then the minimal crossing number should be greater than or equal to 19 from primeness. Under the restriction, if there exists an example L for Question 5.5 (1) with minimal crossing number 19, then L is a 2-component link such that

(i) its components are a knot with minimal crossing number 15 and the unknot,
(ii) $\mathrm{lk}(L) = 0$, and
(iii) on its diagram realizing the minimal crossing number, its components are also realizing the minimal crossing numbers (i.e. 15 and 0).

About (2), our example L_n was a prime component-preservingly $(-,+)$-amphicheiral link with odd minimal crossing number. In general, the linking number of a 2-component (ε)-amphicheiral link is 0. 11^2_{n247} in Fig. 2 is a prime (ε)-amphicheiral link with odd minimal crossing number. However it is not component-preservingly (ε)-amphicheiral.

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References

[1] D. Bar-Natan and S. Morrison: *Knot atlas*, http://katlas.math.toronto.edu/wiki/Main_Page.
[2] R.I. Hartley: *Invertible amphicheiral knots*, Math. Ann. **252** (1980), 103–109.
[3] R. Hartley and A. Kawauchi: *Polynomials of amphicheiral knots*, Math. Ann. **243** (1979), 63–70.
[4] J.A. Hillman: *Symmetries of knots and links, and invariants of abelian coverings*, I, Kobe J. Math. **3** (1986), 7–27.
[5] J. Hoste, M. Thistlethwaite and J. Weeks: *The first* 1,701,936 *knots*, Math. Intelligencer **20** (1998), 33–48.
[6] T. Kadokami: *The link-symmetric groups of* 2*-bridge links*, J. Knot Theory Ramifications **20** (2011), 1129–1144.
[7] T. Kadokami: *Amphicheiral links with special properties*, I, J. Knot Theory Ramifications **21** (2012), 1250048, 17 pp, arXiv:math.GT/1107.0377.
[8] T. Kadokami: *Amphicheiral links with special properties*, II, J. Knot Theory Ramifications **21** (2012), 1250047, 15 pp, arXiv:math.GT/1107.0378.
[9] T. Kadokami and A. Kawauchi: *Amphicheirality of links and Alexander invariants*, Sci. China Math. **54** (2011), 2213–2227.
[10] A. Kawauchi: *The invertibility problem on amphicheiral excellent knots*, Proc. Japan Acad. Ser. A Math. Sci. **55** (1979), 399–402.
[11] A. Kawauchi: A Survey of Knot Theory, Birkhäuser, Basel, 1996.
[12] Y. Kobatake: *Morote-gata karamime no kousei* (*A construction of amphicheiral links*), Master Thesis, Osaka City University (2012), (Japanese).
[13] W. Menasco and M. Thistlethwaite: *The classification of alternating links*, Ann. of Math. (2) **138** (1993), 113–171.
[14] K. Murasugi: *Jones polynomials and classical conjectures in knot theory*, Topology **26** (1987), 187–194.
[15] K. Murasugi: *Jones polynomials and classical conjectures in knot theory*, II, Math. Proc. Cambridge Philos. Soc. **102** (1987), 317–318.
[16] D. Rolfsen: Knots and Links, Publish or Perish, Berkeley, CA, 1976.
[17] A. Stoimenow: *Tait's conjectures and odd crossing number amphicheiral knots*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), 285–291.

[18] A. Stoimenow: *Non-triviality of the Jones polynomial and the crossing number of amphicheiral knots*, arXiv:math.GT/0606255.
[19] W.C. Whitten, Jr.: *Symmetries of links*, Trans. Amer. Math. Soc. **135** (1969), 213–222.

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