TAUBERIAN THEOREM FOR HARMONIC MEAN OF STIELTJES TRANSFORMS AND ITS APPLICATIONS TO LINEAR DIFFUSIONS

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Abstract

When two Radon measures on the half line are given, the harmonic mean of their Stieltjes transforms is again the Stieltjes transform of a Radon measure. We study the relationship between the asymptotic behavior of the resulting measure and those of the original ones. The problem comes from the spectral theory of second–order differential operators and the results are applied to linear diffusions neither boundaries of which is regular.

1. Introduction

Let $\mathcal{H}$ be the totality of the functions on $(0, \infty)$ having the following representation:

$$h(s) = a + \int_{[0,\infty)} \frac{d\sigma(\xi)}{s + \xi}, \quad s > 0 \quad (\exists a \geq 0)$$

where $\sigma \colon \mathcal{R} \to [0,\infty)$ is a nondecreasing, right-continuous function vanishing on $(-\infty, 0)$ such that

$$0 < \int_{[0,\infty)} \frac{d\sigma(\xi)}{1 + \xi} < \infty.$$

Let us call σ the *spectral function* of h (the reason will be clear in Section 5). For $h_1, h_2 \in \mathcal{H}$ define h by

$$(1.1) \qquad \frac{1}{h(s)} = \frac{1}{h_1(s)} + \frac{1}{h_2(s)}.$$

Then as is well known we again have $h \in \mathcal{H}$ (a property of Herglotz functions).

The aim of the present article is to study the relationship between the asymptotic behavior of $\sigma(\lambda)$ as $\lambda \to +0$ and those of $\sigma_i(\lambda)$ ($i = 1, 2$), where σ and σ_i are the spectral functions of h and h_i, respectively. Notice that the equation (1.1) is familiar in the spectral theory of Sturm–Liouville operators and is fundamental in the theory

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of linear diffusions. Indeed, our problem is motivated by a study of the asymptotic behavior of the transition probability in the long term and this problem will be discussed in Section 6. Especially, Example 6.1 will illustrate the motivation of our problem.

To describe our results let us prepare some notation. We define $l_i = h_i(+0)$ ($i = 1, 2$) and $l = h(+0)$ ($\leq \infty$). By (1.1) it holds

$$(1.2) \qquad \frac{1}{l} = \frac{1}{l_1} + \frac{1}{l_2}$$

with the convention that $1/\infty = 0$ (namely, if $l_1 < \infty$ and $l_2 = \infty$ then $l = l_1$, while if $l_1 = l_2 = \infty$ then $l = \infty$).

We also define

$$(1.3) \qquad p = \frac{l_2}{l_1 + l_2} \left(= \frac{l}{l_1} \right), \quad q = \frac{l_1}{l_1 + l_2} \left(= \frac{l}{l_2} \right),$$

when they make sense.

Very roughly speaking our result is as follows: Under a certain regularity condition, it holds

$$(1.4) \qquad \sigma(\lambda) \sim \begin{cases} \dfrac{\sigma_1(\lambda)\sigma_2(\lambda)}{\sigma_1(\lambda) + \sigma_2(\lambda)} & (l_1 = l_2 = \infty), \\[2ex] p^2\sigma_1(\lambda) + q^2\sigma_2(\lambda) & (l_1 < \infty, l_2 < \infty), \\[2ex] l_2^2 \displaystyle\int_0^\lambda \xi \, d\sigma_1^*(\xi) + \sigma_2(\lambda) & (l_1 = \infty, l_2 < \infty), \end{cases}$$

where, σ_1^* is the 'dual' of σ_1.

The precise statement will be given in Section 2 and will be proved in Section 4. In Section 3 we prepare some intermediate results we need in the proofs of the main results. Sections 5 and 6 are devoted to applications of the mains results to linear diffusions. Since we shall repeatedly make use of Tauberian theorems for Lebesgue–Stieltjes transforms, we listed necessary facts in Appendix for the convenience of the reader.

REMARK 1.1.   We shall discuss only the case of (1.1), but the results can easily be extended to the case where

$$\frac{1}{h(s)} = \frac{1}{h_1(s)} + \frac{1}{h_2(s)} + \cdots + \frac{1}{h_n(s)}.$$

So our results may have applications to diffusions on some sort of graphs as well as linear diffusions.

2. Main results

We denote by $R_\alpha(0)$ the totality of ultimately positive functions (defined on some interval $(0, A)$) *varying regularly* at $+0$ with index α $(\in \mathcal{R})$: i.e.,

$$\lim_{s \to +0} \frac{f(cs)}{f(s)} = c^\alpha \quad (\forall c > 0).$$

A regularly varying function with index $\alpha = 0$ is said to be *slowly varying*. Clearly $f \in R_\alpha(0)$ if and only if $f(s) = s^\alpha L(s)$ with slowly varying L.

Theorem 2.1 (Case I). *Suppose that $l_1 < \infty$, $l_2 < \infty$ and let p, q be as in* (1.3). *If $\varphi \in R_\alpha(0)$ $(\alpha \geq 1)$, then*
(i)

$$\sigma(\lambda) \sim \varphi(\lambda) \quad (\lambda \to +0)$$

if and only if

$$\sigma_{pq}(\lambda) := p^2\sigma_1(\lambda) + q^2\sigma_2(\lambda) \sim \varphi(\lambda) \quad (\lambda \to +0).$$

(ii) *As a special case*, *if*

$$\sigma_i(\lambda) \sim c_i\varphi(\lambda) \quad (\lambda \to +0), \quad i = 1, 2,$$

for $c_1, c_2 \geq 0$ $(c_1 + c_2 > 0)$, then

$$(2.1) \qquad\qquad \sigma(\lambda) \sim (c_1p^2 + c_2q^2)\varphi(\lambda) \quad (\lambda \to +0).$$

Here and throughout, $f \sim cg$ means $f/g \to c$ including the case $c = 0$.

Theorem 2.2 (Case II). *Suppose that $l_1 = l_2 = \infty$ and let $\varphi \in R_\alpha(0)$ $(0 \leq \alpha \leq 1)$. If $\sigma_i(\lambda) \sim c_i\varphi(\lambda)$ for $c_i \in (0, \infty)$, $(i = 1, 2)$, then*

$$(2.2) \qquad\qquad \sigma(\lambda) \sim \frac{c_1c_2}{c_1 + c_2}\varphi(\lambda) \quad (\lambda \to +0).$$

The assertion remains valid in the extreme case $0 < c_1 < \infty$, $c_2 = \infty$, with the convention $c_1c_2/(c_1 + c_2) = c_1$.

(The restriction $\alpha \leq 1$ is necessary for $l_1 = l_2 = \infty$.)

Apparently (2.2) may look quite different from (2.1), but (2.2) is in fact the extreme case of (2.1) as $l_1, l_2 \to \infty$ with $l_1/l_2 = c_1/c_2$.

Corollary 2.1 (Lopsided case I). *Suppose that $\sigma_1 \in R_\alpha(0)$ and $\sigma_2 \in R_\beta(0)$.*

(i) *If $0 \leq \beta < \alpha < 1$, then*

$$\sigma(\lambda) \sim \sigma_1(\lambda) \quad (\lambda \to +0).$$

(ii) *If $1 < \alpha < \beta$, then*

$$\sigma(\lambda) \sim p^2 \sigma_1(\lambda) \quad (\lambda \to +0).$$

Proof. (i) Apply Theorem 2.2 with $\varphi(\lambda) = \sigma_1(\lambda)$ and $c_1 = 1$, $c_2 = \infty$.
(ii) Apply Theorem 2.1 (ii) with $c_1 = 1$, $c_2 = 0$. $\qquad\qquad\qquad\square$

It remains to discuss the case where $l_1 = \infty$, $l_2 < \infty$. To this end we need to consider the *dual* of h: For a given $h \in \mathcal{H}$ its dual h^* is defined by

$$h^*(s) = \frac{1}{sh(s)}.$$

As is well known it holds $h^* \in \mathcal{H}$. So let a^* and σ^* correspond to h^*: i.e.,

$$(2.3) \qquad\qquad h^*(s) = a^* + \int_{[0,\infty)} \frac{d\sigma^*(\xi)}{s + \xi}.$$

We also define $\sigma^{\#}: \mathcal{R} \to [0, \infty)$ by

$$\sigma^{\#}(\lambda) = \begin{cases} \displaystyle\int_{[0,\lambda]} \xi \, d\sigma^*(\xi) & (\lambda \geq 0), \\ 0 & (\lambda < 0). \end{cases}$$

Another characterization of $\sigma^{\#}$ will be given in (3.1).

Theorem 2.3 (Case III). *Suppose that $l_1 = \infty$, $l_2 < \infty$ and let $\varphi \in R_\beta(0)$ $(\beta \geq 1)$. Then,*

$$(2.4) \qquad\qquad \sigma(\lambda) \sim \varphi(\lambda) \quad (\lambda \to +0)$$

if and only if

$$(2.5) \qquad\qquad l_2^2 \sigma_1^{\#}(\lambda) + \sigma_2(\lambda) \sim \varphi(\lambda) \quad (\lambda \to +0).$$

(The restriction $\beta \geq 1$ is necessary for the assumption $l_2 < \infty$.)

Corollary 2.2. *Let $\psi \in R_\alpha(0)$ $(0 \leq \alpha < 1)$ and $c_1, c_2 > 0$. If*

$$\sigma_1(\lambda) \sim c_1 \psi(\lambda) \quad (\in R_\alpha(0)), \quad \sigma_2(\lambda) \sim c_2 \lambda^2 / \psi(\lambda) \quad (\in R_{2-\alpha}(0)) \quad (\lambda \to +0),$$

then

$$\sigma(\lambda) \sim \left(\frac{\alpha}{2-\alpha} \frac{l_2^2}{c_1\{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^2} + c_2 \right) \frac{\lambda^2}{\psi(\lambda)} \in R_{2-\alpha}(0) \quad (\lambda \to +0).$$

The assertion remains valid even if $(c_1 = \infty; 0 < c_2 < \infty)$ or $(0 < c_1 < \infty; c_2 = 0; \alpha \neq 0)$ with the convention that $1/\infty = 0$.

Proof. As we shall see in Proposition 3.2, the assumption $\sigma_1(\lambda) \sim c_1 \psi(\lambda)$ implies that

$$\sigma_1^{\#}(\lambda) \sim \frac{\alpha}{2-\alpha} \frac{l_2^2}{c_1\{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^2} \frac{\lambda^2}{\psi(\lambda)} \quad (\lambda \to +0).$$

Therefore, the assertion follows immediately from Theorem 2.3. Note that when $\alpha = 0$ we may use the last part of Proposition 3.2. $\quad\square$

By the last part of Corollary 2.2 we have

Corollary 2.3 (Lopsided case II). *Let $\sigma_1 \in R_\alpha(0)$ and $\sigma_2 \in R_\beta(0)$.*
(i) *If $0 \leq \alpha < 1 < \beta$ and $\alpha + \beta < 2$, then,*

$$\sigma(\lambda) \sim \sigma_2(\lambda) \quad (\lambda \to +0).$$

(ii) *If $0 < \alpha < 1 < \beta$ and $\alpha + \beta > 2$, then*

(2.6) $$\sigma(\lambda) \sim \frac{\alpha}{2-\alpha} \frac{l_2^2}{\{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^2} \cdot \frac{\lambda^2}{\sigma_1(\lambda)} \quad (\lambda \to +0).$$

(In (ii) we excluded the case $\alpha = 0$ because the the right-hand side of (2.6) vanishes.)

Proof of Corollary 2.3. (i) Let $\psi(\lambda) = \lambda^2/\sigma_2(\lambda)$ $(\in R_{2-\beta}(0))$. Then $\sigma_1(\lambda)/\psi(\lambda) \in R_{\alpha+\beta-2}(0)$ and, hence, $\alpha + \beta < 2$ implies $\sigma_1(\lambda)/\psi(\lambda) \to \infty$. Therefore, we can apply Corollary 2.2 with $c_1 = \infty$, $c_2 = 1$.

For the proof of (ii) put $\psi(\lambda) = \sigma_1(\lambda)$, then appeal to Corollary 2.2 with $c_1 = 1$, $c_2 = 0$. $\quad\square$

Let us next consider the case $\alpha = 1$ which we excluded in Corollary 2.3.

Corollary 2.4 (Lopsided case III: $\alpha = 1$). *Suppose that $\sigma_1 \in R_1(0)$ and $\sigma_2 \in R_\beta(0)$.*
(i) *If $0 \leq \beta < 1$, then*

$$\sigma(\lambda) \sim \sigma_1(\lambda) \quad (\lambda \to +0).$$

(ii) *If $\beta > 1$, then*

(2.7) $$\sigma(\lambda) \sim l^2 \frac{\sigma_1(\lambda)}{\hat{L}(\lambda)^2} \quad (\lambda \to +0),$$

where

$$\hat{L}(s) := \int_s^\infty \frac{\sigma_1(u)}{u^2} \, du \in R_0(0).$$

Note that $\hat{L}(s) \to l_1$ (cf. (3.12)). So in (ii), if $l_1 < \infty$, then (2.7) may also be written as

$$\sigma(\lambda) \sim p^2 \sigma_1(\lambda) \quad (\lambda \to +0),$$

because $p = l/l_1$. This means that (ii) of Corollary 2.1 remains valid in the extreme case $\alpha = 1$ when $l_1 < \infty$.

Proof of Corollary 2.4.   (i)   If $l_1 = \infty$, we can apply Theorem 2.2 with $c_2 = \infty$ because $\sigma_1 = o(\sigma_2)$. Next consider the case $l_1 < \infty$. By Theorem 2.3,

$$\sigma(\lambda) \sim l_1^2 \sigma_2^\#(\lambda) + \sigma_1(\lambda).$$

By Proposition 3.2, we have $\sigma_2^\# \in R_{2-\beta}(0)$ so that $\sigma_2^\# = o(\sigma_1)$. Thus we have the assertion.

(ii)   When $l_1 < \infty$, just apply Theorem 2.1 with $\varphi = \sigma_1$, $c_1 = 1$, $c_2 = 0$. When $l_1 = \infty$, as we shall prove in Proposition 3.3, it holds

$$\sigma_1^\#(\lambda) \sim \frac{\sigma_1(\lambda)}{\hat{L}(\lambda)^2} \in R_1(0).$$

On the other hand $\sigma_2 \in R_\beta(0)$ with $\beta > 1$ implies $\sigma_2 = o(\sigma_1^\#)$. Therefore, we deduce the assertion from Theorem 2.3.   □

In Corollary 2.2 we discussed the case where $l_1 = \infty$, $l_2 < \infty$ and $\sigma \in R_\gamma(0)$ with $0 < \gamma < 2$. For the case $\gamma > 2$, we have the following result. Here, notice that the condition $\sigma_1(+0) > 0$ trivially implies $\sigma_1 \in R_0(0)$ and $l_1 \, (= h_1(+0)) = \infty$.

Corollary 2.5.   *Suppose $\sigma_1(+0) > 0$ and $\varphi \in R_\gamma(0)$ with $\gamma > 2$. If*

$$\sigma_2(\lambda) \sim c_2 \varphi(\lambda) \quad (\lambda \to +0)$$

and

(2.8) $$\sigma_1(\lambda) - \sigma_1(+0) \sim c_1 \varphi(\lambda)/\lambda^2 \quad (\lambda \to +0),$$

for $c_1, c_2 \geq 0$ $(c_1 + c_2 > 0)$, then

$$\sigma(\lambda) \sim \left(\frac{c_1 l_2^2}{\sigma_1(+0)^2} \frac{\gamma - 2}{\gamma} + c_2 \right) \varphi(\lambda) \in R_\gamma(0).$$

Proof.   As we shall see in Proposition 3.4 the condition (2.8) is equivalent to

$$\sigma_1^{\#}(\lambda) \sim \frac{\gamma - 1}{\gamma} \frac{c_1}{\sigma_1(+0)^2} \frac{\gamma - 2}{\gamma - 1} \varphi(\lambda) = \frac{c_1}{\sigma_1(+0)^2} \frac{\gamma - 2}{\gamma} \varphi(\lambda).$$

Therefore, the assertion follows from Theorem 2.3.   □

Theorem 2.4 (Case IV).   *Suppose that $\sigma_1(+0) > 0$ and $\sigma_2(+0) > 0$. Then,*

(2.9) $$\sigma(+0) = \frac{\sigma_1(+0)\sigma_2(+0)}{\sigma_1(+0) + \sigma_2(+0)} \quad (> 0).$$

Furthermore, if

(2.10) $$\sigma_i(\lambda) - \sigma_i(+0) \sim c_i \varphi(\lambda) \quad (\lambda \to +0), \quad i = 1, 2$$

for $\varphi \in R_\alpha(0)$ $(\alpha > 0)$ and $c_1, c_2 \geq 0$ $(c_1 + c_2 > 0)$, then

(2.11) $$\sigma(\lambda) - \sigma(+0) \sim (p_*^2 c_1 + q_*^2 c_2)\varphi(\lambda) \quad (\lambda \to +0),$$

where

$$p_* = \frac{\sigma_2(+0)}{\sigma_1(+0) + \sigma_2(+0)}, \quad q_* = \frac{\sigma_1(+0)}{\sigma_1(+0) + \sigma_2(+0)}.$$

3. Intermediate results

In this section we prepare a few propositions we need for the proofs of Theorems 2.1–2.4 in Section 2.

Throughout the paper, we put

$$\hat{h}(s) = \frac{1}{h(s)}, \quad \hat{h}_i(s) = \frac{1}{h_i(s)} \quad (i = 1, 2).$$

Since $\hat{h}(s) = sh^*(s)$, we have

$$\hat{h}'(s) = h^*(s) + sh^{*\prime}(s) = a^* + \int_0^\infty \frac{d\sigma^*(\xi)}{s + \xi} - \int_0^\infty \frac{s \, d\xi^*(\xi)}{(s + \xi)^2}$$

$$= a^* + \int_0^\infty \frac{\xi \, d\sigma^*(\xi)}{(s + \xi)^2}.$$

Therefore,

$$
(3.1) \qquad \hat{h}^{(n)}(s) =
\begin{cases}
a^* + \displaystyle\int_0^\infty \frac{d\sigma^{\#}(\xi)}{(s+\xi)^2} & (n = 1), \\[4mm]
(-1)^{n+1} n! \displaystyle\int_0^\infty \frac{d\sigma^{\#}(\xi)}{(s+\xi)^{n+1}} & (n \geq 2).
\end{cases}
$$

Of course we have similar formulas for $h_1, h_2 \in \mathcal{H}$ and we may define $\sigma_i^{\#}$ ($i = 1, 2$) in the obvious manner.

Since (1.1) can be written as

$$
(3.2) \qquad \hat{h}(s) = \hat{h}_1(s) + \hat{h}_2(s),
$$

it holds

$$
\hat{h}'(s) = \hat{h}'_1(s) + \hat{h}'_2(s)
$$

and hence, by (3.1), we have

$$
(3.3) \qquad \sigma^{\#}(\lambda) = \sigma_1^{\#}(\lambda) + \sigma_2^{\#}(\lambda).
$$

Thus the proofs of the results in the previous section are reduced to the study of the relationship between the asymptotic behavior of σ, σ_1, σ_2 and that of $\sigma^{\#}$, $\sigma_1^{\#}$, $\sigma_2^{\#}$.

Next we define

$$
\#(h) = \inf\{k \geq 0; \, |h^{(k)}(+0)| = \infty\}
$$

$$
\left(= \inf\left\{k = 0, 1, 2, \ldots; \, \int_{[0,1)} d\sigma(\xi)/\xi^{k+1} = \infty\right\}\right)
$$

and

$$
\#(\hat{h}) = \inf\{k \geq 0; \, |\hat{h}^{(k)}(+0)| = \infty\}.
$$

Proposition 3.1. *If $n_0 := \#(h) \geq 1$ (i.e., $l < \infty$), then it holds that $\#(h) = \#(\hat{h})$ and*

$$
(3.4) \qquad \lim_{s \to +0} \frac{\hat{h}^{(n)}(s)}{h^{(n)}(s)} = -\frac{1}{l^2}, \quad \forall n \geq n_0.
$$

For the proof of Proposition 3.1 we prepare

Lemma 3.1. *Suppose $\#(h) = n_0 \geq 1$ (i.e., $l < \infty$). Then;*
(i)

$$
\lim_{s \to +0} |\hat{h}^{(k)}(s)|
\begin{cases}
< \infty & (1 \leq \forall k < n_0), \\
= \infty & (\forall k \geq n_0).
\end{cases}
$$

(ii)

(3.5)
$$h^{(n_0)}(s) \sim -l^2 \hat{h}^{(n_0)}(s) \quad (s \to +0).$$

Proof. (i) By the Leibniz formula we have, for $m \geq 1$,

$$\sum_{k=0}^{m} {}_mC_k h^{(k)}(s) \hat{h}^{(m-k)}(s) = (h(s)\hat{h}(s))^{(m)} = 0.$$

Therefore,

(3.6)
$$\hat{h}^{(m)}(s) = -\frac{1}{h(s)} \sum_{k=1}^{m} {}_mC_k h^{(k)}(s) \hat{h}^{(m-k)}(s).$$

Now we have the assertion by induction on $m = 1, 2, \ldots, n_0$.

(ii) As we have seen in (3.6), it holds that

(3.7)
$$\hat{h}^{(n_0)}(s) = -\frac{1}{h(s)} \left\{ h^{(n_0)}(s)\hat{h}(s) + \sum_{k=1}^{n_0-1} {}_nC_k h^{(k)}(s) \, \hat{h}^{(n_0-k)}(s) \right\}$$

and hence by (i) we have

$$\hat{h}^{(n_0)}(s) = -h^{(n_0)}(s)\frac{\hat{h}(s)}{h(s)} + O(1) = -h^{(n_0)}(s)\frac{1}{h(s)^2} + O(1).$$

Since $|h^{(n_0)}(s)| \to \infty$ by the definition of n_0, we can neglect the $O(1)$ in the right-hand side and deduce the assertion. $\square$

Lemma 3.2. *Let $n \geq 1$. If $l < \infty$ and $|h^{(n)}(+0)| = \infty$, then for $k = 1, 2, \ldots, n-1$ it holds*

(3.8)
$$h^{(k)}(s) = o(|h^{(n)}(s)|^{k/n}) \quad (s \to +0)$$

and

(3.9)
$$h^{(k)}(s)h^{(n-k)}(s) = o(|h^{(n)}(s)|) \quad (s \to +0).$$

Proof. Since (3.9) follows immediately from (3.8), we shall prove (3.8) only. If $|h^{(k)}(+0)| < \infty$, then the assertion is obvious. So we assume that $|h^{(k)}(+0)| = \infty$. For every $\varepsilon > 0$, applying Hölder's inequality to

$$\int_0^\varepsilon \frac{d\sigma(\xi)}{(s+\xi)^{k+1}} = \int_0^\varepsilon \frac{d\sigma(\xi)}{(s+\xi)^{1-(k/n)} \cdot (s+\xi)^{(n+1)k/n}}$$

with $1/p = 1 - (k/n)$ and $1/q = k/n$, we have

$$\int_0^\varepsilon \frac{d\sigma(\xi)}{(s+\xi)^{k+1}} \leq \left(\int_0^\varepsilon \frac{d\sigma(\xi)}{s+\xi}\right)^{1-(k/n)} \left(\int_0^\varepsilon \frac{d\sigma(\xi)}{(s+\xi)^{n+1}}\right)^{k/n}$$

$$\leq \left(\int_0^\varepsilon \frac{d\sigma(\xi)}{\xi}\right)^{1-(k/n)} \left(\int_0^\infty \frac{d\sigma(\xi)}{(s+\xi)^{n+1}}\right)^{k/n}.$$

Therefore, using the condition $|h^{(n)}(+0)| = \infty$, we deduce

$$\limsup_{s \to +0} \int_0^\infty \frac{d\sigma(\xi)}{(s+\xi)^{k+1}} \Big/ \left(\int_0^\infty \frac{d\sigma(\xi)}{(s+\xi)^{n+1}}\right)^{k/n} \leq \left(\int_0^\varepsilon \frac{d\sigma(\xi)}{\xi}\right)^{1-(k/n)}.$$

Since the right-hand side converges to 0 as $\varepsilon \to +0$ because $\int_0^\infty (1/\xi)\, d\sigma(\xi) = l < \infty$ by assumption, we obtain (3.8). $\qquad\square$

Proof of Proposition 3.1.   Let us prove the assertion by induction on n $(\geq n_0)$. The case $n = n_0$ is proved in Lemma 3.1 (ii). Next suppose that (3.4) holds for $n = n_0, n_0 + 1, \ldots, m$, and let us see that (3.4) remains valid for $n = m + 1$. By Lemma 3.1 we have

$$\hat{h}^{(k)}(s) \sim \begin{cases} O(1) & (1 \leq k < n_0), \\ -l^{-2}h^{(k)}(s) & (n_0 \leq k \leq m). \end{cases}$$

So in any case,

$$\hat{h}^{(k)}(s) = O(h^{(k)}(s)), \quad k = 1, 2, \ldots, m.$$

Therefore, for any $k = 1, 2, \ldots, m$,

$$h^{(m+1-k)}\hat{h}^{(k)} = O(h^{(m+1-k)}h^{(k)}),$$

and, hence by Lemma 3.2, we see

$$h^{(m+1-k)}\hat{h}^{(k)} = o(h^{(m+1)}), \quad k = 1, \ldots, m,$$

or, changing the variable k, we have

(3.10) $$h^{(k)}\hat{h}^{(m+1-k)} = o(h^{(m+1)}), \quad k = 1, \ldots, m.$$

Now as in (3.7), we have

$$\hat{h}^{(m+1)}(s) = -\frac{1}{h(s)}\left\{h^{(m+1)}(s)\hat{h}(s) + \sum_{k=1}^m {}_nC_k h^{(k)}(s)\hat{h}^{(m+1-k)}(s)\right\}.$$

So, applying (3.10) to the right-hand side we deduce

$$\hat{h}^{(m+1)}(s) = -\frac{1}{h(s)}h^{(m+1)}(s)\hat{h}(s) + o(h^{(m+1)}(s)).$$

Thus we have

$$\hat{h}^{(m+1)}(s) \sim -l^{-2}h^{(m+1)}(s),$$

completing the induction. □

Proposition 3.2. *Let* $\varphi \in R_\alpha(0)$ $(0 < \alpha < 1)$. *Then* $\sigma(\lambda) \sim \varphi(\lambda)$ *if and only if*

$$\sigma^{\#}(\lambda) \sim \frac{\alpha}{2-\alpha}\frac{1}{\{\Gamma(1+\alpha)\Gamma(1-\alpha)\}^2} \cdot \frac{\lambda^2}{\varphi(\lambda)} \quad (\lambda \to +0).$$

As an extreme case, *if* $\sigma \in R_0(0)$, *then*

$$\sigma^{\#}(\lambda) = o\left(\frac{\lambda^2}{\varphi(\lambda)}\right) \quad (\lambda \to +0).$$

Proof. The assertion follows from Tauberian theorem (Theorem 7.1) as follows. For the definition of $C_{0,\alpha}$ see (7.1).

$$\sigma(\lambda) \sim \varphi(\lambda) \overset{\text{iff}}{\Longleftrightarrow} h(s) \sim C_{0,\alpha}\frac{\varphi(s)}{s} \overset{\text{iff}}{\Longleftrightarrow} h^*(s) = \frac{1}{sh(s)} \sim \frac{1}{C_{0,\alpha}\varphi(s)}$$

$$\overset{\text{iff}}{\Longleftrightarrow} \sigma^*(\lambda) \sim \frac{1}{C_{0,\alpha}C_{0,1-\alpha}}\frac{\lambda}{\varphi(\lambda)} \quad (\in R_{1-\alpha}(0)),$$

and the last one is also equivalent to

$$\sigma^{\#}(\lambda) \sim \frac{1}{C_{0,\alpha}C_{0,1-\alpha}}\frac{1-\alpha}{2-\alpha}\frac{\lambda^2}{\varphi(\lambda)}$$

by Lemma 7.1 (apply with $\beta = 1 - \alpha$).

When $\alpha = 0$, the above argument does not hold because $C_{0,1-\alpha}$ does not make sense. So let us prove directly. If $\sigma \sim \varphi \in R_0(0)$, then

$$h(s) \sim \frac{\sigma(s)}{s}, \quad -h'(s) \sim \frac{\sigma(s)}{s^2}, \quad h''(s) \sim 2\frac{\sigma(s)}{s^3}.$$

Therefore,

$$\frac{h''(s)}{h(s)^2} \sim \frac{2}{s\sigma(s)}, \quad \frac{h'(s)^2}{h(s)^3} \sim \frac{1}{s\sigma(s)}$$

and so

$$\hat{h}''(s) = -\frac{h''(s)}{h(s)^2} + 2\frac{h'(s)^2}{h(s)^3} = o\left(\frac{1}{s\sigma(s)}\right) = o\left(\frac{1}{s\varphi(s)}\right).$$

Thus, recalling (3.1), we have

$$\int_0^\infty \frac{d\sigma^{\#}(\xi)}{(s+\xi)^3} = o\left(\frac{1}{s\varphi(s)}\right)$$

and hence $\sigma^{\#}(\xi) = o(\xi^2/\varphi(\xi))$ (see Theorem 7.1). $\qquad\square$

Proposition 3.2 does not include the extreme case $\alpha = 1$ (the right-hand side diverges). So, in this case we need a slight modification as follows in order to know the exact order of $\sigma^{\#}(\lambda)$:

Proposition 3.3. *Let* $\sigma \in R_1(0)$. *Then*

$$\hat{L}(s) := \int_s^\infty \frac{\sigma(u)}{u^2}\,du, \quad s > 0$$

varies slowly as $s \to +0$ *and*

(3.11) $h(s) \sim \hat{L}(s) \quad (s \to +0).$

Furthermore, it holds

$$\sigma^{\#}(\lambda) \sim \frac{\sigma(\lambda)}{\hat{L}(\lambda)^2} \quad (\lambda \to +0).$$

Proof.   Since

(3.12) $\displaystyle\int_{+0}^\infty \frac{\sigma(u)}{u^2}\,du = \int_{+0}^\infty \frac{d\sigma(u)}{u} \quad (= l),$

we see $\hat{L}(+0) = l$. Therefore, if $l < \infty$ then the slowly varying property of $\hat{L}$ and (3.11) are clear. So let us consider the case where $\hat{L}(+0)\,(= l) = \infty$. Note first that

(3.13) $\displaystyle\lim_{s\to+0} \frac{c\hat{L}'(cs)}{\hat{L}'(s)} = \lim_{s\to+0} \frac{c\sigma(cs)}{(cs)^2}\bigg/\frac{\sigma(s)}{s^2} = \lim_{s\to+0} \frac{\sigma(cs)}{c\sigma(s)} = 1.$

The last equality holds by the assumption $\sigma \in R_1(0)$. Combining (3.13) with the condition $\hat{L}(+0) = \infty$, we deduce

$$\lim_{s\to+0} \frac{\hat{L}(cs)}{\hat{L}(s)} = \lim_{s\to+0} \frac{(\hat{L}(cs))'}{(\hat{L}(s))'} = 1.$$

Thus $\hat{L}$ varies slowly. Next note that, by Tauberian theorem (Corollary 7.1), $\sigma \in R_1(0)$ implies

$$(3.14) \qquad\qquad -h'(s) \sim \frac{\sigma(s)}{s^2} \quad (= -\hat{L}'(s)).$$

Since $l = \infty$, (3.14) implies (3.11).

Next, combining (3.14) and (3.11) we deduce

$$\hat{h}'(s) = (1/h(s))' = -h'(s)/h(s)^2 \sim \frac{\sigma(s)}{s^2} \bigg/ \hat{L}(s)^2$$

namely, by (3.1),

$$(3.15) \qquad\qquad \int_0^\infty \frac{d\sigma^\#(\lambda)}{(s+\lambda)^2} \sim \frac{\sigma(s)}{s^2 \hat{L}(s)^2} \in R_{-1}(0),$$

which proves, by Tauberian theorem (Theorem 7.1),

$$\sigma^\#(\lambda) \sim \frac{1}{C_{1,1}} \frac{\sigma(\lambda)}{\hat{L}(\lambda)^2} = \frac{\sigma(\lambda)}{\hat{L}(\lambda)^2} \quad (\lambda \to +0). \qquad\qquad \square$$

EXAMPLE 3.1.   If $\sigma(\lambda) \sim \lambda$, then by Proposition 3.3 we have

$$\sigma^\#(\lambda) \sim \frac{\sigma(\lambda)}{\hat{L}(\lambda)^2} \sim \frac{\lambda}{(\log(1/\lambda))^2} \quad (\lambda \to +0).$$

For,

$$\hat{L}(s) = \int_s^\infty \frac{\sigma(u)}{u^2}\, du \sim \log\frac{1}{s} \quad (s \to +0).$$

In Propositions 3.2 and 3.3 we studied the case where $\sigma^\# \in R_\beta(0)$ with $1 < \beta < 2$ and $\beta = 1$, respectively. The following proposition is concerned with the case $\beta > 2$.

Proposition 3.4.   *Let $\varphi \in R_\alpha(0)$ ($\alpha > 0$) and $A \geq 0$. If $\sigma(+0) = \sigma_0 > 0$ then, as $\lambda \to +0$,*

$$\sigma(\lambda) - \sigma(+0) \sim A\varphi(\lambda) \overset{iff}{\Longleftrightarrow} \int_0^\lambda \xi\, d\sigma(\xi) \sim A\frac{\alpha}{\alpha+1}\lambda\varphi(\lambda)$$

$$\overset{iff}{\Longleftrightarrow} \sigma^*(\lambda) \sim \frac{A}{\sigma_0^2}\frac{\alpha}{\alpha+1}\lambda\varphi(\lambda)$$

$$\overset{iff}{\Longleftrightarrow} \sigma^\#(\lambda) \sim \frac{A}{\sigma_0^2}\frac{\alpha}{\alpha+2}\lambda^2\varphi(\lambda).$$

For the proof of Proposition 3.4, we prepare

Lemma 3.3. *Suppose $\sigma(+0) = \sigma_0 > 0$ and $n \geq 1$. Then,*

$$(3.16) \qquad (h^*(s))^{(n)} \sim -\frac{1}{\sigma_0^2}(sh(s))^{(n)} \quad (s \to +0)$$

provided that at least one of the two sides diverges to infinity as $s \to +0$.

Proof. Since $(h^*)^* = h$, the assertion can be reduced to Proposition 3.1 as follows. Clearly it holds $sh(s) \to \sigma_0$. Therefore, $h^*(s)$ $(= 1/(sh(s))) \to l^* = 1/\sigma_0 < \infty$ and we can apply Proposition 3.1 to h^* in place of h and (3.4) can be written as

$$(h^*)^{(n)}(s) \sim -l^{*2}\hat{h}^{*(n)}(s) = -\frac{1}{\sigma_0^2}(sh(s))^{(n)} \quad (s \to +0),$$

which proves (3.16). Here we used $\hat{h}^*(s) = 1/h^*(s) = sh(s)$. $\square$

Proof of Proposition 3.4. For the proofs of the first and the last relationship see Lemma 7.1. So we shall prove the second one. Since

$$(sh(s))' = h(s) + sh'(s) = a + \int_{[0,\infty)} \frac{d\sigma(\lambda)}{s + \lambda} - \int_{[0,\infty)} \frac{s\,d\sigma(\lambda)}{(s + \lambda)^2}$$
$$= a + \int_{[0,\infty)} \frac{\lambda\,d\sigma(\lambda)}{(s + \lambda)^2},$$

it holds

$$(sh(s))^{(n)} = (-1)^{n+1}n! \int_0^\infty \frac{\lambda\,d\sigma(\lambda)}{(s + \lambda)^{n+1}}, \quad n \geq 2.$$

On the other hand we have

$$(h^*(s))^{(n)} = (-1)^n n! \int_0^\infty \frac{d\sigma^*(\lambda)}{(s + \lambda)^{n+1}}, \quad n \geq 0.$$

These two combined with (3.16) imply

$$\int_0^\infty \frac{d\sigma^*(\lambda)}{(s + \lambda)^{n+1}} \sim \frac{1}{\sigma_0^2} \int_0^\infty \frac{\lambda\,d\sigma(\lambda)}{(s + \lambda)^{n+1}}$$

for $n \geq 2$. Now appeal to the Tauberian theorem to deduce the assertion. $\square$

4. Proofs of Theorems 2.1–2.4

Proof of Theorem 2.1.   Let

$$(4.1) \qquad\qquad h_{pq}(s) = p^2 h_1(s) + q^2 h_2(s),$$

so that

$$h_{pq}(s) = \int_{[0,\infty)} \frac{d\sigma_{pq}(\xi)}{s+\xi}.$$

By the Tauberian theorem (Corollary 7.2), it suffices to show that

$$(4.2) \qquad\qquad h^{(n)}(s) \sim h_{pq}^{(n)}(s) \quad (s \to +0)$$

for some $n > \alpha - 1$. To begin with let us see that $\#(h_{pq}) = \#(h)$ and (4.2) holds for $n \ge n_0 := \#(h)$.

Since

$$(4.3) \qquad\qquad \hat{h}^{(k)}(s) = \hat{h}_1^{(k)}(s) + \hat{h}_2^{(k)}(s),$$

we see that $\hat{h}^{(k)}(+0) = \infty$ holds if and only if $\hat{h}_1^{(k)}(+0) = \infty$ or $\hat{h}_2^{(k)}(s) = \infty$. So by Lemma 3.1, $\#(h) = n_0$ if and only if $\min\{\#(h_1),\#(h_2)\} = n_0$. Similarly, by the definition of $\#(h)$, we see that $\#(h_{pq}) = \min\{\#(h_1), \#(h_2)\} = n_0$.

For the proof of (4.2) first consider the case where $\#(\hat{h}_1) = \#(\hat{h}_2) = n_0$. By Proposition 3.1, for $n \ge n_0$, it holds

$$h^{(n)}(s) \sim l^2 \hat{h}^{(n)}(s) = l^2(\hat{h}_1^{(n)}(s) + \hat{h}_2^{(n)}(s)) \sim l^2\left(\frac{1}{l_1^2}h_1^{(n)}(s) + \frac{1}{l_2^2}h_2^{(n)}(s)\right) = h_{pq}^{(n)}(s)$$

and hence (4.2) is proved for all $n \ge n_0$.

When $\#(\hat{h}_1) \ne \#(\hat{h}_2)$, we need a slight modification. Consider the case where $\#(\hat{h}_1) = n_0$ and $\#(\hat{h}_2) > n_0$. In this case, for n such that $n_0 \le n < \#(\hat{h}_2)$, the argument above does not hold, but $h_2^{(n)}(s)$ and $\hat{h}_2^{(n)}(s)$ are bounded and hence negligible when compared to $h_1^{(n)}(s)$ and $\hat{h}_1^{(n)}(s)$. Therefore we have the same conclusion.   □

Proof of Theorem 2.2.   There are two cases.

CASE 1 $(0 \le \alpha < 1)$ By Tauberian theorem (Theorem 7.1) we have $h_i(s) \sim C_{0,\alpha}c_i\varphi(s)/s$. Therefore,

$$h(s) = \frac{h_1(s)h_2(s)}{h_1(s) + h_2(s)} \sim \frac{c_1 c_2}{c_1 + c_2}C_{0,\alpha}\frac{\varphi(s)}{s},$$

which proves the assertion by Theorem 7.1. When $c_2 = \infty$, it means $\sigma_1 = o(\sigma_2)$ so that $h_1 = o(h_2)$ and hence $h/h_1 = h_2/(1 + (h_1/h_2)) \to 1$. Thus we have $h \sim h_1$, which proves $\sigma \sim \sigma_1 \sim c_1\varphi$.

Case 2 ($\alpha = 1$)   In this case the above argument is insufficient because $h \sim h_1$ does not necessarily imply $\sigma \sim \sigma_1$. However, $h' \sim h_1'$ implies $\sigma \sim \sigma_1$. So let us prove $h' \sim h_1'$. Observe that $\sigma_1(\lambda) \sim (c_1/c_2)\sigma_2(\lambda)$ implies $h_1'(s) \sim (c_1/c_2)h_2'(s)$, which also implies $h_1(s) \sim (c_1/c_2)h_2(s)$ (by de l'Hospital) and hence $h(s) \sim \{c_1/(c_1 + c_2)\}h_2(s)$ and $h(s) \sim \{c_2/(c_1 + c_2)\}h_1(s)$. Since (1.1) implies

$$\frac{h'(s)}{h(s)^2} = \frac{h_1'(s)}{h_1(s)^2} + \frac{h_2'(s)}{h_2(s)^2},$$

we see

$$\frac{h'}{h_2'} = \frac{h^2}{h_1^2} \cdot \frac{h_1'}{h_2'} + \frac{h^2}{h_2^2} \to \left(\frac{c_2}{c_1 + c_2}\right)^2 \frac{c_1}{c_2} + \left(\frac{c_1}{c_1 + c_2}\right)^2 = \frac{c_1}{c_1 + c_2}.$$

Thus we have $h'/h_2' \to c_1/(c_1 + c_2)$ and by Tauberian theorem as before we can deduce $\sigma(\lambda) \sim \{c_1/(c_1 + c_2)\}\sigma_2(\lambda)$. In the extreme case $c_1 = \infty$, it holds that $h_2 = o(h_1)$ and $h_2' = o(h_1')$. The rest of the proof is the same.   □

Proof of Theorem 2.3.   Since $l_1 = \infty$ and $l_2 < \infty$, it holds $l = l_2 < \infty$ and hence $\#(h), \#(h_2) \geq 1$. Therefore, we have from Lemma 3.1 that $h^{(n)}(s) \sim l^2 \hat{h}^{(n)}(s)$ and $h_2^{(n)}(s) \sim l^2 \hat{h}_2^{(n)}(s)$ for all sufficiently large n. So

$$h^{(n)}(s) \sim l^2 \hat{h}^{(n)}(s) = l^2 \hat{h}_1^{(n)}(s) + l^2 \hat{h}_2^{(n)}(s) \sim l^2 \hat{h}_1^{(n)}(s) + h_2^{(n)}(s).$$

This implies, by (3.1),

$$\int_0^\infty \frac{d\sigma(\lambda)}{(s + \lambda)^{n+1}} \sim l^2 \int_0^\infty \frac{d\sigma_1^\#(\lambda)}{(s + \lambda)^{n+1}} + \int_0^\infty \frac{d\sigma_2(\lambda)}{(s + \lambda)^{n+1}},$$

which proves $\sigma(\lambda) \sim l^2 \sigma_1^\#(\lambda) + \sigma_2(\lambda)$.   □

Proof of Theorem 2.4.   Since

$$\sigma(+0) = \lim_{s \to +0} sh(s), \quad \sigma_i(+0) = \lim_{s \to +0} sh_i(s),$$

we have from (1.1) that

$$\frac{1}{\sigma(+0)} = \frac{1}{\sigma_1(+0)} + \frac{1}{\sigma_2(+0)},$$

which implies (2.9).

By Proposition 3.4 we see that (2.10) is equivalent to

$$\sigma_i^*(\lambda) \sim \frac{1}{\sigma_i(+0)^2} \frac{\alpha}{\alpha + 1} c_i \lambda \varphi(\lambda).$$

Since (1.1) implies $h^*(s) = h_1^*(s) + h_2^*(s)$, it holds $\sigma^*(\lambda) = \sigma_1^*(\lambda) + \sigma_2^*(\lambda)$, and therefore

$$\sigma^*(\lambda) = \sigma_1^*(\lambda) + \sigma_2^*(\lambda) \sim \left(\frac{c_1}{\sigma_1(+0)^2} + \frac{c_2}{\sigma_2(+0)^2} \right) \frac{\alpha}{\alpha + 1} \lambda \varphi(\lambda).$$

Appealing to Proposition 3.4 again, this is equivalent to

$$\sigma(\lambda) - \sigma(+0) \sim \sigma(+0)^2 \left(\frac{c_1}{\sigma_1(+0)^2} + \frac{c_2}{\sigma_2(+0)^2} \right) \varphi(\lambda) \quad (\lambda \to +0). \qquad \square$$

5. An application to positive recurrent linear diffusions

In this section we generalize a result of [5], where the transition density of positive recurrent diffusions is discussed.

Let $X = (X_t)_{t \geq 0}$ be a diffusion on $I = [0, \infty)$ with local generator

$$(5.1) \qquad \mathcal{L} = \frac{1}{2} \left(\frac{d^2}{dx^2} + b(x) \frac{d}{dx} \right), \quad x > 0,$$

$b(x)$ being assumed to be an element of $L^1_{\mathrm{loc}}([0, \infty), dx)$. We put reflecting boundary condition at the left boundary.

Define

$$W(x) = \exp \left(\int_0^x b(u) \, du \right), \quad x \geq 0.$$

Then Feller's canonical form of (5.1) is

$$\mathcal{L} = \frac{d}{dm(x)} \frac{d}{ds(x)},$$

where

$$m(x) = 2 \int_0^x W(u) \, du, \quad s(x) = \int_0^x \frac{du}{W(u)}.$$

Note that the scale-changed process $Y_t := s(X_t)$ corresponds to

$$\mathcal{L} = \frac{d}{d\tilde{m}(s)} \frac{d}{ds}, \quad \text{where} \quad \tilde{m}(\cdot) := m(s^{-1}(\cdot)).$$

It is well known that the transition density $p(t, x, y)$ with respect to $dm(x)$ exists and $p(t, 0, 0)$ has the following spectral representation:

$$(5.2) \qquad p(t, 0, 0) = \int_{[0, \infty)} e^{-\lambda t} \, d\sigma(\lambda), \quad t > 0.$$

The spectral function σ can be characterized by the following formula: If we define

$$G_s(x, y) := \int_0^\infty e^{-st} p(t, x, y)\, dt, \quad s > 0,$$

then (5.2) implies

$$h(s) := G_s(0, 0) = \int_{[0,\infty)} \frac{d\sigma(\xi)}{s + \xi}, \quad s > 0.$$

How to calculate $G_s(x, y)$ (and hence $h(s)$) from $\mathcal{L}$ will be explained in Section 6. We remark that it is known that $l \ (:= h(+0)) = s(+\infty)$. (This fact will easily be seen from (6.2).) Therefore, it holds that

$$(5.3) \qquad\qquad l = \int_0^\infty \frac{du}{W(u)}.$$

The authors recently obtained the following result: We denote by $R_\alpha(\infty)$ the totality of functions varying regularly at ∞ with index α.

Theorem 1 ([6, Theorem 4.2]). *Let $\rho > 0$. If*

$$(5.4) \qquad\qquad W(\,\cdot\,) \in R_{\rho-1}(\infty)$$

then, as $t \to \infty$,

$$(5.5) \qquad\qquad p(t, 0, 0) \sim \frac{1}{2^{\rho/2}\Gamma(\rho/2)} \frac{1}{\sqrt{t}\, W(\sqrt{t})} \in R_{-\rho/2}(\infty).$$

If we recall the canonical representation of slowly varying functions (see e.g., [1, p. 12]), we easily see that a sufficient condition for (5.4) is

$$(5.6) \qquad\qquad xb(x) \to \rho - 1 \quad (x \to \infty),$$

or, equivalently,

$$b(x) = \frac{\rho - 1}{x} + o\!\left(\frac{1}{x}\right) \quad (x \to \infty).$$

We remark that, by (5.2) and Tauberian theorem for Laplace transforms (see e.g. [1, p. 37]), (5.5) is equivalent to

$$(5.7) \qquad\qquad \sigma(\lambda) \sim \frac{1}{2^{(\rho/2)-1}\rho\Gamma(\rho/2)^2} \frac{\sqrt{\lambda}}{W(1/\sqrt{\lambda})} \quad (\lambda \to +0).$$

Now the aim of the present section is to study the case where (5.4) holds for $\rho < 0$. In this case we see

$$\hat{m} := \int_0^\infty dm(x) = 2 \int_0^\infty W(x)\,dx < \infty,$$

which is, probabilistically, equivalent to that the process is positively recurrent.

Since, as is well known,

(5.8)
$$\sigma(+0) = \frac{1}{\hat{m}},$$

we see that $\hat{m} < \infty$ implies $\sigma(+0) > 0$ and therefore,

$$p(t, 0, 0) \to \frac{1}{\hat{m}} \quad (t \to \infty)$$

(cf. [2, pp. 35–37]). So let us evaluate $p(t, 0, 0) - 1/\hat{m}$ as $t \to \infty$. Since (5.8) implies

$$p(t, 0, 0) - \frac{1}{\hat{m}} = \int_{(0,\infty)} e^{-\lambda t}\,d\sigma(\lambda),$$

our problem will be reduced to the study of

$$\sigma(\lambda) - \sigma(+0) \quad (\lambda \to +0).$$

To this end let us consider the *dual process* of (5.1):

$$\mathcal{L}^\bullet := \frac{1}{2}\left(\frac{d^2}{dx^2} - b(x)\frac{d}{dx}\right), \quad x > 0.$$

Note the following argument remains valid under the condition $\rho < 2$ rather than $\rho < 0$. The functions W, s, m, $\tilde{m}$, h, σ corresponding to $\mathcal{L}^\bullet$ will be denoted by $W^\bullet$, $s^\bullet$, $m^\bullet$, $\tilde{m}^\bullet$, $h^\bullet$ and $\sigma^\bullet$, respectively. Since they correspond to $-b$ in place of b, we have

$$W^\bullet(x) = \exp\left(-\int_0^x b(u)\,du\right)$$

so that $W^\bullet = 1/W$, and hence, $W^\bullet \in R_{-(\rho-1)}(\infty) = R_{\rho^\bullet-1}(\infty)$ where $\rho^\bullet := 2 - \rho$. If $\rho < 2$, it holds $\rho^\bullet > 0$. So we can apply Theorem 1 to $-b(x)$ to deduce

(5.9)
$$\sigma^\bullet(\lambda) \sim C_\rho^\bullet \sqrt{\lambda}\, W(1/\sqrt{\lambda}) \in R_{\rho^\bullet/2}(0) \quad (\lambda \to +0),$$

where

$$C_\rho^\bullet = \frac{1}{2^{\rho^\bullet/2-1}\rho^\bullet\Gamma(\rho^\bullet/2)^2} = \frac{2^{\rho/2}}{(2-\rho)\Gamma((2-\rho)/2)^2}.$$

We next consider the relationship between $\sigma^\bullet$ and σ^* defined in (2.3). To this end we recall *Krein's correspondence* (see e.g. [8]): The correspondence between h and $\tilde{m}$ is one-to-one and $\tilde{m}^*(x) := \tilde{m}^{-1}(x)$ corresponds to $h^*(s) = 1/(sh(s))$. Furthermore, $c\tilde{m}(cx)$ corresponds to $(1/c)h(s)$ (and, hence, $c\tilde{m}^*(cx)$ to $(1/c)h^*(s)$). Since $s^\bullet(x) = (1/2)m(x)$ and $m^\bullet(x) = 2s(x)$, we have $\tilde{m}^\bullet(x) = 2\tilde{m}^{-1}(2x) = 2\tilde{m}^*(2x)$. This proves that $h^\bullet(s) = (1/2)h^*(s)$, which implies

$$\sigma^\bullet(\lambda) = \frac{1}{2}\sigma^*(\lambda).$$

So by (5.9) we have

Proposition 5.1. *Suppose that* (5.6) *holds with* $\rho < 2$. *Then,*

$$(5.10) \qquad \sigma^*(\lambda) \sim C_\rho^* \sqrt{\lambda} W(1/\sqrt{\lambda}) \in R_{\rho^\bullet/2}(0) \quad (\lambda \to +0),$$

where

$$C_\rho^* = 2C_\rho^\bullet = \frac{2^{(\rho/2)+1}}{(2-\rho)\Gamma((2-\rho)/2)^2}.$$

REMARK 5.1. When $0 < \rho < 2$ we can confirm (5.10) directly as follows: Since $\sigma \in R_{\rho/2}(0)$, we have $h \in R_{(\rho/2)-1}(0)$, $h^*(s) = 1/(sh(s)) \in R_{(\rho/2)-1}(0)$ and $\sigma^* \in R_{1-(\rho/2)}(0)$. Therefore,

$$\sigma^*(\lambda) \sim \frac{\lambda}{C_{0,1-(\rho/2)}}h^*(\lambda) \sim \frac{1}{C_{0,1-(\rho/2)}}\frac{1}{h(\lambda)} \sim \frac{1}{C_{0,1-(\rho/2)}C_{0,\rho/2}}\frac{\lambda}{\sigma(\lambda)}.$$

Combining this with (5.7) we obtain

$$\sigma(\lambda) \sim \frac{2^{\rho/2}\Gamma(\rho/2)\Gamma((\rho/2)+1)}{C_{0,1-(\rho/2)}C_{0,\rho/2}}\sqrt{\lambda}W(1/\sqrt{\lambda}) = C_\rho^* \sqrt{\lambda}W(1/\sqrt{\lambda}).$$

Now let us return to the case $\rho < 0$ instead of $\rho < 2$. In this case $\alpha := (\rho^\bullet/2) - 1 = -\rho/2 > 0$ and we can apply Proposition 3.4 to (5.10) to obtain

$$\sigma(\lambda) - \sigma(+0) \sim \sigma(+0)^2\frac{\alpha+1}{\alpha}\frac{\sigma^*(\lambda)}{\lambda} \in R_\alpha(0) = R_{-\rho/2}(0).$$

Thus we have the following result, which extends Example 3.8 of [5], where only the case $-2 < \rho < 0$ is discussed.

Theorem 5.1. *Suppose that* (5.6) *holds with* $\rho < 0$. *Then,*

$$(5.11) \qquad \sigma(\lambda) - \frac{1}{\hat{m}} \sim \frac{1}{\hat{m}^2}D_\rho\frac{1}{\sqrt{\lambda}}W\left(\frac{1}{\sqrt{\lambda}}\right) \in R_{-\rho/2}(0) \quad (\lambda \to +0),$$

where

$$D_\rho = \frac{2^{(\rho/2)+1}}{|\rho|\Gamma((2-\rho)/2)^2}.$$

Notice that, by the reason we explained before, (5.11) is equivalent to

$$p(t, 0, 0) - \frac{1}{\hat{m}} \sim \frac{1}{\hat{m}^2} D_\rho \Gamma\left(1 - \frac{\rho}{2}\right) \sqrt{t} W(\sqrt{t}) \in R_{\rho/2}(\infty) \quad (t \to \infty).$$

6. An application to bilateral diffusions

The aim of this section is study how our results in Section 2 work when we wish to apply Theorem A in the previous section to 'bilateral' diffusions. Here, 'bilateral' means that neither boundary of the state space is regular.

To begin with let us quickly review necessary facts on linear diffusions. Let $X = (X_t)_{t \geq 0}$ be a regular, conservative diffusion on an interval in $\mathcal{R}$. For simplicity, we change the scale if necessary so that the local generator is of the form

$$\mathcal{L} = \frac{d}{d\tilde{m}(x)} \frac{d}{dx}, \quad -l_- < x < l_+$$

where $0 < l_-, l_+ \leq \infty$ and $\tilde{m}(x)$ is a nondecreasing right-continuous function defined on $I = (-l_-, l_+)$. (We need not to assume that $\tilde{m}$ is strictly increasing so that generalized diffusions such as birth–death processes are included.)

It is well known that the transition density $p(t, x, y)$ with respect to $d\tilde{m}(x)$ can be computed as follows (see e.g. [4]): For each $\lambda \in \mathbb{C}$, we can define $\varphi_\lambda(x)$ and $\psi_\lambda(x)$ as the unique solutions of

$$-\mathcal{L}u = \lambda u, \quad x \in I$$

with the initial conditions that $(u(0), u'(-0)) = (1, 0)$ and $(u(0), u'(-0)) = (0, 1)$, respectively; or, precisely, the solutions of the following integral equations:

$$(6.1) \quad \begin{cases} \varphi_\lambda(x) = 1 - \lambda \displaystyle\int_{-0}^{x} (x - y)\varphi_\lambda(y) \, d\tilde{m}(y), \\[2mm] \psi_\lambda(x) = x - \lambda \displaystyle\int_{-0}^{x} (x - y)\psi_\lambda(y) \, d\tilde{m}(y) \end{cases}$$

with the convention that $\int_{-0}^{x} = -\int_{x}^{-0}$ if $x < 0$. Then,

$$(6.2) \quad h_+(s) := \lim_{x \uparrow l_+} \frac{\psi_{-s}(x)}{\varphi_{-s}(x)} \quad \left(= \int_0^{l_+ -0} \frac{dx}{\varphi_{-s}(x)^2}\right), \quad s > 0$$

and

$$h_-(s) := -\lim_{x \downarrow -l_-} \frac{\psi_{-s}(x)}{\varphi_{-s}(x)} \quad \left(= \int_0^{l_- -0} \frac{dx}{\varphi_{-s}(-x)^2}\right), \quad s > 0$$

are called the *characteristic functions* of $\tilde{m}$. It holds that $h_+, h_- \in \mathcal{H}$, and hence we have the following representation:

$$(6.3) \qquad h_\pm(s) = a_\pm + \int_{[0,\infty)} \frac{d\sigma_\pm(\xi)}{\xi + s}, \quad s > 0.$$

For every $s > 0$, we put

$$u_1(s; x) = \varphi_{-s}(x) - \frac{1}{h_+(s)} \psi_{-s}(x);$$

$$u_2(s; x) = \varphi_{-s}(x) + \frac{1}{h_-(s)} \psi_{-s}(x).$$

These two are nonnegative solutions of

$$\begin{cases} \mathcal{L}u(x) = su(x), & x \in I, \\ u(0) = 1 \end{cases}$$

such that u_1 is nonincreasing and u_2 nondecreasing. The Wronskian is

$$(6.4) \qquad W[u_1(s; \cdot), u_2(s; \cdot)] \quad (:= u_1 u_2' - u_1' u_2) \quad = \frac{1}{h_-(s)} + \frac{1}{h_+(s)}.$$

So the Green function is given by

$$(6.5) \qquad G_s(x, y) = \begin{cases} h(s) u_2(s; x) u_1(s; y) & (x \le y), \\ h(s) u_1(s; x) u_2(s; y) & (x > y), \end{cases}$$

where $h(s) = 1/W[u_1(s; \cdot), u_2(s; \cdot)]$; namely, by (6.4),

$$(6.6) \qquad \frac{1}{h(s)} = \frac{1}{h_+(s)} + \frac{1}{h_-(s)}.$$

Note that it holds

$$(6.7) \qquad G_s(0, 0) = h(s),$$

which follows immediately from (6.5) because $u_i(s; 0) = 1$.

The transition density $p(t, x, y)$ (with respect to $d\tilde{m}(x)$) can be obtained from $G_s(x, y)$ via the following formula:

$$\int_0^\infty e^{-st} p(t, x, y) \, dt = G_s(x, y) \quad (s > 0) \quad d\tilde{m}(x) \, d\tilde{m}(y)\text{-a.e.}$$

Especially, by (6.7) we have

$$(6.8) \qquad \int_0^\infty e^{-st} p(t, 0, 0) \, dt = h(s) \quad (s > 0)$$

provided that $0 \in \text{Supp}\{d\tilde{m}(x)\}$. Notice that, if σ is the spectral function of $h(s)$, then (6.8) implies

(6.9) $$p(t, 0, 0) = \int_{-0}^{\infty} e^{-s\lambda}\, d\sigma(\lambda), \quad s > 0.$$

In this way the asymptotic behavior of the transition density $p(t, 0, 0)$ will be reduced to those of two diffusions; one is on $(-l_-, 0]$ and the other on $[0, l_+)$.

Let $X = (X_t)_{t\geq 0}$ be a diffusion with generator (5.1) on the whole line $\mathcal{R}$ with $b(\,\cdot\,) \in L^1_{loc}(\mathcal{R})$. Notice that the results in the above are applicable first to the suitably scaled process $Y_t = s(X_t)$ and hence to X. So, for example, there exists the transition density $p(t, x, y)$ with respect to $dm(x) := 2\exp\!\left(\int_0^x b(u)\,du\right) dx$ and (6.9) remains valid if we choose the scale function $s(x)$ so that $s(0) = 0$.

Define

$$\begin{cases} W_+(x) = \exp \displaystyle\int_0^x b(u)\, du, & x \geq 0, \\[2mm] W_-(x) = \exp -\displaystyle\int_0^x b(-u)\, du, & x \geq 0 \end{cases}$$

as in Section 5. The reason why $-b(-x)$ appears in the definition of $W_-(x)$ is simply because the diffusion $(-X_t)_{t\geq 0}$ corresponds to

$$\mathcal{L} = \frac{1}{2}\left(\frac{d^2}{dx^2} - b(-x)\frac{d}{dx} \right)$$

in place of (5.1).

By (6.2) we easily see that $l_\pm$ $(:= h_\pm(+0)) = s_\pm(+\infty)$, where $s_\pm(x)$ is defined in a similar way as in the previous section, and hence it holds that

$$l_\pm = \int_0^{\infty} \frac{dx}{W_\pm(x)} \quad (\leq \infty).$$

So $l := h(+0)$ is obtained by

$$l = \frac{l_+ l_-}{l_+ + l_-},$$

(see (1.2)). Also as in the previous section we have

$$\sigma_\pm(+0) = \frac{1}{\hat{m}_\pm}, \quad \hat{m}_\pm = 2\int_0^{\infty} W_\pm(u)\, du$$

and therefore,

$$\sigma(+0) = \frac{\sigma_+(+0)\sigma_-(+0)}{\sigma_+(+0) + \sigma_-(+0)} = \frac{1}{\hat{m}},$$

(see (2.9)), where

$$\hat{m} = \hat{m}_+ + \hat{m}_- \quad \left(= 2 \int_0^\infty (W_+(u) + W_-(u)) \, du \right).$$

Theorem 6.1 (Balanced case). *Suppose that*

$$\lim_{x \to \infty} xb(x) = \rho_+ - 1 \quad (\rho_+ \neq 0)$$

and

(6.10) $$\lim_{A \to \infty} \int_{-A}^A b(u) \, du = \log \frac{r}{1-r} \quad (0 < r < 1).$$

(i) *If $(0 < \rho_+ < 2)$ or $(\rho_+ = 2; l_+ = \infty)$, then*

$$\sigma(\lambda) \sim r\sigma_+(\lambda) \quad (\lambda \to +0).$$

(ii) *If $(\rho_+ > 2)$ or $(\rho_+ = 2; l_+ < \infty)$, then*

$$\sigma(\lambda) \sim \left(p^2 + q^2 \frac{r}{1-r} \right) \sigma_+(\lambda) \quad (\lambda \to +0),$$

where $p = l/l_+$ and $q = l/l_-$.
(iii) *If $\rho_+ < 0$, then*

$$\sigma(\lambda) - \frac{1}{\hat{m}} \sim \left\{ \left(\frac{\hat{m}_+}{\hat{m}} \right)^2 + \left(\frac{\hat{m}_-}{\hat{m}} \right)^2 \frac{r}{1-r} \right\} \left(\sigma_+(\lambda) - \frac{1}{\hat{m}_+} \right).$$

Proof. (i) It holds $W_+ \in R_{\rho_+ - 1}(\infty)$ and $\sigma_+ \in R_{\rho_+/2}(0)$ as before (see (5.6) and (5.7)). By the balancing condition (6.10), it holds

$$\frac{W_-(x)}{W_+(x)} \to \frac{1-r}{r},$$

which implies, by (5.7),

$$\frac{\sigma_-(\lambda)}{\sigma_+(\lambda)} \to c := \frac{r}{1-r}.$$

Therefore, by Theorem 2.2 we deduce

$$\sigma(\lambda) \sim \frac{c}{c+1} \sigma_+(\lambda) = r\sigma_+(\lambda) \quad (\lambda \to +0).$$

(ii) Similarly, by Theorem 2.1, we see

$$\sigma(\lambda) \sim p^2 \sigma_+(\lambda) + q^2 \sigma_-(\lambda) \sim \left(p^2 + q^2 \frac{r}{1-r} \right) \sigma_+(\lambda).$$

(iii)   The assertion can be shown in a similar way by using Theorems 2.4 and 5.1.
$\square$

Theorem 6.2 (Lopsided case).   *Suppose that*

$$\begin{cases} \lim_{x \to +\infty} xb(x) = \rho_+ - 1, \\ \lim_{x \to -\infty} xb(x) = \rho_- - 1, \end{cases}$$

where $\rho_+ > 0$ and $\rho_- \in \mathcal{R}$. Then;
(i)   *In the following three cases it holds*

(6.11) $$\sigma(\lambda) \sim \sigma_+(\lambda) \in R_{\rho_+/2}(0) \quad (\lambda \to +0).$$

   (1) $\rho_- < \rho_+ < 2$,
   (2) $\rho_- < 2 < \rho_+$, $\rho_- + \rho_+ < 4$,
   (3) $\rho_- < \rho_+ = 2$.
(ii)   *If $2 < \rho_+ < \rho_-$ or $(2 = \rho_+ < \rho_-, \, l_+ < \infty)$, then*

$$\sigma(\lambda) \sim p^2 \sigma_+(\lambda).$$

(iii)   *If $(\rho_- < 2 < \rho_+, \, \rho_- + \rho_+ > 4)$ or $(2 = \rho_- < \rho_+, \, l_- = \infty)$, then*

(6.12) $$\sigma(\lambda) \sim l_+^2 \sigma_-^{\#}(\lambda) \in R_{2-(\rho_-/2)}(0) \quad (\lambda \to +0).$$

Proof.   First we remark that $\sigma_\pm \in R_{(\rho_\pm \vee 0)/2}(0)$.
   (i)   In each of the cases (1)–(3) we can apply Corollary 2.1 (i), Corollary 2.3 (i),
and Corollary 2.4 (i), respectively.
   (ii)   Apply Corollary 2.1 (ii) and Corollary 2.4 (ii), respectively.
   (iii)   Since $\sigma_- \in R_{2-(\rho_-/2)}(0)$ and $\sigma_+ \in R_{\rho_+/2}(0)$, the assertion follows from The-
orem 2.3. $\square$

As we mentioned before the transition density $p(t, x, y)$ with respect to $dm(x) = 2\exp\left(\int_0^x b(u)\, du\right) dx$ exists and especially $p(t, 0, 0)$ satisfies (6.9). So by Karamata–
Tauberian theorem for Laplace transforms the results for the spectral function σ can be
translated into those for $p(t, 0, 0)$. Thus we have

EXAMPLE 6.1.   Let $b(x) = b_0(x) + \eta(x)$ where

(6.13) $$b_0(x) = \begin{cases} \dfrac{\rho_+ - 1}{x} & (x > 1), \\ 0 & (|x| \le 1), \\ \dfrac{\rho_- - 1}{x} & (x < -1) \end{cases}$$

and $\eta(\,\cdot\,) \in L^1(\mathcal{R}, dx)$.

Since $W_\pm(x) \sim \text{const} \cdot x^{\rho_\pm - 1}$ $(x \to \infty)$, we see that $l_\pm < \infty$ if and only if $\rho_\pm > 2$, and also $\hat{m} < \infty$ if and only if $\rho_\pm < 0$.

(1) If $(\rho_- \leq \rho_+ \leq 2; \rho_+ > 0)$ or $2 < \rho_+ \leq \rho_-$, then,

$$p(t, 0, 0) \sim \text{const} \cdot t^{-\rho_+/2} \quad (t \to \infty).$$

(2) If $\rho_- < 2 < \rho_+$, then

$$p(t, 0, 0) \sim \text{const} \cdot \max\{t^{-2+(\rho_-/2)}, t^{-\rho_+/2}\} \quad (t \to \infty).$$

(3) If $2 = \rho_- \leq \rho_+$, then

$$p(t, 0, 0) \sim \text{const} \cdot t^{-1}(\log t)^{-2} \quad (t \to \infty).$$

(4) If $\rho_- \leq \rho_+ < 0$, then

$$p(t, 0, 0) - \frac{1}{\hat{m}} \sim \text{const} \cdot t^{\rho_+/2} \quad (t \to \infty).$$

Since (1) and (2) are immediate from Theorem 6.2 (and Tauberian theorem for Laplace transforms), let us see (3) and (4) only. (3) follows from Theorem 6.2 (iii): Since $\sigma_-(\lambda) \sim \text{const} \cdot \lambda$ and $\sigma_+(\lambda) \sim \text{const} \cdot \lambda^{\rho_+/2}$ (see (5.7)), we see from (6.12) and Example 3.1 that

$$\sigma(\lambda) \sim \text{const} \cdot \sigma_-^\#(\lambda) \sim \text{const} \cdot \frac{\lambda}{(\log(1/\lambda))^2} \quad (\lambda \to +0).$$

Thus we have (3). Similarly, we can deduce (4) by Theorems 6.1 (iii) and Theorem 5.1.

7. Appendix

In this section we briefly sum up some results on Tauberian theorems for Stieltjes transforms.

For a nondecreasing, right-continuous function $\sigma : (-\infty, \infty) \to [0, \infty)$ such that $\sigma(x) = 0$ on $(-\infty, 0)$, we define the *generalized Lebesgue–Stieltjes transform* by

$$H_n(\sigma; s) = \int_{[0,\infty)} \frac{d\sigma(\lambda)}{(s + \lambda)^{n+1}} = \frac{1}{n + 1} \int_{[0,\infty)} \frac{\sigma(\lambda) \, d\lambda}{(s + \lambda)^{n+2}} \quad (n \geq 0)$$

provided that the integral converges. The generalized Lebesgue–Stieltjes transform determines the measure $d\sigma(\lambda)$ uniquely. For an inversion formula see [9, Appendix].

The most important case is the following: Let $0 \leq \alpha < n + 1$. Then

$$\sigma(\lambda) = \lambda^\alpha, \quad \lambda > 0$$

if and only if

$$H_n(\sigma; s) = C_{n,\alpha} s^{\alpha-n-1},$$

where

(7.1) $$C_{n,\alpha} = \int_0^\infty \frac{d\lambda^\alpha}{(1+\lambda)^{n+1}} = \frac{\Gamma(n+1-\alpha)\Gamma(1+\alpha)}{\Gamma(n+1)}.$$

The well-known Karamata's extension of Hardy–Littlewood Tauberian theorem is

Theorem 7.1. *Let $0 \leq \alpha < n+1$, $A \geq 0$, and $\varphi \in R_\alpha(0)$. Then,*

$$\sigma(\lambda) \sim A\varphi(\lambda) \quad (\lambda \to +0)$$

if and only

$$H_n(\sigma; s) \sim AC_{n,\alpha}\, \varphi(s)\, s^{-n-1} \quad (s \to +0).$$

For the proofs we refer to [1, p. 40] and [7, Appendix].

The assertion holds even if $A = 0$ with the convention that $f \sim Ag$ means $f/g \to A$. Also, '$\lambda, s \to +0$' may be replaced by '$\lambda, s \to \infty$'.

Let $h \in \mathcal{H}$. Since $h^{(n)}(s) = (-1)^n n!\, H_n(\sigma; s)$ we have,

Corollary 7.1. *Let $0 \leq \alpha < n+1$. Then $\sigma \in R_\alpha(0)$ if and only if $h^{(n)} \in R_{\alpha-n-1}(0)$, and then,*

$$(-1)^n h^{(n)}(s) \sim \Gamma(n+1-\alpha)\Gamma(1+\alpha)s^{-n-1}\sigma(s), \quad s \to +0.$$

Corollary 7.2. *Let $0 \leq \alpha < n+1$ and $A \geq 0$. If $\sigma_1 \in R_\alpha(0)$ (or, equivalently, if $h_1^{(n)} \in R_{\alpha-n-1}(0)$), then*

$$h_2^{(n)}(s) \sim A\, h_1^{(n)}(s) \ (s \to +0) \overset{\text{iff}}{\Longleftrightarrow} \sigma_2(\lambda) \sim A\, \sigma_1(\lambda) \ (\lambda \to +0).$$

Lemma 7.1. *Let $\varphi \in R_\beta(0)$ ($\beta \geq 0$) and $B \geq 0$.*
(i) *If $\beta > 0$, then*

(7.2) $$\sigma(\lambda) - \sigma(+0) \sim B\varphi(\lambda) \quad (\lambda \to +0)$$

if and only if

$$\int_0^\lambda \xi\, d\sigma(\xi) \sim B\frac{\beta}{\beta+1}\lambda\varphi(\lambda) \quad (\lambda \to +0).$$

(ii) *If $\beta = 0$, then (7.2) implies*

$$\int_0^\lambda \xi\, d\sigma(\xi) = o(\lambda\varphi(\lambda)) \quad (\lambda \to +0).$$

Proof. (i) By Tauberian theorem for Laplace transforms (7.2) holds if and only if

$$\int_{(0,\infty)} e^{-\lambda s} d\sigma(\lambda) \sim B\Gamma(\beta+1)\varphi\left(\frac{1}{s}\right) \quad (s \to \infty).$$

Then, by the monotone density theorem (see e.g. [1, p. 39]) this is equivalent to

$$\int_0^\infty e^{-\lambda s}\lambda\, d\sigma(\lambda) \sim B\Gamma(\beta+1)\beta\frac{1}{s}\varphi\left(\frac{1}{s}\right),$$

which is also equivalent to

$$\int_0^\lambda \xi\, d\sigma(\xi) \sim \frac{B\Gamma(\beta+1)\beta}{\Gamma(\beta+2)}\lambda\varphi(\lambda) = B\frac{\beta}{\beta+1}\lambda\varphi(\lambda).$$

(ii) Without loss of generality, we may assume that $\sigma(+0) = 0$. Since

$$\int_0^\lambda \xi\, d\sigma(\xi) = \lambda\sigma(\lambda) - \int_0^\lambda \sigma(\xi)\, d\xi$$

and the second term of the right-hand side is asymptotically equal to $\lambda\sigma(\lambda)$, we deduce the assertion. $\square$

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