MOMENT-ANGLE MANIFOLDS AND CONNECTED SUMS OF SPHERE PRODUCTS

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Abstract

Corresponding to every finite simplicial complex K, there is a moment-angle complex $\mathcal{Z}_K$; if K is a triangulation of a sphere, $\mathcal{Z}_K$ is a compact manifold. The question of whether $\mathcal{Z}_K$ is a connected sum of sphere products was considered in [3, Section 11]. So far, all known examples of moment-angle manifolds which are homeomorphic to connected sums of sphere products have the property that every product is of exactly two spheres. In this paper, we give a example whose cohomology ring is isomorphic to that of a connected sum of sphere products with one product of three spheres. We also give some general properties of this kind of moment-angle manifolds.

1. Introduction

Throughout this paper, we assume that m is a positive integer and $[m] = \{1, 2, \dots, m\}$. For an abstract simplicial complex K with m vertices labeled by $[m]$ and a sequence $I = (i_1, \dots, i_k) \subseteq [m]$ with $1 \leq i_1 \leq \cdots \leq i_k \leq m$, we denote by K_I the *full subcomplex* of K on I, and $\hat{I} = [m] \setminus I$.

1.1. Moment-angle complex. Given a simple polytope P with m fecets, Davis and Januszkiewicz [7] constructed a manifold $\mathcal{Z}_P$ with an action of a real torus T^m. After that Buchstaber and Panov [4] generalized this definition to any simplicial complex K, that is

$$\mathcal{Z}_K = \bigcup_{\sigma \in K} (D^2)^\sigma \times (S^1)^{[m] \setminus \sigma},$$

and named it the *moment-angle complex* associated to K, whose study connects algebraic geometry, topology, combinatorics, and commutative algebra. This cellular complex is always 2-connected and has dimension $m + n + 1$, where n is the dimension of K.

It turns out that the algebraic topology of a moment-angle complex $\mathcal{Z}_K$, such as the cohomology ring and the homotopy groups is intimately related to the combinatorics of the underlying simplicial complex K.

1.2. Moment-angle manifold. Now suppose that K is an n-dimensional simplicial sphere (a triangulation of a sphere) with m vertices. Then, as shown by Buchstaber and Panov [4], the moment-angle complex $\mathcal{Z}_K$ is a manifold of dimension $n + m + 1$, referred to as a *moment-angle manifold*. In particular, if K is a polytopal sphere (see Definition 1.1), or more generally a starshaped sphere (see Definition 1.2), then $\mathcal{Z}_K$ admits a smooth structure.

DEFINITION 1.1. A *polytopal sphere* is a triangulated sphere isomorphic to the boundary complex of a simplicial polytope.

DEFINITION 1.2. A simplicial sphere K of dimension n is said to be *starshaped* if there is a geometric realization $|K|$ of K in $\mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ with the property that each ray emanating from p meets $|K|$ in exactly one point.

REMARK 1.3. A polytopal sphere is apparently a starshaped sphere, but for $n \geq 3$, there are examples that are starshaped and not polytopal. The easiest such example is given by the *Brückner sphere* (see [9]).

The topology of a moment-angle manifold can be quite complicated. The complexity increases when the dimension n of the associated simplicial sphere K increases. for $n = 0$, $\mathcal{Z}_K$ is S^3. For $n = 1$, K is the boundary of a polygon, and $\mathcal{Z}_K$ is a connected sum of sphere products. In higher dimensions, the situation becomes much more complicated. On the other hand, McGavran [10] showed that, for any $n > 0$, there are infinitely many n-dimensional polytopal spheres whose corresponding moment-angle manifolds are connected sums of sphere products.

Theorem 1.4 (McGavran, see [3, Theorem 6.3])**.** *Let K be a polytopal sphere dual to the simple polytope obtained from the k-simplex by cutting off vertices for l times. Then the corresponding moment-angle manifold is homeomorphic to a connected sum of sphere products*

$$\mathcal{Z}_K \cong \overset{l}{\underset{j=1}{\#}} j \binom{l+1}{j+1} S^{j+2} \times S^{2k+l-j-1}.$$

For $k = 2$ or 3, the above theorem gives all moment-angle manifolds which are homomorphic to connected sums of sphere products (see [3, Proposition 11.6]). Nevertheless, in higher dimension they are not the only ones whose cohomology ring is isomorphic to that of a connected sum of sphere products. Bosio and Meersseman [3,

Section 11] gave many other examples of moment-angle manifolds whose cohomology rings have this property. We notice that all examples of connected sums of sphere products given in [3] have the property that every product is of two spheres, this leads to a question:

QUESTION A.   If $\mathscr{Z}_K$ is a connected sum of sphere products, is it true that every product is of exactly two spheres?

In this paper (Proposition 4.1), we give a negative answer to this question at the aspect of cohomology rings, by constructing a 3-dimensional polytopal sphere, so that the cohomology ring of the corresponding moment-angle manifold is isomorphic to the cohomology ring of the connected sum of sphere products

$$S^3 \times S^3 \times S^6 \,\#\, (8)S^5 \times S^7 \,\#\, (8)S^6 \times S^6.$$

2. Cohomology ring of moment-angle complex

DEFINITION 2.1.   Let K be a simplicial complex with vertex set $[m]$. A *missing face* of K is a sequence $(i_1, \ldots, i_k) \subseteq [m]$ such that $(i_1, \ldots, i_k) \notin K$, but every proper subsequence of $(i_1, \ldots, i_k)$ is a simplex of K. Denote by $\mathrm{MF}(K)$ the set of all missing faces of K.

From definition 2.1, it is easy to see that if K_I is a full subcomplex of K, then $\mathrm{MF}(K_I)$ is a subset of $\mathrm{MF}(K)$. Concretely,

$$\mathrm{MF}(K_I) = \{\sigma \in \mathrm{MF}(K) \colon \sigma \subseteq I\}.$$

Let $R[m] = R[v_1, \ldots, v_m]$ denote the graded polynomial algebra over R, where R is a field or $\mathbb{Z}$, $\deg v_i = 2$. The *face ring* (also known as the *Stanley–Reisner ring*) of a simplicial complex K on the vertex set [m] is the quotient ring

$$R(K) = R[m]/\mathcal{I}_K,$$

where $\mathcal{I}_K$ is the ideal generated by all square free monomials $v_{i_1} v_{i_2} \cdots v_{i_s}$ such that $(i_1, \ldots, i_s) \in \mathrm{MF}(K)$.

The following result is used to calculate the cohomology ring of $\mathscr{Z}_K$, which is proved by Buchstaber and Panov [5, Theorems 7.6] for the case over a field, [2] for the general case; see also [11, Theorem 4.7]. Another proof of Theorem 2.2 for the case over $\mathbb{Z}$ was given by Franz [8].

Theorem 2.2 (Buchstaber-Panov, [11, Theorem 4.7])**.** *Let K be a abstract simplicial complex with m vertices. Then the cohomology ring of the moment-angle com-*

plex $\mathcal{Z}_K$ is given by the isomorphisms

$$H^*(\mathcal{Z}_K; R) \cong \text{Tor}_{R[m]}^{*,*}(R(K), R) \cong \bigoplus_{I \subseteq [m]} \tilde{H}^*(K_I; R)$$

where

$$H^p(\mathcal{Z}_K; R) \cong \bigoplus_{\substack{J \subseteq [m] \\ -i+2|J|=p}} \text{Tor}_{R[m]}^{-i,2|J|}(R(K), R)$$

and

$$\text{Tor}_{R[m]}^{-i,2|J|}(R(K), R) \cong \tilde{H}^{|J|-i-1}(K_J; R).$$

REMARK 2.3.   There is a canonical ring structure on $\bigoplus_{I \subseteq [m]} \tilde{H}^*(K_I)$ (called the *Hochster ring* and denoted by $\mathcal{H}^{*,*}(K)$, where $\mathcal{H}^{i,J}(K) = \tilde{H}^i(K_J)$) given by the maps

$$\eta \colon \tilde{H}^{p-1}(K_I) \otimes H^{q-1}(K_J) \to H^{p+q-1}(K_{I \cup J}),$$

which are induced by the canonical simplicial inclusions $K_{I \cup J} \to K_I * K_J$ (join of simplicial complexes) for $I \cap J = \emptyset$ and zero otherwise. Precisely, Let $\tilde{C}^q(K)$ be the qth reduced simplicial cochain group of K. For a oriented simplex $\sigma = (i_1, \ldots, i_p)$ of K (the orientation is given by the order of vertices of σ), denote by $\sigma^* \in \tilde{C}^{p-1}(K)$ the basis cochain corresponding to σ; it takes value 1 on σ and vanishes on all other simplices. Then for $I, J \in [m]$ with $I \cap J = \emptyset$, we have isomorphisms of reduced simplicial cochains

$$\mu \colon \tilde{C}^{p-1}(K_I) \otimes \tilde{C}^{q-1}(K_J) \to \tilde{C}^{p+q-1}(K_I * K_J), \quad p, q \geq 0,$$

$$\sigma^* \otimes \tau^* \mapsto (\sigma \sqcup \tau)^*$$

where $\sigma \sqcup \tau$ means the juxtaposition of σ and τ. Given two cohomology classes $[c_1] \in \tilde{H}^{p-1}(K_I)$ and $[c_2] \in \tilde{H}^{q-1}(K_J)$, which are represented by the cocycles $\sum_i \sigma_i^*$ and $\sum_j \tau_j^*$ respectively. Then

$$\eta([c_1] \otimes [c_2]) = \varphi^* \left(\left[\mu \left(\sum_{i,j} \sigma_i^* \otimes \tau_j^* \right) \right] \right),$$

where $\varphi \colon K_{I \cup J} \to K_I * K_J$ is the simplicial inclusion.

We denote by $\psi([c])$ the inverse image of a class $[c] \in \bigoplus_{I \subseteq [m]} \tilde{H}^*(K_I)$ by the composition of the two isomorphisms in Theorem 2.2. Given two cohomology classes $[c_1] \in \tilde{H}^p(K_I)$ and $[c_2] \in \tilde{H}^q(K_J)$, define

$$[c_1] * [c_2] = \eta([c_1] \otimes [c_2]).$$

Bosio and Meersseman proved in [3] (see also [6, Proposition 3.2.10]) that, up to sign

$$\psi([c_1]) \smile \psi([c_2]) = \psi([c_1] * [c_2]).$$

REMARK 2.4. Baskakov showed in [1] (see [11, Theorem 5.1]) that the isomorphisms in Theorem 2.2 are functorial with respect to simplicial maps (here we only consider simplicial inclusions). That is, for a simplicial inclusion $i\colon K' \hookrightarrow K$ (suppose the vertex sets of K' and K are $[m']$ and $[m]$ respectively) which induces natural inclusions

$$\phi\colon \mathcal{Z}_{K'} \hookrightarrow \mathcal{Z}_K$$

and

$$i|_{K'_I}\colon K'_I \hookrightarrow K_I, \quad \text{for each} \quad I \subseteq [m'],$$

there is a commutative diagram of algebraic homomorphisms

$$\begin{array}{ccc}
H^*(\mathcal{Z}_K) & \xrightarrow{\ \phi^*\ } & H^*(\mathcal{Z}_{K'}) \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\bigoplus_{I \subseteq [m]} \tilde{H}^*(K_I) & \xrightarrow[\oplus_I (i|_{K'_I})^*]{} & \bigoplus_{I \subseteq [m']} \tilde{H}^*(K'_I).
\end{array}$$

Actually, there are three ways to calculate the integral cohomology ring of a moment-angle complex $\mathcal{Z}_K$.

(1) The first is to calculate the Hochster ring $\mathcal{H}^{*,*}(K)$ of K and apply the isomorphisms in Theorem 2.2.

(2) The second is to calculate $\mathrm{Tor}^{*,*}_{\mathbb{Z}[m]}(\mathbb{Z}(K), \mathbb{Z})$ by means of the Koszul resolution ([5, Theorems 7.6 and 7.7]), that is

$$\mathrm{Tor}^{*,*}_{\mathbb{Z}[m]}(\mathbb{Z}(K), \mathbb{Z}) \cong H(\Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}(K), d),$$

where $\Lambda[u_1, \dots, u_m]$ is the exterior algebra over $\mathbb{Z}$ generated by m generators. On the right side, we have

$$\mathrm{bideg}\, u_i = (-1, 2), \quad \mathrm{bideg}\, v_i = (0, 2), \quad du_i = v_i, \quad dv_i = 0.$$

In fact, there is a simpler way to calculate the cohomology of this differential graded algebra by applying the following result

Proposition 2.5 ([6, Lemma 3.2.6]). *The projection homomorphism*

$$\varrho\colon \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}(K) \to A(K)$$

induces an isomorphism in cohomology, where $A(K)$ is the quotient algebra

$$A(K) = \Lambda[u_1, \dots, u_m] \otimes \mathbb{Z}(K)/(v_i^2 = u_i v_i = 0,\ 1 \le i \le m).$$

(3) The third way is to use the Taylor resolution for $\mathbb{Z}(K)$ to calculate $\mathrm{Tor}_{\mathbb{Z}[m]}(\mathbb{Z}(K), \mathbb{Z})$. This was introduced first by Yuzvinsky in [13]. Wang and Zheng [12] applied this method to toric topology. Concretely, let $\mathbb{P} = \mathrm{MF}(K)$, and let $\Lambda[\mathbb{P}]$ be the exterior algebra generated by $\mathbb{P}$. Given a monomial $u = \sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r}$ in $\Lambda[\mathbb{P}]$, let

$$S_u = \sigma_{k_1} \cup \sigma_{k_2} \cdots \cup \sigma_{k_r}.$$

Define bideg $u = (-r, 2|S_u|)$, and define

$$\partial_i(u) = \sigma_{k_1} \cdots \hat{\sigma}_{k_i} \cdots \sigma_{k_r} = \sigma_{k_1} \cdots \sigma_{k_{i-1}} \sigma_{k_{i+1}} \cdots \sigma_{k_r}.$$

Let $(\Lambda^{*,*}[\mathbb{P}], d)$ be the cochain complex (with a different product structure from $\Lambda[\mathbb{P}]$) induced from the bi-graded exterior algebra on $\mathbb{P}$. The differential $d\colon \Lambda^{-q,*}[\mathbb{P}] \to \Lambda^{-(q-1),*}[\mathbb{P}]$ is given by

$$d(u) = \sum_{i=1}^{q}(-1)^i\partial_i(u)\delta_i,$$

where $\delta_i = 1$ if $S_u = S_{\partial_i(u)}$ and zero otherwise. The product structure in $(\Lambda^{*,*}[\mathbb{P}], d)$ is given by

$$u \times v = \begin{cases} u \cdot v & \text{if} \quad S_u \cap S_v = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $\cdot$ denote the ordinary product in the exterior algebra $\Lambda[\mathbb{P}]$.

Proposition 2.6 (see [12, Theorems 2.6 and 3.2])**.** *There is a algebraic isomorphism*

$$\mathrm{Tor}_{\mathbb{Z}[m]}^{*,*}(\mathbb{Z}(K), \mathbb{Z}) \cong H(\Lambda^{*,*}[\mathbb{P}], d).$$

3. Construction of a polytopal 3-sphere with eight vertices

In this section, we construct a 3-dimensional polytopal sphere K with eight vertices, such that the cohomology ring of the corresponding moment-angle manifold $\mathcal{Z}_K$ is isomorphic to the the cohomology ring of a connected sum of sphere products with one product of three spheres.

CONSTRUCTION 3.1. We construct K by three steps. First give a 2-dimensional simplicial complex K_0 with 4 vertices shown in Fig. 1.

$$\mathrm{MF}(K_0) = \{(1, 2, 3), (1, 3, 4)\}.$$

It has two subcomplex K_1 and K_2 also shown in Fig. 1. Next let $L_1 = K_0 \cup \mathrm{cone}(K_1)$ with a new vertex 5. (i.e., L_1 is the mapping cone of the inclusion map $K_1 \hookrightarrow K_0$),
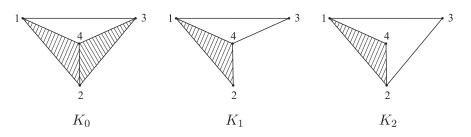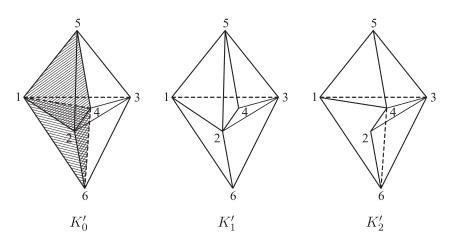
Fig. 1. K_0, K_1 and K_2.

Fig. 2. K'_0, K'_1 and K'_2.

and let $L_2 = K_0 \cup \mathrm{cone}(K_2)$ with a new vertex 6. Let $K'_0 = L_1 \cup L_2$ be a simplicial complex obtained by gluing L_1 and L_2 along K_0 (see Fig. 2). Then

$$\mathrm{MF}(K'_0) = \{(1, 2, 3), (1, 3, 4), (2, 3, 5), (3, 4, 6), (5, 6)\}.$$

Note that K'_0 can be viewed as a "thick" 2-sphere with two 3-simplices $(1, 2, 4, 5)$ and $(1, 2, 4, 6)$, shown shaded in Fig. 2. K'_0 has two subcomplexes K'_1 and K'_2 (see Fig. 2), which are all triangulations of S^2. Let $\mathrm{cone}(K'_2)$ be the cone of K'_2 with a new vertex 8. Then it is easy to see that $K' = K'_0 \cup \mathrm{cone}(K'_2)$ is a triangulation of D^3 and its boundary is K'_1. Finally, let $K = K' \cup \mathrm{cone}(K'_1)$ with a new vertex 7. Clearly, K is a triangulation of S^3, and the missing faces of K are

(3.1)
$$\mathrm{MF}(K) = \{(1, 2, 3), (1, 3, 4), (2, 3, 5), (3, 4, 6), (5, 6),$$
$$(1, 4, 7), (4, 6, 7), (1, 2, 8), (2, 5, 8), (7, 8)\}.$$

Grünbaum and Sreedharan [9] gave a complete enumeration of the simplicial 4-polytopes with 8 vertices. A direct verification shows that K we construct above
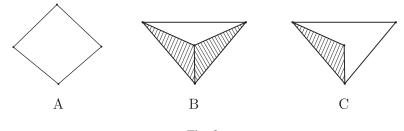
Fig. 3.

is isomorphic to the boundary of P_{28}^8 (a 4-polytope with 18 facets) in [9]. Then K is actually a polytopal sphere. From the construction we know that all 3-simplices of K are

$$(1, 2, 4, 5), \ (1, 2, 4, 6), \ (1, 2, 5, 7), \ (1, 2, 6, 7), \ (1, 3, 5, 7), \ (1, 3, 6, 7),$$

$$(2, 3, 4, 7), \ (2, 3, 6, 7), \ (2, 4, 5, 7), \ (3, 4, 5, 7), \ (1, 4, 5, 8), \ (1, 4, 6, 8),$$

$$(1, 3, 5, 8), \ (1, 3, 6, 8), \ (2, 3, 4, 8), \ (2, 3, 6, 8), \ (2, 4, 6, 8), \ (3, 4, 5, 8).$$

4.   connected sums of sphere products

In the first part of this section, we calculate the cohomology ring of $\mathcal{Z}_K$ corresponding to the polytopal sphere K constructed in the last section. In the second part, we give some general properties for the moment-angle manifolds whose cohomology ring is isomorphic to that of a connected sum of sphere product.

Proposition 4.1.   *For the polytopal sphere K defined in* Construction 3.1, *the cohomology ring of the corresponding moment-angle manifold $\mathcal{Z}_K$ is isomorphic to the cohomology ring of*

$$S^3 \times S^3 \times S^6 \,\#\, (8)S^5 \times S^7 \,\#\, (8)S^6 \times S^6.$$

We will calculate $H^*(\mathcal{Z}_K)$ in the first way introduced in Section 2. Therefore we need first to calculate the reduced cohomology rings of all full subcomplexes of K. Note first the following obvious fact: Let Γ be a simplicial complex with vertex set $[m]$. Define $\mathcal{I} = \bigcup_{\sigma \in \mathrm{MF}(\Gamma)} \sigma$. If $\mathcal{I} \neq [m]$, then $\Gamma = K_{\mathcal{I}} * \Delta^{m-|\mathcal{I}|-1}$, and therefore Γ is contractible.

Now we do this work in 6 cases according to the cardinality of I for K_I.

(1) Since the case $|I| = 1$ is trivial, we start with the case $|I| = 2$. In this case, from (3.1), it is easy to see that $\tilde{H}^*(K_I) \neq 0$ if and only if $I = (5, 6)$ or $(7, 8)$, and if so, $\tilde{H}^*(K_I) \cong \tilde{H}^0(K_I) \cong \mathbb{Z}$. Denote by a_1 (respectively a_2) a generator of $\tilde{H}^0(K_I)$ for $I = (5, 6)$ (respectively $(7, 8)$).

(2) $|I| = 3$. It is easy to see that the union of any two missing faces of K contains at least four vertices. Combining the preceding argument we have that $\tilde{H}^*(K_I) \neq 0$ if

Table 4.1. Non-contractible full subcomplexes of K with four vertices.

	K_{I_0}	$K_{\hat{I}_0}$	K_{I_1}	$K_{\hat{I}_1}$	K_{I_2}	$K_{\hat{I}_2}$
vertex set	{1, 2, 3, 4}	{5, 6, 7, 8}	{1, 2, 3, 5}	{4, 6, 7, 8}	{1, 2, 3, 8}	{4, 5, 6, 7}
missing faces	(1, 2, 3), (1, 3, 4)	(5, 6), (7, 8)	(1, 2, 3), (2, 3, 5)	(4, 6, 7), (7, 8)	(1, 2, 3), (1, 2, 8)	(4, 6, 7), (5, 6)

K_{I_3}	$K_{\hat{I}_3}$	K_{I_4}	$K_{\hat{I}_4}$	K_{I_5}	$K_{\hat{I}_5}$
{1, 2, 5, 8}	{3, 4, 6, 7}	{1, 2, 7, 8}	{3, 4, 5, 6}	{1, 3, 4, 6}	{2, 5, 7, 8}
(1, 2, 8), (2, 5, 8)	(3, 4, 6), (4, 6, 7)	(1, 2, 8), (7, 8)	(3, 4, 6), (5, 6)	(1, 3, 4), (3, 4, 6)	(2, 5, 8), (7, 8)

K_{I_6}	$K_{\hat{I}_6}$	K_{I_7}	$K_{\hat{I}_7}$	K_{I_8}	$K_{\hat{I}_8}$
{1, 3, 4, 7}	{2, 5, 6, 8}	{1, 4, 6, 7}	{2, 3, 5, 8}	{1, 4, 7, 8}	{2, 3, 5, 6}
(1, 3, 4), (1, 4, 7)	(2, 5, 8), (5, 6)	(1, 4, 7), (4, 6, 7)	(2, 3, 5), (2, 5, 8)	(1, 4, 7), (7, 8)	(2, 3, 5), (5, 6)

and only if I is one of the eight missing faces with three vertices in MF(K), and if so, $\tilde{H}^*(K_I) \cong \tilde{H}^1(K_I) \cong \mathbb{Z}$, whose generator we denote by b_i ($1 \leq i \leq 8$).

(3) $|I| = 4$. An easy observation shows that the union of any three missing faces of K contains at least five vertices, and K has no missing face with four vertices. So if K_I is not contractible, then it has exactly two missing faces. Thus from (3.1), the form of MF(K_I) is one of $\{(v_1, v_2), (v_3, v_4)\}$, $\{(v_1, v_2, v_3), (v_2, v_3, v_4)\}$ and $\{(v_1, v_2), (v_1, v_3, v_4)\}$, for which the corresponding simplicial complexes are respectively A, B and C shown in Fig. 3. It is easy to see that they are all homotopic to S^1. In Table 4.1 we list all non-contractible full subcomplexes K_I of K for $|I| = 4$ (each I_j contains vertex 1). Denote by α_j (respectively α'_j) a generator of $\tilde{H}^*(K_{I_j}) \cong \tilde{H}^1(K_{I_j}) \cong \mathbb{Z}$ (respectively $\tilde{H}^*(K_{\hat{I}_j})$) for $0 \leq j \leq 8$.

(4) $|I| = 5$. We need to use the following well known fact: Let Γ be a simplicial complex on $[m]$, Γ_J a full subcomplex on $J \subseteq [m]$. Then $\Gamma_{\hat{J}}$ is a deformation retract of $\Gamma \setminus \Gamma_J$. From this and Alexander duality on K we have that $\tilde{H}^j(K_I) \cong \tilde{H}_{2-j}(K_{\hat{I}})$. Since $|I| = 5$, $|\hat{I}| = 3$. From the arguments in case (2), $H_*(K_{\hat{I}})$ are all torsion free, so $\tilde{H}^*(K_{\hat{I}}) \cong \tilde{H}_*(K_{\hat{I}})$. Thus $\tilde{H}^*(K_I)$ is non-trivial if and only if $\hat{I}$ is one of the eight missing faces with three vertices, and if so, $\tilde{H}^*(K_I) \cong \tilde{H}^1(K_I) \cong \mathbb{Z}$, whose generator we denote by β_i ($1 \leq i \leq 8$).

(5) $|I| = 6$. The same argument as in (4) shows that $\tilde{H}^*(K_I)$ is non-trivial if and only if $\hat{I}$ is (5, 6) or (7, 8), and if so, $\tilde{H}^*(K_I) \cong \tilde{H}^2(K_I) \cong \mathbb{Z}$. Denote by λ_1 (respectively λ_2) a generator of $\tilde{H}^2(K_I)$ for $\hat{I} = (5, 6)$ (respectively (7, 8)).

(6) $|I| \geq 7$. If $|I| = 7$, $\tilde{H}^*(K_I) = 0$ is clear. If $|I| = 8$, $K_I = K$, so $\tilde{H}^*(K) \cong$ $H^3(K) \cong \mathbb{Z}$. Denote by ξ a generator of it.

Proof of Proposition 4.1. Theorem 2.2 and the preceding arguments give the cohomology group of $\mathscr{Z}_K$:

i	$\tilde{H}^i(\mathscr{Z}_K) \cong$
1, 2, 4, 8, 10, 11	0
3	$\mathbb{Z} \cdot \psi(a_1) \oplus \mathbb{Z} \cdot \psi(a_2)$
5	$\bigoplus_{1 \leq i \leq 8} \mathbb{Z} \cdot \psi(b_i)$
6	$\bigoplus_{0 \leq i \leq 8}(\mathbb{Z} \cdot \psi(\alpha_i) \oplus \mathbb{Z} \cdot \psi(\alpha_i'))$
7	$\bigoplus_{1 \leq i \leq 8} \mathbb{Z} \cdot \psi(\beta_i)$
9	$\mathbb{Z} \cdot \psi(\lambda_1) \oplus \mathbb{Z} \cdot \psi(\lambda_2)$
12	$\mathbb{Z} \cdot \psi(\xi)$

Now we give the cup product structure of $H^*(\mathscr{Z}_K)$. First by Poincaré duality on $\mathscr{Z}_K$ and Remark 2.3, up to sign

$$(4.1) \qquad \psi(a_i) \smile \psi(\lambda_i) = \psi(\xi), \quad i = 1, 2;$$

$$(4.2) \qquad \psi(b_i) \smile \psi(\beta_i) = \psi(\xi), \quad i = 1, \ldots, 8;$$

$$(4.3) \qquad \psi(\alpha_i') \smile \psi(\alpha_i) = \psi(\xi), \quad i = 0, \ldots, 8.$$

Note that $K_{\hat{I}_0} = K_{(5,6)} * K_{(7,8)}$, so up to sign $\psi(a_1) \smile \psi(a_2) = \psi(\alpha_0')$ (see Remark 2.3), and so

$$(4.4) \qquad \psi(a_1) \smile \psi(a_2) \smile \psi(\alpha_0) = \psi(\xi)$$

Since $a_2 * \alpha_0 \in \tilde{H}^*(K_{(\hat{5,6})})$, $\psi(a_2) \smile \psi(\alpha_0) = \psi(a_2 * \alpha_0) = p \cdot \psi(\lambda_1)$ for some $p \in \mathbb{Z}$. From formulae (4.1) and (4.4) we have $p = 1$. Similarly, $\psi(a_1) \smile \psi(\alpha_0) = -\psi(\lambda_2)$. Moreover from the arguments in case (5), we have that $\psi(a_i) \smile \psi(\alpha_j) = 0$ for $1 \leq j \leq 8$, and $\psi(a_i) \smile \psi(\alpha_j') = 0$ for $0 \leq j \leq 8$; $i = 1, 2$. By an observation on the dimension of the non-trivial cohomology groups of $\mathscr{Z}_K$, it is easy to verify that any other products between these generators are trivial. Combining all the product relations above we get the desired result. $\square$

There are other two different polytopal spheres from K (corresponding to the two 4-polytopes P_{27}^8 and P_{29}^8 in [9]), so that the corresponding moment-angle manifolds have the same cohomology rings as $\mathscr{Z}_K$. The proof of this is the same as Proposition 4.1.

For a moment-angle manifold corresponding to a simplicial 2-sphere, if its cohomology ring is isomorphic to the one of a connected sum of sphere products, then

it is actually diffeomorphic to this connected sum of sphere products ([3], Proposition 11.6). This leads to the following conjecture:

Conjecture 4.2. *$\mathcal{Z}_K$ is diffeomorphic to the connected sum of sphere products in Proposition 4.1.*

Note that the connected sum of sphere products in Proposition 4.1 only has one product of three spheres, we then ask: Is there a moment-angle manifold (corresponding to a simplicial 3-sphere) whose cohomology ring is isomorphic to the one of a connected sum of sphere products with more than one product of three spheres? The following theorem gives a negative answer to this question.

Theorem 4.3. *Let K be a n-dimensional simplicial sphere ($n \geq 2$) satisfies $H^*(\mathcal{Z}_K) \cong H^*(M)$, where $M \cong M_1 \# \cdots \# M_k$, and each M_i is a product of spheres. Let q_i be the number of sphere factors of M_i. Then*
(a) *If $q_i = n + 1$ for some i, then $k = 1$, and $\mathcal{Z}_K \cong M \cong S^3 \times S^3 \cdots \times S^3$.*
(b) *Let $I = \{i : q_i \geq [n/2] + 2\}$ (where $[\cdot]$ denotes the integer part). Then $|I| \leq 1$.*

Lemma 4.4. *Let K be a simplicial complex on $[m]$. Given two classes $[a], [b] \in \mathcal{H}^{0,*}(K)$, if $[a] * [b] \neq 0$, then there must be a full subcomplex K_I ($|I| \geq 4$) which is isomorphic to the boundary of a polygon, and satisfying $[a] * [b] \in \tilde{H}^1(K_I)$.*

Proof. Let $\mathcal{M} = \{I \in \mathrm{MF}(K) : |I| \neq 3\}$, and let K' be a simplicial complex on $[m]$ so that $\mathrm{MF}(K') = \mathcal{M}$. Clearly, K is a subcomplex of K'. Note that K' and K have the same 1-skeleton, so if we can prove that for some $I \in [m]$, K'_I is isomorphic to the boundary of a polygon (K'_I can not be the boundary of a triangle by the definition of $\mathcal{M}$), then the result holds. From Remark 2.4, there is a ring homomorphism $i^* : \mathcal{H}^{*,*}(K') \to \mathcal{H}^{*,*}(K)$ induced by the simplicial inclusion $i : K \hookrightarrow K'$. It is easy to see that i^* is a isomorphism when restricted to $\mathcal{H}^{0,*}(K')$. Suppose $i^*([a']) = [a]$ and $i^*([b']) = [b]$. By assumption, $i^*([a'] * [b']) \neq 0$, so $[a'] * [b'] \in \mathcal{H}^{1,*}(K') \neq 0$. Without loss of generality, we can assume $[a'] * [b'] \in \tilde{H}^1(K'_J)$ for some $J \subset [m]$. The lemma follows once we prove the following assertion:

ASSERTION. For any simplicial complex Γ satisfies $\tilde{H}^1(\Gamma) \neq 0$, there must be a full subcomplex Γ_I which is isomorphic to the boundary of a polygon, satisfying that

$$j^* : \tilde{H}^1(\Gamma) \to \tilde{H}^1(\Gamma_I)$$

is an epimorphism, where $j : \Gamma_I \to \Gamma$ is the inclusion map.

Now we prove this. Since $H^1(\Gamma) \neq 0$, $H_1(\Gamma) \neq 0$, then there is a nonzero homology class $[c] \in H_1(\Gamma)$ represented by the 1-cycle

$$c = (v_1, v_2) + (v_2, v_3) + \cdots + (v_{k-1}, v_k) + (v_k, v_1),$$

where v_i is a vertex of Γ. Without loss of generality, we assume $v_1, v_2, \ldots, v_k$ are all different and the vertex number k is minimal among all $[c]$'s and their representations. Let $I = (v_1, \ldots, v_k)$, we claim that Γ_I is isomorphic to the boundary of a polygon. If this is not true, then there must be a 1-simplex, say $(v_1, v_j) \in \Gamma_I$ such that $j \neq 2, k$. Let

$$c_1 = (v_1, v_2) + (v_2, v_3) + \cdots + (v_j, v_1);$$
$$c_2 = (v_1, v_j) + (v_j, v_{j+1}) + \cdots + (v_k, v_1).$$

Then $c = c_1 + c_2$, and therefore $[c_1] \neq 0$ or $[c_2] \neq 0$. In either case, the vertex number of c_i ($i = 1, 2$) is less than k, a contradiction. Apparently, $j_*([c])$ is the fundamental class of Γ_I.   □

Lemma 4.5. *Let K be a simplicial sphere satisfies $H^*(\mathcal{Z}_K)$ is isomorphic to the cohomology ring of a connected sum of sphere products. If a proper full subcomplex is isomorphic to the boundary of a m-gon, then $m \leq 4$.*

Proof. Suppose on the contrary that there is a proper full subcomplex K_I isomorphic to the boundary of a m-gon with $m \geq 5$. Then $H^*(\mathcal{Z}_{K_I})$ is a proper subring and a direct summand of $H^*(\mathcal{Z}_K)$. By Theorem 1.4 we can find five elements a_1, a_2, b_1, b_2, c of $H^*(\mathcal{Z}_K)$, where $\dim(a_1) = 3$, $\dim(a_2) = 4$, $\dim(b_1) = m-1$, $\dim(b_2) = m-2$ and $\dim(c) = m+2$, such that each of them is a generator of a $\mathbb{Z}$ summand of $H^*(\mathcal{Z}_K)$, and the cup product relations between them are given by:

$$a_1 \smile b_1 = a_2 \smile b_2 = c,$$

all other products are zero. Clearly, $\dim(c)$ is not equal to the top dimension of $H^*(\mathcal{Z}_K)$. Suppose $H^*(\mathcal{Z}_K)$ is isomorphic to the cohomology ring of

$$S_{1,1}^{f(1,1)} \times S_{1,2}^{f(1,2)} \times \cdots \times S_{1,k(1)}^{f(1,k(1))} \# \cdots \# S_{n,1}^{f(n,1)} \times S_{n,2}^{f(n,2)} \times \cdots \times S_{n,k(n)}^{f(n,k(n))},$$

where f is a function of $(\mathbb{Z}^+)^2 \to \mathbb{Z}^+$ ($f(i, j) \geq 3$ for all i, j), $S_{i,j}^{f(i,j)} = S^{f(i,j)}$, $k(i) \in \mathbb{Z}^+$ denote the number of spheres in the i-th summand of sphere product. Denote by $e_{ij}^{(k)}$ a generator of $H^k(\mathcal{Z}_K)$ corresponding to $S_{i,j}^k$ ($f(i, j) = k$). Then we can write

$$a_1 = \sum_{f(i,j)=3} \lambda_{ij} e_{ij}^{(3)}, \quad a_2 = \sum_{f(i,j)=4} \lambda_{ij}' e_{ij}^{(4)}$$

where $\lambda_{ij}, \lambda'_{ij} \in \mathbb{Z}$. It is easy to see that $e_{ij}^{(k)} \smile e_{rs}^{(t)} \neq 0$ if and only if $i = r$ and $j \neq s$. Since $a_1 \smile a_2 = 0$, we have that if $\lambda_{ij}, \lambda'_{i'j'} \neq 0$, then $i \neq i'$. However this implies that $a_1 \smile b_1 \neq a_2 \smile b_2$ since $\dim(c)$ is not equal to the top dimension of $H^*(\mathcal{Z}_K)$, a contradiction. $\square$

Lemma 4.6.   *Let K be a n-dimensional simplicial sphere satisfies $H^*(\mathcal{Z}_K)$ is isomorphic to the cohomology ring of a connected sum of sphere products. If there is a full subcomplex isomorphic to the boundary of a quadrangle, then for any full subcomplex K_I satisfies $\tilde{H}^0(K_I) \neq 0$, we have $|I| = 2$. Moreover, if I_1, I_2 are two different such sequences, then $K_{I_1 \cup I_2}$ is isomorphic to the boundary of a quadrangle.*

Proof.   The case $n = 1$ are trivial, so we assume $n > 1$. If we can prove the statement that for any two different missing faces $\sigma_1, \sigma_2 \in \mathrm{MF}(K)$, which contain two vertices, we have $\sigma_1 \cap \sigma_2 = \emptyset$, then the lemma holds.

Suppose $J = (1, 2, 3, 4)$, and $\mathrm{MF}(K_J) = \{(1, 3), (2, 4)\}$ (i.e., K_J is isomorphic to the boundary of a quadrangle) by assumption. First we will prove that for any vertex $v \notin J$ and any $j \in J$, (j, v) is a simplex of K. Without loss of generality, suppose on the contrary that $(1, 5) \in \mathrm{MF}(K)$. Let Γ be a simplicial complex with vertex set $\{1, 3, 5\}$ such that $\mathrm{MF}(\Gamma) = \{(1, 3), (1, 5)\}$. Then $K_{(1,3,5)}$ is a subcomplex of Γ. Clearly, $\tilde{H}^0(\Gamma) \cong \mathbb{Z}$, denote by c_1 a generator of it. Let $L = \Gamma * K_{(2,4)}$. Denote by c_2 a generator of $\tilde{H}^0(K_{(2,4)}) \cong \mathbb{Z}$, then an easy calculation shows that (see Remark 2.3) $c_1 * c_2$ is a generator of $\tilde{H}^1(L) \cong \mathbb{Z}$. Let $J' = (1, 2, 3, 4, 5)$. Then $K_{J'}$ is a subcomplex of L, and the inclusion map induces a monomorphism $\tilde{H}^1(L) \xrightarrow{\mu} \tilde{H}^1(K_{J'})$ (actually, $\mu(H^1(L))$ is a direct summand of $H^1(K_{J'})$). There is a commutative diagram

$$
\begin{array}{ccc}
\tilde{H}^0(\Gamma) \otimes \tilde{H}^0(K_{(2,4)}) & \xrightarrow{\ \eta\ } & \tilde{H}^1(L) \\
{\scriptstyle \phi \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \mu} \\
\tilde{H}^0(K_{(1,3,5)}) \otimes \tilde{H}^0(K_{(2,4)}) & \xrightarrow{\ \eta\ } & \tilde{H}^1(K_{J'}),
\end{array}
$$

where ϕ is induced by the inclusion map. So $\mu(c_1 * c_2) = \phi(c_1) * c_2$ is a generator of $\tilde{H}^1(K_{J'})$. Thus by Poincaré duality on $\mathcal{Z}_K$, there is an element c_0 of $\tilde{H}^{n-2}(K_{\hat{j'}})$ such that $c_0 * \phi(c_1) * c_2$ is a generator of $\tilde{H}^n(K) \cong \mathbb{Z}$. On the other hand, let e_1 be a generator of $\tilde{H}^0(K_{(1,3)})$. Clearly $e_1 * c_2$ is a generator of $\tilde{H}^1(K_J) \cong \mathbb{Z}$, so there is a element e_0 of $\tilde{H}^{n-2}(K_{\hat{j}})$ such that $e_0 * e_1 * c_2 = c_0 * \phi(c_1) * c_2$. Since $e_0 * e_1, c_0 * \phi(c_1) \in \tilde{H}^{n-1}(K_{(\hat{2,4})}) \cong \mathbb{Z}$, we have $e_0 * e_1 = c_0 * \phi(c_1)$. Since $\dim(\psi(e_1)) = 3$, $\dim(\psi\phi(c_1)) = 4$, and $e_1 * \phi(c_1) = 0$, then we get a contradiction by applying the arguments as in the proof of Lemma 4.5.

Now suppose $v_1, v_2, v_3 \in \hat{J}$ such that $(v_1, v_2), (v_1, v_3) \in \mathrm{MF}(K)$. Let $J_0 = (v_1, v_2, 1, 3)$. Then from the result in the last paragraph we have K_{J_0} is isomorphic to the boundary

of a quadrangle. Thus by applying the same arguments as in the last paragraph, we have that (v_1, v_3) is a simplex of K, a contradiction. $\qquad\square$

Now let us use the preceding results to complete the proof of Theorem 4.3

Proof of Theorem 4.3.   (a)   From the assumption and Theorem 2.2, we have that there are $n + 1$ elements $c_i \in \tilde{H}^{k_i}(K_{J_i})$, $1 \le i \le n + 1$, such that $\prod_{i=1}^{n+1} c_i \ne 0 \in \tilde{H}^n(K)$ (clearly, $J_i \cap J_k = \emptyset$ for $i \ne k$ and $\bigcup_{i=1}^{n+1} J_i$ is the vertex set of K). From Remark 2.3, the cohomology dimension of the class $\prod_{i=1}^{n+1} c_i$ is $n + \sum_{i=1}^{n+1} k_i$. Thus $k_i = 0$ for all $1 \le i \le n + 1$. Combine all the preceding lemmas, we have that $|J_i| = 2$ for all $1 \le i \le n + 1$, so $J_i \in \mathrm{MF}(K)$, and so K is a subcomplex of $K_{J_1} * \cdots * K_{J_{n+1}}$. Since $K_{J_1} * \cdots * K_{J_{n+1}}$ is a triangulation of S^n itself, then $K \cong K_{J_1} * \cdots * K_{J_{n+1}}$, and then the conclusion follows.

(b)   Suppose there is a M_u with $q_u \ge [n/2] + 2$, then as in (a) there are q_u elements $c_i \in \tilde{H}^{k_i}(K_{J_i})$, $1 \le i \le q_u$, such that $\prod_{i=1}^{q_u} c_i \ne 0 \in \tilde{H}^n(K)$. The cohomology dimension of the class $\prod_{i=1}^{q_u} c_i$ is $q_u - 1 + \sum_{i=1}^{q_u} k_i$, then from the inequality $q_u \ge [n/2] + 2$, there are at least two k_i's with $k_i = 0$. Then K satisfies the conditions in all of the three Lemmas above. From the first statement of Lemma 4.6, we have that for any $a \in \mathcal{H}^{0,*}(K)$, $\dim(\psi(a)) = 3$. So there are at least two S^3 factors in M_u. From the second statement of Lemma 4.6, we have that for any two linear independent element $a_1, a_2 \in \mathcal{H}^{0,*}(K)$, $a_1 * a_2 \ne 0$. This implies that all S^3 factors in the expression of M are in M_u. Then there can not be another M_v with $q_v \ge [n/2] + 2$. The conclusion holds. $\qquad\square$

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