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RECONSTRUCTIBLE GRAPHS, SIMPLICIAL FLAG COMPLEXES OF HOMOLOGY MANIFOLDS AND ASSOCIATED RIGHT-ANGLED COXETER GROUPS

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Abstract

In this paper, we investigate a relation between finite graphs, simplicial flag complexes and right-angled Coxeter groups, and we provide a class of reconstructible finite graphs. We show that if Γ is a finite graph which is the 1-skeleton of some simplicial flag complex L which is a homology manifold of dimension $n \geq 1$, then the graph Γ is reconstructible.

1. Introduction

In this paper, we investigate a relation between finite graphs, simplicial flag complexes and right-angled Coxeter groups, and we provide a class of reconstructible finite graphs. This paper treats only "simplicial" graphs. We show that if Γ is a finite graph which is the 1-skeleton of some simplicial flag complex L which is a homology manifold of dimension $n \geq 1$, then the graph Γ is reconstructible.

A graph Γ is said to be *reconstructible*, if any graph Γ' with the following property $(*)$ is isomorphic to Γ.

$(*)$ Let S and S' be the vertex sets of Γ and Γ' respectively. Then there exists a bijection $f \colon S \to S'$ such that the subgraphs $\Gamma_{S-\{s\}}$ and $\Gamma'_{S'-\{f(s)\}}$ are isomorphic for any $s \in S$, where $\Gamma_{S-\{s\}}$ and $\Gamma'_{S'-\{f(s)\}}$ are the full subgraphs of Γ and Γ' whose vertex sets are $S-\{s\}$ and $S'-\{f(s)\}$ respectively.

The following open problem is well-known as the reconstruction conjecture.

PROBLEM (Reconstruction conjecture). Every finite graph with at least three vertices will be reconstructible?

Some classes of reconstructible graphs are known (cf. [3], [20], [21], [22], [23], [26]) as follows: Let Γ be a finite graph with at least three vertices.

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(i) If Γ is a regular graph, then it is reconstructible.
(ii) If Γ is a tree, then it is reconstructible.
(iii) If Γ is not connected, then it is reconstructible.
(iv) If Γ has at most 11 vertices, then it is reconstructible.

Our motivation to consider graphs of the 1-skeletons of some simplicial flag complexes comes from the following idea on right-angled Coxeter groups and their nerves.

Details of Coxeter groups and Coxeter systems are found in [4], [6] and [19], and details of flag complexes, nerves, Davis complexes and their boundaries are found in [8], [9] and [24].

Let Γ be a finite graph and let S be the vertex set of Γ. Then the graph Γ uniquely determines a finite simplicial flag complex L whose 1-skeleton $L^{(1)}$ coincide with Γ. Here a simplicial complex L is a *flag complex*, if the following condition holds:
$(**)$ For any vertex set $\{s_0, \dots, s_n\}$ of L, if $\{s_i, s_j\}$ spans 1-simplex in L for any $i, j \in \{0, \dots, n\}$ with $i \neq j$ then the vertex set $\{s_0, \dots, s_n\}$ spans n-simplex in L.

Also every finite simplicial flag complex L uniquely determines a right-angled Coxeter system (W, S) whose nerve $L(W, S)$ coincide with L (cf. [1], [8], [9], [10], [12]). Here for any subset T of S, T spans a simplex of L if and only if the parabolic subgroup W_T generated by T is finite (such a subset T is called a *spherical subset of* S).

Moreover it is known that every right-angled Coxeter group W uniquely determines its right-angled Coxeter system (W, S) up to isomorphisms ([28], [18]).

By this corresponding, we can identify a finite graph Γ, a finite simplicial flag complex L, a right-angled Coxeter system (W, S) and a right-angled Coxeter group W.

Let Γ and Γ' be finite graphs, let L and L' be the corresponding flag complexes, let (W, S) and (W', S') be the corresponding right-angled Coxeter systems, and let W and W' be the corresponding right-angled Coxeter groups, respectively. Then the following statements are equivalent:
(1) Γ and Γ' are isomorphic as graphs;
(2) L and L' are isomorphic as simplicial complexes;
(3) (W, S) and (W', S') are isomorphic as Coxeter systems;
(4) W and W' are isomorphic as groups.

Also, for any subset T of the vertex set S of the graph Γ, the full subgraph Γ_T of Γ with vertex set T corresponds the full subcomplex L_T of L with vertex set T, the parabolic Coxeter system (W_T, T) generated by T, and the parabolic subgroup W_T of W generated by T.

Hence we can consider the reconstruction problem as the problem on simplicial flag complexes and also as the problem on right-angled Coxeter groups.

Moreover, the right-angled Coxeter system (W, S) associated by the graph Γ defines the Davis complex Σ which is a CAT(0) space and we can consider the ideal boundary $\partial\Sigma$ of the CAT(0) space Σ (cf. [1], [2], [5], [8], [9], [10], [12], [15], [16], [24]). Then the topology of the boundary $\partial\Sigma$ is determined by the graph Γ, and the topology of $\partial\Sigma$ is also a graph invariant.

Based on the observations above, we can obtain the following lemma from results of F.T. Farrell [13, Theorem 3], M.W. Davis [10, Theorem 5.5] and [17, Corollary 4.2] (we introduce details of this argument in Section 3).

Lemma 1.1. *Let (W,S) be an irreducible Coxeter system where W is infinite and let $L = L(W,S)$ be the nerve of (W,S). Then the following statements are equivalent*:
(1) *W is a virtual Poincaré duality group.*
(2) *L is a generalized homology sphere.*
(3) *$\tilde{H}^i(L_{S-T}) = 0$ for any i and any non-empty spherical subset T of S.*

Here a generalized homology n-sphere is a polyhedral homology n-manifold with the same homology as an n-sphere $\mathbb{S}^n$ (cf. [10, Section 5], [11], [25, p. 374], [27]). Also detail of (virtual) Poincaré duality groups is found in [7], [10], [11], [13].

In Lemma 1.1, we particularly note that the statement (3) is a local condition of L which determines a global structure of L as the statement (2). From this observation, it seems that the following theorem holds. (However the proof is not so obvious.)

Theorem 1.2. *Let Γ be a finite graph with at least 3 vertices and let (W,S) be the right-angled Coxeter system associated by Γ (i.e. the 1-skeleton of the nerve $L(W,S)$ of (W,S) is Γ). If the Coxeter group W is an irreducible virtual Poincaré duality group, then the graph Γ is reconstructible. Hence,*
(i) *if Γ is the 1-skeleton of some simplicial flag complex L which is a generalized homology sphere, then the graph Γ is reconstructible, and*
(ii) *in particular, if Γ is the 1-skeleton of some flag triangulation L of some n-sphere $\mathbb{S}^n$ $(n \geq 1)$, then the graph Γ is reconstructible.*

Here, based on this motivation, we investigate a finite graph which is the 1-skeleton of some simplicial flag complex which is a *homology manifold* as an extension of a generalized homology sphere, and we prove the following theorem. (Hence as a corollary, we also obtain Theorem 1.2.)

Theorem 1.3. *Let Γ be a finite graph with at least 3 vertices.*
(i) *If Γ is the 1-skeleton of some simplicial flag complex L which is a homology n-manifold $(n \geq 1)$, then the graph Γ is reconstructible.*
(ii) *In particular, if Γ is the 1-skeleton of some flag triangulation L of some n-manifold $(n \geq 1)$, then the graph Γ is reconstructible.*

Here detail of homology manifolds is found in [10, Section 5], [11], [25, p. 374], [27].

2. Proof of Theorem 1.3

We prove Theorem 1.3.

Proof of Theorem 1.3. Let Γ be a finite graph with at least 3 vertices which is the 1-skeleton of some simplicial flag complex L which is a homology manifold of dimension $n \geq 1$. Then we show that the graph Γ is reconstructible.

Let Γ' be a finite graph and let L' be the finite simplicial flag complex associated by Γ'. Also let S and S' be the vertex sets of the graphs Γ and Γ' respectively.

Now we suppose that the condition $(*)$ holds:

$(*)$ There exists a bijection $f\colon S \to S'$ such that the subgraphs $\Gamma_{S-\{s\}}$ and $\Gamma'_{S'-\{f(s)\}}$ are isomorphic for any $s \in S$.

To show that the graph Γ is reconstructible, we prove that the two graphs Γ and Γ' are isomorphic, i.e., the two simplicial flag complexes L and L' associated by Γ and Γ' respectively are isomorphic.

Let $v_0 \in S$ and let $v_0' = f(v_0)$. Then the two subgraphs $\Gamma_{S-\{v_0\}}$ and $\Gamma'_{S'-\{v_0'\}}$ are isomorphic by the assumption $(*)$, and the two subcomplexes $L_{S-\{v_0\}}$ and $L'_{S'-\{v_0'\}}$ are isomorphic. Let ϕ be an isomorphism from $L_{S-\{v_0\}}$ to $L'_{S'-\{v_0'\}}$.

If for any $a \in \mathrm{Lk}(v_0, L)^{(0)}$, $\phi(a) \in \mathrm{Lk}(v_0', L')^{(0)}$ then we obtain an isomorphism $\bar{\phi}\colon L \to L'$ from $\bar{\phi}|_{L_{S-\{v_0\}}} = \phi$ and $\bar{\phi}(v_0) = v_0'$ (since $\deg v_0 = \deg v_0'$), hence L and L' are isomorphic.

Now we suppose that there exists $a_0 \in S - \{v_0\}$ such that $a_0 \notin \mathrm{Lk}(v_0, L)^{(0)}$ and $a_0' := \phi(a_0) \in \mathrm{Lk}(v_0', L')^{(0)}$.

Here if there does not exist $u_0' \in S' - \mathrm{St}(a_0', L')^{(0)}$, then $\mathrm{St}(a_0', L')^{(0)} = S'$, where $\mathrm{St}(a_0', L')$ means the closed star of a_0' in L'. Hence $[a_0', b'] \in L'^{(1)}$ for any $b' \in S' - \{a_0'\}$. Since $\deg a_0 = \deg a_0'$ and $|S| = |S'|$, $[a_0, b] \in L^{(1)}$ for any $b \in S - \{a_0\}$. This particularly implies $[a_0, v_0] \in L^{(1)}$. This is a contradiction because it means $a_0 \in \mathrm{Lk}(v_0, L)^{(0)}$.

Thus we suppose that there exists $u_0' \in S' - \mathrm{St}(a_0', L')^{(0)}$.

Let $u_0 := f^{-1}(u_0')$. Then by the assumption $(*)$, the two subcomplexes $L_{S-\{u_0\}}$ and $L'_{S'-\{u_0'\}}$ are isomorphic and let ψ be an isomorphism from $L_{S-\{u_0\}}$ to $L'_{S'-\{u_0'\}}$.

Then

$$\begin{aligned}\mathrm{Lk}(\psi^{-1}(a_0'), L_{S-\{u_0\}}) &\cong \mathrm{Lk}(a_0', L'_{S'-\{u_0'\}})\\ &\cong \mathrm{Lk}(a_0', L'),\end{aligned}$$

since ψ is an isomorphism and $u_0' \notin \mathrm{St}(a_0', L')$. Also we obtain

$$\begin{aligned}\mathrm{St}(\psi^{-1}(a_0'), L_{S-\{u_0\}}) &\cong \mathrm{St}(a_0', L'_{S'-\{u_0'\}})\\ &\cong \mathrm{St}(a_0', L').\end{aligned}$$

Then

$$\mathrm{St}(a_0', L'_{S'-\{v_0'\}}) \subsetneqq \mathrm{St}(a_0', L') \cong \mathrm{St}(\psi^{-1}(a_0'), L_{S-\{u_0\}}).$$

Here we note that $\mathrm{St}(\psi^{-1}(a_0'), L_{S-\{u_0\}})$ is either
(a) the closed star $\mathrm{St}(\psi^{-1}(a_0'), L)$ of the vertex $\psi^{-1}(a_0')$ in the homology n-manifold L, or
(b) $\mathrm{St}(\psi^{-1}(a_0'), L) - u_0$ where $u_0 \in \mathrm{Lk}(\psi^{-1}(a_0'), L)$,
and also note that $\mathrm{St}(a_0', L'_{S'-\{v_0'\}}) = \mathrm{St}(a_0', L') - v_0'$. Hence we obtain that
(I) $\mathrm{St}(a_0', L'_{S'-\{v_0'\}})$ is isomorphic to some closed star deleted one or two vertices from its link in the homology n-manifold L.

On the other hand,

$$\mathrm{St}(a_0', L'_{S'-\{v_0'\}}) \cong \mathrm{St}(a_0, L_{S-\{v_0\}}) \cong \mathrm{St}(a_0, L),$$

since ϕ is an isomorphism and $a_0 \notin \mathrm{St}(v_0, L)$. Here we note that $\mathrm{St}(a_0, L)$ is the closed star in the homology n-manifold L. Hence we obtain that
(II) $\mathrm{St}(a_0', L'_{S'-\{v_0'\}})$ is isomorphic to some closed star in the homology n-manifold L.

Then (I) and (II) imply the contradiction. Indeed the following claim holds.

Claim. *Let $A = \mathrm{St}(a)$ be a closed star of a vertex a in a homology n-manifold and let $B = \mathrm{St}(b) - \{c_1, c_2\}$ be a closed star of a vertex b deleted one or two vertices $\{c_1, c_2\} \subset \mathrm{Lk}(b)$ in a homology n-manifold. Then the simplicial complexes A and B are not isomorphic.*

We first note that every triangulated homology n-manifold is a union of n-simplexes ([25, Corollary 63.3 (a)]). Hence $A = \mathrm{St}(a)$ and $\mathrm{St}(b)$ are unions of n-simplexes containing a and b respectively. Then there exists an n-simplex σ_0 such that $c_1 \in \sigma_0 \subset \mathrm{St}(b)$.

Here if $c_1 \neq c_2$ then we can take σ_0 as $c_2 \notin \sigma_0$. Indeed if $c_1 \neq c_2$ and $c_2 \in \sigma_0$ then $[c_1, c_2] \subset \sigma_0$ and we can consider $(n-1)$-simplex τ as $\tau^{(0)} = \sigma_0^{(0)} - \{c_2\}$. Then by [25, Corollary 63.3 (b)], there exist precisely two n-simplexes containing τ as a face. Hence we can take an n-simplex σ_0' containing τ as a face and $\sigma_0' \neq \sigma_0$. Then $c_1 \in \sigma_0' \subset \mathrm{St}(b)$ and $c_2 \notin \sigma_0'$. Hence in this case we retake σ_0 as σ_0'.

Now σ_0 is an n-simplex such that $c_1 \in \sigma_0 \subset \mathrm{St}(b)$ and if $c_1 \neq c_2$ then $c_2 \notin \sigma_0$. Let τ_0 be the $(n-1)$-simplex as $\tau_0^{(0)} = \sigma_0^{(0)} - \{c_1\}$. Then we note that $\tau_0 \subset \mathrm{St}(b) - \{c_1, c_2\} = B$.

Now we suppose that A and B are isomorphic and there exists an isomorphism $g\colon B \to A$. Then $g(\tau_0)$ is an $(n-1)$-simplex in A. By [25, Corollary 63.3 (b)], there exist precisely two n-simplexes $\bar{\sigma}_1$ and $\bar{\sigma}_2$ containing $g(\tau_0)$ as a face in A. Then $g^{-1}(\bar{\sigma}_1)$ and $g^{-1}(\bar{\sigma}_2)$ are n-simplexes containing τ_0 as a face in B, since $g\colon B \to A$ is an isomorphism. Here $g^{-1}(\bar{\sigma}_1)$, $g^{-1}(\bar{\sigma}_2)$ and σ_0 are distinct n-simplexes containing τ_0 as a face in $\mathrm{St}(b)$. This contradicts to [25, Corollary 63.3 (b)].

Thus the simplicial complexes A and B are not isomorphic.

Hence, there does not exist $a_0 \in S - \{v_0\}$ such that $a_0 \notin \mathrm{Lk}(v_0, L)^{(0)}$ and $\phi(a_0) \in \mathrm{Lk}(v_0', L')^{(0)}$, that is, for $a \in S - \{v_0\}$, $a \in \mathrm{Lk}(v_0, L)^{(0)}$ if and only if $\phi(a) \in \mathrm{Lk}(v_0', L')^{(0)}$, since $\deg v_0 = \deg v_0'$. Hence the map $\bar{\phi}\colon S \to S'$ defined by $\bar{\phi}|_{S-\{v_0\}} = \phi$ and $\bar{\phi}(v_0) = v_0'$ induces an isomorphism of the two graphs Γ and Γ'.

Therefore the graph Γ is reconstructible. □

3. Virtual Poincaré duality Coxeter groups and reconstructible graphs

We introduce a relation of virtual Poincaré duality Coxeter groups and reconstructible graphs, which is our motivation of this paper.

DEFINITION 3.1 (cf. [7], [10], [11], [13]). A torsion-free group G is called an *n-dimensional Poincaré duality group*, if G is of type FP and if

$$H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & (i \neq n), \\ \mathbb{Z} & (i = n). \end{cases}$$

Also a group G is called a *virtual Poincaré duality group*, if G contains a torsion-free subgroup of finite-index which is a Poincaré duality group.

On Coxeter groups and (virtual) Poincaré duality groups, the following results are known.

Theorem 3.2 (Farrell [13, Theorem 3]). *Suppose that G is a finitely presented group of type FP, and let n be the smallest integer such that $H^n(G; \mathbb{Z}G) \neq 0$. If $H^n(G; \mathbb{Z}G)$ is a finitely generated abelian group, then G is an n-dimensional Poincaré duality group.*

REMARK. It is known that every infinite Coxeter group W contains some torsion-free subgroup G of finite-index in W which is a finitely presented group of type FP and $H^*(G; \mathbb{Z}G)$ is isomorphic to $H^*(W; \mathbb{Z}W)$. Hence if n is the smallest integer such that $H^n(W; \mathbb{Z}W) \neq 0$ and if $H^n(W; \mathbb{Z}W)$ is finitely generated (as an abelian group), then W is a virtual Poincaré duality group of dimension n.

Theorem 3.3 (Davis [10, Theorem 5.5]). *Let (W, S) be a Coxeter system. Then the following statements are equivalent*:

(1) *W is a virtual Poincaré duality group of dimension n.*

(2) *W decomposes as a direct product $W = W_{T_0} \times W_{T_1}$ such that T_1 is a spherical subset of S and the simplicial complex $L_{T_0} = L(W_{T_0}, T_0)$ associated by (W_{T_0}, T_0) is a generalized homology $(n-1)$-sphere.*

Theorem 3.4 ([17, Corollary 4.2]). *Let (W, S) be an infinite irreducible Coxeter system, let $L = L(W, S)$ and let $0 \leq i \in \mathbb{Z}$. Then the following statements are equivalent*:

(1) *$H^i(W; \mathbb{Z}W)$ is finitely generated.*

(2) *$H^i(W;\mathbb{Z}W)$ is isomorphic to $\tilde{H}^{i-1}(L)$.*
(3) *$\tilde{H}^{i-1}(L_{S-T}) = 0$ for any non-empty spherical subset T of S.*
Here $L_{S-T} = L(W_{S-T}, S-T)$.

We obtain the following lemma from results above.

Lemma 3.5. *Let (W,S) be an irreducible Coxeter system where W is infinite and let $L = L(W, S)$. Then the following statements are equivalent*:
(1) *W is a virtual Poincaré duality group.*
(2) *L is a generalized homology sphere.*
(3) *$\tilde{H}^i(L_{S-T}) = 0$ for any i and any non-empty spherical subset T of S.*

Proof. (1) $\Leftrightarrow$ (2): We obtain the equivalence of (1) and (2) from Theorem 3.3, since (W, S) is irreducible.

(1) $\Rightarrow$ (3): We obtain this implication from Theorem 3.4, because if W is a virtual Poincaré duality group then $H^i(W;\mathbb{Z}W)$ is finitely generated for any i.

(3) $\Rightarrow$ (1): Suppose that $\tilde{H}^i(L_{S-T}) = 0$ for any i and any non-empty spherical subset T of S. Then by Theorem 3.4, $H^{i+1}(W;\mathbb{Z}W)$ is finitely generated for any i. Since W is infinite, $H^{i_0}(W;\mathbb{Z}W)$ is non-trivial for some i_0 (cf. [7], [14]). Hence by Theorem 3.2, W is a virtual Poincaré duality group. □

We obtain Theorem 1.2 from Theorem 1.3. In particular, we obtain the following.

Theorem 3.6. *Let Γ be a finite graph with at least 3 vertices and let (W, S) be the right-angled Coxeter system associated by Γ. If the Coxeter group W is an irreducible virtual Poincaré duality group, then the graph Γ is reconstructible.*

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