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BASIC VECTOR VALUED SIEGEL MODULAR FORMS OF GENUS TWO

In memoriam Jun-Ichi Igusa (1924–2013)

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Abstract

We consider the space $[\Gamma, rr_0, v^r, \varrho]$ of all vector-valued holomorphic modular forms $f\colon \mathbb{H}_n \to \mathcal{Z}$ of transformation type

$$f(MZ) = v^r(M)\det(CZ + D)^{r_0 r}\varrho(CZ + D)f(Z).$$

$\varrho\colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(\mathcal{Z})$ is a rational representation on a finite dimensional complex vector space $\mathcal{Z}$. These spaces can be collected in a graded $A(\Gamma)$-module

$$\mathcal{M} = \mathcal{M}_\Gamma(r_0, v, \varrho) := \bigoplus_{r\in\mathbb{Z}} [\Gamma, rr_0, v^r, \varrho].$$

We treat in this paper some special cases in genus 2. The first one is essentially due to Wieber. Here the starting weight is $1/2$, the starting multiplier system is the multiplier system v_Θ and for ϱ we take the second symmetric power of standard representation. Thus we consider a variant of this case and a new example. In this final case the starting weight is $1/2$, the starting multiplier system is the theta multiplier system v_ϑ and for ϱ we take the standard representation. In all these cases we will determine the structure of $\mathcal{M}$.

Introduction

For a congruence subgroup $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ of the Siegel modular group of genus n, a starting weight $r_0 > 0$, $2r_0 \in \mathbb{Z}$, and a starting multiplier system v of weight r_0 one can consider the ring of modular forms (for definitions we refer to Section 1)

$$A(\Gamma) = A(\Gamma, r_0, r) = \bigoplus_{r\in\mathbb{Z}} [\Gamma, rr_0, v^r].$$

This is a finitely generated algebra. In addition, let $\varrho\colon \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(\mathcal{Z})$ be a rational representation on a finite dimensional complex vector space $\mathcal{Z}$. We always will assume that ϱ is irreducible and polynomial and does not vanish on the determinant surface $\det A = 0$. Then we consider the space $[\Gamma, rr_0, v^r, \varrho]$ of all vector-valued holomorphic

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modular forms $f : \mathbb{H}_n \to \mathcal{Z}$ of transformation type

$$f(MZ) = v^r(M)\det(CZ + D)^{r_0 r}\varrho(CZ + D)f(Z).$$

We collect these spaces to the graded $A(\Gamma)$-module

$$\mathcal{M} = \mathcal{M}_\Gamma(r_0, v, \varrho) := \bigoplus_{r\in\mathbb{Z}} [\Gamma, rr_0, v^r, \varrho].$$

There are twisted variants of these modules [16]. For a character χ on Γ one can consider

$$\mathcal{M}^\chi = \bigoplus_{r\in\mathbb{Z}} [\Gamma, rr_0, \chi v^r, \varrho].$$

These modules are finitely generated. It is a natural task to look for examples where the structure of this module can be determined.

Meanwhile there appeared several papers getting results into this direction using different methods, [1, 2, 3, 9, 14, 15, 16]. Our method is a further development of Wieber's geometric method [16] which he used to solve certain $\Gamma_2[2, 4]$-cases.

The geometric method of Wieber rests on the fact that vector valued modular forms sometimes can be interpreted as Γ-invariant tensors and hence as (usually rational) tensors on the Siegel modular variety $\mathbb{H}_n/\Gamma$. In some cases the structure of this variety is known which enables to study tensors on it in detail. A vector valued modular form can define a tensor only if the representation ϱ up to a power of the determinant occurs in some tensor power of the representation Sym^2. This is not always the case. For example the standard representation of $\mathrm{GL}(2, \mathbb{C})$ does not have this property. In this paper we describe a modification of Wieber's method which allows to recover his main results in [16] in a quick way and which applies to more cases as for example the standard representation.

Recall that the principal congruence subgroup is defined as

$$\Gamma_n[q] = \mathrm{kernel}(\mathrm{Sp}(n, \mathbb{Z}) \to \mathrm{Sp}(n, \mathbb{Z}/q\mathbb{Z}))$$

and Igusa's subgroup as

$$\Gamma_n[q, 2q] := \{M \in \Gamma_n[q],\ (C\,{}^tD)_0 \equiv (A\,{}^tB)_0 \equiv 0 \bmod 2q\}.$$

Here S_0 denotes the column built of the diagonal of a square matrix S.

Besides Wieber's known results we treat in this paper a new example that belongs to the group $\Gamma_2[4, 8]$. The starting weight is $1/2$, the starting multiplier system is the theta multiplier system v_ϑ and for ϱ we take the standard representation. In this case we will determine the structure of $\mathcal{M}$ completely (Theorem 7.1). It will turn out that $\mathcal{M}$ can be generated by the Γ_2 orbits of two specific modular forms. We will describe the relations and—as a consequence—we will obtain the Hilbert function of this module (Theorem 7.2).

We want to thank Wieber for fruitful discussion and for his help with quite involved computer calculations.

1. Vector valued modular forms

We consider the Siegel modular group $\Gamma_n = \mathrm{Sp}(n, \mathbb{Z})$ of genus n. It consists of all integral $2n \times 2n$-matrices M such that ${}^tMIM = I$, where $I = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ (E denotes the unit matrix) is the standard alternating matrix. Let $\Gamma \subset \mathrm{Sp}(n, \mathbb{Z})$ be a congruence subgroup, r be an integer, v a multiplier system of weight $r/2$ on Γ and $\varrho\colon \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathcal{Z})$ a rational representation on a finite dimensional complex vector space $\mathcal{Z}$. We assume that ϱ is reduced which means that it is polynomial and does not vanish along the determinant surface $\det(A) = 0$. Then we can consider the space $[\Gamma, r/2, v, \varrho]$. It consists of holomorphic functions $f : \mathbb{H}_n \to \mathcal{Z}$ on the Siegel upper half-plane

$$\mathbb{H}_n = \{Z \in \mathbb{C}^{n \times n};\ \mathrm{Im}\, Z > 0 \text{ (positive definite)}\}$$

with the transformation property

$$f(MZ) = v(M)\sqrt{\det(CZ + D)}^{r}\varrho(CZ + D)f(Z) \quad (M \in \Gamma).$$

(In the case $n = 1$ a condition at the cusps has to be added.) We can also consider meromorphic solutions f and call them meromorphic modular forms if they satisfy a meromorphicity condition at the cusps (which in most cases will be automatically true). What we demand is that there exists a non-vanishing holomorphic scalar valued form g that fg is holomorphic. We denote the space of meromorphic modular forms by $\{\Gamma, r/2, v, \varrho\}$. It is a vector space of dimension $\leq \mathrm{Rank}(\varrho)$ over the field of modular functions

$$K(\Gamma) = \{\Gamma, 0, \mathrm{triv}, \mathrm{triv}\},$$

in fact the wedge product of $k = \mathrm{Rank}(\varrho)$ meromorphic vector valued modular forms is a, possibly zero, scalar modular form. When ϱ is the one-dimensional trivial representation, we simply write $\{\Gamma, r/2, v\}$ instead of $\{\Gamma, r/2, v, \varrho\}$ and similarly we skip v if r is even and v trivial. The same convention is used for the spaces of holomorphic modular forms.

2. Thetanullwerte

An element $m \in \{0, 1\}^{2n}$ is called a theta characteristic of genus n. Usually it is considered as column and divided into two columns a, b of length n. It is called even if ${}^tab$ is even and odd else. We use the classical theta series

$$\vartheta[m](Z, z) = \sum_{g \in \mathbb{Z}^n} \exp \pi\mathrm{i}(Z[g + a/2] + 2\,{}^t(g + a/2)(z + b/2).$$

We are interested in the nullwerte

$$\vartheta[m](Z) = \vartheta[m](Z, 0)$$

and in the nullwerte of the derivatives

$$\frac{\partial}{\partial z_i}\vartheta[m](Z, z)\Big|_{z=0}.$$

We collect them in a column which we denote by grad $\vartheta[m](Z)$. We recall that the theta nullwerte are non-zero only for even and the gradients for odd characteristics.

Besides the nullwerte of first kind $\vartheta[m]$ the nullwerte of second kind

$$f_a(Z) := \vartheta\begin{bmatrix} a \\ 0 \end{bmatrix}(2Z), \quad a \in (\mathbb{Z}/2\mathbb{Z})^n,$$

will play a role.

We recall that $\vartheta[0](Z)$ is a modular form of weight $1/2$ for the *theta group*

$$\Gamma_{n,\vartheta} := \Gamma_n[1, 2]$$

with respect to a certain multiplier system v_ϑ on this group. Since $\Gamma_{n,\vartheta} \supset \Gamma_n[2]$ we have in particular

$$\vartheta[0] \in [\Gamma_n[2], 1/2, v_\vartheta].$$

For each characteristic there exists a character χ_m on $\Gamma_n[2]$ which is trivial on $\Gamma_n[4, 8]$ and quadratic on the group $\Gamma_n[2, 4]$ such that

$$\vartheta[m] \in [\Gamma_n[2], 1/2, v_\vartheta \chi_m] \quad \text{and} \quad \operatorname{grad} \vartheta[m] \in [\Gamma_n[2], 1/2, v_\vartheta \chi_m, \mathrm{St}].$$

In particular, all thetanullwerte $\vartheta[m]$ are contained in $[\Gamma_n[4, 8], 1/2, v_\vartheta]$. For details we refer to [13].

Similar results hold for the thetas of second kind f_a. They are modular forms for $\Gamma_n[2, 4]$ with respect to a certain multiplier system v_Θ on this group,

$$f_a \in [\Gamma_n[2, 4], 1/2, v_\Theta].$$

We consider the rings

$$A(\Gamma_n[4, 8]) = \bigoplus_{r\in\mathbb{Z}} [\Gamma_n[4, 8], r/2, v_\vartheta^r], \quad A(\Gamma_n[2, 4]) = \bigoplus_{r\in\mathbb{Z}} [\Gamma_n[2, 4], r/2, v_\Theta^r].$$

So the starting weights are $1/2$ in both cases and the starting multiplier system is v_ϑ for $\Gamma_n[4, 8]$ but v_Θ for $\Gamma_n[2, 4]$. We mention that the two multiplier systems are different on $\Gamma_n[4, 8]$. The following results are basic. The first one has proved by Igusa 1964 [10] the second by Runge 1994 [11, 12].

Theorem 2.1 (Igusa, Runge)**.**

$$A(\Gamma_n[4,8]) = \mathbb{C}[\ldots, \vartheta[m], \ldots] \quad \textit{for} \quad n \leq 2,$$
$$A(\Gamma_n[2,4]) = \mathbb{C}[\ldots, f_a, \ldots] \quad \textit{for} \quad n \leq 3.$$

In the case $n = 2$ Runge obtained an even better result. It is known that the square of v_Θ is a non-trivial quadratic character on $\Gamma_2[2,4]$. The kernel of v_Θ^2 is a subgroup of index two of $\Gamma_2[2,4]$. We use Runge's notation $\Gamma_2^*[2,4]$ for it.

Recall that Igusa's modular form χ_5 is the unique cusp form of weight 5 for the full Siegel modular form. It can be defined as the product of the ten theta constants of first kind. Its character is trivial on $\Gamma_2[2]$.

We denote the 4 functions f_a in the ordering $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ by $f_0, \ldots, f_3$.

Theorem 2.2 (Runge)**.**

$$A(\Gamma_2^*[2,4]) = \bigoplus_{r\in\mathbb{Z}} [\Gamma_2^*[2,4], r/2, v_\Theta^r] = \mathbb{C}[f_0, \ldots, f_3] \oplus \chi_5\mathbb{C}[f_0, \ldots, f_3].$$

3. A first example due to Wieber

From now on we shall assume $n = 2$. We consider the module $\mathcal{M}$ introduced in the introduction, for the group $\Gamma_2[2,4]$, starting weight $1/2$ and starting multiplier system v_Θ and for the representation Sym^2. We will write $\mathcal{M}^+$ instead of $\mathcal{M}$ since, in the next section, we shall treat a twisted variant $\mathcal{M}^-$.

The representation Sym^2 of $\mathrm{GL}(2,\mathbb{C})$ can be realized on the space of symmetric 2×2-matrices and the action of $\mathrm{GL}(2,\mathbb{C})$ is given by $AW\,{}^tA$.

This means that we have to consider symmetric 2×2-matrices f of holomorphic functions with the transformation property

$$f(MZ) = v_\Theta(M)^r \sqrt{CZ+D}^r (CZ+D) f(Z)\,{}^t(CZ+D).$$

They define a vector space

$$\mathcal{M}^+(r) = [\Gamma_2[2,4], r/2, v_\Theta^r, \mathrm{Sym}^2].$$

We consider the direct sum

$$\mathcal{M}^+ = \bigoplus_{r\in\mathbb{Z}} \mathcal{M}^+(r).$$

This is a graded module over the ring

$$A(\Gamma_2[2,4]) := \bigoplus [\Gamma_2[2,4], r/2, v_\Theta^r] = \mathbb{C}[f_0, \ldots, f_3].$$

Elements of $\mathcal{M}^+$ can be constructed as follows. Let $f, g \in [\Gamma, r/2, v_\Theta^r]$, $g \neq 0$. Then f/g is a modular function and $d(f/g)$ is a meromorphic differential. It can be considered as element of $\{\Gamma_2[2, 4], 0, \mathrm{Sym}^2\}$. If we multiply it by g^2 we get a holomorphic form

$$\{f, g\} = g^2\, d(f/g) \in \mathcal{M}^+(2r).$$

Theorem 3.1 (Wieber)**.** *We have*

$$\mathcal{M}^+ = \sum_{0 \le i < j \le 3} \mathbb{C}[f_0, \ldots, f_3]\{f_i, f_j\}.$$

Defining relations of this module are

$$f_k[f_i, f_j] = f_j[f_i, f_k] + f_i[f_k, f_j], \quad [f_i, f_j] + [f_j, f_i] = 0.$$

Proof. We give a new simple proof for this result. We consider $\{\Gamma[2,4], 1, \mathrm{Sym}^2\}$ as vector space over the field of modular functions. The three forms $\{f_0, f_i\}$, $1 \le i \le 3$, give a basis of this vector space.

Lemma 3.2. *With some non-zero constant C we have*

$$f_0^4\, d(f_1/f_0) \wedge d(f_2/f_0) \wedge d(f_3/f_0) = C\chi_5\, dz_0 \wedge dz_1 \wedge dz_2, \quad Z = \begin{pmatrix} z_0 & z_1 \\ z_1 & z_2 \end{pmatrix}.$$

Corollary. *If we consider the three elements*

$$\{f_0, f_1\}, \quad \{f_0, f_2\}, \quad \{f_0, f_3\}$$

as a 3×3-matrix, its determinant is up to a constant factor $f_0^2\chi_5$.

This lemma is well-known. For sake of completeness we give the argument. The left hand side is holomorphic and it defines a modular form of weight 5 with respect to the full Siegel modular form. So it must be a constant multiple of χ_5. □

We mentioned already in the introduction that $\{\Gamma, r/2, v, \varrho\}$ is a vector space of dimension $\le \mathrm{Rank}(\varrho)$ on the field of modular functions relative to Γ. We can get a basis of $\{\Gamma_2[2, 4], r/2, v_\Theta^r, \varrho\}$ if we multiply the three $\{f_0, f_i\}$ by f_0^{r-2}. Hence an arbitrary $T \in \{\Gamma, r/2, v, \varrho\}$ can be written in the form

$$T = g_1\{f_0, f_1\} + g_2\{f_0, f_2\} + g_3\{f_0, f_3\}$$

where g_i are meromorphic modular forms in $\{\Gamma_2[2,4], (r-2)/2, v_\Theta^{r-2}\}$. They are rational functions in the f_i. If T is holomorphic then $f_0^2\chi_5 g_i$ must be holomorphic. The form

$f_0^2\chi_5 g_i$ is contained in $A(\Gamma_2^*[2,4])$ and $f_0^2 g_i$ are also rational in the f_i. From Runge's result (Theorem 2.1) follows that $f_0^2 g_i$ is holomorphic. In other words

$$\mathcal{M}^+ \subset \frac{1}{f_0^2}\sum_{i=1}^{3}\mathbb{C}[f_0,\ldots,f_3]\{f_0, f_i\}.$$

The rest is just Wieber's argument that we can permute the variables and obtain that $\mathcal{M}$ is contained in the intersection of 4 modules,

$$\mathcal{M}^+ \subset \bigcap_{i=0}^{3}\frac{1}{f_i^2}\sum_{j=1}^{3}\mathbb{C}[f_0,\ldots,f_3]\{f_i, f_j\}.$$

Using the fact that $\mathcal{M}^+$ is contained in the free module generated by df_i (as symbols), it is easy to show (compare [16]) that this intersection equals

$$\sum_{0\le i<j\le 3}\mathbb{C}[f_0,\ldots,f_3]\{f_i, f_j\}.$$

which is Wieber's result.

4. A second example of Wieber

Always with respect to the group $\Gamma_2[2,4]$, Wieber also considers the twist of $\mathcal{M}^+$ with the quadratic character v_Θ^{-2}. To be precise, he introduces the spaces $\mathcal{M}^-(r)$ consisting of holomorphic forms of the type

$$f(MZ) = v_\Theta(M)^{r-2}\sqrt{CZ+D)}^{r}(CZ+D)f(Z)\,{}^t(CZ+D).$$

They can be collected to

$$\mathcal{M}^- = \bigoplus \mathcal{M}^-(r)$$

which is also a graded module over $\mathbb{C}[f_0,\ldots,f_4]$. There are obvious inclusions

$$\chi_5\mathcal{M}^+ \subset \mathcal{M}^-, \quad \chi_5\mathcal{M}^- \subset \mathcal{M}^+.$$

Following more general constructions of Ibukiyama [9], Wieber defined elements of $\mathcal{M}^-$ in a different way. He considers three homogeneous elements f, g, h of $\mathbb{C}[f_0,\ldots,f_3]$ of degree r and considers then the differential form

$$d(g/f)\wedge d(h/f) = h_0\,dz_1\wedge dz_2 + h_1\,dz_0\wedge dz_2 + h_2\,dz_0\wedge dz_1.$$

Then

$$\begin{pmatrix} h_2 & -h_1 \\ -h_1 & h_0 \end{pmatrix} \in \{\Gamma_2[2,4], 1, \mathrm{Sym}^2\}.$$

(The multiplier system is trivial.) We set

$$\{f, g, h\} = f^3\begin{pmatrix} h_2 & -h_1 \\ -h_1 & h_0 \end{pmatrix}.$$

It is easy to see that this is holomorphic. It is contained in

$$[\Gamma_2[2, 4], (3r + 2)/2, v_\Theta^{3r}, \mathrm{Sym}^2] = \mathcal{M}^-(3r + 2).$$

If f, g, h have weight $1/2$, then this form is contained in $\mathcal{M}^-(5)$. An arbitrary $T \in [\Gamma, (r + 5)/2, v, \varrho]$ can be written in the form

$$T = g_1\{f_0, f_1, f_2\} + g_2\{f_0, f_1, f_3\} + g_3\{f_0, f_2, f_3\}$$

where g_i are meromorphic modular forms in $\{\Gamma_2[2, 4], r/2, v_\Theta^r\}$. They are expressible as quotients of homogeneous polynomials in the variables f_i. We have to work out that this form, or equivalently

$$f_0^3(g_1\, d(f_1/f_0) \wedge d(f_2/f_0) + g_2 d(f_1/f_0) \wedge d(f_3/f_0) + g_3 d(f_2/f_0) \wedge d(f_3/f_0)),$$

is holomorphic. We take the wedge product with $f_0^2\, df(f_i/f_0)$ and obtain from Lemma 3.2 that $f_0\chi_5 g_i$ are holomorphic. Hence the argument of the previous section shows that $f_0 g_i$ is a polynomial in the f_i. Similar to the previous section we obtain

$$\mathcal{M}^- \subset \bigcap_{i=0}^{3} \frac{1}{f_i} \sum_{i<j<k} \mathbb{C}[f_0, \ldots, f_3]\{f_i, f_j, f_k\}.$$

A simple argument now gives the following result.

Theorem 4.1 (Wieber)**.** *We have*

$$\mathcal{M}^- = \sum_{0\le i<j<k\le 3} \mathbb{C}[f_0, \ldots, f_3]\{f_i, f_j, f_k\}.$$

Defining relation of this module is

$$f_3\{f_0, f_1, f_2\} = f_0\{f_1, f_2, f_3\} - f_1\{f_0, f_2, f_3\} + f_2\{f_0, f_1, f_3\}.$$

We observe that the relation is an immediate consequence of the fact that we have four generators and it is induced by the standard properties of the differentiation.

5. The standard representation

In this section we study the module $\mathcal{M}$ for the group $\Gamma_2[4,8]$, starting weight $1/2$, starting multiplier system v_ϑ and the standard representation $\mathrm{St} = \mathrm{id}$,

$$\mathcal{M} = \bigoplus_{r\in\mathbb{Z}} [\Gamma_2[4,8],\, r/2,\, v_\vartheta^r,\, \mathrm{St}].$$

This is a module over the ring $A(\Gamma_2[4,8])$ which, by Igusa's result, is generated by the ten even theta nullwerte. We will order them as follows:

$$(m^{(1)},\ldots,m^{(10)}) = \begin{pmatrix} 0&0&0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&0&0&1&1\\ 0&0&1&1&0&1&0&0&0&1\\ 0&1&0&1&0&0&0&1&0&1 \end{pmatrix}.$$

The associated theta series are denote by $\vartheta_1,\ldots,\vartheta_{10}$ (in this ordering). They satisfy the quartic Riemann relations which are defining relations:

$$\begin{aligned}
&\vartheta_6^2\vartheta_8^2 - \vartheta_4^2\vartheta_9^2 + \vartheta_1^2\vartheta_{10}^2 = 0, && \vartheta_4^2\vartheta_5^2 - \vartheta_2^2\vartheta_6^2 - \vartheta_7^2\vartheta_{10}^2 = 0,\\
&\vartheta_5^2\vartheta_8^2 - \vartheta_2^2\vartheta_9^2 + \vartheta_3^2\vartheta_{10}^2 = 0, && \vartheta_3^2\vartheta_5^2 - \vartheta_1^2\vartheta_6^2 - \vartheta_8^2\vartheta_{10}^2 = 0,\\
&\vartheta_7^4 - \vartheta_8^4 - \vartheta_9^4 + \vartheta_{10}^4 = 0, && \vartheta_2^2\vartheta_5^2 - \vartheta_4^2\vartheta_6^2 - \vartheta_8^2\vartheta_9^2 = 0,\\
&\vartheta_6^2\vartheta_7^2 - \vartheta_3^2\vartheta_9^2 + \vartheta_2^2\vartheta_{10}^2 = 0, && \vartheta_1^2\vartheta_5^2 - \vartheta_3^2\vartheta_6^2 - \vartheta_7^2\vartheta_9^2 = 0,\\
&\vartheta_5^2\vartheta_7^2 - \vartheta_1^2\vartheta_9^2 + \vartheta_4^2\vartheta_{10}^2 = 0, && \vartheta_3^4 - \vartheta_4^4 - \vartheta_6^4 + \vartheta_{10}^4 = 0,\\
&\vartheta_4^2\vartheta_7^2 - \vartheta_3^2\vartheta_8^2 - \vartheta_5^2\vartheta_{10}^2 = 0, && \vartheta_2^2\vartheta_3^2 - \vartheta_1^2\vartheta_4^2 + \vartheta_9^2\vartheta_{10}^2 = 0,\\
&\vartheta_3^2\vartheta_7^2 - \vartheta_4^2\vartheta_8^2 - \vartheta_6^2\vartheta_9^2 = 0, && \vartheta_1^2\vartheta_3^2 - \vartheta_2^2\vartheta_4^2 - \vartheta_5^2\vartheta_6^2 = 0,\\
&\vartheta_2^2\vartheta_7^2 - \vartheta_1^2\vartheta_8^2 - \vartheta_6^2\vartheta_{10}^2 = 0, && \vartheta_2^4 - \vartheta_4^4 - \vartheta_8^4 + \vartheta_{10}^4 = 0,\\
&\vartheta_1^2\vartheta_7^2 - \vartheta_2^2\vartheta_8^2 - \vartheta_5^2\vartheta_9^2 = 0, && \vartheta_1^2\vartheta_2^2 - \vartheta_3^2\vartheta_4^2 - \vartheta_7^2\vartheta_8^2 = 0,\\
&\vartheta_5^4 - \vartheta_6^4 - \vartheta_9^4 + \vartheta_{10}^4 = 0, && \vartheta_1^4 - \vartheta_2^4 - \vartheta_6^4 - \vartheta_9^4 = 0.
\end{aligned}$$

We order the 6 odd characteristics as follows:

$$(n^{(1)},\ldots,n^{(6)}) = \begin{pmatrix} 0&0&1&1&1&1\\ 1&1&0&0&1&1\\ 0&1&1&1&0&1\\ 1&1&0&1&1&0 \end{pmatrix}.$$

We use the notation

$$e(m) = (-1)^{{}^t ab} \qquad \left(m = \tbinom{a}{b}\right).$$

Recall that a triplet m_1, m_2, m_3 of characteristics is called *azygetic* if they are pairwise different and if

$$e(m_1)e(m_2)e(m_3)e(m_1+m_2+m_3)=-1.$$

The following result has been stated without proof by Rosenhain and proved by Thomae and Weber. A proof can be found in [4].

Lemma 5.1. *For two different odd characteristics m, n there exist* 4 *even characteristics $n_1, \dots, n_4$ such that m, n, n_i is azygetic. If we consider the pair* $(\operatorname{grad}\vartheta[m], \operatorname{grad}\vartheta[n])$ *as a* 2×2 *matrix then*

$$\det(\operatorname{grad}\vartheta[m], \operatorname{grad}\vartheta[n]) = \pm\pi^2\vartheta[n_1]\cdots\vartheta[n_4].$$

Since the signs are essential for us, we collect them in a table. This table can be found in [4]. (One sign in [4] had to be corrected.) We use the abbreviation

$$D(i,j)=\pi^{-2}\det(\operatorname{grad}\vartheta[n^{(i)}], \operatorname{grad}\vartheta[n^{(j)}]),\quad 1\le i<j\le 6,$$

$$\begin{aligned}
&D(1,2)=\vartheta_7\vartheta_8\vartheta_9\vartheta_{10}, && D(1,3)=\vartheta_2\vartheta_3\vartheta_5\vartheta_7, && D(1,4)=\vartheta_1\vartheta_4\vartheta_5\vartheta_8,\\
&D(1,5)=-\vartheta_3\vartheta_4\vartheta_6\vartheta_{10}, && D(1,6)=\vartheta_1\vartheta_2\vartheta_6\vartheta_9, && D(2,3)=\vartheta_1\vartheta_4\vartheta_6\vartheta_7,\\
&D(2,4)=\vartheta_2\vartheta_3\vartheta_6\vartheta_8, && D(2,5)=-\vartheta_1\vartheta_2\vartheta_5\vartheta_{10}, && D(2,6)=\vartheta_3\vartheta_4\vartheta_5\vartheta_9,\\
&D(3,4)=-\vartheta_5\vartheta_6\vartheta_9\vartheta_{10}, && D(3,5)=-\vartheta_1\vartheta_3\vartheta_8\vartheta_9, && D(3,6)=\vartheta_2\vartheta_4\vartheta_8\vartheta_{10},\\
&D(4,5)=-\vartheta_2\vartheta_4\vartheta_7\vartheta_9, && D(4,6)=\vartheta_1\vartheta_3\vartheta_7\vartheta_{10}, && D(5,6)=\vartheta_5\vartheta_6\vartheta_7\vartheta_8.
\end{aligned}$$

We describe 20 relations between the 6 generators grad_i of the module $\mathcal{M}$. We use the notation

$$\operatorname{grad}_i=\operatorname{grad}\vartheta[n^{(i)}],\quad 1\le i\le 6.$$

Lemma 5.2. *For* $1\le i<j<k\le 6$ *the relation*

$$D(i,j)\operatorname{grad}_k = D(i,k)\operatorname{grad}_j - D(j,k)\operatorname{grad}_i$$

holds. Each of them is divisible by one of the ϑ_i. Hence we obtain 20 *relations where a typical one is*

$$\vartheta_1\vartheta_4\vartheta_6\operatorname{grad}_1-\vartheta_2\vartheta_3\vartheta_5\operatorname{grad}_2-\vartheta_8\vartheta_9\vartheta_{10}\operatorname{grad}_3=0.$$

Proof. The proof is trivial. Just notice that the occurring D-s are just the 2×2 sub-determinants of the 2×3-matrix $(\operatorname{grad}_i, \operatorname{grad}_j, \operatorname{grad}_k)$. Now Lemma 5.2 is just a consequence of the known fact that the cross product $a\times b$ of two vectors in $\mathbb{C}^3$ is orthogonal to both a, b. □

We fix two odd characteristics m, n. Every homogeneous element of $\mathcal{M}^+$ can be written in the form

$$T = g_m \operatorname{grad} \vartheta[m] + g_n \operatorname{grad} \vartheta[n]$$

with two meromorphic modular forms from $A(\Gamma[4, 8])$. From Lemma 5.1 we can deduce that $g_i \vartheta[n_1] \cdots \vartheta[n_4]$ are holomorphic. Hence $\mathcal{M}$ is contained in

$$\mathcal{M}(m, n) := \frac{1}{\vartheta[n_1] \cdots \vartheta[n_4]} (A(\Gamma[4, 8]) \operatorname{grad} \vartheta[m] + A(\Gamma[4, 8]) \operatorname{grad} \vartheta[n]).$$

We can vary m, n and obtain

$$\mathcal{M} \subset \bigcap_{m,n} \mathcal{M}(m, n).$$

The elements in the right hand side have no poles outside the zeros of the forms $\vartheta[n_1] \cdots \vartheta[n_4]$. These 15 forms have no joint zero in $\mathbb{H}_2$, in fact they are, relatively to the genus 2 case, the Plucker's coordinates of the map studied in [8], cf. [4] also. Hence the elements of the intersection are holomorphic. So we obtain the following proposition.

Proposition 5.3. *We have*

$$\mathcal{M} = \bigcap_{m,n} \mathcal{M}(m, n).$$

We consider the submodule $\mathcal{N}$ of $\mathcal{M}$ that is generated by all grad_i over $A(\Gamma[4, 8])$. Proposition 5.3 shows that $\mathcal{M}$ is a submodule of $(1/\chi_5)\mathcal{N}$. It is described as finite intersection of certain submodules which are defined by means of finitely many generators. As soon as we understand the structure of $(1/\chi_5)\mathcal{N}$, or equivalently, of $\mathcal{N}$, we have a chance to determine this intersection. In the next section we shall describe all relations between the six grad_i.

6. Relations

In Lemma 5.2 we described some of the relations between the grad_i. It will turn out they do not generate all relations. To describe all relations we introduce the free module of rank 6 over $A(\Gamma[4, 8])$. We denote the generators by $T_1, \ldots, T_6$. We have to describe the kernel of the natural homomorphism

$$\mathcal{F} \to \mathcal{N}, \quad T_i \mapsto \operatorname{grad}_i.$$

We denote by $\mathcal{K}$ the submodule of $\mathcal{F}$ that is generated by the 20 elements which arise in Lemma 5.2. A typical example is

$$\vartheta_1\vartheta_4\vartheta_6 T_1 - \vartheta_2\vartheta_3\vartheta_5 T_2 - \vartheta_8\vartheta_9\vartheta_{10} T_3.$$

Lemma 6.1. *Let T be an element of the kernel of $\mathcal{F} \to \mathcal{N}$. Then $\chi_5 T$ is contained in $\mathcal{K}$, and the kernel of $\mathcal{F} \to \mathcal{N}$ equals the kernel of*

$$\mathcal{F} \xrightarrow{\cdot\chi_5} \mathcal{F} \to \mathcal{F}/\mathcal{K}.$$

Proof. Let $P_1 \operatorname{grad}_1 + \cdots + P_6 \operatorname{grad}_6 = 0$ be a (homogeneous) relation. After multiplication by χ_5 we can use the relations in Lemma 5.2 to eliminate in this relation all grad_i, $i > 2$. Then we obtain a relation between grad_1, grad_2. But these two forms are independent due to Lemma 5.1. Hence the above relation, after multiplication by χ_5, is a consequence of the relations in Lemma 5.2. □

In principle, Lemma 6.1 is a complete description of the module $\mathcal{N}$. We can use it to work out a finite generating system of relations. For this, we describe some extra relations between the grad_i.

Lemma 6.2. *For each ordered pair of two different odd characteristics there is a relation which is determined by this pair up to the sign. The relation that belongs to pair* (5, 6) *in our numbering is*

$$\vartheta_6^2 D(1,5)\operatorname{grad}_1 - \vartheta_5^2 D(2,5)\operatorname{grad}_2 - \vartheta_8^2 D(3,5)\operatorname{grad}_3 + \vartheta_7^2 D(4,5)\operatorname{grad}_4 = 0.$$

The full modular group acts transitively on these 30 *relations* (*counted up to the sign*). *In general the relation for a pair* (α, β) *is the sum of four* $\pm\vartheta_k^2 D(i,\alpha)\operatorname{grad}_i$, $i \neq \alpha, \beta$, *where* ϑ_k *is the only theta that divides* $D(i,\alpha)$ *and* $D(\alpha,\beta)$.

Proof. Along the lines of the proof of Lemma 6.1, one multiplies the claimed relation by χ_5 and eliminates grad_i, $i > 2$. Then one obtains an expression $P_1 \operatorname{grad}_1 + P_2 \operatorname{grad}_2$ with explicitly given polynomials in the ϑ_i. One has to show $P_1 = P_2 = 0$. We omit the straightforward calculation and mention only that for this one has to use the Riemann relations. □

There is a second kind of extra relations between the grad_i. To explain them, we need some facts about theta characteristics. In [7] it has been proved that each odd characteristic n can written in 12 different ways (up to ordering) as a sum of five pairwise different even characteristics

$$n = m_1 + \cdots + m_5.$$

The full modular group acts transitively on the set of all $\{n, m_1, \ldots, m_5\}$. For each of them we define the modular form

$$S := S(n, m_1, \ldots, m_5) = \vartheta[m_1] \cdots \vartheta[m_5] \operatorname{grad} \vartheta[n].$$

Hence we obtain 72 modular forms.

As we mentioned, the forms $\vartheta[m]$ and $\operatorname{grad} \vartheta[n]$ are modular forms with respect to the group $\Gamma_2[2, 4]$. As a consequence, we get

$$S \in [\Gamma_2[2, 4], 3, \chi_S, \mathrm{St}]$$

with a certain quadratic character χ_S on $\Gamma_2[2, 4]$. The information about these characters which we need can be taken from the paper [13].

Lemma 6.3. *The* 72 *forms* S *are modular forms with respect to the group* $\Gamma_2[2, 4]$ *and a certain quadratic character* χ_S. *In this way there arise* 12 *different characters and to the each associated space of modular forms belong six of the forms* S. *Each of the* 72 *forms* S *is uniquely determined by its odd characteristic* n *and the character* χ_S.

We denote the form S which belongs to n and χ by $S(n, \chi)$.

Lemma 6.4. *We fix one of the* 12 *characters* χ. *Let* $S_i = S(i, \chi)$ *for* $i = 1, \ldots 6$ *be the six functions* $S(n, \chi)$. *If one cancels one of the six, say* S_6, *one gets a relation between the other five of the following type.*

$$\sum_{i=1}^{5} \pm \vartheta[m_i]^2 S_i = 0.$$

Here m_i *are certain even characteristics which are uniquely determined and also the signs* (*up to a common sign change*) *are uniquely determined.*

The rule how the m_i can be found is a little complicated. We explain how m_1 can be found.

1) There are three even characteristics q_1, q_2, q_3 that occur in S_1 but not in S_6 (the form which has been cancelled).

2) There is one pair in $\{q_1, q_2, q_3\}$, say $\{q_2, q_3\}$, such that $\vartheta[q_2]\vartheta[q_3]$ does not occur in any of the four $S_2, \ldots, S_5$.

Then one has to use $m_1 := q_1$ in Lemma 6.4.

We explain this in an example. The six forms

$$S_1 = \vartheta_3\vartheta_5\vartheta_6\vartheta_8\vartheta_9 \operatorname{grad}_1, \quad S_2 = \vartheta_1\vartheta_2\vartheta_4\vartheta_8\vartheta_9 \operatorname{grad}_2, \quad S_3 = \vartheta_2\vartheta_5\vartheta_7\vartheta_8\vartheta_{10} \operatorname{grad}_4,$$

$$S_4 = \vartheta_4\vartheta_6\vartheta_7\vartheta_9\vartheta_{10} \operatorname{grad}_6, \quad S_5 = \vartheta_1\vartheta_3\vartheta_4\vartheta_5\vartheta_{10} \operatorname{grad}_3, \quad S_6 = \vartheta_1\vartheta_2\vartheta_3\vartheta_6\vartheta_7 \operatorname{grad}_5$$

belong to the same character. The characteristic that occur in S_1 but not in S_6 are m_5, m_8, m_9. The pair m_5, m_9 does not occur in $S_2, \ldots, S_5$. Hence the theta square which has to be added is $\vartheta[m_8]^2$. The whole relation will be

$$\vartheta[m_8]^2 S_1 - \vartheta[m_9]^2 S_2 + \vartheta[m_5]^2 S_3 - \vartheta[m_4]^2 S_4 + \vartheta[m_{10}]^2 S_5.$$

The determination of the signs needs some extra work.

We do not give details of the proof of Lemma 6.4. We only mention that it is similar to the proof of Lemma 6.2.

The relations that we described so far are defining relations.

Proposition 6.5. *The module of relations between the six forms* grad_i, *i.e. the kernel of the natural homomorphism* $\mathcal{F} \to \mathcal{N}$, *is generated by the* 20 *relations described in* Lemma 5.2, *the* 30 *relations described in* Lemma 6.2 *and the* 72 *relations described in* Lemma 6.4.

Proof. Using Lemma 6.1, the proof can be given with the help of a computer. □

7. A structure theorem

Now we have the possibility to determine the structure of $\mathcal{M}$. From Proposition 5.3 we get

$$\chi_5 \mathcal{M} = \bigcap_{m,n} \frac{\chi_5}{\vartheta[n_1] \cdots \vartheta[n_4]} (A(\Gamma[4, 8]) \operatorname{grad} \vartheta[m] + A(\Gamma[4, 8]) \operatorname{grad} \vartheta[n]).$$

The right-hand side is a submodule of $\mathcal{N}$ which we understand completely (Proposition 6.5). Hence it is possible to compute the intersection with the help of a computer. We did this by means of the computer algebra system SINGULAR. In this way we could determine a finite system of generators of $\mathcal{M}$ and we also could get the Hilbert function. We mention that $\mathcal{M}$ is bigger than $\mathcal{N}$. We have to describe now the extra generators.

Proposition 7.1. *The modular form*

$$\frac{(\vartheta_4 \vartheta_6^4 \vartheta_8 + \vartheta_4 \vartheta_8 \vartheta_9^4) \operatorname{grad}_1 - \vartheta_1 \vartheta_6 \vartheta_9 \vartheta_{10}^3 \operatorname{grad}_3}{\vartheta_2 \vartheta_5}$$

is holomorphic, hence contained in $[\Gamma[4, 8], 5/2, \mathrm{St}]$. *It is not contained in the submodule* $\mathcal{N}$. *The orbit under the full modular form consists, up to constant factors, of* 360 *modular forms that span a space of dimension* 90.

We omit the relations among the above forms since they are long and do not seem to have a nice combinatorial description.

By means of SINGULAR one can verify that the $A(\Gamma[4,8])$-module generated by the six grad_i and the Γ_2-orbit of the form described in Proposition 7.1 equals the module $\mathcal{M}$. SINGULAR also gives the Hilbert function.

Theorem 7.2. *The* $A(\Gamma[4,8])$*-module*

$$\mathcal{M} = \bigoplus [\Gamma[4,8], r/2, v_\vartheta^r, \mathrm{St}]$$

is generated by the six grad_i *and the* Γ_2*-orbit of the form described in* Proposition 7.1. *The Hilbert function is*

$$\begin{aligned}&\sum_{r=0}^{\infty} [\Gamma[4,8], r/2, v_\vartheta^r, \mathrm{St}]t^r \\ &= \frac{60t^9 - 60t^8 - 318t^7 + 252t^6 + 606t^5 + 316t^4 + 126t^3 + 36t^2 + 6t}{(1-t)^4} \\ &= 6t + 60t^2 + 330t^3 + 1300t^4 + 4060t^5 + 9952t^6 + 20000t^7 + 35168t^8 + \cdots.\end{aligned}$$

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