

PRIME AND PRIMARY IDEALS IN SEMIRINGS

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Abstract

We study zero divisors and minimal prime ideals in semirings, notably those of characteristic one. Thereafter we find a counterexample to the most obvious version of primary decomposition, but are able to establish a weaker version. Lastly, we study Evans' condition in this context.

1. Introduction

Primary decomposition was first established in polynomial rings (over \mathbf{Z} or over a field) in Lasker's classical paper ([13]); another proof was later given by Macaulay ([17]). In her famous paper of 1921 ([18]), Emmy Noether established the result for the class of rings that now bears her name. Therefore Lasker's theorem led to the discovery of two of the main concepts of modern algebra: *noetherian rings* and *Cohen–Macaulay rings*.

The decomposition of an arbitrary ideal as an intersection of primary ones is, *via* the proof of Krull's theorem, an essential tool in algebraic geometry (see e.g. [22], pp.47–48). The Riemann hypothesis is arguably the most important open problem in mathematics; its natural analogue, Weil's conjecture ([23]), was finally established by Deligne ([4]) using the whole strength of Grothendieck's *theory of schemes*.

It has therefore long been expected (see e.g. [2] and [21]) that an “algebraic geometry in characteristic one” might provide the natural framework for an approach of the Riemann hypothesis. Many such theories have been propounded, including Deitmar's theory of *F_1 -schemes* ([3]) and Zhu's *characteristic one algebra* ([24]). In [14], §5, I have shown that part of Deitmar's theory embeds in a functorial way into Zhu's; the basic objects are B_1 -algebras, i.e. *characteristic one semirings*, that is unitary semirings A such that

$$1_A + 1_A = 1_A.$$

We have resolved to develop systematically and as far as possible the study of these objects.

As usual, we shall denote by B_1 the set $\{0, 1\}$ equipped with the usual multiplication and addition, with the slight change that $1 + 1 = 1$.

In three previous articles ([14], [15], [16]), we have shown that B_1 -algebras (i.e. characteristic 1 semirings) behave, in many respects, like ordinary rings. In particular, one may define polynomial algebras over B_1 ([14], Theorem 4.5) and classify the maximal congruences on them ([14], Theorem 4.8).

There is a natural definition of a prime ideal (see [15], Definition 2.2) in such a semiring; the set $Pr_s(A)$ of *saturated* ([15], Theorem 3.7) prime ideals of A can be endowed with a natural Zariski-type topology, to which most of the usual topological properties of ring spectra carry over (see [16], Proposition 6.2). In [15], we discussed the relationship between congruences and ideals in B_1 -algebras; the two concepts are not equivalent, but *excellent* congruences correspond bijectively to *saturated* ideals. The set $Pr_s(A)$ of saturated prime ideals of a B_1 -algebra A is in bijection with the set $MaxSpec(A)$ of maximal (nontrivial) congruences on A ; that bijection is even a homeomorphism for the natural Zariski-type topologies ([16], Theorem 3.1), and it is functorial ([16], Theorem 4.2).

It is therefore natural to examine whether higher results of commutative algebra have valid analogs in the setting of B_1 -algebras, or, more generally, in the setting of semirings with 0 and 1. Without any extra hypothesis, this is the case for the fundamental properties of minimal (saturated) prime ideals (§3). Actually, modulo an hypothesis of noetherian flavour, it appears that all minimal prime ideals (more generally, all associated prime ideals) are saturated (§4).

The next natural question concerns a possible primary decomposition. The basic properties of primary ideals carry over (§5), but Lasker–Noether primary decomposition need not hold, even though a weaker version can be established (§6). In other words, a (weakly) noetherian semiring (even if it is a B_1 -algebra) is not necessarily laskerian.

But it turns out (§7) that if the semiring is *either* laskerian *or* weakly noetherian, it has the Evans property (first introduced in [5]).

In §8 we specialize the previous results to the characteristic 1 case.

2. Some definitions

Up to and including in §7, we shall denote by A an arbitrary commutative semiring with 0 and 1. The following concepts and results are adapted from [9].

A k -ideal I of A is by definition ([9], p. 220) an ideal I of A such that, whenever $x + i = j$ with $i \in I$ and $j \in I$, then $x \in I$ (such ideals are called *subtractive* in [12], p. 3). For each ideal I of A , there is a smallest k -ideal C_I containing I ; it is given by

$$C_I = \{x \in A \mid \exists(i, j) \in I^2 \mid x + i = j\};$$

in [1], it is denoted $cl(I)$. The equivalence relation \mathcal{R}_I on A given by

$$x\mathcal{R}_I y \equiv (\exists(i, j) \in I^2)x + i = y + j$$

is compatible with the semiring operations, i.e. a *congruence* on A , and therefore the

quotient set A/\mathcal{R}_I inherits a structure of semiring with 0 and 1. We shall denote it by A/I ; it is easily seen that

$$A/I = A/C_I.$$

For I an ideal of A , we set

$$\sqrt{I} := \{x \in A \mid (\exists n \geq 1) x^n \in I\};$$

in the characteristic 1 setting, this was denoted by $r(I)$ in [16] (Definition 5.3). The ideal I will be termed *radical* if $I = \sqrt{I}$; one may see that a k -ideal is radical if and only if it is an intersection of prime (k -) ideals (in the characteristic 1 case, this was proved in [16], Proposition 5.5).

$Pr(A)$ will denote the set of prime ideals in A , and $Pr_k(A)$ the set of prime k -ideals in A . $Min Pr(A)$ and $Min Pr_k(A)$ will denote the sets of minimal elements (for inclusion) of $Pr(A)$ and $Pr_k(A)$, respectively. Classical arguments (see e.g. [10], Proposition II.6, p.69) establish that $(Pr(A), \supseteq)$ and $(Pr_k(A), \supseteq)$ are inductive. Therefore Zorn's Lemma implies that each prime (resp. prime k -ideal) contains a minimal prime ideal (resp. minimal prime k -ideal).

By $Max_k(A)$ we denote the set of maximal elements among proper k -ideals of A . The following two results are sometimes useful.

Proposition 2.1.

$$Max_k(A) \subseteq Pr_k(A).$$

Proof. Let $\mathcal{M} \in Max_k(A)$, and let us assume $u \notin \mathcal{M}$, $v \notin \mathcal{M}$, and $uv \in \mathcal{M}$. Then the maximality of \mathcal{M} yields $C_{\mathcal{M}+Au} = C_{\mathcal{M}+Av} = A$. Therefore one may find $(y, y') \in (\mathcal{M} + Av)^2$ such that $1 + y = y'$. Let us write $y = m' + bv$ and $y' = m + av$ ($(m, m') \in \mathcal{M}^2$, $(a, b) \in A^2$); then

$$uy = um' + b(uv) \in \mathcal{M},$$

and, similarly,

$$uy' \in \mathcal{M};$$

but

$$u + uy = u(1 + y) = uy',$$

whence $u \in C_{\mathcal{M}} = \mathcal{M}$, a contradiction: \mathcal{M} is prime. Arguments such as the above will often recur in this paper.

In the characteristic one case, we might also have used Theorem 3.3 from [16]. \square

Lemma 2.2. *Let I and J denote ideals of A ; then*

$$\sqrt{C_{I \cap J}} = \sqrt{C_I \cap C_J} = \sqrt{C_I} \cap \sqrt{C_J}.$$

Proof. The inclusions

$$\sqrt{C_{I \cap J}} \subseteq \sqrt{C_I \cap C_J} \subseteq \sqrt{C_I} \cap \sqrt{C_J}$$

are clearly valid. Let now $\sqrt{C_I} \cap \sqrt{C_J}$; then $u^m \in C_I$ for some $m \geq 1$ and $u^n \in C_J$ for some $n \geq 1$. Thus one may find $(i, i') \in I^2$ and $(j, j') \in J^2$ with $u^m + i = i'$ and $u^n + j = j'$. Then $u^m j + ij = i' j \in I \cap J$ and $ij \in I \cap J$, whence $u^m j \in C_{I \cap J}$; similarly, $u^n j' \in C_{I \cap J}$. But

$$u^{m+n} + u^m j = u^m j'$$

whence

$$u^{m+n} \in C_{C_{I \cap J}} = C_{I \cap J},$$

and

$$u \in \sqrt{C_{I \cap J}}.$$

Therefore

$$\sqrt{C_I} \cap \sqrt{C_J} \subseteq \sqrt{C_{I \cap J}},$$

and the result follows. \square

For $s \in A$ we define the *annihilator* of s by

$$(0 : s) = \{x \in A \mid sx = 0\}.$$

It is clearly an ideal of A ; furthermore, from $(y, y') \in (0 : s)^2$ and $x + y = y'$ follows

$$sx = sx + 0 = sx + sy = s(x + y) = sy' = 0,$$

thus $x \in (0 : s)$: $(0 : s)$ is a k -ideal.

For S a subset of A , we define

$$(0 : S) := \bigcap_{s \in S} (0 : s);$$

as an intersection of k -ideals of A , it is a k -ideal of A .

For $x \in A \setminus \{0\}$, let

$$\tilde{A}_x := A/(0 : x),$$

and let $\pi_x : A \twoheadrightarrow \tilde{A}_x$ denote the canonical projection.

DEFINITION 2.3. An ideal \mathcal{P} of A is termed *associated* to $x \in A \setminus \{0\}$ if it can be expressed as $\mathcal{P} = \pi_x^{-1}(\mathcal{Q})$ for some minimal prime ideal \mathcal{Q} of \tilde{A}_x ; it is termed *associated* if it is associated to some $x \in A \setminus \{0\}$.

$Ass(A)$ will denote the set of associated ideals of A ; clearly,

$$Ass(A) \subseteq Pr(A).$$

Obviously, each minimal prime ideal of A is associated ($x = 1$ is suitable), whence $Min Pr(A) \subseteq Ass(A)$.

The elements of the set

$$\mathcal{D}_A := \{a \in A \setminus \{0\} \mid (\exists b \in A \setminus \{0\})ab = 0\}$$

are called *zero-divisors* in A . Clearly, for A non-trivial, one has

$$\mathcal{D}_A \cup \{0\} = \bigcup_{s \in A \setminus \{0\}} (0 : s).$$

DEFINITION 2.4. The A is *noetherian* if every ascending chain of ideals of A is ultimately stationary.

By standard arguments (see e.g. [11], Proposition I.2, p.47), this is equivalent to the assertion that every ideal of A is finitely generated.

DEFINITION 2.5. A is *weakly noetherian* if every ascending chain of k -ideals of A is ultimately stationary.

It is obvious that, if A is noetherian, then it is weakly noetherian. The converse is false, even for B_1 -algebras; in fact, $B_1[x]$ is weakly noetherian (it follows from the reasoning used in the proof of [14], Theorem 4.2 that its k -ideals are $\{0\}$ and the $x^n B_1[x] (n \in \mathbf{N})$) but not noetherian (one may even find, in $B_1[x]$, a strictly increasing sequence of *prime* ideals: cf. [12], Chapter 3, p.65).

REMARK 2.6. It is clear that A is weakly noetherian if and only if each k -ideal I of A is finitely generated *as a k -ideal*, that is there is a finite family (a_1, \dots, a_n) of elements of A such that $I = C_{(a_1, \dots, a_n)}$; for that, it is enough that each k -ideal be finitely generated *as an ideal*, but the condition is not necessary. For example, let A be the characteristic one semiring given by

$$A = \{0, 1, y\} \cup \{x_n \mid n \geq 1\}$$

such that

$$x_i + x_j = y$$

whenever $i \neq j$,

$$y + 1 = 1$$

and

$$ab = 0$$

except whenever $a = 1$ or $b = 1$.

Then A is weakly noetherian (its k -ideals are $\{0\}$, the $\{0, x_i\}$, $A \setminus \{1\}$ and A) but not noetherian (the ideal $A \setminus \{0\}$ is not of finite type).

DEFINITION 2.7. We call the B_1 -algebra A *standard* if $\mathcal{D}_A \cup \{0\}$ is a finite union of saturated prime ideals of A .

DEFINITION 2.8. For J an ideal of A and $x \in A$, let

$$\mathcal{C}_x(J) := \{y \in A \mid xy \in J\}.$$

Clearly, $\mathcal{C}_x(J)$ is an ideal of A , and a k -ideal whenever J itself is one; furthermore,

$$\mathcal{C}_x(\{0\}) = (0 : x).$$

For saturated J , the ideals of the form $\mathcal{C}_x(J)$ ($x \notin J$) will be termed *J-conductors*.

Lemma 2.9. *Let J be a saturated ideal of A , $y \notin J$, and assume that*

$$\mathcal{P} := \mathcal{C}_y(J)$$

is a maximal element of the set of J -conductors. Then $\mathcal{P} \in \text{Pr}(A)$.

Proof. One has $1 \notin \mathcal{P}$ (as $y \notin J$), whence $\mathcal{P} \neq A$. Let us assume $uv \in \mathcal{P}$ and $u \notin \mathcal{P}$; then $uy \notin J$ and $\mathcal{C}_y(J) \subseteq \mathcal{C}_{uy}(J)$. It follows that

$$\mathcal{P} = \mathcal{C}_y(J) = \mathcal{C}_{uy}(J);$$

but

$$v(uy) = (uv)y \in J,$$

whence

$$v \in \mathcal{C}_{uy}(J) = \mathcal{P} :$$

\mathcal{P} is prime. □

3. Minimal prime ideals

Theorem 3.1. *Let $\mathcal{P} \in \text{Min Pr}(A) \cup \text{Min Pr}_k(A)$; then each nonzero element of \mathcal{P} is a zero-divisor in A .*

Proof. Let $x \in \mathcal{P} \in \text{Min Pr}(A)$, $x \neq 0$, and assume that x is not a zero-divisor; then, for each $a \in A \setminus \{0\}$ and $n \in \mathbf{N}$, $ax^n \neq 0$. In particular

$$\forall n \in \mathbf{N} \quad \forall a \in A \setminus \mathcal{P} \quad ax^n \neq 0.$$

Let \mathcal{E} denote the set of ideals I of A such that

$$(*) \quad \forall n \in \mathbf{N} \quad \forall a \in A \setminus \mathcal{P} \quad ax^n \notin I.$$

Then $\mathcal{E} \neq \emptyset$ (as $\{0\} \in \mathcal{E}$), and \mathcal{E} is inductive for \subseteq , whence \mathcal{E} contains a maximal element I . As $1 = 1 \cdot x^0 \notin I$, $I \neq A$.

Let us suppose for a moment that $uv \in I$, $u \notin I$ and $v \notin I$; then $I + Au$ and $I + Av$ are ideals of A strictly containing I , whence $I + Au \notin \mathcal{E}$ and $I + Av \notin \mathcal{E}$. Thus one may find $(a, b) \in (A \setminus \mathcal{P})^2$, $(i, j) \in I^2$, $(c, d) \in A^2$ and $(m, n) \in \mathbf{N}^2$ with $ax^m = i + cu$ and $bx^n = j + dv$.

Then $ab \in A \setminus \mathcal{P}$ (as \mathcal{P} is prime) and

$$\begin{aligned} abx^{m+n} &= (ax^m)(bx^n) \\ &= (i + cu)(j + dv) \\ &= i(j + dv) + (cu)j + (cd)(uv) \in I, \end{aligned}$$

a contradiction. Therefore I is prime. But, by definition,

$$\forall a \in A \setminus \mathcal{P} \quad a = ax^0 \notin I,$$

whence $A \setminus \mathcal{P} \subseteq A \setminus I$, or $I \subseteq \mathcal{P}$. The minimality of \mathcal{P} now implies that

$$I = \mathcal{P},$$

whence $1 \cdot x^1 = x \in \mathcal{P} = I$, contradicting the definition of I (we have essentially followed [7], Corollary 1.2, and [11], p.34, Lemma 3.1).

In case $\mathcal{P} \in \text{Min Pr}_k(A)$, the same argument applies modulo a slight complication: by defining \mathcal{E} to be the set of k -ideals I of A satisfying $(*)$, we find a maximal element I of \mathcal{E} , and have $I \neq A$. Assuming $uv \in I$, $u \notin I$ and $v \notin I$, we see that that $C_{I+Au} \notin \mathcal{E}$ and $C_{I+Av} \notin \mathcal{E}$. Therefore we may find $(a, a') \in (A \setminus \mathcal{P})^2$ and $(m, n) \in \mathbf{N}^2$ such that $ax^m \in C_{I+Au}$ and $a'x^n \in C_{I+Av}$. Therefore $ax^m + y = y'$ for some $(y, y') \in (I + Au)^2$, and $a'x^n + z = z'$ for some $(z, z') \in (I + Av)^2$. Set $y = i + cu$ ($i \in I$) and $z = i' + dv$ ($i' \in I$); then

$$yv = iv + c(uv) \in I,$$

and similarly $y'v \in I$. As

$$\begin{aligned} ax^m v + yv &= (ax^m + y)v \\ &= y'v \end{aligned}$$

and I is a k -ideal, it appears that $ax^m v \in I$. Then

$$\begin{aligned} ax^m z &= ax^m(i' + dv) \\ &= ax^m i' + d(ax^m v) \\ &\in I, \end{aligned}$$

for the same reason, $ax^m z' \in I$ as

$$\begin{aligned} aa'x^{m+n} + ax^m z &= ax^m(a'x^n + z) \\ &= ax^m z', \end{aligned}$$

it follows as above that $aa'x^{m+n} \in I$. But $aa' \in A \setminus \mathcal{P}$, contradicting the definition of I . \square

4. The weakly noetherian case

Theorem 4.1. *In case A is weakly noetherian, each associated prime ideal of A is of the form $(0 : u)$ for some $u \in A \setminus \{0\}$; in particular, it is a k -ideal.*

Proof. Let \mathcal{P} denote a prime ideal of A associated to $x \in A \setminus \{0\}$; then

$$\mathcal{P} = \pi_x^{-1}(\mathcal{Q})$$

for some $\mathcal{Q} \in \text{Min Pr}(\tilde{A}_x)$. We define

$$\mathcal{W}(\mathcal{P}) := \{z \in A \mid (0 : zx) \subseteq \mathcal{P}\}.$$

$\mathcal{W}(\mathcal{P})$ is non-empty, as $1 \in \mathcal{W}(\mathcal{P})$. For $y \in \mathcal{W}(\mathcal{P})$, let

$$I_{\mathcal{P}}(y) := \bigcup_{s \in A \setminus \mathcal{P}} (0 : sxy).$$

As

$$\begin{aligned} \forall (s, s') \in (A \setminus \mathcal{P})^2, \\ (0 : sxy) \cup (0 : s'xy) \subseteq (0 : ss'xy) \end{aligned}$$

and $ss' \in A \setminus \mathcal{P}$, $I_{\mathcal{P}}(y)$ is the union of a filtering family of k -ideals, whence it is itself a k -ideal.

By definition, whenever $y \in \mathcal{W}(\mathcal{P})$, $(0 : xy) \subseteq \mathcal{P}$, therefore from $s \in A \setminus \mathcal{P}$ and $z \in (0 : sxy)$ follows

$$(sz)(xy) = (sxy)z = 0,$$

thus $sz \in (0 : xy) \subseteq \mathcal{P}$, $sz \in \mathcal{P}$ and $z \in \mathcal{P}$. We have shown that

$$I_{\mathcal{P}}(y) \subseteq \mathcal{P}.$$

Let now $J := I_{\mathcal{P}}(y_0)$ denote a maximal element of

$$\{I_{\mathcal{P}}(y) \mid y \in \mathcal{W}(\mathcal{P})\}$$

(the existence of such an element follows from the weak noetherianity hypothesis). As seen above, $J \subseteq \mathcal{P}$, whence $J \neq A$. Let us suppose $ab \in J$ and $a \notin J$; then, for each $s \in A \setminus \mathcal{P}$, $a \notin (0 : sxy_0)$, whence

$$s(xy_0a) = (sxy_0)a \neq 0.$$

Therefore $(0 : xy_0a) \subseteq \mathcal{P}$, i.e. $y_0a \in \mathcal{W}(\mathcal{P})$. Clearly $I_{\mathcal{P}}(y_0) \subseteq I_{\mathcal{P}}(y_0a)$, whence $I_{\mathcal{P}}(y_0) = I_{\mathcal{P}}(y_0a)$ according to the definition of y_0 .

As $ab \in J = I_{\mathcal{P}}(y_0)$, there exists $s \in A \setminus \mathcal{P}$ such that $(sxy_0)ab = 0$; but then $s(xy_0a)b = 0$, whence $b \in (0 : sx(y_0a)) \subseteq I_{\mathcal{P}}(y_0a) = I_{\mathcal{P}}(y_0) = J$. We have shown that $ab \in J$ implies $a \in J$ or $b \in J$: J is prime.

As $J \subseteq \mathcal{P}$ and

$$(0 : x) \subseteq (0 : xy_0) \subseteq I_{\mathcal{P}}(y_0) = J,$$

$\pi_x(J)$ is a prime ideal of \tilde{A}_x and $\pi_x(J) \subseteq \pi_x(\mathcal{P}) = \mathcal{Q}$, it now follows from the minimality of \mathcal{Q} that $\pi_x(J) = \mathcal{Q}$.

Let now $u \in \mathcal{P}$; then $\pi_x(u) \in \pi_x(\mathcal{P}) = \mathcal{Q} = \pi_x(J)$, therefore $\pi_x(u) = \pi_x(j)$ for some $j \in J$. Then there are $(y, y') \in (0 : x)^2$ such that $u + y = j + y'$, hence $u + y \in J$, and $u \in C_J = J$ (as J is a k -ideal). It follows that $\mathcal{P} \subseteq J$, whence $\mathcal{P} = J = I_{\mathcal{P}}(y_0)$; in particular, \mathcal{P} is a k -ideal.

As A is weakly noetherian, there is a finite family (p_1, \dots, p_n) of elements of \mathcal{P} such that

$$\mathcal{P} = C_{\{p_1, \dots, p_n\}}.$$

Each p_j belongs to $\mathcal{P} = J = I_{\mathcal{P}}(y_0)$, whence there is an $s_j \in A \setminus \mathcal{P}$ such that $p_j \in (0 : s_jxy_0)$. Let $s_0 := s_1 \cdots s_n$ and

$$u := s_0xy_0 = s_1 \cdots s_nxy_0;$$

then each p_j belongs to $(0 : u)$, whence

$$\begin{aligned} \mathcal{P} &= C_{\{p_1, \dots, p_n\}} \\ &\subseteq C_{(0:u)} \\ &= (0 : u) \end{aligned}$$

(as $(0 : u)$ is a k -ideal).

On the other hand, $s_0 \in A \setminus \mathcal{P}$, therefore

$$\begin{aligned} (0 : u) &= (0 : s_0xy_0) \\ &\subseteq I_{\mathcal{P}}(y_0) \\ &= \mathcal{P}, \end{aligned}$$

whence $\mathcal{P} = (0 : u)$. □

Corollary 4.2. *If A is weakly noetherian, then $\text{Min Pr}(A) = \text{Min Pr}_k(A)$.*

Proof. Let $\mathcal{P} \in \text{Min Pr}(A)$. As seen in §2, \mathcal{P} is associated, whence, by Theorem 4.1, \mathcal{P} is a k -ideal, hence $\mathcal{P} \in \text{Min Pr}_s(A)$.

Conversely, let $\mathcal{P} \in \text{Min Pr}_k(A)$; then \mathcal{P} is prime, hence contains some minimal prime ideal \mathcal{P}_0 (cf. §2). Now \mathcal{P}_0 is a k -ideal whence (as $\mathcal{P}_0 \subseteq \mathcal{P}$)

$$\mathcal{P} = \mathcal{P}_0 \in \text{Min Pr}(A). \quad \square$$

5. Definition and first properties of primary ideals

The usual theory generalizes without major problem to semirings with 0 and 1.

DEFINITION 5.1. An ideal \mathcal{Q} of A is termed *primary* if $\mathcal{Q} \neq A$ and

$$\forall (x, y) \in A^2 \quad [xy \in \mathcal{Q} \implies x \in \mathcal{Q} \text{ or } (\exists n \geq 1) y^n \in \mathcal{Q}].$$

Obviously, a prime ideal is primary.

Proposition 5.2. *If \mathcal{Q} is primary, then $\sqrt{\mathcal{Q}}$ is prime.*

Proof. Let $\mathcal{P} = \sqrt{\mathcal{Q}}$. As $\mathcal{Q} \neq A$, $1 \notin \mathcal{Q}$, thus $1 \notin \mathcal{P}$. Let us assume that $uv \in \mathcal{P}$; then, for some $n \geq 1$, $(uv)^n \in \mathcal{Q}$, i.e. $u^n v^n \in \mathcal{Q}$, whence (as \mathcal{Q} is primary) either $u^n \in \mathcal{Q}$ or there exists $m \geq 1$ with $v^{nm} = (v^n)^m \in \mathcal{Q}$. Therefore either u or v belongs to $\sqrt{\mathcal{Q}} = \mathcal{P}$: \mathcal{P} is prime. □

REMARK 5.3. As seen in [16], Lemma 5.4(ii) in the context of B_1 -algebras, if \mathcal{Q} is a k -ideal then so is $\mathcal{P} = \sqrt{\mathcal{Q}}$; with some modifications, our proof goes through in the general case.

DEFINITION 5.4. The primary ideal \mathcal{Q} will be termed *\mathcal{P} -primary* if $\mathcal{P} = \sqrt{\mathcal{Q}}$.

Lemma 5.5. *Let $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ be \mathcal{P} -primary ideals for the same prime ideal \mathcal{P} ; then $\mathcal{Q} := \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_n$ is also \mathcal{P} -primary.*

Proof. Let us assume $xy \in \mathcal{Q}$ and $x \notin \mathcal{Q}$. As $x \notin \mathcal{Q}$, there is a $k \in \{1, \dots, n\}$ such that $x \notin \mathcal{Q}_k$; as $xy \in \mathcal{Q} \subseteq \mathcal{Q}_k$, we have $xy \in \mathcal{Q}_k$, whence (as \mathcal{Q}_k is primary) there exists $n \geq 1$ such that $y^n \in \mathcal{Q}_k$. Then $y \in \sqrt{\mathcal{Q}_k} = \mathcal{P}$. As all \mathcal{Q}_i 's are \mathcal{P} -primary, one has, for each i , $y \in \sqrt{\mathcal{Q}_i}$, whence there exists $m_i \geq 1$ such that $y^{m_i} \in \mathcal{Q}_i$. Let $m' := \max_{1 \leq i \leq n}(m_i)$; then

$$y^{m'} \in \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_n = \mathcal{Q} :$$

\mathcal{Q} is primary.

Incidentally, we have established that $\mathcal{P} \subseteq \sqrt{\mathcal{Q}}$; but $\sqrt{\mathcal{Q}} \subseteq \sqrt{\mathcal{Q}_1} = \mathcal{P}$, whence $\mathcal{P} = \sqrt{\mathcal{Q}}$: \mathcal{Q} is \mathcal{P} -primary. \square

6. Weak primary decomposition

DEFINITION 6.1. The B_1 -algebra A is termed *laskerian* if any k -ideal of A can be expressed as a finite intersection of primary k -ideals.

It is natural to conjecture that each weakly noetherian B_1 -algebra is laskerian, but this is false, as shown by the following example.

EXAMPLE 6.2. Let $A = \{0, z, x, y, u, 1\}$; it is easily seen that there is a unique structure of B_1 -algebra on A such that $z + x = x$, $z + y = y$, $x + y = u$, $u + 1 = 1$, $x^2 = x$, $y^2 = y$, $z^2 = 0$, $u^2 = u$, $xy = xz = yz = uz = 0$, $xu = x$ and $yu = y$.

Each primary k -ideal of A contains $0 = xy$, therefore it contains either x or a power of y , whence it contains x or y , thus it contains z . Thus any intersection of primary k -ideals contains z , and $\{0\}$ is not an intersection of primary k -ideals: A is not laskerian.

REMARK 6.3. This would seem to contradict Theorem 4 from [1], which asserts that in an arbitrary noetherian semiring primary decomposition holds for k -ideals. But the proof given in [1] is incorrect: Lemma 6 (the proof of which is declared “trivial”!) need not hold except if “irreducible” is interpreted as meaning *irreducible as a k -ideal*, but then the proof or Proposition 1 will not hold. Indeed we are referred to [20], Proposition 4.34; but in that argument appear some ideals (e.g. $I + Ra^n$, cf. p. 78) that need not be k -ideals, even when I itself is.

REMARK 6.4. In the recent preprint [6], Flores and Weibel establish primary decomposition in noetherian monoids (cf. [6], Theorem 1.3). From this one may deduce the validity of primary decomposition for k -ideals in B_1 -algebras of the shape $B_1[M]$, for M a noetherian monoid. Indeed, prime k -ideals in $B_1[M]$ correspond bijectively to prime ideals (including \emptyset) of M ([15], Theorem 4.2), and the same holds with “primary” in place of “prime”, as is easily seen. But the semiring constructed above is not isomorphic to one of that type.

Nevertheless a weaker property holds true.

Theorem 6.5. *Let I denote a radical k -ideal in A ; then I can be written as a finite intersection of prime k -ideals.*

Proof. Let us proceed by contradiction, and let J denote a maximal element of the set of radical k -ideals that cannot be written as a finite intersection of prime k -ideals; in particular, $J \neq A$ and J is not prime. Therefore one may find $u \notin J$ and $v \notin J$ with $uv \in J$. Let $K = \sqrt{C_{J+Au}}$ and $L = \sqrt{C_{J+Av}}$; then K and L are k -ideals of A strictly containing J , whence each is a finite intersection of prime saturated ideals. Clearly, $J \subseteq K \cap L$. Let now $x^m \in K \cap L$; then $x^m + y = y'$ for some $(y, y') \in (J + Au)^2$ and $x^n + z = z'$ for some $(z, z') \in (J + Av)^2$; writing $y = j + au$ ($j \in J$) and $z = j' + bv$ ($j' \in J$) we get

$$vx^m + vy = v(x^m + y) = vy';$$

but $vy = v(j + au) = vj + a(uv) \in J$, and similarly $vy' \in J$, whence $vx^m \in C_J = J$. But then

$$x^{m+n}z = x^{m+n}(j' + bv) = x^{m+n}j' + bx^n(x^m v) \in J,$$

and

$$x^{m+2n} + x^{m+n}z = x^m(x^n + z) = x^{m+n}z'.$$

It follows that $x^{m+2n} \in C_J = J$, whence $x \in \sqrt{J} = J$. We have shown that $J = K \cap L$, hence J is a finite intersection of prime k -ideals, a contradiction. \square

Corollary 6.6. *If A is weakly noetherian and I is a k -ideal of A , there are prime k -ideals $\mathcal{P}_1, \dots, \mathcal{P}_n$ of A such that*

$$\sqrt{I} = \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n.$$

Proof. As seen above, \sqrt{I} is a k -ideal; obviously it is radical, and we may then apply Theorem 6.5. \square

Proposition 6.7. *If A is weakly noetherian, then $\text{Min Pr}(A)$ is finite.*

Proof. Let us apply Corollary 6.6 to $I = \{0\}$; we obtain the existence of a finite family $\mathcal{P}_1, \dots, \mathcal{P}_n$ of prime k -ideals of A such that

$$\text{Nil}(A) = \sqrt{\{0\}} = \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n.$$

Let us suppose that no \mathcal{P}_j ($1 \leq j \leq n$) be contained in \mathcal{P} ; then one may find, for each $j \in \{1, \dots, n\}$, $x_j \in \mathcal{P}_j$, $x_j \notin \mathcal{P}$. It ensues that

$$x_1 \cdots x_n \in \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n = \text{Nil}(A) \subseteq \mathcal{P}$$

and each $x_j \notin \mathcal{P}$, contradicting the definition of \mathcal{P} . Therefore, for some j , $\mathcal{P}_j \subseteq \mathcal{P}$, whence (by the minimality of \mathcal{P}) $\mathcal{P}_j = \mathcal{P}$. We have shown that

$$\text{Min Pr}(A) \subseteq \{\mathcal{P}_1, \dots, \mathcal{P}_n\};$$

in particular, $\text{Min Pr}(A)$ is finite. □

REMARK 6.8. Incidentally, we have reestablished Corollary 4.2, as all \mathcal{P}_j 's are k -ideals.

7. The Evans condition

For A a B_1 -algebra and I an ideal of A , let

$$\begin{aligned} \mathcal{D}_A(I) &:= \{x \in A \mid (\exists y \notin I) \ xy \in I\} \\ &= \bigcup_{y \notin I} \mathcal{C}_y(I). \end{aligned}$$

Obviously, if A is nontrivial,

$$\mathcal{D}_A(\{0\}) = \mathcal{D}_A \cup \{0\}.$$

DEFINITION 7.1 (see [5], and also [11], Chapter 3, pp. 121–122). *A has the Evans property* if, for each k -ideal I of A , $\mathcal{D}_A(I)$ is a finite union of prime k -ideals of A .

REMARK 7.2 (to be compared with Theorem 3.1). If A has the Evans property, then A is standard (take $I = \{0\}$).

Theorem 7.3. *If A is laskerian, then it has the Evans property.*

Proof. We follow closely the proof of [5], Proposition 7. Let I denote a k -ideal of A ; then one may write

$$I = \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_n \quad (n \in \mathbf{N})$$

where each \mathcal{Q}_i is saturated and primary. Let us choose such a decomposition with n minimal, and, for each j , set $\mathcal{P}_j = \sqrt{\mathcal{Q}_j}$; according to Proposition 5.2, \mathcal{P}_j is prime (and a k -ideal).

Let $y \in \mathcal{D}_A(I)$; there is $x \notin I$ such that $yx \in I$. As $x \notin I = \mathcal{Q}_1 \cap \dots \cap \mathcal{Q}_n$, there exists $j \in \{1, \dots, n\}$ such that $x \notin \mathcal{Q}_j$.

As $xy = yx \in I \subseteq \mathcal{Q}_j$, $xy \in \mathcal{Q}_j$; therefore, as \mathcal{Q}_j is primary, there is a $m \geq 1$ such that $y^m \in \mathcal{Q}_j$, whence $y \in \sqrt{\mathcal{Q}_j} = \mathcal{P}_j$. We have shown that

$$\mathcal{D}_A(I) \subseteq \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n.$$

Conversely, let $y \in \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n$; then $y \in \mathcal{P}_j$ for some j . Let

$$K_j = \bigcap_{i=1; i \neq j}^n \mathcal{Q}_i;$$

according to our choice of n , $K_j \neq I$; as $I = K_j \cap \mathcal{Q}_j$, one has $K_j \not\subseteq \mathcal{Q}_j$, whence there exists $b \in K_j$, $b \notin \mathcal{Q}_j$. As $y \in \mathcal{P}_j$, $y^m \in \mathcal{Q}_j$ for some $m \geq 1$, therefore $y^m b \in K_j \cap \mathcal{Q}_j = I$.

But $y^0 b = b \notin I$ (as $b \notin \mathcal{Q}_j$); therefore there is a (unique) $k \in \mathbf{N}$ such that $y^k b \notin I$ and $y^{k+1} b \in I$. Let $z := y^k b$; then $z \notin I$ and $yz = y^{k+1} b \in I$, hence $y \in \mathcal{D}_A(I)$. Thus

$$\mathcal{D}_A(I) = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_n,$$

as desired. □

Theorem 7.4. *If A is weakly noetherian, then it has the Evans property.*

Proof. We shall adapt the reasoning used in the proof of Lemma 7 from [19].

Let I denote a saturated ideal of A ; according to the weak noetherianity hypothesis, each I -conductor is contained in a maximal one.

Let \mathcal{E} denote the set of $y \in A \setminus I$ such that $\mathcal{C}_y(I)$ is maximal, and let

$$R = C_{\{\mathcal{E}\}}.$$

Using once more the weak noetherianity hypothesis, one finds a finite family $(y_1, \dots, y_n) \in \mathcal{E}^n$ such that

$$R = C_{\langle y_1, \dots, y_n \rangle}.$$

By definition of \mathcal{E} and Lemma 2.9, each $\mathcal{P}_j := \mathcal{C}_{y_j}(I)$ is prime. Let

$$u \in \mathcal{P}_1 \cap \dots \cap \mathcal{P}_n;$$

then, by definition, $uy_j \in I$ for each j , whence

$$\langle y_1, \dots, y_n \rangle \subseteq \mathcal{C}_u(I)$$

and

$$R = C_{\langle y_1, \dots, y_n \rangle} \subseteq \mathcal{C}_u(I)$$

(as $\mathcal{C}_u(I)$ is saturated).

Let now $x \in \mathcal{E}$ and $\mathcal{P} = \mathcal{C}_x(I)$; then $x \in R$, whence $x \in \mathcal{C}_u(I)$ and $ux \in I$. It follows that $u \in \mathcal{C}_x(I) = \mathcal{P}$. Therefore

$$\mathcal{P}_1 \cap \dots \cap \mathcal{P}_n \subseteq \mathcal{P},$$

thus

$$\mathcal{P}_1 \cdots \mathcal{P}_n \subseteq \mathcal{P};$$

as in the proof of Proposition 6.7, it follows that, for some j , $\mathcal{P}_j \subseteq \mathcal{P}$, whence (by maximality) $\mathcal{P}_j = \mathcal{P}$. Therefore the set of maximal I -conductors is contained in $\{\mathcal{P}_1, \dots, \mathcal{P}_n\}$; in particular, it is finite.

Thus, the maximal elements of \mathcal{E} are finite in number; but they are prime ideals (Lemma 2.9), and $\mathcal{D}_A(I)$ is their union. \square

8. The characteristic one case

Let us now consider a B_1 -algebra A , and let I denote an ideal of A ; if $x \in A$ and $(i, j) \in I^2$ are such $x + i = j$, then $i + j = i + (i + x) = (i + i) + x = i + x = j$ and $x + j = x + (i + j) = (x + i) + j = j + j = j$, whence

$$C_I \subseteq \{x \in A \mid (\exists j \in I) x + j = j\};$$

the opposite inclusion being trivial, one has

$$C_I = \{x \in A \mid (\exists j \in I) x + j = j\} = \bar{I}$$

(see [15], Theorem 3.7, for the definition of \bar{I}). Then I is a k -ideal if and only if $I = C_I$, that is $I = \bar{I}$, in other words if and only if I is *saturated* in the sense of [15], p.1786. All of the above results therefore apply to B_1 -algebras modulo the replacement of “ k -ideal” by “saturated ideal” and of \sqrt{I} by $r(I)$.

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