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VARIETIES OF PICARD RANK ONE AS COMPONENTS OF AMPLE DIVISORS

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Abstract

Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ and denote by $\mathcal{L}$ an ample line bundle on $\mathcal{V}$. By imposing that the linear system $|\mathcal{L}|$ contains an element $A = A_1 + \cdots + A_r$, $r \geq 1$, where all the A_i's are distinct effective Cartier divisors with $\mathrm{Pic}(A_i) = \mathbb{Z}$, we show that such a $\mathcal{V}$ is as special as the components A_i of $A \in |\mathcal{L}|$. After making a list of some consequences about the positivity of the components A_i, we characterize pairs $(\mathcal{V}, \mathcal{L})$ as above when either $A_1 \cong \mathbb{P}^{n-1}$ and $\mathrm{Pic}(A_j) = \mathbb{Z}$ for $j = 2, \dots, r$, or $\mathcal{V}$ is smooth and each A_i is a variety of small degree with respect to $[H_i]_{A_i}$, where $[H_i]_{A_i}$ is the restriction to A_i of a suitable line bundle H_i on $\mathcal{V}$.

1. Introduction

Projective manifolds with an irreducible hyperplane section being a special variety have been studied since longtime (see, e.g., [2] and [6]), but the corresponding study for a reducible hyperplane section consisting of a simple normal crossing divisor whose components are special varieties started only recently by Chandler, Howard and Sommese [3]. Therefore, we continue here the study of varieties in terms of a hyperplane section A which is not irreducible, assuming that A is a union of distinct irreducible components $A_1, \dots, A_r$, with $r \geq 1$. More precisely, let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle $\mathcal{L}$. Assume that

($\diamondsuit$) $|\mathcal{L}|$ contains an element $A = A_1 + \cdots + A_r$, $r \geq 1$, where all the components A_i are distinct and effective Cartier divisors with $\mathrm{Pic}(A_i)$ of rank one.

Let us observe here that this assumption is a natural generalization of the classical hypothesis $\mathrm{Pic}(A') \cong \mathbb{Z}$ on a hyperplane section A' of $\mathcal{V}$. Furthermore, if ($\diamondsuit$) holds then every component A_i of $A \in |\mathcal{L}|$ does not admit a non-trivial morphism onto a variety W_i with $0 < \dim W_i < n - 1$ and in general, also for the smooth case, the main known results on reducible hyperplane sections (see, e.g., [3]) can not be applied on any of the A_i's. However, we can show that if a reducible subvariety A as in ($\diamondsuit$) is contained in $\mathcal{V}$ as an ample divisor, then it imposes severe restrictions to $\mathcal{V}$, that is, the topological and geometric structures of $\mathcal{V}$ are very closely related to those of each

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component of A. So, in §3 we prove first the following

Theorem 1.1. *Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Assume that ($\diamondsuit$) holds. Then all the A_i's are nef and big Cartier divisors on $\mathcal{V}$ and for any $i = 1, \ldots, r$ there exist proper birational morphisms $f_i : \mathcal{V} \to \mathcal{V}_i$ from $\mathcal{V}$ to a projective normal variety $\mathcal{V}_i$ given by the map associated to $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ for some $m_i \gg 0$. Furthermore, each map f_i contracts at most a finite number of curves on $\mathcal{V}$ and it is an isomorphism in a neighborhood of A_i such that $f_i(A_i)$ is an effective ample divisor on $\mathcal{V}_i$.*

The above result shows that assumption ($\diamondsuit$) implies that all the components A_i of A are either ample, or at worst big and 1-ample in the sense of [15, (1.3)]. By imposing some restrictions on the singularities of $\mathcal{V}$, we are able to deduce that all the A_i's are in fact ample Cartier divisors on $\mathcal{V}$.

Theorem 1.2. *Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen–Macaulay singularities. Let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$ and assume that ($\diamondsuit$) holds. Furthermore, suppose that $\mathcal{V} - F$ is a locally complete intersection for some finite, possibly empty, set $F \subset \mathcal{V} - \mathrm{Irr}(\mathcal{V})$ with* $\dim \mathrm{Irr}(\mathcal{V}) \leq 0$, *where* $\mathrm{Irr}(\mathcal{V})$ *is the set of irrational singularities of $\mathcal{V}$. Then all the A_i's are ample Cartier divisors on $\mathcal{V}$ and the maps $f_i : \mathcal{V} \to \mathcal{V}_i$ of* Theorem 1.1 *are all isomorphisms.*

Furthermore, under some additional hypotheses on $\mathcal{V}$ and on some A_i, we can finally obtain that $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$ for an ample line bundle Λ on $\mathcal{V}$ (Corollary 3.1).

All of these results allow us to list in §4 some consequences about the positivity of the A_i's and to obtain similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on $\mathcal{V}$ (Propositions 5.2 and 5.3).

We would like to note that the above results make use of weak hypotheses on $\mathcal{V}$ and on each A_i, and that ($\diamondsuit$) seems optimal a priori for Theorems 1.1 and 1.2, since easy examples show that these results do not hold assuming that $\mathrm{Pic}(A_i) \neq \mathbb{Z}\langle\mathcal{H}_i\rangle$ for some $i = 1, \ldots, r$, also when $r = 2$ and $\mathcal{V}$ is smooth (Remark 3.2). Moreover, the techniques we employ in §3 leave out of account any special polarization on each A_i by a (very) ample line bundle on $\mathcal{V}$ and they allow us to assume that $\mathcal{L}$ is simply ample and not necessarily ample and spanned or very ample on $\mathcal{V}$.

Finally, as a by-product of §4 and some results obtained by many other authors about smooth complex projective variety X containing ample divisors of special type (e.g., [1], [2], [6], [7], [9], [10], [11], [14]), in §5.1 we obtain similar results as in [1, Theorem 1] and [14, Proposition VI] (Propositions 5.2 and 5.3), and in §5.2 we classify smooth polarized pairs (X, L) which admit an ample divisor $A \in |L|$ such that $A = A_1 + \cdots + A_r$, $r \geq 1$, and all the components A_i have small degree with respect to suitable line bundles H_i on X for every $i = 1, \ldots, r$.

Proposition 1.3. *Let L be an ample line bundle on a smooth complex projective variety X of dimension n with $n \geq 5$. Assume that there is a divisor $A = A_1 + \cdots + A_r \in |L|$, $r \geq 1$, where each A_i is an irreducible and reduced normal Gorenstein projective variety with* $\dim \mathrm{Irr}(A_i) \leq 0$, *where* $\mathrm{Irr}(A_i)$ *is the set of irrational singularities of A_i. Suppose that for any $k = 1, \ldots, r$ there exist ample and spanned line bundles H_k on X such that $[H_k]_{A_k}$ is very ample and $[H_k]^{n-1}_{A_k} \leq 4$. Then one of the following possibilities holds*:

(1) *$r \geq 1$, $H_1 = \cdots = H_r = H$ and (X, H) is one of the following pairs*:

(a) *$(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{P}^n}(a_i)|$ with $1 \leq a_i \leq 4$ for every $i = 1, \ldots, r$;*

(b) *$(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{Q}^n}(a_i)|$ with $a_i = 1, 2$ for every $i = 1, \ldots, r$;*

(c) *$X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree* 3 *or* 4, *and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+1}}(1)_X|$ for every $i = 1, \ldots, r$;*

(d) *$X = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$ is a complete intersection of two quadric hypersurfaces $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$ for $i = 1, 2$, and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+2}}(1)_X|$ for every $i = 1, \ldots, r$;*

(e) *$\pi\colon X \to \mathbb{P}^n$ is a double cover of $\mathbb{P}^n$ with branch locus $\Delta \in |\mathcal{O}_{\mathbb{P}^n}(2b)|$, $b = 1,2$, $H \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(1)|$ and $A_i \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(a_i)|$, $a_i = 1, 2$, for every $i = 1, \ldots, r$;*

(f) *$\pi : X \to \mathbb{Q}^n$ is a double cover of a quadric hypersurface $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ with branch locus $\Delta \in |\mathcal{O}_{\mathbb{Q}^n}(2b')|$, $b' = 1, 2$, and $A_i = H \in |\pi^*\mathcal{O}_{\mathbb{Q}^n}(1)|$ for every $i = 1, \ldots, r$;*

(g) *$\pi : X \to \mathbb{P}^n$ is a d-cover of $\mathbb{P}^n$ with $d = 3, 4$, and $A_i = H \in |\pi^*\mathcal{O}_{\mathbb{P}^n}(1)|$ for any $i = 1, \ldots, r$;*

(2) *$r \geq 2$, $X \cong \mathbb{P}^1 \times \mathbb{P}^4$ and after renaming $(A_1, [H_1]_{A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 1))$, with $A_1 \in |\mathcal{O}_X(0, 1)|$ and $H_1 \in |\mathcal{O}_X(1, 1)|$, $(A_2, [H_2]_{A_2}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ with $A_2 \in |\mathcal{O}_X(1, 0)|$ and $H_2 \in |\mathcal{O}_X(t, 1)|$ for some integer $t \geq 1$, and the remaining polarized pairs $(A_k, [H_k]_{A_k})$ are of these two types.*

2. Notation

Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle $\mathcal{L}$. All notation and terminology used here are standard in algebraic geometry. We adopt the additive notation for line bundles and the numerical equivalence is denoted by $\equiv$, while the linear equivalence by $\simeq$. The pull-back $\iota^*\mathcal{F}$ of a line bundle $\mathcal{F}$ on $\mathcal{V}$ by an embedding $\iota\colon Y \hookrightarrow \mathcal{V}$ is sometimes denoted by $\mathcal{F}_Y$. We denote by $K_{\mathcal{V}}$ the canonical bundle of $\mathcal{V}$. For such a polarized variety $(\mathcal{V}, \mathcal{L})$, we will use the adjunction theoretic terminology of [2] and we say that $\mathcal{V}$ is an *n-fold* (denoted by X) when $\mathcal{V}$ is smooth.

3. Proof of Theorems 1.1 and 1.2

First of all, let us show that ($\Diamond$) implies that all the components of A are actually nef and big Cartier divisors on $\mathcal{V}$, obtaining the following

Proof of Theorem 1.1. Suppose that $A_1 \cap A_2 := W$ is nonempty. Define integers a_{ij} putting $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} := \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$, where $\mathcal{H}_j$ is the ample generator of $\mathrm{Pic}(A_j)$ (mod torsion). Then

$$\mathcal{O}_W(a_{11}\mathcal{H}_1) = \mathcal{O}_W(A_1) = \mathcal{O}_W(a_{12}\mathcal{H}_2)$$

is an ample line bundle on W. Hence a_{11} is positive. Since A is a connected divisor on $\mathcal{V}$ (see [8, III 7.9]), for every $i = 1, \ldots, r$ there exists a component A_j of $A \in |\mathcal{L}|$ with $j \neq i$ such that $A_i \cap A_j \neq \emptyset$. This shows that $\mathcal{O}_{\mathcal{V}}(A_i)_{A_i}$ is ample for any $i = 1, \ldots, r$, i.e. A_i is nef and big. From [7, III 4.2] (see also [2, (2.6.5)] or [12, (1.2.30)]), it follows that there exists a proper birational morphism $f_i : \mathcal{V} \to \mathcal{V}_i$ from $\mathcal{V}$ to a projective normal variety $\mathcal{V}_i$ given by the map associated to $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ for some $m_i \gg 0$, which is an isomorphism in a neighborhood of A_i and such that $f_i(A_i)$ is an effective ample divisor on $\mathcal{V}_i$.

Claim. *The morphism f_i contracts at most a finite number of curves on $\mathcal{V}$.*

By [2, (2.5.5)] note that $A_i \simeq H_i + D_i$ for any $i = 1, \ldots, r$, where H_i is $\mathbb{Q}$-ample and D_i is $\mathbb{Q}$-effective. This shows that $A_i \cdot A_j$ is nonzero and so $A_i \cap A_j$ can not be empty. Let $W_{ij} := A_i \cap A_j$ and note that W_{ij} is an ample divisor both in A_i and in A_j. Let Z be an irreducible variety of dimension greater than or equal to two. Since $\mathcal{L}$ is ample, we have that $Z \cap A_k \neq \emptyset$ for some $k = 1, \ldots, r$, and then $W_{ik} \cap Z \neq \emptyset$ since $\dim Z \cap A_k \geq 1$. Put $R := Z \cap A_i$. Thus R is nonempty, $\dim R \geq 1$ and $\mathcal{O}(R)_R$ is ample. By [7, III 4.2] (or [12, (1.2.30)]), the linear system $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ restricted to Z can not contract Z to a lower dimensional variety. □

Proof of Theorem 1.2. Put $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} = \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$ and $l_t := \mathcal{O}_{A_t}(\mathcal{H}_t)^{n-1} > 0$ for any $i, j, t \in \{1, \ldots, r\}$. From Theorem 1.1 we know that $a_{ii} > 0$ for every $i = 1, \ldots, r$. Moreover, since A is a connected divisor on X (see [8, III 7.9]), for every $j = 1, \ldots, r$ there exists a component A_k of $A \in |L|$ with $k \neq j$ such that $A_j \cap A_k \neq \varnothing$. So we get the following expressions

$$\begin{aligned} A_k^s A_j^{n-s} &= \mathcal{O}_{A_k}(a_{kk}\mathcal{H}_k)^{s-1}\mathcal{O}_{A_k}(a_{jk}\mathcal{H}_k)^{n-s} = a_{kk}^{s-1}a_{jk}^{n-s}l_k, \\ A_k^s A_j^{n-s} &= \mathcal{O}_{A_j}(a_{kj}\mathcal{H}_j)^{s}\mathcal{O}_{A_j}(a_{jj}\mathcal{H}_j)^{n-s-1} = a_{kj}^{s}a_{jj}^{n-s-1}l_j, \end{aligned} \tag{1}$$

with $1 \leq s \leq n-1$. Note that $a_{jj} \neq 0$, $a_{kk} \neq 0$, $a_{kj} > 0$ and $a_{jk} > 0$. Moreover, from the equations (1) with $s = 1, 2$ we deduce that

$$a_{jk}(a_{kj}^2 a_{jj}^{n-3} l_j) = a_{jk}(a_{kk}a_{jk}^{n-2}l_k) = a_{kk}(a_{jk}^{n-1}l_k) = a_{kk}(a_{kj}a_{jj}^{n-2}l_j),$$

that is,

$$a_{jk}a_{kj} = a_{jj}a_{kk}. \tag{2}$$

So we get

$$
\begin{aligned}
A_s^2\mathcal{L}^{n-2} &= \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1!\cdots h_r!} A_1^{h_1}\cdots A_s^{h_s+2}\cdots A_r^{h_r} \\
&= \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1!\cdots h_r!} a_{1s}^{h_1}\cdots a_{ss}^{h_s+1}\cdots a_{rs}^{h_r} l_s \\
&= a_{ss}^{n-1} l_s + (\text{non-negative terms}) > 0,
\end{aligned}
$$

for every $s = 1, \ldots, r$. Furthermore, for $j \neq k$ we have also the following equations

$$
(3)\quad A_j^2\mathcal{L}^{n-2} = [A_j]_{A_j}\mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\cdots+k_r=n-2} \frac{(n-2)!}{k_1!\cdots k_r!} a_{1j}^{k_1}\cdots a_{jj}^{k_j+1}\cdots a_{rj}^{k_r} l_j,
$$

$$
(4)\quad A_k^2\mathcal{L}^{n-2} = [A_k]_{A_k}\mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1!\cdots h_r!} a_{1k}^{h_1}\cdots a_{kk}^{h_k+1}\cdots a_{rk}^{h_r} l_k,
$$

$$
(5)\quad A_jA_k\mathcal{L}^{n-2} = [A_k]_{A_j}\mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\cdots+k_r=n-2} \frac{(n-2)!}{k_1!\cdots k_r!} a_{1j}^{k_1}\cdots a_{jj}^{k_j}\cdots a_{kj}^{k_k+1}\cdots a_{rj}^{k_r} l_j,
$$

$$
(6)\quad A_jA_k\mathcal{L}^{n-2} = [A_j]_{A_k}\mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\cdots+h_r=n-2} \frac{(n-2)!}{h_1!\cdots h_r!} a_{1k}^{h_1}\cdots a_{jk}^{h_j+1}\cdots a_{kk}^{h_k}\cdots a_{rk}^{h_r} l_k.
$$

Since by (2) we get

$$
\begin{aligned}
&(a_{1j}^{k_1}\cdot\cdots\cdot a_{jj}^{k_j+1}\cdot\cdots\cdot a_{rj}^{k_r})(a_{1k}^{h_1}\cdot\cdots\cdot a_{kk}^{h_k+1}\cdot\cdots\cdot a_{rk}^{h_r}) \\
&= a_{jj}a_{kk}(a_{1j}^{k_1}\cdot\cdots\cdot a_{jj}^{k_j}\cdot\cdots\cdot a_{rj}^{k_r})(a_{1k}^{h_1}\cdot\cdots\cdot a_{kk}^{h_k}\cdot\cdots\cdot a_{rk}^{h_r}) \\
&= a_{jk}a_{kj}(a_{1j}^{k_1}\cdot\cdots\cdot a_{jj}^{k_j}\cdot\cdots\cdot a_{rj}^{k_r})(a_{1k}^{h_1}\cdot\cdots\cdot a_{kk}^{h_k}\cdot\cdots\cdot a_{rk}^{h_r}) \\
&= (a_{1j}^{k_1}\cdot\cdots\cdot a_{jj}^{k_j}\cdot\cdots\cdot a_{kj}^{k_k+1}\cdot\cdots\cdot a_{rj}^{k_r})(a_{1k}^{h_1}\cdot\cdots\cdot a_{jk}^{h_j+1}\cdot\cdots\cdot a_{kk}^{h_k}\cdot\cdots\cdot a_{rk}^{h_r}),
\end{aligned}
$$

from (3), (4), (5) and (6), we obtain that

$$
(A_kA_j\mathcal{L}^{n-2})^2 = (A_k^2\mathcal{L}^{n-2})(A_j^2\mathcal{L}^{n-2}) > 0
$$

for every k and j such that $A_j \cap A_k \neq \emptyset$.

Thus, by [2, (2.5.4)] we see that there exists a rational number λ_{jk} such that A_j is numerically equivalent to $\lambda_{jk}A_k$. Since from (2) it follows that A_k meets all the other components of A, by an inductive argument we get

$$
(7)\qquad \mathcal{L} \simeq A_1 + \cdots + A_r \equiv (\lambda_{1k} + \cdots + \widehat{\lambda_{kk}} + \cdots + \lambda_{rk} + 1)A_k = \mu_k A_k,
$$

where $\mu_k := \lambda_{1k} + \cdots + \widehat{\lambda_{kk}} + \cdots + \lambda_{rk} + 1$ and the symbol $\widehat{\ }$ denotes suppression.

Moreover, since

$$
0 < \mathcal{L}_{A_j}^{n-1} = A_j\mathcal{L}^{n-1} = \lambda_{jk}A_k\mathcal{L}^{n-1} = \lambda_{jk}\mathcal{L}_{A_k}^{n-1},
$$

we have that $\lambda_{jk} > 0$ for every $j \neq k$ and from (7) it follows that $\mu_k \geq 1$, i.e. A_k is an ample Cartier divisor on $\mathcal{V}$ for any $k = 1, \dots, r$. By combining this with Theorem 1.1, we obtain that every f_i is an isomorphism. □

Corollary 3.1. *Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen–Macaulay singularities. Let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$ and assume that ($\diamond$) holds. Furthermore, suppose that one of the following conditions holds*:
(a) *$\mathcal{V}$ is a locally complete intersection*;
(b) *$n = 3$, $\mathcal{V} - A_k$ and $\mathcal{V} - F$ are locally complete intersections for some $k = 1, \dots, r$ and some finite set $F \subset \mathcal{V} - \mathrm{Irr}(\mathcal{V})$.*
Then $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$, *where Λ is an ample line bundle on $\mathcal{V}$. In particular, all the A_i's are ample Cartier divisors on $\mathcal{V}$ and the maps $f_i : \mathcal{V} \to \mathcal{V}_i$ of* Theorem 1.1 *are all isomorphisms.*

Proof. In both cases (a) and (b), we simply use Theorem 1.2 and [2, (2.3.4)]. □

REMARK 3.2. Let $\mathcal{V} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))$ with $0 < d < n$ and denote by $\pi : \mathcal{V} \to \mathbb{P}^{n-1}$ the projection map. Note that there exists a smooth divisor E on $\mathcal{V}$ which is a section of π such that $E \cong \mathbb{P}^{n-1}$ and $E_E \in |\mathcal{O}_{\mathbb{P}^{n-1}}(-d)|$. Put $\mathcal{L} := E + \pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$. Then, for a suitable integer $a \gg 0$, we have that $\mathcal{L}$ is ample on $\mathcal{V}$ and that there exists a divisor $A \in |\mathcal{L}|$ such that $A = A_1 + A_2$, where $A_1 = E$ and $A_2 \in |\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)|$ is a smooth divisor on $\mathcal{V}$. This example shows that if $r \geq 2$ then the above results can not be improved by assuming in ($\diamond$) that $\mathrm{Pic}(A_i) \neq \mathbb{Z}$ for some $i = 1, \dots, r$, also when $r = 2$ and $\mathcal{V}$ is a Fano n-fold with $n \geq 3$.

4. Some immediate consequences

Let us collect here some results due to the different positivity of the components A_i of $A \in |\mathcal{L}|$ under the hypothesis ($\diamond$).

4.0.1. Nefness and bigness of the A_i's. From Theorem 1.1, we obtain the following

Corollary 4.1. *Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Assume that ($\diamond$) holds. Set $D = \sum A_{i_h}$, where all the $i_h \in \{1, \dots, r\}$ are not necessarily distinct indexes. Moreover, let* $\mathrm{Irr}(\mathcal{V})$ *be the set of irrational singularities of $\mathcal{V}$. Then we have the following properties*:

- (Vanishing type theorems).

  (1) *Let $\varphi : \mathcal{V} \to Y$ be a morphism from $\mathcal{V}$ to a projective variety Y. Then*

$$\varphi_{(i)}(K_{\mathcal{V}} + D) = 0 \quad \textit{for} \quad i \geq \max_{y \in \varphi(\mathrm{Irr}(\mathcal{V}))} \dim(\varphi^{-1}(y) \cap \mathrm{Irr}(\mathcal{V})) + 1,$$

where $\varphi_{(i)}(\mathcal{T})$ *is the* i*-th higher derived functor of the direct image* $\varphi_*(\mathcal{T})$ *of a sheaf* $\mathcal{T}$ *on* $\mathcal{V}$;

(2) $H^1(\mathcal{V}, -D) = 0$;

(3) *assuming that* $\mathrm{Irr}(\mathcal{V})$ *is finite and nonempty, we have that*

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}} + D) \geq \#(\mathrm{Irr}(\mathcal{V})) > 0,$$

where $\#(\mathrm{Irr}(\mathcal{V}))$ *is the number of points in* $\mathrm{Irr}(\mathcal{V})$; *in particular, if* $\mathcal{A} \in |D|$ *lies in the set of Cohen–Macaulay points of* $\mathcal{V}$, *then*

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}}) + \dim H^0(\mathcal{A}, K_{\mathcal{A}}) \geq \#(\mathrm{Irr}(\mathcal{V})) > 0.$$

- (Lefschetz type theorems).

  (4) *Let* $\mathcal{A} \in |D|$ *be a divisor such that* $\mathcal{V} - \mathcal{A}$ *is a local complete intersection. Then under the restriction map it follows that*

  $$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \textit{for} \quad j \leq n-3,$$

  and

  $$H^j(\mathcal{V}, \mathbb{Z}) \to H^j(\mathcal{A}, \mathbb{Z}) \quad \textit{for} \quad j = n-2,$$

  is injective with torsion free cokernel; *moreover, we have that* $\mathrm{Pic}(\mathcal{V}) \cong \mathrm{Pic}(\mathcal{A})$ *for* $n \geq 5$, *and the restriction mapping* $\mathrm{Pic}(\mathcal{V}) \to \mathrm{Pic}(\mathcal{A})$ *is injective with torsion free cokernel for* $n = 4$; *in particular, if* $\mathcal{V} - A_i$ *is a local complete intersection for some* $i = 1, \ldots, r$, *then* $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$ *for* $n \geq 4$ *and some ample line bundle* Λ *on* $\mathcal{V}$;
- (The Albanese mapping).

  (5) *Assume that* $\mathcal{A} \in |D|$ *is normal. Moreover, suppose that* $\mathcal{A}$ *and* $\mathcal{V}$ *have at worst rational singularities. Then the map* $\mathrm{Alb}(\mathcal{A}) \to \mathrm{Alb}(\mathcal{V})$ *induced by inclusion is an isomorphism.*
- (Hodge index type theorems).

  (6) *For* $n_{i_1} + \cdots + n_{i_k} = n - 1$ *and* $n_{i_1} \geq 1$, *we have*

  $$(A_{i_0} \cdot A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^2 \geq (A_{i_0}^2 \cdot A_{i_1}^{n_{i_1}-1} \cdots A_{i_k}^{n_{i_k}})(A_{i_1}^{n_{i_1}+1} \cdots A_{i_k}^{n_{i_k}}),$$

  and for $n_{i_1} + \cdots + n_{i_k} = n$, *we get also*

  $$(A_{i_1}^{n_{i_1}} \cdots A_{i_k}^{n_{i_k}})^n \geq (A_{i_1}^n)^{n_{i_1}} \cdots (A_{i_k}^n)^{n_{i_k}} > 0,$$

  where $i_h \in \{1, \ldots, r\}$;

  (7) *we have the following inequality*: $(A_i^{n-1} \cdot A_j)(A_i \cdot A_j^{n-1}) \geq A_i^n A_j^n > 0$;

  (8) *for* $t \geq 1$ *and any nef and big line bundle* H_i *on* $\mathcal{V}$ *with* $1 \leq i \leq t$, *it follows that* $\mathcal{O}_{\mathcal{V}}(A_j)^{n-t} \cdot \prod_{i=1}^t H_i > 0$, *where* $j \in \{1, \ldots, r\}$;

(9) *let* H *be a line bundle such that* $\dim H^0(\mathcal{V}, NH) \geq 2$ *for some* $N \geq 1$. *Then* $H \cdot \mathcal{O}_{\mathcal{V}}(A_{i_1}) \cdots \mathcal{O}_{\mathcal{V}}(A_{i_{n-1}}) > 0$, *where* $i_h \in \{1, \ldots, r\}$.

Proof. First of all, from Theorem 1.1 we deduce that the effective divisor $D = \sum A_{i_h}$ is nef and big on $\mathcal{V}$. Thus (1), (2) and (3) of the statement follow from [2, (2.2.5), (2.2.7), (2.2.8)]. Finally, cases (4) to (9) follow from [2, (2.3.3), (2.3.4), (2.4.4), (2.5.1), (2.5.3), (2.5.2), (2.5.8) and (2.5.9)]. □

By applying Theorem 1.1 also to the zero locus of special sections of k-ample vector bundles on an n-fold X for $k \geq 0$ (see [15, §1] and [2, §2.1]), we obtain the following

Corollary 4.2 (Lefschetz–Sommese type theorem)**.** *Let* $\mathcal{E}$ *be a* k*-ample vector bundle of rank* $r \geq 1$ *on an* n*-fold* X *with* $n \geq 4$. *Assume that there exists a section* $s \in \Gamma(\mathcal{E})$ *whose zero locus* $Z = (s)_0$ *is an integral normal complex variety such that* $\dim Z \geq 3$. *Suppose that there exists a divisor* $A = A_1 + \cdots + A_s$, $s \geq 1$, *on* Z *which satisfies* ($\diamondsuit$). *If* $Z - A_i$ *is a local complete intersection for some* $i = 1, \ldots, s$, *then*

$$H^i(X, \mathbb{Z}) \cong H^i(A_i, \mathbb{Z}) \quad \textit{for} \quad i \leq \min\{\dim Z - 3, n - r - k - 1\},$$

and the restriction maps $H^i(X, \mathbb{Z}) \to H^i(A_i, \mathbb{Z})$ *are injective with torsion free cokernel for* $i = \min\{\dim Z - 2, n - r - k\}$.

Proof. From Theorem 1.1, we know that each A_k is at worst 1-ample on Z. Thus by Corollary 4.1 (4), we have that

$$H^j(Z, \mathbb{Z}) \cong H^j(A_i, \mathbb{Z}) \quad \text{for} \quad j \leq \dim Z - 3,$$

and the restriction maps $H^j(Z, \mathbb{Z}) \to H^j(A_i, \mathbb{Z})$ are injective with torsion free cokernel for $j = \dim Z - 2$. Moreover, from the Lefschetz–Sommese's theorem for k-ample vector bundles on an n-fold X (see [15, (1.16)] and [13, (7.1.1), (7.1.9)]), we deduce that

$$H^j(Z, \mathbb{Z}) \cong H^j(X, \mathbb{Z}) \quad \text{for} \quad j \leq n - r - k - 1,$$

and that the restriction maps $H^j(X, \mathbb{Z}) \to H^j(Z, \mathbb{Z})$ are injective with torsion free cokernel for $j = n - r - k$. □

4.0.2. Ampleness of the A_i's. Under the same assumption of Theorem 1.2, we first deduce the following

Corollary 4.3. *Let* $\mathcal{V}$ *be an integral normal complex projective variety of dimension* $n \geq 3$ *with at worst Cohen–Macaulay singularities. Let* $\mathcal{L}$ *be an ample line bundle on* $\mathcal{V}$ *and assume that* ($\diamondsuit$) *holds. Moreover, suppose that* $\mathcal{V} - F$ *is a local complete intersection for some finite, possibly empty, set* $F \subset \mathcal{V} - \mathrm{Irr}(\mathcal{V})$ *with* $\dim \mathrm{Irr}(\mathcal{V}) \leq 0$,

where Irr($\mathcal{V}$) *is the set of irrational singularities of* $\mathcal{V}$. *Set* $D = \sum A_{i_h}$, *where all the* $i_h \in \{1, \dots, r\}$ *are not necessarily distinct indexes. Then we have the following properties*:
(I) (Fujita's vanishing type theorem). *Given any coherent sheaf* $\mathcal{F}$ *on* $\mathcal{V}$, *there exists an integer* $m(\mathcal{F}, D)$ *such that*

$$H^i(\mathcal{V}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{V}}(mD + N)) = 0 \quad \textit{for} \quad i > 0,\ m \geq m(\mathcal{F}, D),$$

where N *is any nef divisor on* $\mathcal{V}$;
(II) $\dim H^0(\mathcal{V}, D) \leq D^n + n$, *with equality if and only if* $\mathcal{V}$ *is one of the following*:
(a) $\mathbb{P}^n$;
(b) *a quadric hypersurface* $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$;
(c) *a* $\mathbb{P}^{n-1}$*-bundle over* $\mathbb{P}^1$;
(d) *a generalized cone over a smooth submanifold* $V \subset \mathcal{V}$ *as in* (a), (b), (c);
(III) (Lefschetz type theorems). *Let* $\mathcal{A} \in |D|$ *be a divisor such that* $\mathcal{V} - \mathcal{A}$ *is a local complete intersection. Then under the restriction map it follows that*

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \textit{for} \quad j \leq n - 2,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \to H^j(\mathcal{A}, \mathbb{Z}) \quad \textit{for} \quad j = n - 1,$$

is injective with torsion free cokernel; *moreover, we have that* $\mathrm{Pic}(\mathcal{V}) \cong \mathrm{Pic}(\mathcal{A})$ *for* $n \geq 4$, *and the restriction mapping* $\mathrm{Pic}(\mathcal{V}) \to \mathrm{Pic}(\mathcal{A})$ *is injective with torsion free cokernel for* $n = 3$; *in particular, if* $\mathcal{V} - A_i$ *is a local complete intersection for some* $i = 1, \dots, r$, *then* $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$ *for* $n \geq 3$ *and some ample line bundle* Λ *on* $\mathcal{V}$.

Proof. By Nakai–Moishezon–Kleiman criterion and Theorem 1.2, we see that $D = \sum A_{i_h}$ is ample on $\mathcal{V}$. Thus case (I) of the statement follows from [5, §1, Theorem 1] (see also [12, (1.4.35)]. Finally, we obtain case (II) by [6, I (4.2), (5.10), (5.15)], while (III) follows from [2, (2.3.3)] and [2, (2.3.4)] respectively. □

Finally, let us deduce also the following result for ample vector bundles on a smooth variety.

Corollary 4.4. *Let* $\mathcal{E}$ *be an ample vector bundle of rank* $r \geq 1$ *on an* n*-fold* X *with* $n \geq 4$. *Assume that there exists a section* $s \in \Gamma(\mathcal{E})$ *whose zero locus* $Z = (s)_0$ *is a smooth submanifold of* X. *Suppose that there exists a divisor* $A = A_1 + \cdots + A_s, s \geq 1$, *on* Z *which satisfies* ($\diamondsuit$). *If* $r < n - 2$ *then both* $\mathrm{Pic}(X)$ *and* $\mathrm{Pic}(Z)$ *have rank one.*

Proof. It follows easily from Corollary 4.3 (III) and [15, (1.16)], or [13, (7.1.5) (ii)]. □

5. Some applications

Here are two applications.

5.1. All the A_i's are Fano varieties of Picard rank one. First of all, let us prove the following

Lemma 5.1. *Let $\mathcal{L}$, $\mathcal{V}$, A_i, $\mathcal{V}_i$, f_i be as in* Theorem 1.1. *If $\mathcal{V}_k$ is $\mathbb{Q}$-factorial for some $k = 1, \ldots, r$, then A_k is ample on $\mathcal{V}$ and $f_k\colon \mathcal{V} \to \mathcal{V}_k$ is in fact an isomorphism.*

Proof. Take any line bundle $\mathcal{D}$ on $\mathcal{V}$ and consider the following commutative diagram (∗):

$$\begin{array}{ccc} U & \xrightarrow{f_k|_U} & U' \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j_k} \\ \mathcal{V} & \xrightarrow[f_k]{} & \mathcal{V}_k \end{array}$$

where $j\colon U \to \mathcal{V}$ and $j_k\colon U' \to \mathcal{V}_k$ are the inclusion maps and $f_k|_U\colon U \to U'$ is the isomorphism induced by $f_k\colon \mathcal{V} \to \mathcal{V}_k$. Since $\mathcal{V}_k$ is $\mathbb{Q}$-factorial, we see that $f_{k*}(N\mathcal{D}) = \mathcal{L}'$ is a line bundle on $\mathcal{V}_k$ for some positive integer N. Write $\mathcal{L}' = \sum_h a_h \mathcal{L}'_h$, where $\mathcal{L}'_h$ are the generators of $\mathrm{Pic}(\mathcal{V}_k)$. Then by (∗) we get

$$\begin{aligned} N\mathcal{D}|_U &= j^*(N\mathcal{D}) = f_k|_{U*} j^*(N\mathcal{D}) = {j_k}^* f_{k*}(N\mathcal{D}) \\ &= \sum_h a_h {j_k}^* \mathcal{L}'_h = \sum_h a_h (f_k|_U)^* {j_k}^* \mathcal{L}'_h = \sum_h a_h j^* {f_k}^* \mathcal{L}'_h \\ &= j^*\left(\sum_h a_h {f_k}^* \mathcal{L}'_h\right) = \left(\sum_h a_h {f_k}^* \mathcal{L}'_h\right)\Bigg|_U . \end{aligned}$$

By Hartogs' lemma (see, e.g., [4, (11.4)]), this gives $N\mathcal{D} = \sum_h a_h f_k^*(\mathcal{L}'_h)$, i.e. $\mathcal{D} = \sum_h (a_h/N) f_k^*(\mathcal{L}'_h)$. Therefore, if A_k is not ample, then we deduce that there exists an irreducible curve $\Gamma \subset \mathcal{V}$ such that $m_k A_k \cdot \Gamma = 0$ for any positive integer m_k, i.e. the map f_k contracts the curve Γ. Hence $f_k^*(\mathcal{L}'_h) \cdot \Gamma = 0$ for any h, i.e. $\mathcal{D} \cdot \Gamma = 0$, but this leads to a contradiction by taking $\mathcal{D} = \mathcal{L}$. □

Similar results as in [1, Theorem 1] (see also [14, Proposition VI]) for the case of reducible ample divisors on $\mathcal{V}$ can be now proved.

Proposition 5.2. *Let $\mathcal{V}$ be an integral normal complex projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Assume that (◇) holds. If, up to renaming, $A_1 \cong \mathbb{P}^{n-1}$, then $\mathcal{V}$ is the cone $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ on $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ with $A_1 = v_s(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$, where v_s is the s^{th} Veronese embedding of $\mathbb{P}^{n-1}$.*

Proof. Since ($\diamondsuit$) holds and $A_1 \cong \mathbb{P}^{n-1}$, by Theorem 1.1 we have that $f_1(A_1) \cong \mathbb{P}^{n-1}$ is ample on $\mathcal{V}_1$. By [1, Theorem 1] we see that $\mathcal{V}_1$ is the cone $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ over $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, where s is a positive integer such that $\mathcal{N}_{f_1(A_1)/\mathcal{V}_1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$.

Since $\mathcal{V}_1$ is $\mathbb{Q}$-factorial and $\mathrm{Pic}(\mathcal{V}_1) = \mathbb{Z}$, by Lemma 5.1 we deduce that f_1 is an isomorphism and $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}$. Therefore, A_1 is an ample divisor on $\mathcal{V}$ and by applying now [1, Theorem 1] to the pair $(\mathcal{V}, A_1)$, we get that $\mathcal{V} \cong \mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, $A_1 \cong v_s(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$. □

Proposition 5.3. *Let $\mathcal{V}$ be an integral normal Gorenstein projective variety of dimension $n \geq 3$ and let $\mathcal{L}$ be an ample line bundle on $\mathcal{V}$. Suppose that ($\diamondsuit$) holds and that $\dim \mathrm{Irr}(\mathcal{V}) \leq 0$, where $\mathrm{Irr}(\mathcal{V})$ is the set of irrational singularities of $\mathcal{V}$. Assume that each A_i is a normal Gorenstein variety such that $K_{A_i} + \tau_i \mathcal{H}_i \simeq \mathcal{O}_{A_i}$ for some integer τ_i. If $\mathcal{V} - A_k$ is a local complete intersection for some $k = 1, \dots, r$ and either $n \geq 4$, or $n = 3$ and $\mathcal{V} - F$ is a local complete intersection for some finite, possibly empty set $F \subset \mathcal{V} - \mathrm{Irr}(\mathcal{V})$, then $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$, where Λ is an ample line bundle on $\mathcal{V}$, $K_{\mathcal{V}} = \rho\Lambda$, $A_i = a_i\Lambda$ and $\Lambda_{A_i} = h_i\mathcal{H}_i$ with $\tau_i = -h_i(\rho + a_i)$, where ρ, $a_i > 0$ and $h_i > 0$ are integers. In particular, for $n \geq 5$ we have $h_i = 1$ for every $i = 1, \dots, r$.*

Proof. Assume that $n \geq 4$. From case (4) of Corollary 4.1 it follows that either (a) $n \geq 5$ and $\mathrm{Pic}(\mathcal{V}) \cong \mathrm{Pic}(A_i) = \mathbb{Z}$ for any $i = 1, \dots, r$, or (b) $n = 4$ and $\mathrm{Pic}(\mathcal{V})$ restricts injectively into $\mathrm{Pic}(A_i) = \mathbb{Z}$. In both situations, we see that $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$ for some ample line bundle Λ on $\mathcal{V}$. Moreover, in (a) we have that $\Lambda_{A_i} \simeq \mathcal{H}_i$, while in (b) we have that $\Lambda_{A_i} \simeq h_i\mathcal{H}_i$ for some positive integer h_i. By adjunction formula, we obtain that

$$\mathcal{O}_{A_i} \simeq K_{A_i} + \tau_i\mathcal{H}_i \simeq \begin{cases} (K_{\mathcal{V}} + A_i + \tau_i\Lambda)_{A_i} & \text{in case (a),} \\ \left(K_{\mathcal{V}} + A_i + \dfrac{\tau_i}{h_i}\Lambda\right)_{A_i} & \text{in case (b).} \end{cases}$$

i.e. $K_{\mathcal{V}} + A_i + (\tau_i/h_i)\Lambda \simeq \mathcal{O}_{\mathcal{V}}$ for some $h_i \geq 1$. If we put $K_{\mathcal{V}} = \rho\Lambda$ and $A_i = a_i\Lambda$ for some integers ρ and $a_i \geq 1$, then we see that $\tau_i = -h_i(\rho + a_i)$, with $h_i = 1$ when $n \geq 5$.

As to the case $n = 3$, from (III) of Corollary 4.3 it follows that $\mathrm{Pic}(\mathcal{V})$ restricts injectively into $\mathrm{Pic}(A_i) = \mathbb{Z}$. Then by arguing as above, we conclude that also in this situation $\mathrm{Pic}(\mathcal{V}) = \mathbb{Z}\langle\Lambda\rangle$ for some ample line bundle Λ on $\mathcal{V}$ and, with the same notation as above, $\tau_i = -h_i(\rho + a_i)$, where h_i and a_i are positive integers. □

REMARK 5.4. When $n \geq 6$, Proposition 5.3 generalizes [16, Theorem 1].

5.2. All the A_i's have small degrees. Denote now by X an n-fold with $n \geq 3$ and by L an ample line bundle on X. In this subsection we prove the last result stated in the Introduction.

Proof of Proposition 1.3. Since $[H_i]_{A_i}^{n-1} \leq 4$, by [2, (8.10.1)] we see that $(A_i, [H_i]_{A_i})$ satisfies one of the following two conditions:
(a) $\mathrm{Pic}(A_i) = \mathbb{Z}\langle [H_i]_{A_i} \rangle$;
(b) $(A_i, [H_i]_{A_i}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 1))$.

STEP (I). First of all, assume that $r = 1$. In Case (a), by Lefschetz's theorem we see that $\mathrm{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$. Write $A_1 = a_1 H_1$ for a positive integer a_1. Since $a_1 H_1^n = A_1 H_1^{n-1} = [H_1]_{A_1}^{n-1} \leq 4$, we deduce that $a_1 H_1^n \leq 4$. Consider the map $\varphi\colon X \to \mathbb{P}^N$ associated to $|H_1|$. Since $|H_1|$ is ample and spanned, the morphism φ is finite and such that

$$[H_1]^n = \deg \varphi \cdot \deg \varphi(X) \leq 4.$$

This gives the following three possibilities:
(1) $\deg \varphi = 1$, $\deg \varphi(X) \leq 4$;
(2) $\deg \varphi = 2$, $\deg \varphi(X) \leq 2$;
(3) $\deg \varphi = 3, 4$ and $\deg \varphi(X) = 1$.
In (1), since $n \geq 5$ and $\mathrm{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$, by [2, (8.10.1)] (see also [6], [9], [11]) we deduce that (X, H_1) is one of the following pairs:
- $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, where $A_1 \in |\mathcal{O}_{\mathbb{P}^n}(a_1)|$ with $0 < a_1 \leq 4$;
- $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, where $A_1 \in |\mathcal{O}_{\mathbb{Q}^n}(a_1)|$ with $0 < a_1 \leq 2$;
- $(V_d, \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{V_d})$ with $A_1 \in |H_1|$, where $V_d \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d = 3, 4$;
- $(W, \mathcal{O}_{\mathbb{P}^{n+2}}(1)_W)$ with $A_1 \in |H_1|$, where $W = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$ is a complete intersection of two quadric hypersurfaces $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$ for $i = 1, 2$.

In (2), the map φ is a double cover of either (i) $\mathbb{P}^n$, or (ii) $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$. In case (i), we see that $H_1 = \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$ and $A_1 \in |a_1 H_1|$ with $a_1 = 1, 2$. In case (ii), we get $H_1 = \varphi^*\mathcal{O}_{\mathbb{Q}^n}(1)$ and $A_1 \in |H_1|$. Finally, in (3) the morphism φ is a d-cover of $\mathbb{P}^n$ with $d = 3, 4$, and $A_1 = H_1 = \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$.

Finally, consider Case (b). Note that $\mathrm{Pic}(X) \neq \mathbb{Z}$. Moreover, since A_1 is ample, for any ample line bundle H on $\mathbb{P}^1 \times \mathbb{P}^3$, we have

$$\begin{aligned} 0 &= \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 0) \cdot \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(2, 0) \cdot H^2 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 0) \cdot (K_{A_1} + 4H_{1A_1}) \cdot H^2 \\ &= \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 0) \cdot (K_X + 4H_1 + A_1)_{A_1} \cdot H^2 \geq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 0) \cdot (K_X + 5H_1)_{A_1} \cdot H^2, \end{aligned}$$

i.e. $K_X + 5H_1$ can not be ample on X. So by [10], [6] and [2, §7.2], we see that (X, H_1) is a scroll over $\mathbb{P}^1$ of dimension five, i.e. $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_5)$ with $1 \leq a_1 \leq \cdots \leq a_5$, and H_1 is the tautological line bundle ξ of $\mathbb{P}(\mathcal{E})$. Put $\xi' = \xi - a_1 F$. Since A_1 is ample on X, write $A_1 = a\xi' + bF$ for some positive integers a and b (see [2, (3.2.4)]). Then

$$[H_1]_{A_1}^4 = (a\xi' + bF)(\xi)^4 = a(\xi - a_1 F)(\xi)^4 + b = a(a_2 + \cdots + a_5) + b \geq 5,$$

but this is absurd. Thus Case (b) can not occur for $r = 1$.

STEP (II). Suppose now that $r \geq 2$. First of all, let us prove the following

Claim. *If* $(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1,1))$, *then* $K_X + 4H_i$ *is not nef for some* $i = 1, \ldots, r$.

Proof. Suppose that $K_X + 4H_k$ is nef for every $k = 1, \ldots, r$. Let $\Phi_1 \colon A_1 \to \mathbb{P}^1$ be the nefvalue morphism of (A_1, H_{1A_1}). Up to renaming, assume that $A_2, \ldots, A_s$ are the only components of A such that $A_1 \cap A_h \neq \varnothing$ for $h = 2, \ldots, s$ with $s \leq r$. Put $h_k := [A_k]_{A_1}$ for $k = 2, \ldots, r$ and note that h_t is trivial for $t = s+1, \ldots, r$.

First of all, if $\Phi_1(h_k)$ is a union of points of $\mathbb{P}^1$ for every $k = 2, \ldots, s$, then for a general fiber $F \cong \mathbb{P}^3$ of Φ_1 we have that

$$(L_{A_1})_F = ([A_1]_{A_1})_F + [h_2]_F + \cdots + [h_s]_F = ([A_1]_{A_1})_F = \mathcal{O}_F(a),$$

for a suitable integer $a \geq 1$. Moreover, note that $K_{A_1} + 4H_{1A_1} \simeq 2F$.

Thus by adjunction we obtain that

$$\begin{aligned}[K_X + 4H_1]_F H_{1F}^2 &= ([K_X + 4H_1]_{A_1})_F (H_{1A_1})_F^2 = (2F - [A_1]_{A_1})_F (H_{1A_1})_F^2 \\ &= -([A_1]_{A_1})_F (H_{1A_1})_F^2 = -\mathcal{O}_F(a)\mathcal{O}_F(1)^2 < 0,\end{aligned}$$

but this is absurd, since $K_X + 4H_1$ is nef and H_1 is ample on X.

Assume now that $\Phi_1(h_t) = \mathbb{P}^1$ for some $t = 2, \ldots, s$. If $\mathrm{Pic}(A_t) = \mathbb{Z}$, then by [3, (4.2)] we get a contradiction by taking $B = A_t$ and $A = A_1$. Thus we can assume that $(A_t, H_{tA_t}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 1))$. Note that there exists a general fiber F of Φ_1 such that $F \not\subseteq h_t$ and $[h_t]_F$ is effective and not trivial. Moreover, we have that $\mathrm{Pic}(F) \cong \mathbb{Z}$ and $F \not\subseteq A_t$, since otherwise $F \subseteq A_1 \cap A_t = h_t$. Therefore $F \cap A_t \neq \varnothing$ and $F \cap A_t \neq F$. Since

$$\dim F + \dim A_t - \dim \Phi_1(A_1) = 3 + 4 - 1 = 6 > 5 = \dim X,$$

by the same argument as in the proof of [3, (4.2)] with $B = F$ and $A = A_t$, we can conclude that $F \subseteq A_t$, but this gives a contradiction. □

By the above Claim, if one of the A_k, say A_1, is such that

$$(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^3}(1, 1)),$$

then $n = 5$ and $K_X + 4H_i$ is not nef for some $i = 1, \ldots, r$. Since $\mathrm{Pic}(A_1) \neq \mathbb{Z}$, by [10], [6] and [2, §7] we deduce that (X, H_i) is a $\mathbb{P}^4$-bundle over a smooth curve C. Moreover, A_1 dominates C. So $C \cong \mathbb{P}^1$ and $X \cong \mathbb{P}(\mathcal{E})$ for some vector bundle $\mathcal{E}$ of rank-5 on $\mathbb{P}^1$ such that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\bar{a}_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(\bar{a}_4)$ with $0 \leq \bar{a}_1 \leq \cdots \leq \bar{a}_4$. Let ξ be the tautological line bundle of $\mathbb{P}(\mathcal{E})$. Write $A_1 = a_1\xi + b_1 F$ and $H_1 = \alpha_1\xi + \beta_1 F$ with $b_1 \geq 0$ and a_1, α_1, β_1 positive integers (see [2, (3.2.4)]). Furthermore, put $F_{A_1} \simeq$

$\mathcal{O}_{A_1}(a, 0)$ and $\xi_{A_1} \simeq \mathcal{O}_{A_1}(b, c)$ with $a > 0$ and $b, c \geq 0$. By the adjunction formula we have

$$\begin{aligned}\mathcal{O}_{A_1}(-2, -4) \simeq K_{A_1} \simeq (K_X + A_1)_{A_1} &\simeq [-5\xi + (\deg \mathcal{E} - 2)F + a_1\xi + b_1 F]_{A_1} \\ &= (a_1 - 5)\xi_{A_1} + (b_1 + \deg \mathcal{E} - 2)F_{A_1} \\ &\simeq \mathcal{O}_{A_1}(b(a_1 - 5) + a(b_1 + \deg \mathcal{E} - 2), c(a_1 - 5)).\end{aligned}$$

This gives the following two equations:

$$(\flat) \qquad 4 = c(5 - a_1),$$

$$(\natural) \qquad 2 = b(5 - a_1) - a(b_1 + \deg \mathcal{E} - 2).$$

Note that from $(\flat)$ it follows that $1 \leq a_1 \leq 4$ and then

$$4 \geq a_1 = [A_1]_F \cdot [\xi]_F^3 = F_{A_1} \cdot \xi_{A_1}^3 = \mathcal{O}_{A_1}(a, 0) \cdot \mathcal{O}_{A_1}(b, c)^3 = ac^3.$$

Thus we deduce that $c = 1$ and $a_1 = a$. By $(\flat)$ we have also that $a = a_1 = 1$ and the equation $(\natural)$ gives $4b = b_1 + \deg \mathcal{E}$. Since

$$\mathcal{O}_{A_1}(1, 1) \simeq [H_1]_{A_1} = [\alpha_1\xi + \beta_1 F]_{A_1} = \mathcal{O}_{A_1}(\alpha_1 b + \beta_1, \alpha_1),$$

we see that $\alpha_1 = 1$ and $1 = b + \beta_1 \geq b + 1 \geq 1$, i.e. $b = 0$ and $\beta_1 = 1$. Therefore, by $(\natural)$ we get $0 = 4b = b_1 + \deg \mathcal{E} \geq \deg \mathcal{E}$, i.e. $\bar{a}_1 = \cdots = \bar{a}_4 = 0$ and $b_1 = 0$. This shows that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 5}) \cong \mathbb{P}^1 \times \mathbb{P}^4$, $A_1 \in |\mathcal{O}_X(0, 1)|$ and $H_1 \in |\mathcal{O}_X(1, 1)|$. Consider $A_i \in |\mathcal{O}_X(d, e)|$ and $H_i \in |\mathcal{O}_X(t, s)|$ for some $i = 2, \ldots, r$, where $d, e \geq 0$ and $t, s > 0$. Then we have

$$4 \geq [H_i]_{A_i}^4 = \mathcal{O}_X(d, e) \cdot \mathcal{O}_X(t, s)^4 = s^3(ds + 4et) \geq s^3.$$

This gives $s = 1$ and $0 \leq e \leq 1$. If $e = 1$, then we get $d = 0$, $t = s = e = 1$ and $(A_i, [H_i]_{A_i})$ is of the same type as $(A_1, [H_1]_{A_1})$. If $e = 0$, then we see that $1 \leq d \leq 4$, $t \geq 1$ and $s = 1$. This shows that $A_i \in |\mathcal{O}_X(d, 0)|$ and $H_i \in |\mathcal{O}_X(t, 1)|$ with $t \geq 1$ and $1 \leq d \leq 4$, i.e. $A_i \to \mathbb{P}^4$ is a d-cover of $\mathbb{P}^4$ with $1 \leq d \leq 4$. Note that in this case we have

$$|\mathcal{O}_X(d, 0)| = \underbrace{|\mathcal{O}_X(1, 0)| + \cdots + |\mathcal{O}_X(1, 0)|}_{d\text{-times}}.$$

Thus, since $A_i \in |\mathcal{O}_X(d, 0)|$ is irreducible and reduced, we conclude that $d = 1$.

Finally, since the above argument works independently from the choice of the component A_1 of $A \in |L|$, we can assume, without loss of generality, that every component A_i of A is such that $\mathrm{Pic}(A_i) = \mathrm{Z}\langle [H_i]_{A_i} \rangle$. Then by Proposition 5.3 and Corollary 4.3 (III), we conclude that $\mathrm{Pic}(X) = \mathbb{Z}\langle \Lambda \rangle$ for some ample line bundle Λ on X. Write

$A_i = a_i\Lambda$ and $H_i = b_i\Lambda$ for suitable positive integers a_i and b_i. Thus we obtain that

$$a_i b_i^{n-1}\Lambda^n = (a_i\Lambda)(b_i\Lambda)^{n-1} = [H_i]_{A_i}^{n-1} \le 4,$$

i.e. $H_1 = \cdots = H_r = \Lambda$ and $a_i\Lambda^n \le 4$. Consider the map $\varphi\colon X \to \mathbb{P}^N$ associated to $|\Lambda|$. Since Λ is ample and spanned, we have $\Lambda^n = \deg\varphi \cdot \deg\varphi(X) \le 4$ and by arguing as in case $r = 1$, we obtain the statement. □

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References

[1] L. Bădescu: *On ample divisors*, Nagoya Math. J. **86** (1982), 155–171.
[2] M.C. Beltrametti and A.J. Sommese: The Adjunction Theory of Complex Projective Varieties, de Gruyter Expositions in Mathematics **16**, de Gruyter, Berlin, 1995.
[3] K.A. Chandler, A. Howard and A.J. Sommese: *Reducible hyperplane sections*, I, J. Math. Soc. Japan **51** (1999), 887–910.
[4] D. Eisenbud: Commutative Algebra, Graduate Texts in Mathematics **150**, Springer, New York, 1995.
[5] T. Fujita: *Vanishing theorems for semipositive line bundles*; in Algebraic Geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math. **1016**, Springer, Berlin, 1983, 519–528.
[6] T. Fujita: Classification Theories of Polarized Varieties, London Mathematical Society Lecture Note Series **155**, Cambridge Univ. Press, Cambridge, 1990.
[7] R. Hartshorne: Ample Subvarieties of Algebraic Varieties, Lecture Notes in Mathematics **156**, Springer, Berlin, 1970.
[8] R. Hartshorne: Algebraic Geometry, Graduate Texts in Mathematics **52**, Springer, New York, 1977.
[9] P. Ionescu: *Embedded projective varieties of small invariants*; in Algebraic Geometry, Bucharest 1982 (Bucharest, 1982), Lecture Notes in Math., 1056, Springer, Berlin, 1984, 142–186.
[10] P. Ionescu: *Generalized adjunction and applications*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 457–472.
[11] P. Ionescu: *On manifolds of small degree*, Comment. Math. Helv. **83** (2008), 927–940.
[12] R. Lazarsfeld: Positivity in Algebraic Geometry, I, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, A Series of Modern Surveys in Mathematics **48**, Springer, Berlin, 2004.
[13] R. Lazarsfeld: Positivity in Algebraic Geometry, II, Ergebnisse der Mathematik und ihrer Grenzgebiete 3 Folge, A Series of Modern Surveys in Mathematics **49**, Springer, Berlin, 2004.
[14] A.J. Sommese: *On manifolds that cannot be ample divisors*, Math. Ann. **221** (1976), 55–72.
[15] A.J. Sommese: *Submanifolds of Abelian varieties*, Math. Ann. **233** (1978), 229–256.
[16] A.L. Tironi: *Ample normal crossing divisors consisting of two del Pezzo manifolds*, Forum Math. **22** (2010), 667–682.

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