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HYPERCYCLICITY OF TRANSLATION OPERATORS IN A REPRODUCING KERNEL HILBERT SPACE OF ENTIRE FUNCTIONS INDUCED BY AN ANALYTIC HILBERT-SPACE-VALUED KERNEL

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Abstract

The study of the hypercyclicity of an operator is an old problem in mathematics; it goes back to a paper of Birkhoff in 1929 proving the hypercyclicity of the translation operators in the space of all entire functions with the topology of uniform convergence on compact subsets. This article studies the hypercyclicity of translation operators in some general reproducing kernel Hilbert spaces of entire functions. These spaces are obtained by duality in a complex separable Hilbert space $\mathcal{H}$ by means of an analytic $\mathcal{H}$-valued kernel. A link with the theory of de Branges spaces is also established. An illustrative example taken from the Hamburger moment problem theory is included.

1. Introduction

The goal in this paper is to study the hypercyclicity of the translation operators in a reproducing kernel Hilbert space of entire functions $\mathcal{H}_K$ obtained by duality by means of a kernel K. Namely, let $\mathcal{H}$ be a complex, separable Hilbert space with inner product $\langle\cdot,-\rangle_{\mathcal{H}}$ and suppose K is an analytic $\mathcal{H}$-valued function defined on $\mathbb{C}$. For each $x\in\mathcal{H}$, define the function $f_x(z)=\langle K(z),x\rangle_{\mathcal{H}}$ on $\mathbb{C}$, and let $\mathcal{H}_K$ denote the collection of all such functions f_x. One can transfer the hilbertian structure of $\mathcal{H}$ to obtain a reproducing kernel Hilbert space $\mathcal{H}_K$ of entire functions since the kernel K is analytic on $\mathbb{C}$ (see, for instance, [14]).

These spaces $\mathcal{H}_K$ are ubiquitous in mathematics; see [14] and the references therein. These spaces are also familiar in sampling theory (see, for instance, [7, 10]). In particular, choosing $K\colon\mathbb{C}\to L^2[-\pi,\pi]$ as $K(z)(w):=\mathrm{e}^{izw}/\sqrt{2\pi}$, $w\in[-\pi,\pi]$, the corresponding $\mathcal{H}_K$ space coincides with the well-known Paley–Wiener space PW_π of bandlimited functions to $[-\pi,\pi]$.

Recall that a vector f is said to be hypercyclic for a continuous linear operator T in a Fréchet space $\mathcal{F}$ (a complete metrizable convex space) if its orbit $\{T^n f\}_{n=0}^{\infty}$ is

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dense in $\mathcal{F}$. In this case the operator T is also said to be hypercyclic. The first example of a hypercyclic operator was obtained by Birkhoff [3] who showed that the translation operators $T_w\colon f \mapsto f(\cdot - w)$, $w \in \mathbb{C}$, $w \neq 0$, are hypercyclic in the space of all entire functions with the topology of uniform convergence on compact subsets of the plane. It was shown by MacLane [13] that the differentiation operator is also hypercyclic in this space, and in Shapiro–Godefroy [11] this result was extended to all operators commuting with differentiation except the scalar multiples of identity. Thus, spaces of entire functions have proved to be an important source of hypercyclic operators.

Besides, Chan and Shapiro studied in [5] the hypercyclicity of translations in the setting of Hilbert spaces of entire functions of "slow growth"; in [5] the posed question was whether translations in a "reasonable" space of entire functions are always hypercyclic, and it was shown that this is not true. For example, differentiation and translation operators in the Paley–Wiener space PW_a are bounded but not hypercyclic; in particular, T_w is an isometry of PW_a if $w \in \mathbb{R}$.

In the present paper we prove, under appropriate hypotheses, hypercyclicity of translation operators T_w, $w \neq 0$, in an $\mathcal{H}_K$ space (see Theorem 2.12 in Section 2). In so doing we apply a result from Chan–Shapiro [5] requiring the density of the set of polynomials $\mathcal{P}(\mathbb{C})$ in $\mathcal{H}_K$; also, the well-definedness and boundedness of the differentiation operator on $\mathcal{H}_K$ are needed. All these preliminaries are included in Section 2.

Translations' hypercyclicity in a de Branges space $\mathcal{H}(E)$ has been studied by Baranov in [2]; this study is carried out in the case that the polynomials are dense in $\mathcal{H}(E)$, obtaining sufficient conditions for special structure functions E. In Section 3, we study when a space $\mathcal{H}_K$ is equal isometrically to a de Branges space $\mathcal{H}(E)$ where the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense; under suitable hypotheses we give a necessary and sufficient condition (see Theorem 3.2). We also prove a result on the hypercyclicity of the translation operators T_w, $w \neq 0$, in these spaces (see Corollary 3.6).

Finally, in Section 4 we illustrate our results with an example taken from the indeterminate Hamburger moment problem theory.

2. Hypercyclicity of translation operators in $\mathcal{H}_K$ spaces

In order to prove our result on the hypercyclicity of translation operators in spaces $\mathcal{H}_K$ we first introduce these spaces, and we derive some needed properties.

2.1. Some preliminaries on $\mathcal{H}_K$ spaces. Suppose we are given a separable complex Hilbert space and an abstract kernel K which is a $\mathcal{H}$-valued function on $\mathbb{C}$. For each $z \in \mathbb{C}$, set $f_x(z) := \langle K(z), x\rangle_{\mathcal{H}}$ and denote by $\mathcal{H}_K$ the collection of all such functions f_x, $x \in \mathcal{H}$, and let $\mathcal{T}_K$ be the mapping

$$\mathcal{H} \ni x \overset{\mathcal{T}_K}{\mapsto} f_x \in \mathcal{H}_K \tag{2.1}$$

Notice that the mapping $\mathcal{T}_K$ is an antilinear mapping from $\mathcal{H}$ onto $\mathcal{H}_K$ and $\ker \mathcal{T}_K$ is a closed subspace of $\mathcal{H}$. The norm $\|f\|_{\mathcal{H}_K} := \inf\{\|x\|_{\mathcal{H}} : f = \mathcal{T}_K x\}$ defined in $\mathcal{H}_K$, induces the inner product

$$\langle f, g\rangle_{\mathcal{H}_K} = \langle P_{(\ker \mathcal{T}_K)^\perp} y, P_{(\ker \mathcal{T}_K)^\perp} x\rangle_{\mathcal{H}}$$

where $P_{(\ker \mathcal{T}_K)^\perp}$ is the orthogonal projection on $(\ker \mathcal{T}_K)^\perp$, $\mathcal{T}_K x = f$ and $\mathcal{T}_K y = g$. The closed linear subspace $(\ker \mathcal{T}_K)^\perp$ equipped with the inner product induced by the inner product of $\mathcal{H}$ is a Hilbert space. By this way we obtain that $\mathcal{H}_K$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$k(z, w) = \langle K(z), K(w)\rangle_{\mathcal{H}}, \quad z, w \in \mathbb{C}.$$

(See [14] for the details). It is one-to-one if and only if the set $\{K(z)\}_{z\in\mathbb{C}}$ is complete in $\mathcal{H}$. In this case, the mapping $\mathcal{T}_K$ is an anti-linear isometry from $\mathcal{H}$ onto $\mathcal{H}_K$.

The convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which is uniform on those subsets of $\mathbb{C}$ where the function $z \mapsto \|K(z)\|_{\mathcal{H}}$ is bounded; in particular, in compact subsets of $\mathbb{C}$ whenever K is a continuous kernel. As a consequence, the topology in $\mathcal{H}_K$ is stronger than the topology which induces the uniform convergence on compact subsets.

The space $\mathcal{H}_K$ is a reproducing kernel Hilbert space of entire functions if and only if the $\mathcal{H}$-valued kernel K is analytic in $\mathbb{C}$ ([16, p. 266]). A characterization of the analyticity of the functions in $\mathcal{H}_K$ in terms of Riesz bases can be found in [9].

From now on, $\mathcal{H}_K$ will denote a reproducing kernel Hilbert space of entire functions associated with an analytic $\mathcal{H}$-valued kernel K. Next, we obtain some properties of the Hilbert space $\mathcal{H}_K$ derived from the sequence of Taylor coefficients of the entire kernel K at a point $a \in \mathbb{C}$, say $a = 0$. Indeed, for $a = 0$ we have the Taylor expansion

$$K(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{C},$$

where the coefficient $c_n \in \mathcal{H}$ for each $n \in \mathbb{N}_0$. By using Cauchy's integral formula for derivatives (see [16, p. 268] we have

$$c_n = \frac{1}{n!} K^{(n)}(0) = \frac{1}{2\pi i} \int_{|z|=R} \frac{K(z)}{z^{n+1}}\, dz, \quad n = 0, 1, \ldots,$$

from which

$$\|c_n\|_{\mathcal{H}} \le \frac{1}{R^{n+1}} \sup_{|z|=R} \|K(z)\|_{\mathcal{H}} = \frac{M_R(0)}{R^{n+1}}.$$

Taking $R > 1$, the above inequality shows that the sequence $\{\|c_n\|\}_{n=0}^{\infty}$ belongs to $l^1(\mathbb{N}_0) \subset l^2(\mathbb{N}_0)$ (from now on $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$).

Proposition 2.1. *Assume that the anti-linear operator $\mathcal{T}_K$ in* (2.1) *is one-to-one. Then the sequence $\{c_n\}_{n=0}^\infty$ of Taylor coefficients of K at* 0 *is a complete sequence in $\mathcal{H}$.*

Proof. Suppose that $\langle c_n, x\rangle = 0$ for all $n \in \mathbb{N}_0$; for the function $f(z) := \langle K(z), x\rangle$, $z \in \mathbb{C}$, we have $f(z) = \sum_{n=0}^\infty \langle c_n, x\rangle z^n = 0$ for all $z \in \mathbb{C}$. Since the anti-linear mapping $\mathcal{T}_K$ is one-to-one we deduce that $x = 0$. □

From now on we assume that the anti-linear operator $\mathcal{T}_K$ is injective; consequently, the sequence $\{c_n\}_{n=0}^\infty$ of Taylor coefficients of K at 0 will be a complete sequence in $\mathcal{H}$.

2.2. Polynomials and $\mathcal{H}_K$ spaces. Here, we give a necessary and sufficient condition for the inclusion of the set of polynomials $\mathcal{P}(\mathbb{C})$ in $\mathcal{H}_K$, and also we give a necessary and sufficient condition for its density in $\mathcal{H}_K$. This is done via the Taylor coefficients $\{c_n\}_{n=0}^\infty$ of the kernel K at 0.

DEFINITION 2.2. A sequence $\{c_n\}_{n=0}^\infty$ is said to be minimal in $\mathcal{H}$ if $c_m \notin \overline{\text{span}}\{c_n\}_{n\neq m}$ for each $m \in \mathbb{N}_0$.

Each complete and minimal sequence $\{c_n\}_{n=0}^\infty$ in $\mathcal{H}$ admits a unique biorthogonal sequence $\{c_n^*\}_{n=0}^\infty$ in $\mathcal{H}$, i.e., $\langle c_m^*, c_n\rangle = \delta_{n,m}$. Note that the sequence $\{c_n^*\}_{n=0}^\infty$ is also minimal (see [12, p. 29]). Obviously, each minimal sequence $\{c_n\}_{n=0}^\infty$ is linearly independent in $\mathcal{H}$.

Proposition 2.3. *The set of polynomials $\mathcal{P}(\mathbb{C})$ is contained in $\mathcal{H}_K$ if and only if the sequence $\{c_n\}_{n=0}^\infty$ of Taylor coefficients of K at* 0 *is minimal in $\mathcal{H}$.*

Proof. For each $n \in \mathbb{N}_0$ the monomial z^n belongs to $\mathcal{H}_K$ if and only if there exists $x_n \in \mathcal{H}$ such that $\langle c_m, x_n\rangle = \delta_{m,n}$, where $\delta_{m,n}$ denotes the Kronecker delta. Equivalently, $\{z^n\}_{n=0}^\infty \subset \mathcal{H}_K$ if and only if there exists a biorthogonal sequence $\{x_n\}_{n=0}^\infty \subset \mathcal{H}$ of $\{c_n\}_{n=0}^\infty$. This is known to be equivalent to the minimality of $\{c_n\}_{n=0}^\infty$ (see [17]). □

The density of the set of polynomials $\mathcal{P}(\mathbb{C})$ in $\mathcal{H}_K$ involves the concept of Markusevich basis (M-basis) (see [12, p. 30]):

DEFINITION 2.4. An M-basis for $\mathcal{H}$ is a complete, minimal sequence $\{c_n\}_{n=0}^\infty$ in $\mathcal{H}$ such that its biorthogonal sequence $\{c_n^*\}_{n=0}^\infty$ is also complete in $\mathcal{H}$.

Proposition 2.5. *Assume that the anti-linear operator $\mathcal{T}_K$ given in* (2.1) *is injective. Then, the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense in $\mathcal{H}_K$ if and only if the sequence $\{c_n\}_{n=0}^\infty$ is an M-basis for $\mathcal{H}$.*

Proof. Assume that the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense in $\mathcal{H}_K$. By Proposition 2.3 the sequence $\{c_n\}_{n=0}^{\infty}$ is minimal in $\mathcal{H}$. Since $\{c_n\}_{n=0}^{\infty}$ is complete there exists a unique sequence $\{c_n^*\}_{n=0}^{\infty}$ biorthogonal to $\{c_n\}_{n=0}^{\infty}$. Consequently $\{c_n\}_{n=0}^{\infty}$ is an M-basis for $\mathcal{H}$. Reciprocally, assume that the sequence $\{c_n\}_{n=0}^{\infty}$ is an M-basis for $\mathcal{H}$. For any $n \in \mathbb{N}_0$ we have that $z^n = \langle K(z), c_n^* \rangle$. As a consequence of completeness of the sequence $\{c_n^*\}_{n=0}^{\infty}$ we have that set of polynomials $\mathcal{P}(\mathbb{C}) = \operatorname{span}\{z^n\}_{n=0}^{\infty}$ is dense in $\mathcal{H}_K$. □

2.3. The differentiation operator in $\mathcal{H}_K$ spaces. This section is devoted to the differentiation operator $\mathcal{D}(f) = f'$ in $\mathcal{H}_K$; concretely we study when $\mathcal{D}\colon \mathcal{H}_K \to \mathcal{H}_K$ is a well-defined bounded operator. To this end, let f be a function in $\mathcal{H}_K$; there exists $x \in \mathcal{H}$ such that $f(z) = \langle K(z), x \rangle$, for any $z \in \mathbb{C}$, and $f(z) = \sum_{n=0}^{\infty} \langle c_n, x \rangle z^n$. Therefore,

$$f'(z) = \sum_{n=1}^{\infty} \langle c_n, x \rangle n z^{n-1}, \quad z \in \mathbb{C}.$$

The derivative f' of the entire function f belongs to $\mathcal{H}_K$ if and only if there exists $y \in \mathcal{H}$ such that

$$\langle c_n, y \rangle = (n+1) \langle c_{n+1}, x \rangle \quad \text{for any} \quad n \in \mathbb{N}_0. \tag{2.2}$$

For the sake of completeness we include the following general moment problem result whose proof can be found in [17, p. 126]:

Theorem 2.6. *Let $\{f_1, f_2, f_3, \dots\}$ be a sequence of vectors belonging to a Hilbert space $\mathcal{H}$ and $\{d_1, d_2, d_3, \dots\}$ a sequence of scalars. In order that the system of equations*

$$\langle f, f_n \rangle = d_n, \quad n \in \mathbb{N}$$

has at least one solution $f \in \mathcal{H}$ with $\|f\| \le M$ for some positive constant M, it is necessary and sufficient that

$$\left| \sum_n a_n \overline{d}_n \right| \le M \left\| \sum_n a_n f_n \right\|$$

for every finite sequence of scalars $\{a_n\}$. If the sequence $\{f_1, f_2, f_3, \dots\}$ is complete in $\mathcal{H}$, then the solution is unique.

Assume that the sequence $\{c_n\}_{n=0}^{\infty}$ is complete and minimal in $\mathcal{H}$. As a consequence of the above result the differentiation operator $\mathcal{D}(f) = f'$ is a well-defined operator $\mathcal{D}\colon \mathcal{H}_K \to \mathcal{H}_K$ if and only if, for each $x \in \mathcal{H}$ the linear functional μ_x defined on $\operatorname{span}\{c_n\}_{n=0}^\infty$ as

$$\mu_x\left(\sum_n a_n c_n\right) = \sum_n a_n \langle (n+1)c_{n+1}, x\rangle$$

for every finite sequence of scalars $\{a_n\}$, is bounded. For each fixed $x \in \mathcal{H}$, the linear functional $\mu_x\colon \operatorname{span}\{c_n\}_{n=0}^\infty \to \mathbb{C}$ can be decomposed as the product $T_x\mathfrak{D}$, where $T_x\colon \operatorname{span}\{c_n\}_{n=0}^\infty \to \mathbb{C}$ is the linear operator given by

$$T_x\left(\sum_n a_n c_n\right) = \sum_n a_n \langle c_n, x\rangle$$

and $\mathfrak{D}\colon \operatorname{span}\{c_n\}_{n=0}^\infty \to \operatorname{span}\{c_n\}_{n=0}^\infty$, is the linear operator given by

$$\mathfrak{D}\left(\sum_n a_n c_n\right) = \sum_n a_n (n+1) c_{n+1} \tag{2.3}$$

for every finite sequence of scalars $\{a_n\}$. Observe that the operator $\mathfrak{D}$ is well-defined since the sequence $\{c_n\}_{n\in\mathbb{N}_0}$ is minimal (consequently, linearly independent).

The operator T_x is obviously bounded since

$$T_x\left(\sum_n a_n c_n\right) = \sum_n a_n \langle c_n, x\rangle = \left\langle \sum_n a_n c_n, x\right\rangle.$$

Thus, if the operator $\mathfrak{D}$ given by (2.3) is bounded, then the differentiation operator is a well-defined operator $\mathcal{D}\colon \mathcal{H}_K \to \mathcal{H}_K$. Moreover, the boundedness of the operator $\mathfrak{D}$ implies the boundedness of the differentiation operator $\mathcal{D}$. Indeed, if $\mathfrak{D}$ is bounded on $\operatorname{span}\{c_n\}_{n=0}^\infty$ then it can be extended by continuity to the whole space $\mathcal{H}$. In this case, the adjoint operator of $\mathfrak{D}$, $\mathfrak{D}^*\colon \mathcal{H} \to \mathcal{H}$ is bounded and it is straightforward to prove that $\mathcal{D} = \mathcal{T}_K \mathfrak{D}^* \mathcal{T}_K^{-1}$ where $\mathcal{T}_K\colon \mathcal{H} \to \mathcal{H}_K$ is the anti-linear isometry defined in (2.1).

Next we give a sufficient condition for the boundedness of the differentiation operator $\mathcal{D}\colon \mathcal{H}_K \to \mathcal{H}_K$ assuming the minimality of the sequence $\{c_n\}_{n=0}^\infty$ in $\mathcal{H}$. Following [12, p. 27], the minimality of the sequence $\{c_n\}_{n=0}^\infty$ in $\mathcal{H}$ implies that the numbers δ_k given by

$$\delta_k := \inf_{\theta\in\mathbb{R}} \rho\left(e^{i\theta}\frac{c_k}{\|c_k\|}, \overline{\operatorname{span}}\{c_n\}_{n\neq k}\right), \quad k \in \mathbb{N}_0, \tag{2.4}$$

are strictly positive for every $k \in \mathbb{N}_0$; here ρ denotes the distance with respect to the metric given by the norm in $\mathcal{H}$. Note that the number δ_k measures the inclination in

$\mathcal{H}$ of the straight line spanned by c_k to the closed subspace $\overline{\text{span}}\{c_n\}_{n\neq k}$. Besides, for any $x = \sum_k \alpha_k c_k$ (finite or convergent sum) the inequality

$$|\alpha_k| \leq \frac{\|x\|}{\delta_k \|c_k\|} \quad \text{holds for each} \quad k \in \mathbb{N}_0. \tag{2.5}$$

(See [12] for the details)

Proposition 2.7. *Suppose that the sequence $\{c_n\}_{n=0}^{\infty}$ is complete and minimal in $\mathcal{H}$. Suppose that*

$$\sum_{n=0}^{\infty} \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}\|}{\|c_n\|} < \infty, \tag{2.6}$$

where $\{\delta_n\}_{n=0}^{\infty}$ is the sequence of positive numbers given by (2.4)*. Then the differentiation operator $\mathcal{D}$ is a well-defined bounded operator on $\mathcal{H}_K$.*

Proof. Proceeding as above, the differentiation operator $\mathcal{D}(f) = f'$ is well-defined and bounded on $\mathcal{H}_K$ if and only if the operator $\mathfrak{D}$: $\text{span}\{c_n\}_{n=0}^{\infty} \to \text{span}\{c_n\}_{n=0}^{\infty}$ defined by (2.3) is bounded. Let $x = \sum_{n=0}^{L} \alpha_n c_n \in \text{span}\{c_n\}_{n=0}^{\infty}$ with $\|x\|_{\mathcal{H}} = 1$. By using inequality in (2.5), we obtain

$$\|\mathfrak{D}x\| = \left\| \sum_{n=0}^{L} \alpha_n (n+1) c_{n+1} \right\| \leq \sum_{n=0}^{L} (n+1)|\alpha_n| \|c_{n+1}\| \leq \sum_{n=0}^{L} \frac{(n+1)}{\delta_n} \frac{\|c_{n+1}\|}{\|c_n\|}.$$

Hence, the convergence of the series $\sum_{n=0}^{\infty}((n+1)/\delta_n)(\|c_{n+1}\|/\|c_n\|)$ implies continuity of the operator $\mathfrak{D}$ in $\text{span}\{c_n\}_{n=0}^{\infty}$; by density it will be continuous on $\mathcal{H}$. □

2.4. The hypercyclicity of translation operators in $\mathcal{H}_K$. First, recall that a translation operator $T_w \colon \mathcal{H}_K \to \mathcal{H}_K$, $w \neq 0$, is a hypercyclic operator in $\mathcal{H}_K$ if there exists a vector f in $\mathcal{H}_K$ whose orbit $\{T_w^n f\}_{n=0}^{\infty}$ is dense in $\mathcal{H}_K$.

In order to prove that, under suitable hypotheses, any translation T_w, $w \neq 0$, is hypercyclic on $\mathcal{H}_K$ we will use a result from Chan–Shapiro [5] whose statement is included below. It involves an auxiliary Hilbert space $E^2(\gamma)$ associated with an admissible comparison function γ: An entire function $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ is said to be an admissible comparison function if $\gamma_n > 0$ for each $n \in \mathbb{N}_0$, and the sequence $n\gamma_n/\gamma_{n-1}$ decreases when n tends to infinity. For an admissible comparison function γ, let $E^2(\gamma)$ be the set of all entire functions $f(z) = \sum_{n=0}^{\infty} f_n z^n$ for which

$$\|f\|_{2,\gamma}^2 := \sum_{n=0}^{\infty} \frac{|f_n|^2}{\gamma_n^2} < \infty.$$

Endowed with the norm $\|\cdot\|_{2,\gamma}$, the space $E^2(\gamma)$ becomes a Hilbert space of entire functions.

Theorem 2.8 (Chan–Shapiro [5]). *Suppose that X is a Fréchet space of entire functions with the following properties*:
(1) $\mathcal{P}(\mathbb{C}) \subset X$ *and* $\overline{\mathcal{P}(\mathbb{C})} = X$;
(2) *the topology of X is stronger than the topology of uniform convergence on compact subsets of* $\mathbb{C}$;
(3) *the translation operator T_w is continuous on X for every $w \neq 0$*;
(4) $E^2(\gamma) \subset X$ *for some admissible comparison function γ.*
Then, any translation operator T_w, $w \neq 0$, is hypercyclic in X.

Lemma 2.9. *Suppose that the differentiation operator $\mathcal{D}$ defined as $\mathcal{D}(f) = f'$ is a well-defined bounded operator $\mathcal{D}\colon \mathcal{H}_K \to \mathcal{H}_K$. Then, for each $w \in \mathbb{C}$, the translation operator $T_w\colon \mathcal{H}_K \to \mathcal{H}_K$ is a well-defined bounded operator. Moreover, we have the following expansion for T_w converging in the operator norm*

$$T_w = \sum_{n=0}^{\infty} \frac{(-w)^n}{n!} \mathcal{D}^n. \tag{2.7}$$

Proof. It is a well-known result that (2.7) holds in $\mathcal{E}$, the space of entire functions endowed with the topology of the uniform convergence on compact sets (see, for instance, [5]). Since the differentiation operator $\mathcal{D}$ is bounded on the Hilbert space $\mathcal{H}_K$, the series on the right side of (2.7) converges absolutely, and hence in the operator norm to a bounded operator on $\mathcal{H}_K$. As the convergence in $\mathcal{H}_K$ implies convergence in the space $\mathcal{E}$, this operator must be T_w. □

Next, we give a criterion to ensuring when an entire function g belongs to $\mathcal{H}_K$:

Lemma 2.10. *Assume that the sequence $\{c_n\}_{n=0}^\infty$ of Taylor coefficients of K at 0 is minimal in $\mathcal{H}$. Let $g(z) = \sum_{n=0}^\infty g_n z^n$ be an entire function such that $\sum_{n=0}^\infty |g_n|/(\delta_n \|c_n\|) < \infty$ where the sequence $\{\delta_n\}_{n=0}^\infty$ of positive numbers is given by* (2.4). *Then the function g belongs to $\mathcal{H}_K$.*

Proof. The entire function $g(z) = \sum_{n=0}^\infty g_n z^n$ belongs to $\mathcal{H}_K$ if and only if there exists $x \in \mathcal{H}$ such that

$$\langle c_n, x\rangle = g_n \quad \text{for each} \quad n \in \mathbb{N}_0.$$

Following Theorem 2.6, this is equivalent to the boundedness of the linear functional

$$\mu_g\left(\sum_n a_n c_n\right) = \sum_n a_n g_n$$

defined on $\operatorname{span}\{c_n\}_{n=0}^{\infty}$. Let $y = \sum_n a_n c_n$ be a vector in $\operatorname{span}\{c_n\}_{n=0}^{\infty}$; by using inequalities (2.5) we obtain

$$|\mu_g(y)| \le \sum_n |a_n||g_n| \le \left(\sum_n \frac{|g_n|}{\delta_n \|c_n\|}\right)\|y\| \le \left(\sum_{n=0}^{\infty} \frac{|g_n|}{\delta_n \|c_n\|}\right)\|y\|.$$

Hence, the linear functional μ_g is bounded on $\operatorname{span}\{c_n\}_{n=0}^{\infty}$ and consequently, the entire function g belongs to $\mathcal{H}_K$. □

In order to apply Theorem 2.8 to $\mathcal{H}_K$ we need to prove the existence of an auxiliary Hilbert space $E^2(\gamma) \subset \mathcal{H}_K$

Lemma 2.11. *Assume that the sequence $\{c_n\}_{n=0}^{\infty}$ is minimal in $\mathcal{H}$. There exists an admissible comparison function γ such that $E^2(\gamma) \subset \mathcal{H}_K$.*

Proof. Let $\gamma(z) = \sum_{n=0}^{\infty} \gamma_n z^n$ be an admissible comparison function such that, in addition, the sequence $\{\gamma_n/(\delta_n\|c_n\|)\}_{n=0}^{\infty}$ belongs to $l^2(\mathbb{N}_0)$, where the sequence of positive numbers $\{\delta_n\}_{n=0}^{\infty}$ is defined in (2.4).

Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be an entire function belonging to $E^2(\gamma)$. By using the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{|g_n|}{\delta_n\|c_n\|} &= \left(\sum_{n=0}^{\infty} \frac{\gamma_n}{\delta_n\|c_n\|}\frac{|g_n|}{\gamma_n}\right) \le \left(\sum_{n=0}^{\infty}\left(\frac{\gamma_n}{\delta_n\|c_n\|}\right)^2\right)^{1/2}\left(\sum_{n=0}^{\infty}\left(\frac{|g_n|}{\gamma_n}\right)^2\right)^{1/2}\\ &= \left(\sum_{n=0}^{\infty}\left(\frac{\gamma_n}{\delta_n\|c_n\|}\right)^2\right)^{1/2}\|g\|_{\gamma,2}.\end{aligned}$$

Hence, by Lemma 2.10 we obtain that g belongs to $\mathcal{H}_K$. □

Thus, we can prove the following hypercyclicity result for the translation operators T_w on $\mathcal{H}_K$:

Theorem 2.12. *Let $K\colon \mathbb{C} \to \mathcal{H}$ be an analytic kernel whose Taylor expansion around $z = 0$ is $K(z) = \sum_{n=0}^{\infty} c_n z^n$. Assume that*

i) *the sequence $\{c_n\}_{n=0}^{\infty}$ is an M-basis for $\mathcal{H}$, and*

ii) *the sequence $\{((n+1)/\delta_n)(\|c_{n+1}\|/\|c_n\|)\}_{n=0}^{\infty}$ belongs to $l^1(\mathbb{N}_0)$.*

Then, for any $w \in \mathbb{C} \setminus \{0\}$, the translation operator $T_w\colon \mathcal{H}_K \to \mathcal{H}_K$ is hypercyclic.

Proof. This result is a corollary of Theorem 2.8. Proposition 2.5 guarantees that the set of complex polynomials, $\mathcal{P}(\mathbb{C})$, is dense in $\mathcal{H}_K$. The topology of $\mathcal{H}_K$ is stronger than the topology of uniform convergence on compact sets of $\mathbb{C}$. By Proposition 2.7 and Lemma 2.9 the translation operator T_w is continuous on $\mathcal{H}_K$. Lemma 2.11 gives us the last condition in Theorem 2.8. As a consequence, we conclude that for each $w \neq 0$ in $\mathbb{C}$ the translation operator T_w is hypercyclic on $\mathcal{H}_K$. □

3. The spaces $\mathcal{H}_K$ as de Branges spaces

In [10] one can find some characterizations of a space $\mathcal{H}_K$ as a de Branges space; this means that the space $\mathcal{H}_K$ is equal isometrically to some de Branges space. In this section the goal is to characterize a space $\mathcal{H}_K$ as a de Branges space where the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense. First, we introduce a de Branges space $\mathcal{H}(E)$ with structure function E: Let E be an entire function verifying $|E(x-iy)| < |E(x+iy)|$ for all $y > 0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions f such that

$$\| f \|_E^2 := \int_{-\infty}^{\infty} \left| \frac{f(t)}{E(t)} \right|^2 dt < \infty,$$

and such that both ratios f/E and f^*/E, where $f^*(z) := \overline{f(\overline{z})}$, $z \in \mathbb{C}$, are of bounded type and of non-positive mean type in the upper half-plane. The structure function or de Branges function E has no zeros in the upper half plane. A de Branges function E is said to be strict if it has no zeros on the real axis. We require f/E and f^*/E to be of bounded type and nonpositive mean type in $\mathbb{C}^+$. A function is of bounded type if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$ and it is of nonpositive mean type if it grows no faster than $e^{\varepsilon y}$ for each $\varepsilon > 0$ as $y \to \infty$ on the positive imaginary axis $\{iy : y > 0\}$. Note that the Paley–Wiener space PW_π is a de Branges space with strict structure function $E_\pi(z) = \exp(-i\pi z)$.

It is known that any de Branges space $\mathcal{H}(E)$ can be considered as a $\mathcal{H}_K$ space where the kernel K is given by

$$[K(z)](w) = \frac{B(w)A(z) - A(w)B(z)}{\pi(w-z)}, \quad z, w \in \mathbb{C},$$

where $E(z) = A(z) - iB(z)$, $z \in \mathbb{C}$ (see [10]).

We will use the following classical characterization of a de Branges space which can be found in [4, p. 57]: A Hilbert space $\mathcal{H}$ of entire functions is equal isometrically to some de Branges space $\mathcal{H}(E)$ if and only if the following conditions hold:

B1. Whenever $f \in \mathcal{H}$ and ω is a nonreal zero of f, the function

$$g(z) := \frac{z - \overline{\omega}}{z - \omega} \; f(z)$$

belongs to $\mathcal{H}$ and $\|g\| = \|f\|$.

B2. For each $\omega \notin \mathbb{R}$ the linear mapping $\mathcal{H} \ni f \mapsto f(\omega) \in \mathbb{C}$ is continuous.

B3. For any $f \in \mathcal{H}$ the function f^* defined as $f^*(z) := \overline{f(\overline{z})}$, $z \in \mathbb{C}$, belongs to the space, and $\| f^* \| = \| f \|$.

The main result in this section depends on the symmetry of the multiplication by z operator on the space $\mathcal{H}_K$:

Lemma 3.1. *Assume that $\overline{\mathcal{P}(\mathbb{C})} = \mathcal{H}_K$. Then the densely defined closed operator*

$$\begin{aligned} S\colon\ D_S &\to \mathcal{H}_K \\ f(z) &\mapsto zf(z), \end{aligned}$$

where $D_S := \{f(z) = g(z)/z \in \mathcal{H}_K \mid g \in \mathcal{H}_K\}$, is symmetric if and only if the Gram matrix associated with the biorthogonal sequence $\{c_n^*\}_{n=0}^\infty$ of $\{c_n\}_{n=0}^\infty$ is a semi-infinite Hankel matrix, i.e., $\langle c_{m+1}^*, c_n^*\rangle = \langle c_m^*, c_{n+1}^*\rangle$ for any $m, n \in \mathbb{N}_0$.

Proof. Since $\overline{\mathcal{P}(\mathbb{C})} = \mathcal{H}_K$ and $\mathcal{P}(\mathbb{C}) \subset D_S$ the operator S is a densely defined closed operator. For the closedness, note that the convergence in $\mathcal{H}_K$ implies pointwise convergence in $\mathbb{C}$. Since operator S is injective, let $\mathcal{J} := S^{-1}$ be defined in the range R_S. Clearly, S is a symmetric operator if and only if $\mathcal{J}$ is a symmetric operator. Assume that the Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^\infty$ is a semi-infinite Hankel matrix and let $p(z) = \sum_n a_n z^n$ and $q(z) = \sum_m b_m z^m$ be two polynomials in the range of S, R_S. We have

$$\langle \mathcal{J}p, q\rangle = \sum_{n,m} a_n \overline{b}_m \langle z^{n-1}, z^m\rangle = \sum_{n,m} a_n \overline{b}_m \langle z^n, z^{m-1}\rangle = \langle p, \mathcal{J}q\rangle.$$

Since R_S is included in $\mathcal{H}_0 := \{f \in \mathcal{H}_K \mid f(0) = 0\}$ and $\mathcal{H}_0 = \mathcal{T}_K\{c_0^\perp\} = \overline{\operatorname{span}}\{c_n^*\}_{n=1}^\infty$ we have that for any $f \in R_S$ there exists a sequence of polynomials $\{p_{f,n}\}$ with constant coefficient zero such that $f = \lim_{n\to\infty} p_{f,n}$. Using a density argument it is straightforward to prove that, for any $f, g \in R_S$ one gets $\langle \mathcal{J}f, g\rangle = \langle f, \mathcal{J}g\rangle$.

On the other hand, if $\mathcal{J}$ is a symmetric operator then for any $n, m = 1, 2, \ldots$ we have

$$\langle z^{n-1}, z^m\rangle = \langle \mathcal{J}z^n, z^m\rangle = \langle z^n, \mathcal{J}z^m\rangle = \langle z^n, z^{m-1}\rangle.$$

Since $z^n = \mathcal{T}_K c_n^*$ and $z^m = \mathcal{T}_K c_m^*$ for any $n, m \in \mathbb{N}_0$, we obtain that the Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^\infty$ is a semi-infinite Hankel matrix. □

Theorem 3.2. *Consider an $\mathcal{H}_K$ space such that $K(z) = \sum_{n=0}^\infty c_n z^n$ where the sequence $\{c_n\}_{n=0}^\infty$ forms an M-basis for $\mathcal{H}$. Then, the space $\mathcal{H}_K$ is equal isometrically to a de Branges space where the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense if and only if the two following requirements hold*:

1) *The Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^\infty$ is a semi-infinite Hankel matrix, i.e., $\langle c_{m+1}^*, c_n^*\rangle = \langle c_m^*, c_{n+1}^*\rangle$, $m, n \in \mathbb{N}_0$;*

2) *For any f in $\mathcal{H}_K$ such that $f(w) = 0$, where $w \in \mathbb{C}\setminus\mathbb{R}$, the function $f(z)/(z-w)$ belongs to $\mathcal{H}_K$.*

Proof. For the sufficiency part: From Lemma 3.1 it follows that the multiplication by z operator, S, is a densely defined symmetric operator defined on $D_S := \{f(z) = g(z)/z \in \mathcal{H}_K \mid g \in \mathcal{H}_K\}$. Moreover, for any $w \in \mathbb{C} \setminus \mathbb{R}$ we have that $R(S - wI) = \mathcal{H}_w$, where $\mathcal{H}_w = \{f \in \mathcal{H}_K \mid f(w) = 0\}$. We denote by $U_w := (S - \overline{w}I)(S - wI)^{-1}$ the Cayley transform associated to S. Property 2) implies that the domain of the operator $(S - wI)^{-1}$ is $\mathcal{H}_w$. Let $f \in \mathcal{H}_w$, then

$$\left\| \frac{z - \overline{w}}{z - w} f \right\| = \|U_w f\| = \|f\|.$$

Hence, property B1 is fullfilled by the space $\mathcal{H}_K$.

Let $f = \mathcal{T}_K(x)$ be in $\mathcal{H}_K$ with $x \in \mathcal{H}$. Then, $f^*(z) = \sum_{n=0}^{\infty} \overline{\langle c_n, x \rangle} z^n$, $z \in \mathbb{C}$. As a consequence, f^* belongs to $\mathcal{H}_K$ if and only if there exists $y \in \mathcal{H}$ such that

$$\langle c_n, y \rangle = \overline{\langle c_n, x \rangle}, \quad n = 0, 1, \ldots .$$

Consider the anti-linear operator $\mathbf{U}\colon \operatorname{span}\{c_n^*\}_{n=0}^{\infty} \to \operatorname{span}\{c_n^*\}_{n=0}^{\infty}$ defined by

$$\mathbf{U}\left(\sum_n a_n c_n^*\right) = \sum_n \overline{a}_n c_n^*,$$

for every finite sequence of scalars $\{a_n\}$. For any $p = \sum_n a_n c_n^* \in \operatorname{span}\{c_n^*\}_{n=0}^{\infty}$ we have

$$\langle \mathbf{U}p, \mathbf{U}p \rangle = \sum_{m,n} \overline{a}_m a_n \langle c_m^*, c_n^* \rangle = \sum_{m,n} \overline{a}_m a_n \langle c_n^*, c_m^* \rangle = \langle p, p \rangle,$$

where we are used that the Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^{\infty}$ is a semi-infinite Hankel matrix. Therefore, the operator $\mathbf{U}$ is an anti-linear isometry on $\operatorname{span}\{c_n^*\}_{n=0}^{\infty}$ and since $\overline{\operatorname{span}}\{c_n^*\}_{n=0}^{\infty} = \mathcal{H}$ the operator $\mathbf{U}$ can be extended to an anti-linear isometry defined on $\mathcal{H}$ which we also denote as $\mathbf{U}$. Thus, for any $p \in \operatorname{span}\{c_n^*\}_{n=0}^{\infty}$ and $n \in \mathbb{N}_0$ we have that $\langle c_n, \mathbf{U}p \rangle = \overline{\langle c_n, p \rangle}$. Now, let $x \in \mathcal{H}$, since $\overline{\operatorname{span}}\{c_n^*\}_{n=0}^{\infty} = \mathcal{H}$, there exists a sequence $\{p_{x,m}\} \subset \operatorname{span}\{c_n^*\}_{n=0}^{\infty}$ such that $x = \lim_{m\to\infty} p_{x,m}$. It is straightforward to prove that $\langle c_n, \mathbf{U}x \rangle = \overline{\langle c_n, x \rangle}$. Hence, given $f \in \mathcal{H}_K$ the entire function f^* also belongs to $\mathcal{H}_K$, since $f = \mathcal{T}_K(x)$ implies $f^* = \mathcal{T}_K(\mathbf{U}x)$. Moreover, since $\mathbf{U}$ is a isometry, $\|f\| = \|x\| = \|\mathbf{U}x\| = \|f^*\|$. This proves the property B3. Since the space $\mathcal{H}_K$ is a RKHS, the property B2 is clearly satisfied.

The necessity part is straightforward: If the space $\mathcal{H}_K$ is a de Branges space and the set of polynomials is dense then the multiplication by z operator is closed and symmetric [4, p. 314] and, therefore, from Lemma 3.1 we have that the Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^{\infty}$ is a semi-infinite Hankel matrix.

Moreover, if $f \in \mathcal{H}_K$ and w is a nonreal zero of f then from condition B1 the function $(z-\bar{w})/(z-w)f(z)$ belongs to $\mathcal{H}_K$ and hence, the function

$$\frac{f(z)}{z-w} = \frac{1}{w-\bar{w}}\left(\frac{z-\overline{w}}{z-w}f(z) - f(z)\right)$$

belongs to $\mathcal{H}_K$. □

Closing this section we carry out a deeper study of property 2) in Theorem 3.2. This property concerns the stability of the functions belonging to the space $\mathcal{H}_K$ on removing a finite number of their zeros (this property appears in [11] under the name of *division property*). The connection between this property and sampling theory can be found in [6, 7].

DEFINITION 3.3 (Zero-removing property). A set $\mathcal{A}$ of entire functions has the zero-removing property (ZR property hereafter) if for any $g \in \mathcal{A}$ and any zero w of g the function $g(z)/(z-w)$ belongs to $\mathcal{A}$. A set $\mathcal{A}$ of entire functions has the zero-removing property at a fixed point $w \in \mathbb{C}$ (ZR_w property hereafter) if for any $g \in \mathcal{A}$ with $g(w) = 0$ the function $g(z)/(z-w)$ belongs to $\mathcal{A}$.

Firstly, we study conditions under which ZR_0 property holds in $\mathcal{H}_K$. Reducing the ZR_0 property to a moment problem, a sufficient condition assuring that the ZR_0 property holds involves the continuity of a shift related operator.

Consider a function $f \in \mathcal{H}_K$, i.e., $f(z) = \langle K(z), x\rangle_{\mathcal{H}}$, $z \in \mathbb{C}$, for some $x \in \mathcal{H}$, such that $f(0) = 0$. Then $\langle c_0, x\rangle = 0$ and

$$\frac{f(z)}{z} = \sum_{n=0}^{\infty} \langle c_{n+1}, x\rangle z^n, \quad z \in \mathbb{C}.$$

As a consequence, the space $\mathcal{H}_K$ satisfies the property ZR_0 if and only if for each $x \in \{c_0\}^{\perp}$ there exists $y \in \mathcal{H}$ such that

$$\langle c_n, y\rangle = \langle c_{n+1}, x\rangle, \quad n \in \mathbb{N}_0.$$

Proceeding as in the study of the differentiation operator in Section 2.3 we obtain:

Proposition 3.4. *Consider an $\mathcal{H}_K$ space such that $K(z) = \sum_{n=0}^{\infty} c_n z^n$ where the sequence $\{c_n\}_{n=0}^{\infty}$ is complete and minimal in $\mathcal{H}$. If the operator $R\colon \mathrm{span}\{c_n\}_{n=0}^{\infty} \to \mathrm{span}\{c_n\}_{n=0}^{\infty}$ given by*

$$R\left(\sum_n a_n c_n\right) = \sum_n a_n c_{n+1}, \tag{3.1}$$

for every finite sequence $\{a_n\}$*, is bounded, then the space* $\mathcal{H}_K$ *satisfies the* ZR_0 *property.*

Now, proceeding as in Proposition 2.7, we give a sufficient condition on the continuity of the operator R involving the sequence of positive numbers $\{\delta_n\}_{n=0}^{\infty}$ defined in (2.4).

Theorem 3.5. *Assume that the sequence* $\{c_n\}_{n=0}^{\infty}$ *is complete and minimal in* $\mathcal{H}$*. The convergence of the series*

$$\sum_{n=0}^{\infty} \frac{1}{\delta_n} \frac{\|c_{n+1}\|}{\|c_n\|} \tag{3.2}$$

implies that the operator R *is bounded on* $\mathcal{H}$*.*

Proof. For any finite sum $x = \sum_{n=0}^{L} \alpha_n c_n$, inequalities (2.5) give

$$\|Rx\| \le \sum_{n=0}^{L} |\alpha_n| \|c_{n+1}\| \le \left(\sum_{n=0}^{L} \frac{1}{\delta_n} \frac{\|c_{n+1}\|}{\|c_n\|} \right) \|x\| \le M\|x\|,$$

where M denotes the sum of the series in (3.2). This proves that the operator R is bounded on $\operatorname{span}\{c_n\}_{n=0}^{\infty}$; the completeness of $\{c_n\}_{n=0}^{\infty}$ proves that R is bounded on $\mathcal{H}$. □

It remains the open question whether, in general, the ZR_0 property at the point 0 implies the ZR_w property for each $w \neq 0$. However, the result is true under the hypothesis of well-definedness of the translation operators in $\mathcal{H}_K$. Suppose that, for every $w \in \mathbb{C}$, the translation operator T_w defined as $T_w f(z) = f(z-w)$, $z \in \mathbb{C}$, is a well-defined operator $T_w \colon \mathcal{H}_K \to \mathcal{H}_K$. Then, the ZR_0 property implies the global ZR property in $\mathcal{H}_K$. Indeed, assume that the ZR_0 property holds; for $w \in \mathbb{C}$, let g be an entire function in $\mathcal{H}_K$ such that $g(w) = 0$. The entire function $f = T_{-w} g$ belongs to $\mathcal{H}_K$ and $f(0) = g(w) = 0$. Since the ZR_0 property holds we have

$$h(z) = \frac{f(z)}{z} = \frac{g(z+w)}{z} \in \mathcal{H}_K.$$

Hence $g(z)/(z-w) = (T_w h)(z) \in \mathcal{H}_K$.

A sufficient condition for the well-definedness (and boundedness) of the translation operator $T_w \colon \mathcal{H}_K \to \mathcal{H}_K$ for each $w \in \mathbb{C}$ is given in Lemma 2.9. This results involves the boundedness of the differentiation operator, $\mathcal{D}$, on $\mathcal{H}_K$. A sufficient condition assuring that the differentiation operator $\mathcal{D}$ is a well-defined bounded operator in $\mathcal{H}_K$ is given in Proposition 2.7. As a consequence we obtain the following result:

Corollary 3.6. *Consider an $\mathcal{H}_K$ space such that $K(z) = \sum_{n=0}^{\infty} c_n z^n$ where the sequence $\{c_n\}_{n=0}^{\infty}$ forms an M-basis for $\mathcal{H}$. Assume that the two following statements hold:*
(1) *The Gram matrix associated with the sequence $\{c_n^*\}_{n=0}^{\infty}$ is a semi-infinite Hankel matrix, i.e., $\langle c_{m+1}^*, c_n^* \rangle = \langle c_m^*, c_{n+1}^* \rangle$, $m, n \in \mathbb{N}_0$;*
(2) *The sequence $\{(n+1)\|c_{n+1}\|/(\delta_n \|c_n\|)\}_{n=0}^{\infty}$ belongs to $l^1(\mathbb{N}_0)$.*
Then, the space $\mathcal{H}_K$ is equal isometrically to a de Branges space, and any translation $T_w : \mathcal{H}_K \to \mathcal{H}_K$, $w \neq 0$, is hypercyclic.

4. An illustrative example

In what follows we show an example taken from the Hamburger moment problem theory (see, for instance, [1, 15]). Let $s = \{s_n\}_{n=0}^{\infty}$ be an indeterminate Hamburger moment sequence and let V_s be the set of positive Borel measures μ on $\mathbb{R}$ satisfying $\int_{-\infty}^{\infty} x^n \, d\mu(x) = s_n$, $n \geq 0$. The functional $\mathcal{L}$ defined on the vector space $\mathbb{C}[x]$ of polynomials $p(x) = \sum_{k=0}^{n} p_k x^k$ by

$$\mathcal{L}(p) = \sum_{k=0}^{n} p_k s_k = \int_{-\infty}^{\infty} p(x) \, d\mu(x)$$

is independent of $\mu \in V_s$. Let $\{P_n\}_{n=0}^{\infty}$ be the corresponding orthonormal polynomials satisfying

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) \, d\mu(x) = \delta_{nm}, \quad \text{for each} \quad \mu \in V_s.$$

We assume that P_n has degree n with positive leading coefficient. Recall that $\{P_n(x)\}$ satisfy the three–term recurrence relation

$$x P_n(x) = a_n P_{n+1}(x) + b_n P_n(x) + a_{n-1} P_{n-1}(x), \quad n \geq 0$$

where $P_{-1}(x) = 0$ and $P_0(x) = 1$. The two sequences $\{b_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=0}^{\infty}$ of real and positive numbers, respectively, form the semi-infinite Jacobi matrix associated with the indeterminate Hamburger moment problem (see, for instance, [15]):

$$\mathcal{A} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & \ddots \\ 0 & 0 & a_2 & b_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4.1}$$

For the sequence of moments $\{s_n\}_{n=0}^{\infty}$ we have $s_n = \langle \sigma_0, \mathcal{A}^n \sigma_0 \rangle$, $n \in \mathbb{N}_0$, where σ_0 stands for the sequence $(1, 0, 0, \dots)$ [15, p. 93]. Since we are dealing with an indeterminate

Hamburger moment problem it is known that $\sum_{m=0}^{\infty}|P_m(z)|^2 < \infty$ for each $z \in \mathbb{C}$ [15, p. 94]. Thus, we can consider the analytic kernel

$$\mathbb{C} \ni z \overset{K}{\mapsto} K(z) := \{P_m(z)\}_{m=0}^{\infty} \in l^2(\mathbb{N}_0),$$

and its corresponding $\mathcal{H}_K$ space. Assume that

$$P_m(z) = a_{m,m}z^m + a_{m,m-1}z^{m-1} + \cdots + a_{m,0} \quad \text{with} \quad a_{m,m} > 0.$$

In order to obtain the Taylor expansion $K(z) = \sum_{n=0}^{\infty} c_n z^n$ of the kernel K at $z = 0$ we have:

$$[K(z)](m) = P_m(z) = a_{m,m}z^m + a_{m,m-1}z^{m-1} + \cdots + a_{m,0} = \sum_{n=0}^{\infty} c_n(m)z^n,$$

from which we derive the sequence of Taylor coefficients $\{c_n\}_{n=0}^{\infty}$ of K at 0. Writing the sequences $c_n \in l^2(\mathbb{N}_0)$, $n \in \mathbb{N}_0$, as the rows of a semi-infinite matrix we obtain

$$\mathcal{C} := \begin{pmatrix} a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} & \cdots \\ 0 & a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ 0 & 0 & a_{2,2} & a_{3,2} & \cdots \\ 0 & 0 & 0 & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where its columns are the coefficients of the polynomials P_m, $m \in \mathbb{N}_0$. Clearly, the sequence $\{c_n\}_{n=0}^{\infty}$ is minimal in $l^2(\mathbb{N}_0)$ and, as a consequence, there exists a unique biorthogonal sequence $\{c_n^*\}_{n=0}^{\infty}$ in $l^2(\mathbb{N}_0)$. Since $a_{m,m} > 0$ for all $m \in \mathbb{N}_0$, by using an inductive argument we can get a semi-infinite matrix $\mathcal{C}^*$ with entries in $\mathbb{R}$, and having the form:

$$\mathcal{C}^* = \begin{pmatrix} \alpha_{0,0} & \alpha_{1,0} & \alpha_{2,0} & \alpha_{3,0} & \cdots \\ 0 & \alpha_{1,1} & \alpha_{2,1} & \alpha_{3,1} & \cdots \\ 0 & 0 & \alpha_{2,2} & \alpha_{3,2} & \cdots \\ 0 & 0 & 0 & \alpha_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $\alpha_{m,m} \neq 0$ for all $m \in \mathbb{N}_0$, such that $\mathcal{C}\mathcal{C}^* = \mathcal{I}$. Hence, the columns of the matrix $\mathcal{C}^*$ are precisely the sequences c_n^*, $n \in \mathbb{N}_0$. Hence, the sequence $\{c_n^*\}_{n=0}^{\infty}$ is also minimal and complete in $l^2(\mathbb{N}_0)$. As a consequence, the sequence $\{c_n\}_{n=0}^{\infty}$ is an M-basis for $l^2(\mathbb{N}_0)$, and Proposition 2.5 says us that the set of polynomials $\mathcal{P}(\mathbb{C})$ is dense in $\mathcal{H}_K$ (see [8] for another proof).

Besides, the Gram matrix $\{\langle c_n^*, c_m^* \rangle_{l^2}\}$ is a semi-infinite Hankel matrix. Indeed, let $\mathcal{A}$ be the operator defined by means of the Jacobi matrix (4.1) on $D(\mathcal{A})$, the set of sequences with finite support. This operator is closable since it is symmetric and densely defined; we denote again by $\mathcal{A}$ its closure. Let $\mathcal{A}^*$ be its adjoint operator whose domain $D(\mathcal{A}^*)$ is the set of sequences $s \in l^2(\mathbb{N}_0)$ such that the formal product of the Jacobi matrix by s belongs to $l^2(\mathbb{N}_0)$ [15, p.105]. If the Hamburger moment problem is indeterminate then $K(z) \in l^2(\mathbb{N}_0)$ for each $z \in \mathbb{C}$, and $\mathcal{A}^* K(z) = zK(z)$ [15, p. 94]. Therefore,

$$\langle K(z), \mathcal{A}^n \sigma_0 \rangle_{l^2} = \langle (\mathcal{A}^*)^n K(z), \sigma_0 \rangle_{l^2} = z^n \langle K(z), \sigma_0 \rangle_{l^2} = z^n P_0(z) = z^n,$$

where $\sigma_0 := (1, 0, 0, \dots)$. Hence, $\mathcal{A}^n \sigma_0 = c_n^*$, $n \in \mathbb{N}_0$. As a consequence, the Gram matrix associated to the sequence $\{c_n^*\}_{n=0}^{\infty}$ is the semi-infinite Hankel matrix associated to the moments. Indeed, since the operator $\mathcal{A}$ is symmetric

$$\langle c_{n+1}^*, c_m^* \rangle_{l^2} = \langle \mathcal{A}^{n+1} \sigma_0, \mathcal{A}^m \sigma_0 \rangle_{l^2} = \langle \sigma_0, \mathcal{A}^{n+m+1} \sigma_0 \rangle_{l^2} = s_{n+m+1} = \langle c_n^*, c_{m+1}^* \rangle_{l^2}.$$

Thus, having in mind Theorem 3.2 we deduce that $\mathcal{H}_K$ is equal isometrically to a de Branges space; for property 2 in Theorem 3.2, see [6, 8]. For a completely different proof of the fact that $\mathcal{H}_K$ is a de Branges space, see [10].

In order to establish a condition assuring the hypercyclicity of the translations in $\mathcal{H}_K$ following Theorem 2.12, next we obtain the value of the constants $\delta_k := \widehat{(c_k, L^k)}$, that is, the inclination in $l^2(\mathbb{N}_0)$ of the straight line spanned by c_k to the closed subspace $L^k := \overline{\operatorname{span}}\{c_0, c_1, \dots, c_{k-1}, c_{k+1}, \dots\}$. First note that

$$L^k = \overline{\operatorname{span}}\{e_0, e_1, \dots, e_{k-1}, e_{k+1}, \dots\} \quad \text{where, for} \quad j \in \mathbb{N}_0,\ e_j := \{\delta_{j,n}\}_{n=0}^{\infty}.$$

Thus, by using the orthogonal projection, we get

$$\delta_k = \inf_{\theta \in \mathbb{R}} \rho\left(\mathrm{e}^{i\theta} \frac{c_k}{\|c_k\|}, L^k \right) = \frac{a_{k,k}}{\|c_k\|}, \quad k \in \mathbb{N}_0.$$

Hence, by using Theorem 2.12, if the sequence $\{(k+1)\|c_{k+1}\|/a_{k,k}\}_{k=0}^{\infty}$ belongs to $l^1(\mathbb{N}_0)$, then any translation operator T_w, $w \neq 0$, in $\mathcal{H}_K$ is hypercyclic. Certainly, the above condition is not easy to evaluate. It remains an open question to give a sufficient condition for the hypercyclicity of the translation operators in $\mathcal{H}_K$ involving either the coefficients of the Jacobi matrix $\mathcal{A}$ or the measures solution of the indeterminate Hamburger moment problem.

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