Makino, T.
Osaka J. Math.
52 (2015), 545–580

ON SPHERICALLY SYMMETRIC MOTIONS OF A GASEOUS STAR GOVERNED BY THE EULER–POISSON EQUATIONS

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(Received October 1, 2013, revised February 3, 2014)

Abstract

We consider spherically symmetric motions of a polytropic gas under the self-gravitation governed by the Euler–Poisson equations. The adiabatic exponent (= the ratio of the specific heats) γ is assumed to satisfy $6/5 < \gamma \leq 2$. Then there are equilibria touching the vacuum with finite radii, and the linearized equation around one of the equilibria has time-periodic solutions. To justify the linearization, we should construct true solutions for which this time-periodic solution plus the equilibrium is the first approximation. We solve this problem by the Nash–Moser theorem. The result will realize the so-called physical vacuum boundary. But the present study restricts γ to the case in which $\gamma/(\gamma-1)$ is an integer. Other cases are reserved to the future as an open problem. The time-local existence of smooth solutions to the Cauchy problems is also discussed.

1. Introduction

We consider spherically symmetric motions of a gaseous star governed by the Euler–Poisson equations:

$$
\begin{aligned}
&\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2}{r}\rho u = 0,\\
&\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) + \frac{\partial P}{\partial r} = -\rho\frac{\partial \Phi}{\partial r} \quad (0 < t,\ 0 < r),\\
&\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) = 4\pi g_0\rho.
\end{aligned} \tag{1}
$$

Here ρ is the density, u the velocity, P the pressure, Φ the gravitational potential, and g_0 is the gravitational constant. In this work we assume

$$
P = A\rho^\gamma, \tag{2}
$$

where A and γ are positive constants, and we assume $1 < \gamma \leq 2$.

2010 Mathematics Subject Classification. 35L05, 35L52, 35L57, 35L70, 76L10.

Introducing the mass

$$m := 4\pi \int_0^r \rho(t, r')r'^2\, dr',$$

we can write the equations as

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + \frac{2}{r}\rho u = 0, \\ &\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) + \frac{\partial P}{\partial r} = -g_0\frac{\rho m}{r^2}. \end{aligned} \tag{3}$$

On the other hand, equilibria for the equations (1) are governed by the ordinary differential equation

$$-\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = 4\pi g_0 \rho.$$

In order to normalize this equation, we put

$$\rho = \rho_c \theta^{1/(\gamma-1)}$$

and

$$r = \rho_c^{(\gamma-2)/2} K^{-1/2}\xi \quad \text{with} \quad K := \frac{4\pi g_0(\gamma-1)}{A\gamma},$$

where ρ_c is an arbitrary positive number, say, the central density. Then the equation for equilibria turns out to be

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} + \theta^{1/(\gamma-1)} = 0,$$

which is called the 'Lane–Emden equation'. The solution $\theta(\xi)$ of the equation such that

$$\theta|_{\xi=0} = 1, \quad \left.\frac{d\theta}{d\xi}\right|_{\xi=0} = 0$$

is called the 'Lane–Emden function of polytropic index $1/(\gamma-1)$'. It is known that if and only if $6/5 < \gamma$ there is a finite ξ_1 such that $\theta(\xi) > 0$ for $0 \le \xi < \xi_1$ and $\theta(\xi_1) = 0$, and the radius R and the total mass

$$M := 4\pi \int_0^R \rho(r)r^2\, dr$$

of the equilibrium $\rho(r)$ are given by

$$R = \rho_c^{(\gamma-2)/2}K^{-1/2}\xi_1, \quad \text{and} \quad M = 4\pi\rho_c^{(3\gamma-4)/2}K^{-3/2}\left(-\xi^2\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}.$$

A numerical table of ξ_1, $(-\xi^2\, d\theta/d\xi)_{\xi=\xi_1}$ for various γ can be found in [2, p. 96].

Anyway we have

Lemma 1. *Assume $6/5 < \gamma \leq 2$. For any positive number ρ_c given, there is an equilibrium $\rho = \bar\rho(r)$ with positive numbers R, ρ_1 such that $\bar\rho(r)$ is positive and analytic in $0 < r < R$ and*

$$\bar\rho(r) = \rho_c(1 + [r^2]_1) \quad \textit{as} \quad r \to 0,$$
$$\bar\rho(r) = \rho_1(R-r)^{1/(\gamma-1)}(1 + [R-r, (R-r)^{\gamma/(\gamma-1)}]_1) \quad \textit{as} \quad r \to R-0.$$

NOTATIONALREMARK. Here and hereafter $[X]_q$ denotes a power series of the form $\sum_{j\geq q} a_j X^j$ with positive radius of convergence, and $[X, Y]_q$ a convergent power series of the form $\sum_{j+k\geq q} a_{jk} X^j Y^k$.

For a proof of Lemma 1, see, e.g., [10], and [13, Chapter V] or [24, Chapter IX] and Appendix A.

REMARK. In the expansion of $\bar\rho(r)$ as $r \to R$, the terms including $(R-r)^{\gamma/(\gamma-1)}$ actually appear if $\gamma/(\gamma-1)$ is not an integer. Let us prove it. Otherwise we would have

$$\bar\rho(r) = \rho_1(R-r)^{1/(\gamma-1)}(1 + [R-r]_1)$$

and the function

$$U(r) := \bar\rho(r)^{\gamma-1} = \rho_1^{\gamma-1}(R-r)(1 + [R-r]_1)$$

would be analytic at $r = R$. Now U satisfies

$$-\frac{d^2U}{dr^2} - \frac{2}{r}\frac{dU}{dr} = KU^{1/(\gamma-1)}, \quad K := \frac{4\pi g_0(\gamma-1)}{A\gamma}.$$

Since U is analytic, the left-hand side is analytic, and so, the right-hand side

$$K\rho_1(R-r)^{1/(\gamma-1)}(1 + [R-r]_1)$$

would be analytic at $r = R$. Then $1/(\gamma-1)$ should be an integer. This contradicts to that $\gamma/(\gamma-1) = 1/(\gamma-1) + 1$ is not an integer.

In fact we can find that, if $\gamma/(\gamma-1) \notin \mathbb{N}$, then

$$\bar\rho^{\gamma-1} = U = C(R-r)\biggl(1 + \frac{1}{R}(R-r) - \frac{(\gamma-1)^2KC^{(2-\gamma)/(\gamma-1)}}{\gamma(2\gamma-1)}(R-r)^{\gamma/(\gamma-1)} + [R-r, (R-r)^{\gamma/(\gamma-1)}]_2\biggr)$$

and

$$\frac{1}{\bar{\rho}}\frac{d\bar{P}}{dr} = \frac{A\gamma}{\gamma-1}\frac{dU}{dr}$$
$$= -\frac{A\gamma}{\gamma-1}C\Big(1+\frac{2}{R}(R-r)-\frac{(\gamma-1)KC^{(2-\gamma)/(\gamma-1)}}{\gamma}(R-r)^{\gamma/(\gamma-1)}$$
$$+[R-r,(R-r)^{\gamma/(\gamma-1)}]_2\Big),$$

where $C=\rho_1^{\gamma-1}$ and $\bar{P}(r)=A\bar{\rho}(r)^\gamma$.

In the following discussion we assume that $6/5<\gamma\leq 2$ and we fix an equilibrium $\bar{\rho}(r)$ with the properties in the above lemma.

We are going to construct solutions around this fixed equilibrium.

Here let us glance at the history of researches of this problem.

Of course there were a lot of works on the Cauchy problem to the compressible Euler equations. But there were gaps if we consider density distributions which contain vacuum regions.

As for local-in-time existence of smooth density with compact support, [17] treated the problem under the assumption that the initial density is non-negative and the initial value of

$$\omega := \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{(\gamma-1)/2}$$

is smooth, too. By the variables (ω, u) the equations are symmetrizable continuously including the region of vacuum. Hence the theory of quasi-linear symmetric hyperbolic systems can be applied. However, since

$$\omega \propto \left(\frac{1}{r}-\frac{1}{R}\right)^{1/2} \sim \text{Const.}(R-r)^{1/2} \quad \text{as} \quad r\to R-0$$

for equilibria, ω is not smooth at the boundary $r=R$ with the vacuum. Hence the class of solutions considered in [17] cannot cover equilibria. (See [18] for the discussion on non-isentropic cases. The situation is similar.)

On the other hand, possibly discontinuous weak solutions with compactly supported density can be constructed. The article [20] gave local-in-time existence of bounded weak solutions under the assumption that the initial density is bounded and non-negative, provided that the gas is confined to the domain outside a solid ball. The proof by the compensated compactness method is due to [19], and [5]. Of course the class of weak solutions can cover equilibria, but the concrete structures of solutions were not so clear.

Therefore we wish to construct solutions whose regularities are weaker than solutions with smooth ω and stronger than possibly discontinuous weak solutions. The present result is an answer to this wish. More concretely speaking, the solution $(\rho(t, r), u(t, r))$ constructed in this article should be continuous on $0 \le t \le T$, $0 \le r < \infty$ and there should be found a continuous curve $r = R_F(t)$, $0 \le t \le T$, such that $|R_F(t) - R| \ll 1$, $\rho(t,r) > 0$ for $0 \le t \le T$, $0 \le r < R_F(t)$ and $\rho(t,r) = 0$ for $0 \le t \le T$, $R_F(t) \le r < \infty$. The curve $r = R_F(t)$ is the free boundary at which the density touches the vacuum. It will be shown that the solution satisfies

$$\rho(t, r) = C(t)(R_F(t) - r)^{1/(\gamma-1)}(1 + O(R_F(t) - r))$$

as $r \to R_F(t) - 0$. Here $C(t)$ is positive and smooth in t. This situation is "physical vacuum boundary" so-called by [9] and [4]. This concept can be traced back to [15], [16], [25]. Of course this singularity is just that of equilibria.

Since the major difficulty comes from the free boundary touching the vacuum, which moves along time. So, we take the Lagrangian mass coordinate m as the independent variable instead of r. Then we can write the equations as

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + 4\pi\rho^2(r^2u)_m = 0, \\ &\frac{\partial u}{\partial t} + 4\pi r^2 P_m = -g_0\frac{m}{r^2}, \\ &r = \left(\frac{3}{4\pi}\int_0^m \frac{dm}{\rho}\right)^{1/3}. \end{aligned}$$

Since

$$\frac{\partial r}{\partial t} = u, \quad \frac{\partial r}{\partial m} = \frac{1}{4\pi\rho r^2},$$

the equations are reduced to the single second order equation

$$r_{tt} + 4\pi r^2 P_m = -g_0\frac{m}{r^2}, \tag{4}$$

where

$$P = A\left(4\pi r^2\frac{\partial r}{\partial m}\right)^{-\gamma}.$$

Now we derive the equation for the perturbation y defined by

$$r(t, m) = \bar{r}(m)(1 + y(t, \bar{r}(m))). \tag{5}$$

Here $m \mapsto \bar{r}(m)$ is the function of the Lagrangian mass variable m associated with the fixed equilibrium. In other words, it is the inverse function of

$$\bar{r} \mapsto m = 4\pi\int_0^{\bar{r}} \bar{\rho}(r')r'^2\,dr'.$$

Keeping in mind

$$\frac{\partial r}{\partial m} = \frac{\partial \bar{r}}{\partial m}\left(1 + y + \frac{\bar{r}}{\bar{r}_m}\frac{\partial y}{\partial m}\right),$$

we have

$$P = \bar{P}\left(1 - G\left(y, \frac{\bar{r}}{\bar{r}_m}\frac{\partial y}{\partial m}\right)\right).$$

Here $G(y, v) = 3\gamma y + \gamma v + [y, v]_2$ is defined by

$$(1+y)^{-2\gamma}(1+y+v)^{-\gamma} = 1 - G(y, v).$$

Then the equation is reduced to

$$\bar{r} y_{tt} + \frac{1}{\bar{\rho}}(1+y)^2 \frac{\partial}{\partial \bar{r}}\left(\bar{P}\left(1 - G\left(y, \bar{r}\frac{\partial y}{\partial \bar{r}}\right)\right)\right) + g_0 \frac{m}{\bar{r}^2(1+y)^2} = 0,$$

where we have used

$$\frac{\partial}{\partial m} = \bar{r}_m \frac{\partial}{\partial \bar{r}} = \frac{1}{4\pi \bar{\rho}\bar{r}^2}\frac{\partial}{\partial \bar{r}}.$$

We note that the equilibrium satisfies

$$\frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial \bar{r}} + g_0 \frac{m}{\bar{r}^2} = 0.$$

Let us introduce $H(y) = 4y + [y]_2$ by

$$H(y) = (1+y)^2 - \frac{1}{(1+y)^2}.$$

Then the equation can be written as

$$\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho r}(1+y)^2 \frac{\partial}{\partial r}\left(PG\left(y, r\frac{\partial y}{\partial r}\right)\right) + \frac{1}{\rho r}\frac{dP}{dr}H(y) = 0. \tag{6}$$

Here we have used the abbreviations r, ρ, P, dP/dr instead of $\bar{r}$, $\bar{\rho}$, $\bar{P}$, $d\bar{P}/d\bar{r}$. We consider this nonlinear wave equation.

It is easy to verify by a scale transformation of variables that we can assume that $A = 1/\gamma$ so that $P = \rho^\gamma/\gamma$ without loss of generality. Hence we assume so.

Here let us propose the main goal of this study roughly. Let us fix an arbitrarily large positive number T. Then, under the condition that $\gamma/(\gamma-1)$ is an integer, we have

Main Goal. *For sufficiently small* $\varepsilon > 0$ *there is a solution* $y = y(t, r; \varepsilon)$ *of* (6) *in* $C^2([0, T] \times [0, R])$ *such that*

$$y(t, r; \varepsilon) = \varepsilon y_1(t, r) + O(\varepsilon^2).$$

The same estimates $O(\varepsilon^2)$ *hold between the higher order derivatives of* y *and* εy_1.

Here $y_1(t,r)$ is a time-periodic function specified in Section 2, which is of the form

$$y_1(t, r) = \sin(\sqrt{\lambda} t + \theta_0) \cdot \Phi(r),$$

where λ is a positive number, θ_0 a constant, and $\Phi(r)$ is an analytic function of $0 \le r \le R$.

Once the solution $y(t,r;\varepsilon)$ is given, then the corresponding motion of gas particles can be expressed by the Lagrangian coordinate as

$$\begin{aligned} r(t, m) &= \bar{r}(m)(1 + y(t, \bar{r}(m); \varepsilon)) \\ &= \bar{r}(m)(1 + \varepsilon y_1(t, \bar{r}(m)) + O(\varepsilon^2)). \end{aligned}$$

The curve $r = R_F(t)$ of the free vacuum boundary is given by

$$R_F(t) = r(t, M) = R(1 + \varepsilon \sin(\sqrt{\lambda} t + \theta_0)\Phi(R) + O(\varepsilon^2)).$$

The free boundary $R_F(t)$ *oscillates around* R *with time-period* $2\pi/\sqrt{\lambda}$ *approximately.*

The solution (ρ, u) of the original problem (1) (2) is given by

$$\rho = \bar{\rho}(\bar{r})\left((1 + y)^2\left(1 + y + \bar{r}\frac{\partial y}{\partial \bar{r}}\right)\right)^{-1}, \quad u = \bar{r}\frac{\partial y}{\partial t}$$

implicitly by

$$\begin{aligned} &\bar{r} = \bar{r}(m), \quad y = y(t, \bar{r}(m); \varepsilon), \\ &\frac{\partial y}{\partial \bar{r}} = \partial_r y(t, \bar{r}(m); \varepsilon), \quad \frac{\partial y}{\partial t} = \partial_t y(t, \bar{r}(m); \varepsilon), \end{aligned}$$

where $m = m(t, r)$ for $0 \le r \le R_F(t)$. Here $r \mapsto m = m(t, r)$ is given as the inverse function of the function

$$m \mapsto r = r(t, m) = \bar{r}(m)(1 + y(t, \bar{r}(m); \varepsilon)).$$

We note that

$$R_F(t) - r(t, m) = R(1 + y(t, R; \varepsilon)) - \bar{r}(m)(1 + y(t, \bar{r}(m); \varepsilon))$$

implies

$$\frac{1}{\kappa}(R - \bar{r}) \le R_F(t) - r \le \kappa(R - \bar{r})$$

with $0 < \kappa - 1 \ll 1$, since $|y| + |\partial_r y| \leq \varepsilon C$. Therefore

$$y(t, \bar{r}(m); \varepsilon) = y(t, R; \varepsilon) + O(R_F(t) - r(t, m)),$$

and so on. Hence we get the "physical vacuum boundary", that is, the corresponding density distribution $\rho = \rho(t, r)$, where r denotes the original Euler coordinate, satisfies

$$\rho(t, r) > 0 \text{ for } 0 \leq r < R_F(t), \quad \rho(t, r) = 0 \text{ for } R_F(t) \leq r,$$

and, since $y(t, r)$ is smooth on $0 \leq r \leq R$, we have

$$\rho(t, r) = C(t)(R_F(t) - r)^{1/(\gamma-1)}(1 + O(R_F(t) - r))$$

as $r \to R_F(t) - 0$. Here $C(t)$ is positive and smooth in t.

2. Analysis of the linearized equation

The linearized equation is

$$\frac{\partial^2 y}{\partial t^2} + \mathcal{L}y = 0, \tag{7}$$

$$\begin{aligned} \mathcal{L}y &:= -\frac{1}{\rho r}\frac{\partial}{\partial r}\left(P\left(3\gamma y + \gamma r\frac{\partial y}{\partial r}\right)\right) + \frac{1}{\rho r}\frac{dP}{dr}\cdot(4y) \\ &= -\frac{1}{\rho r^4}\frac{\partial}{\partial r}\left(\gamma r^4 P\frac{\partial y}{\partial r}\right) + \frac{1}{\rho r}(4 - 3\gamma)\frac{dP}{dr}y, \end{aligned} \tag{8}$$

and the associated eigenvalue problem is $\mathcal{L}y = \lambda y$.

This eigenvalue problem was first wrote down in [6, p. 10, (12)] (1918). But the spectral property of the operator, whose coefficients are singular, had been long believed as a Sturm–Liouville type without proof. A mathematically rigorous discussion was first done by [1] (1995). The essential point is as follows.

Let us use the Liouville transformation:

$$\xi := \int_0^r \sqrt{\frac{\rho}{\gamma P}}\, dr, \quad \eta := r^2(\gamma P\rho)^{1/4} y.$$

Through this transformation the equation

$$\mathcal{L}y = \lambda y + f$$

turns out to be the standard form

$$-\frac{d^2\eta}{d\xi^2} + q\eta = \lambda\eta + \hat{f},$$

where

$$q = \frac{\gamma P}{\rho}\left(\frac{2}{r^2} + \left(\frac{7-3\gamma}{2} + \frac{1+\gamma}{4}\frac{rm_r}{m}\right)\frac{1}{r\rho}\frac{d\rho}{dr} + \frac{(\gamma+1)(3-\gamma)}{16}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2\right),$$
$$\hat{f} = r^2(\gamma P\rho)^{1/4} f.$$

The variable ξ runs on the interval $(0, \xi_+)$, where

$$\xi_+ := \int_0^R \sqrt{\frac{\rho}{\gamma P}}\, dr < \infty.$$

Since

$$\xi \sim \sqrt{\frac{\rho_c}{\gamma P_c}} r \quad \text{as} \quad r \to 0,$$

we see

$$q \sim \frac{\gamma P_c}{\rho_c}\frac{2}{r^2} \sim \frac{2}{\xi^2} \quad \text{as} \quad \xi \to 0.$$

Since

$$\frac{1}{\rho}\frac{d\rho}{dr} \sim -\frac{1}{\gamma-1}(R-r)^{-1}, \quad \frac{\gamma P}{\rho} \sim \rho_1^{\gamma-1}(R-r) \quad \text{as} \quad r \to R,$$

and

$$R - r \sim \frac{1}{4}\rho_1^{\gamma-1}(\xi_+ - \xi)^2 \quad \text{as} \quad \xi \to \xi_+,$$

we see

$$q \sim \frac{\gamma P}{\rho}\frac{(\gamma+1)(3-\gamma)}{16}\left(\frac{1}{\rho}\frac{d\rho}{dr}\right)^2 \sim \frac{1}{4}\frac{(1+\gamma)(3-\gamma)}{(\gamma-1)^2}\frac{1}{(\xi_+ - \xi)^2}$$

as $\xi \to \xi_+$. It follows from $1 < \gamma < 2$ that

$$\frac{1}{4}\frac{(1+\gamma)(3-\gamma)}{(\gamma-1)^2} > \frac{3}{4}.$$

Of course q is bounded from below, but it is difficult to know whether its minimum is positive or not. Anyway, the both boundary points $\xi = 0, \xi_+$ are of limit point type, provided that $1 < \gamma < 2$. See, e.g., [22, p. 159, Theorem X.10]. The exceptional case $\gamma = 2$ will be discussed later. See the discussion after Lemma 2 below. Hence we have the following conclusion:

Proposition 1. *The operator* $\mathfrak{T}_0$, $\mathcal{D}(\mathfrak{T}_0) = C_0^\infty(0, \xi_+)$, $\mathfrak{T}_0\eta = -\eta_{\xi\xi} + q\eta$, *in* $L^2(0,\xi_+)$ *has the Friedrichs extension* $\mathfrak{T}$, *a self-adjoint operator, whose spectrum consists of simple eigenvalues* $\lambda_1 < \cdots < \lambda_n < \lambda_{n+1} < \cdots \to +\infty$. *In other words, the operator* $\mathfrak{S}_0$, $\mathcal{D}(\mathfrak{S}_0) = C_0^\infty(0, R)$, $\mathfrak{S}_0 y = \mathcal{L}y$ *in* $L^2((0, R), r^4\rho\, dr)$ *has the Friedrichs extension* $\mathfrak{S}$, *a self-adjoint operator with eigenvalues* $(\lambda_n)_n$.

The domain $\mathcal{D}(\mathfrak{T})$ of the Friedrichs extension $\mathfrak{T}$ is, by definition,

$$\begin{aligned}\mathcal{D}(\mathfrak{T}) = \{\eta \in L^2(0, \xi_+) \mid \exists \phi_n \in C_0^\infty(0, \xi_+),\ Q[\phi_m - \phi_n] \to 0 \\ \text{as } m, n \to \infty,\ \phi_n \to \eta \text{ in } L^2(0, \xi_+) \\ \text{and } -\eta_{\xi\xi} + q\eta \in L^2(0, \xi_+) \text{ in distribution sense}\},\end{aligned}$$

where

$$Q[\phi] := \int_0^{\xi_+} \left(\left| \frac{d\phi}{d\xi} \right|^2 + (q + c)|\phi|^2 \right) d\xi,$$

and c is a constant $> |\min q|$. But $\mathcal{D}(\mathfrak{T})$ is characterized as follows:

$$\mathcal{D}(\mathfrak{T}) = \{\eta \in C[0, \xi_+] \mid \eta(0) = \eta(\xi_+) = 0,\ -\eta_{\xi\xi} + q\eta \in L^2(0, \xi_+)\}.$$

Let us prove it, denoting by M the right-hand side. Let $\eta \in \mathcal{D}(\mathfrak{T})$. Then there are $\phi_n \in C_0^\infty(0, \xi_+)$ such that $\phi_n \to \eta$ in L^2 and $Q[\phi_m - \phi_n] \to 0$. Since

$$|\phi_m(\xi) - \phi_n(\xi)| \le \sqrt{\xi} \left(\int_0^\xi ((\phi_m - \phi_n)_\xi)^2\, d\xi \right)^{1/2} \le \sqrt{\xi} (Q[\phi_m - \phi_n])^{1/2} \to 0,$$

we have $\phi_n \to \eta$ uniformly on $[0, \xi_+]$. Hence $\eta \in C[0, \xi_+]$ and $\eta(0) = 0$. Similarly $\eta(\xi_+) = 0$. Thus $\mathcal{D}(\mathfrak{T}) \subset M$. Let $\eta \in M$. Put $f := -\eta_{\xi\xi} + q\eta \in L^2$. Then $-\eta_{\xi\xi} + (q + c)\eta = g := f + c\eta \in L^2$. Since 0 belongs to the resolvent set of $\mathfrak{T} + c$, we have $v := (\mathfrak{T} + c)^{-1} g \in \mathcal{D}(\mathfrak{T})$. Hence $w := \eta - v \in C[0, \xi_+]$ and $w(0) = w(\xi_+) = 0$, $-w_{\xi\xi} + (q + c)w = 0$, for $\mathcal{D}(\mathfrak{T}) \subset M$. Using $q + c > 0$, we can deduce that $w \equiv 0$ and $\eta = v \in \mathcal{D}(\mathfrak{T})$, that is, $M \subset \mathcal{D}(\mathfrak{T})$. (In fact, if w did not vanish identically, there would exist $a \in (0, \xi_+)$ such that $Dw(a) = 0$ and $w(a) \neq 0$. If $w(a) > 0$, then

$$Dw(\xi) = \int_a^\xi D^2 w(\xi')\, d\xi' = \int_a^\xi (q + c) w(\xi')\, d\xi'$$

implies $Dw(\xi) > 0$ for $a < \xi < \xi_+$ and it contradicts to $w(\xi_+) = 0$. If $w(a) < 0$, then

$$Dw(\xi) = -\int_\xi^a (q + c) w(\xi')\, d\xi'$$

implies $Dw(\xi) > 0$ for $0 < \xi < a$ and it contradicts to $w(0) = 0$.)

Although it is not easy to judge the signature of $\min q$, we have

Proposition 2 ([14], 1997). *If and only if* $4/3 < \gamma \le 2$, *the least eigenvalue* λ_1 *is positive.*

Proof. The function $y \equiv 1$ satisfies

$$\mathcal{L} y = \frac{1}{\rho r}(4 - 3\gamma)\frac{dP}{dr} =: f > 0.$$

Let us consider the corresponding function

$$\eta_1 = r^2(\gamma P\rho)^{1/4}$$

through the Liouville transformation. It is easy to show that η_1 and $d\eta_1/d\xi$ vanish at $\xi = 0, \xi_+$ and $\eta_1 \in \mathcal{D}(\mathfrak{T})$. Let $\phi_1(\xi)$ be the eigenfunction of $-d^2/d\xi^2 + q$ associated with the least eigenvalue λ_1. We can assume $\phi_1(\xi) > 0$ for $0 < \xi < \xi_+$ and ϕ_1 and $d\phi_1/d\xi$ vanishes at $\xi = 0, \xi_+$. Then the integration by parts gives

$$\lambda_1 \int_0^{\xi_+} \phi_1\eta_1 \, d\xi = \int_0^{\xi_+} \phi_1(-\eta_{1,\xi\xi} + q\eta_1)\, d\xi.$$

Since

$$-\eta_{1,\xi\xi} + q\eta_1 = \hat{f} = r(\gamma P)^{1/4}\rho^{-3/4}(4-3\gamma)\frac{dP}{dr}$$

and $dP/dr < 0$, we have the assertion. □

REMARK. Assume that $3/4 < \gamma \leq 2$. Then the least eigenvalue, which is positive, is given by the variational formula

$$\lambda_1 = \min \frac{(\mathcal{L}y \mid y)_{\mathfrak{X}}}{\|y\|^2_{\mathfrak{X}}},$$

where $\mathfrak{X} = L^2((0, R), \rho r^4\, dr)$ endowed with $(u|v)_{\mathfrak{X}} = \int_0^R uv\rho r^4\, dr$. From this we can deduce the following Ritter–Eddington's law of the period-density relation: *Let us consider equilibria $\rho(r)$ with $\rho(0) = \rho_c$ and the corresponding least eigenvalue λ_1 or the "period" $\Pi := 2\pi/\sqrt{\lambda_1}$; then $\Pi\sqrt{\rho_c}$ is a constant depending only upon g_0, A, γ.*

In fact we can consider the one parameter family of equilibria

$$\rho(r) = \rho_\kappa(r) := \kappa^{2/(\gamma-2)}\bar{\rho}(r/\kappa)$$

which has radius $R = \kappa\bar{R} \propto \kappa$ and the central density $\rho_c = \kappa^{2/(\gamma-2)}\bar{\rho}_c \propto \kappa^{2/(\gamma-2)}$. Here $\bar{\rho}$ is a fixed equilibrium with radius $\bar{R}$ and central density $\bar{\rho}_c$. Then it is easy to see that $(\mathcal{L}y \mid y)_{\mathfrak{X}} = \kappa^{(5\gamma-6)/(\gamma-2)}(\bar{\mathcal{L}}y_\kappa \mid y_\kappa)_{\bar{\mathfrak{X}}}$, where $y_\kappa(\bar{r}) = y(\kappa\bar{r})$ and $\bar{\mathfrak{X}} = L^2((0, \bar{R}), \bar{\rho}\bar{r}^4 d\bar{r})$, and $\|y\|^2_{\mathfrak{X}} = \kappa^{(5\gamma-8)/(\gamma-2)}\|y_\kappa\|^2_{\bar{\mathfrak{X}}}$. Hence we have $\lambda_1 \propto \kappa^{2/(\gamma-2)} \propto \rho_c$. This completes the proof. (Note that the mean density $M/(4\pi R^3/3) \propto \kappa^{2/(\gamma-2)} \propto$ the central density ρ_c.) This fact was stated in [6, p. 15], as a result that the pulsation theory conforms with observation of variable stars. As for the priority of A. Ritter (1879), see [23].

Let us introduce the variable x defined by

$$x := \frac{\tan^2\theta}{1+\tan^2\theta}, \quad \theta := \frac{\kappa\xi}{2} = \frac{\kappa}{2}\int_0^r \sqrt{\frac{\rho}{\gamma P}}\, dr, \tag{9}$$

where $\kappa = \pi/\xi_+$. Then x runs over the interval [0, 1] while r runs over [0, R], and

$$\frac{dx}{dr} = \kappa\sqrt{x(1-x)}\sqrt{\frac{\rho}{\gamma P}} = \kappa\sqrt{x(1-x)}\rho^{(-\gamma+1)/2}.$$

Since

$$\begin{aligned}
\frac{d}{dr} &= \kappa\sqrt{x(1-x)}\rho^{(-\gamma+1)/2}\frac{d}{dx},\\
\frac{d^2}{dr^2} &= \kappa^2 x(1-x)\rho^{-\gamma+1}\frac{d^2}{dx^2}\\
&\quad + \left(\frac{1}{2}\kappa^2(1-2x)\rho^{-\gamma+1} + \frac{-\gamma+1}{2}\kappa\sqrt{x(1-x)}\rho^{(-\gamma-1)/2}\frac{d\rho}{dr}\right)\frac{d}{dx},
\end{aligned}$$

we have

$$\begin{aligned}
&\kappa^{-2}\mathcal{L}y\\
&= -x(1-x)\frac{d^2y}{dx^2}\\
&\quad - \left(\frac{1}{2}(1-2x) + \frac{\gamma+1}{2}\frac{1}{\kappa}\sqrt{x(1-x)}\rho^{(\gamma-3)/2}\frac{d\rho}{dr} + \frac{4}{r}\frac{1}{\kappa}\sqrt{x(1-x)}\rho^{(\gamma-1)/2}\right)\frac{dy}{dx}\\
&\quad + \frac{1}{\kappa^2}\frac{\rho^{\gamma-2}}{r}\frac{d\rho}{dr}(4-3\gamma)y.
\end{aligned}$$

As $r \to 0$ ($x \to 0$) we have

$$\begin{aligned}
x &= \frac{\kappa^2}{4}\rho_c^{-\gamma+1}r^2(1+[r^2]_1),\\
r &= \frac{2}{\kappa}\rho_c^{(\gamma-1)/2}\sqrt{x}(1+[x]_1),\\
\frac{d\rho}{dr} &= r[r^2]_0,\\
\frac{4}{r}\frac{1}{\kappa}&\sqrt{x(1-x)}\rho^{(\gamma-1)/2} = 2+[x]_1.
\end{aligned}$$

Then it follows that

$$\kappa^{-2}\mathcal{L}y = -x(1-x)\frac{d^2y}{dx^2} - \left(\frac{5}{2}+[x]_1\right)\frac{dy}{dx} + [x]_0 y.$$

On the other hand, as $r \to R(x \to 1)$, we have

$$\begin{aligned}
1-x &= \kappa^2\rho_1^{-\gamma+1}(R-r)(1+[R-r,(R-r)^{\gamma/(\gamma-1)}]_1),\\
R-r &= \frac{1}{\kappa^2}\rho_1^{\gamma-1}(1-x)(1+[1-x,(1-x)^{\gamma/(\gamma-1)}]_1),\\
\frac{d\rho}{dr} &= -\frac{\rho_1}{\gamma-1}(R-r)^{(2-\gamma)/(\gamma-1)}(1+[R-r,(R-r)^{\gamma/(\gamma-1)}]_1).
\end{aligned}$$

Then it follows that

$$\kappa^{-2}\mathcal{L}y = -x(1-x)\frac{d^2y}{dx^2} + \left(\frac{\gamma}{\gamma-1} + [1-x, (1-x)^{\gamma/(\gamma-1)}]_1\right)\frac{dy}{dx} + [1-x, (1-x)^{\gamma/(\gamma-1)}]_0 y.$$

Changing the scale of t, we can and shall assume that $\kappa = 1$ without loss of generality.

Summing up, we have:

Proposition 3. *We can write*

$$\mathcal{L}y = -x(1-x)\frac{d^2y}{dx^2} - \left(\frac{5}{2}(1-x) - \frac{N}{2}x\right)\frac{dy}{dx} + L_1(x)\frac{dy}{dx} + L_0(x)y, \tag{10}$$

where

$$L_1(x) = \begin{cases} [x]_1 & as \quad x \to +0, \\ [1-x, (1-x)^{N/2}]_1 & as \quad x \to 1-0, \end{cases}$$

$$L_0(x) = \begin{cases} [x]_0 & as \quad x \to +0, \\ [1-x, (1-x)^{N/2}]_0 & as \quad x \to 1-0. \end{cases}$$

Here N is the parameter defined by

$$N = \frac{2\gamma}{\gamma-1} \Leftrightarrow \gamma = 1 + \frac{2}{N-2}. \tag{11}$$

Now let us fix a positive eigenvalue $\lambda = \lambda_n$ and an associated eigenfunction $\Phi(r)$ of $\mathcal{L}$. Then

$$y_1(t, r) = \sin(\sqrt{\lambda}t + \theta_0)\Phi(r)$$

is a time-periodic solution of the linearized problem.

Moreover we can claim

Proposition 4. *We have*

$$\begin{aligned}\Phi(r) &= C_0(1 + [r^2]_1) \quad as \quad r \to 0, \\ &= C_0(1 + [x]_1) \quad as \quad x \to 0\end{aligned}$$

and

$$\begin{aligned}\Phi(r) &= C_1(1 + [R-r, (R-r)^{\gamma/(\gamma-1)}]_1) \quad as \quad r \to R, \\ &= C_1(1 + [1-x, (1-x)^{N/2}]_1) \quad as \quad x \to 1.\end{aligned}$$

Here C_0 and C_1 are non-zero constants. Other independent solutions of $\mathcal{L}y = \lambda y$ do not belong to $L^2(r^4\rho\, dr)$ at $r \sim R$.

To prove this, we use the following lemma:

Lemma 2. *Let us consider the equation*

$$z\frac{d^2y}{dz^2} + b(z, z^a)\frac{dy}{dz} = c(z, z^a)y,$$

where

$$b(z, z^a) = a + [z, z^a]_1, \quad c(z, z^a) = [z, z^a]_0,$$

and let the positive number a satisfy $a \geq 2$. Then
1) *there is a solution y_1 of the form*

$$y_1 = 1 + [z, z^a]_1,$$

and
2) *there is a solution y_2 of the form*

$$y_2 = z^{-a+1}(1 + [z, z^a]_1)$$

provided $a \notin \mathbb{N}$, or

$$y_2 = z^{-a+1}(1 + [z, z^a]_1) + hy_1 \log z$$

provided $a \in \mathbb{N}$. Here h is a constant which can vanish in some cases.

For a proof, see [3, Chapter 4].

We apply this lemma for $a = \gamma/(\gamma-1) = N/2$ (≥ 2) and $z = 1-x$. Even if $N = 4$ $(\gamma = 2)$, $y_2 \sim z^{-N/2+1}$ does not belong to $L^2(r^4\rho\, dr) = L^2(x^{3/2}(1-x)^{N/2-1}\, dx)$, and the boundary point $r = R$ is of the limit point type.

3. Statement of the main result

We rewrite the equation (6) by using the linearized operator $\mathcal{L}$ defined by (8) as

$$\frac{\partial^2 y}{\partial t^2} + \left(1 + G_{\mathrm{I}}\left(y, r\frac{\partial y}{\partial r}\right)\right)\mathcal{L}y + G_{\mathrm{II}}\left(r, y, r\frac{\partial y}{\partial r}\right) = 0, \tag{12}$$

where

$$G_{\mathrm{I}}(y, v) = (1+y)^2\left(1 + \frac{1}{\gamma}\partial_v G_2(y, v)\right) - 1,$$

$$G_{\mathrm{II}}(r, y, v) = \frac{P}{\rho r^2}G_{\mathrm{II}0}(y, v) + \frac{1}{\rho r}\frac{dP}{dr}G_{\mathrm{II}1}(y, v),$$

$$\begin{aligned} G_{\mathrm{II}0}(y, v) &= (1+y)^2(3\partial_v G_2 - \partial_y G_2)v \\ &= -2\gamma(1+y)^{-2\gamma+1}(1+y+v)^{-\gamma-1}v^2, \\ G_{\mathrm{II}1}(y, v) &= \frac{(1+y)^2}{\gamma}\partial_v G_2 \cdot ((-4+3\gamma)y + \gamma v) \\ &\quad + H - 4y(1+y)^2 - (1+y)^2 G_2. \end{aligned}$$

Here

$$\begin{aligned} G_2(y, v) &:= G(y, v) - (3\gamma y + \gamma v) = [y, v]_2, \\ \partial_v G_2 &:= \frac{\partial}{\partial v}G_2 = \frac{\partial G}{\partial v} - \gamma = [y, v]_1. \end{aligned}$$

We have fixed a solution y_1 of the linearized equation $y_{tt} + \mathcal{L}y = 0$, and we seek a solution y of (6) or (12) of the form

$$y = \varepsilon y_1 + \varepsilon w,$$

where ε is a small positive parameter. Then the equation which w should satisfy turns out to be

$$\begin{aligned} &\frac{\partial^2 w}{\partial t^2} + \left(1 + \varepsilon a\left(t, r, w, r\frac{\partial w}{\partial r}, \varepsilon\right)\right)\mathcal{L}w + \varepsilon b\left(t, r, w, r\frac{\partial w}{\partial r}, \varepsilon\right) \\ &= \varepsilon c(t, r, \varepsilon), \end{aligned} \tag{13}$$

where

$$\begin{aligned} a(t, r, w, \Omega, \varepsilon) &= \varepsilon^{-1}G_{\mathrm{I}}(\varepsilon y_1 + \varepsilon w, \varepsilon v_1 + \varepsilon\Omega), \\ b(t, r, w, \Omega, \varepsilon) &= -(F_{\mathrm{I}} + F_{\mathrm{II}}) + (F_{\mathrm{I}} + F_{\mathrm{II}})|_{w=\Omega=0} \\ c(t, r, \varepsilon) &= (F_I + F_{II})|_{w=\Omega=0}. \end{aligned}$$

Here v_1 stands for $r\partial y_1/\partial r$ and

$$\begin{aligned} F_I &:= -\varepsilon^{-1}G_I(\varepsilon y_1 + \varepsilon w, \varepsilon v_1 + \varepsilon\Omega)\mathcal{L}y_1, \\ F_{II} &:= -\varepsilon^{-2}G_{II}(r, \varepsilon y_1 + \varepsilon w, \varepsilon v_1 + \varepsilon\Omega). \end{aligned}$$

It follows from Proposition 4 that a, b, c are smooth functions of t, x, $(1-x)^{N/2}$, w and $\partial w/\partial x$. Here and hereafter x *denotes the variable defined by* (9), *which is equivalently used instead of* r.

Then the main result of this study can be stated as follows:

Theorem 1. *Assume that* $6/5 < \gamma \leq 2$ $(\Leftrightarrow 4 \leq N < 12)$ *and that* $\gamma/(\gamma - 1)$ $(= N/2)$ *is an integer, that is,* γ *is either* 2, 3/2, 4/3 *or* 5/4. *Then for any given*

$T > 0$ there is a sufficiently small positive $\varepsilon_0 = \varepsilon_0(T)$ such that, for $|\varepsilon| \leq \varepsilon_0$, there is a solution $w \in C^\infty([0, T] \times [0, R])$ of (13) *such that*

$$\sup_{j+k\leq n} \left\| \left(\frac{\partial}{\partial t}\right)^j \left(\frac{\partial}{\partial r}\right)^k w \right\|_{L^\infty([0,T]\times[0,R])} \leq C_n \varepsilon,$$

or a solution $y \in C^\infty([0, T] \times [0, R])$ of (6) *or* (12) *of the form*

$$y(t, r) = \varepsilon y_1(t, r) + O(\varepsilon^2),$$

or a motion which can be expressed by the Lagrangian coordinates as

$$r(t, m) = \bar{r}(m)(1 + \varepsilon y_1(t, \bar{r}(m)) + O(\varepsilon^2))$$

for $0 \leq t \leq T$, $0 \leq m \leq M$.

Our task is to find the inverse image $\mathfrak{P}^{-1}(\varepsilon c)$ of the nonlinear mapping $\mathfrak{P}$ defined by

$$\mathfrak{P}(w) := \frac{\partial^2 w}{\partial t^2} + (1 + \varepsilon a)\mathcal{L}w + \varepsilon b. \tag{14}$$

Note $\mathfrak{P}(0) = 0$. It requires a property of the Fréchet derivative of $\mathfrak{P}$:

$$D\mathfrak{P}(w)h = h_{tt} + (1 + \varepsilon a_1)\mathcal{L}h + \varepsilon a_{20}h + \varepsilon a_{21} r h_r, \tag{15}$$

where

$$a_1(t, r) = a\left(t, r, w, r\frac{\partial w}{\partial r}, \varepsilon\right),$$
$$a_{20}(t, r) = \frac{\partial a}{\partial w}\mathcal{L}w + \frac{\partial b}{\partial w},$$
$$a_{21}(t, r) = \frac{\partial a}{\partial \Omega}\mathcal{L}w + \frac{\partial b}{\partial \Omega}.$$

Here Ω is the dummy of $r\, \partial w/\partial r$. We shall use the following observation:

Proposition 5. *We have*

$$a_{21} = \frac{\gamma P}{\rho}(1+y)^{-2\gamma+2}(1+y+v)^{-\gamma-2}\left((\gamma+1)\frac{\partial^2 Y}{\partial r^2} + \frac{4\gamma}{r}\frac{\partial Y}{\partial r} + \frac{2\varepsilon(\gamma-1)}{1+y}\left(\frac{\partial Y}{\partial r}\right)^2\right),$$

where

$$Y = y_1 + w, \quad y = \varepsilon Y, \quad v = r\frac{\partial y}{\partial r} = \varepsilon r\frac{\partial Y}{\partial r}.$$

Proof. Since

$$\frac{\partial a}{\partial \Omega} = \frac{\partial G_{\mathrm{I}}}{\partial v} = \frac{(1+y)^2}{\gamma}\partial_v^2 G_2,$$
$$\frac{\partial b}{\partial \Omega} = \frac{\partial G_{\mathrm{I}}}{\partial v}\mathcal{L}y_1 + \varepsilon^{-1}\frac{\partial G_{\mathrm{II}}}{\partial v},$$

we have

$$\varepsilon a_{21} = -(\partial_v G_{\mathrm{I}})\frac{\gamma P}{\rho}\left(\frac{\partial^2 y}{\partial r^2} + \frac{4}{r}\frac{\partial y}{\partial r}\right) + \frac{P}{\rho r^2}\partial_v G_{\mathrm{II0}} + \frac{1}{\rho r}\frac{dP}{dr}[U],$$

where

$$[U] = -(\partial_v G_I)((3\gamma - 4)y + \gamma v) + \partial_v G_{\mathrm{II1}}.$$

Since

$$\partial_v G_{\mathrm{I}} = \frac{(1+y)2}{\gamma}\partial_v^2 G_2, \quad \partial_v G_{\mathrm{II1}} = \frac{(1+y)^2}{\gamma}\partial_v^2 G_2((-4+3\gamma)y + \gamma v),$$

we have $[U] = 0$. Using

$$\partial_v^2 G_2 = -\gamma(\gamma+1)(1+y)^{-2\gamma}(1+y+v)^{-\gamma-2},$$
$$\partial_v G_{\mathrm{II0}} = -2\gamma(1+y)^{-2\gamma+1}(1+y+v)^{-\gamma-2}\cdot(2(1+y)+(-\gamma+1)v)v,$$

we get the result. □

Hereafter we use the variable x defined by (9) instead of $r = \bar{r}$.

We note that

$$\frac{\gamma P}{\rho} = \rho_1^{2(\gamma-1)}(1-x)(1+[1-x,(1-x)^{N/2}]_1).$$

Hence the function $\hat{a}_{21}$ defined by

$$\hat{a}_{21} := \frac{r}{x(1-x)}\frac{dx}{dr}a_{21} = \frac{r}{\sqrt{x(1-x)}}\rho^{(-\gamma+1)/2}a_{21}$$

is smooth in t, x, $(1-x)^{N/2}$, w, $\partial w/\partial x$, $\partial^2 w/\partial x^2$ including $x = 0, 1$. Therefore

Proposition 6. *The derivative $D\mathfrak{P}$ can be written as*

$$D\mathfrak{P}(w)h = \frac{\partial^2 h}{\partial t^2} + (1+\varepsilon a_1)\mathcal{L}h + \varepsilon\hat{a}_{21}x(1-x)\frac{\partial h}{\partial x} + \varepsilon a_{20}h, \tag{16}$$

where a_1, $\hat{a}_{21}$, a_{20} are smooth functions of t,x, $(1-x)^{N/2}$, w, $\partial w/\partial x$ and $\partial^2 w/\partial x^2$.

4. Proof of the main result

Hereafter we assume that $N/2$ is an integer so that $(1-x)^{N/2}$ is analytic at $x = 1$.

We are going to apply the Nash–Moser theorem formulated by R. Hamilton ([7, p. 171, III.1.1.1.]) as [21], that is:

Theorem (Nash–Moser(–Hamilton) theorem)**.** *Let $\mathfrak{E}_0$ and $\mathfrak{E}$ be tame spaces, U an open subset of $\mathfrak{E}_0$ and $\mathfrak{P}\colon U \to \mathfrak{E}$ a smooth tame map. Suppose that the equation for the derivative $D\mathfrak{P}(w)h = g$ has a unique solution $h = V\mathfrak{P}(w)g$ in $\mathfrak{E}_0$ for all w in U and all g in $\mathfrak{E}$, and $V\mathfrak{P}\colon U \times \mathfrak{E} \to \mathfrak{E}_0$ is a smooth tame map. Then $\mathfrak{P}$ is locally invertible.*

For the definitions of 'tame spaces' and 'tame maps', see [7] or [21]. We shall use the discussions of [21] without repeating the details.

We consider the spaces of functions of t and x:

$$\mathfrak{E} := C^\infty([0, T] \times [0, 1]),$$
$$\mathfrak{E}_0 := \left\{ w \in \mathfrak{E} \;\middle|\; w = \frac{\partial w}{\partial t} = 0 \text{ at } t = 0 \right\}.$$

Let U be the set of all functions w in $\mathfrak{E}_0$ such that $|w| + |\partial w/\partial x| < 1$. Then, for $w \in U$, $y = \varepsilon y_1 + \varepsilon w$ and its derivative re r are small, provided that $|\varepsilon| \le \varepsilon_1$. Then we can consider the mapping

$$\mathfrak{P}\colon w \mapsto \partial_t^2 w + (1 + \varepsilon a)\mathcal{L}w + \varepsilon b$$

maps U into $\mathfrak{E}$, since the coefficients a, b are smooth functions of t, x, w, $\partial w/\partial x$ and the coefficients L_0, L_1 of $\mathcal{L}$ are analytic on $0 \le x \le 1$.

The inverse image $\mathfrak{P}^{-1}(\varepsilon c)$ is a desired smooth solution of (13).

We should introduce gradings of norms on $\mathfrak{E}$ so that $\mathfrak{E}$, $\mathfrak{E}_0$ become tame spaces in the Hamilton's sense. To do so, we use a cut off function $\omega \in C^\infty([0, 1])$ such that $\omega(x) = 1$ for $0 \le x \le 1/3$, $0 < \omega(x) < 1$ for $1/3 < x < 2/3$ and $\omega(x) = 0$ for $2/3 \le x \le 1$. For a function y of $0 \le x \le 1$, we shall denote

$$y^{[0]}(x) = \omega(x)y(x), \quad y^{[1]}(x) = (1 - \omega(x))y(x). \tag{17}$$

We consider the tame spaces

$$\mathfrak{E}_{[0]} = \left\{ y \in C^\infty([0, T] \times [0, 1]) \;\middle|\; y = 0 \text{ for } \frac{5}{6} \le x \le 1 \right\},$$
$$\mathfrak{E}_{[1]} = \left\{ y \in C^\infty([0, T] \times [0, 1]) \;\middle|\; y = 0 \text{ for } 0 \le x \le \frac{1}{6} \right\},$$

endowed with the equivalent gradings of norms $(\|\cdot\|^{(\infty)}_{[\mu]n})_n$, $(\|\cdot\|^{(2)}_{[\mu]n})_n$, $\mu = 0, 1$, by the same manner as in [21], that is, denoting

$$\triangle_{[0]} = x\frac{d^2}{dx^2} + \frac{5}{2}\frac{d}{dx}, \quad \triangle_{[1]} = z\frac{d^2}{dz^2} + \frac{N}{2}\frac{d}{dz}, \quad (z = 1 - x),$$

we put

$$\|y\|^{(\infty)}_{[\mu]n} = \sup_{j+k\le n}\left\|\left(-\frac{\partial^2}{\partial t^2}\right)^j(-\triangle_{[\mu]})^k y\right\|_{L^\infty},$$

$$\|y\|^{(2)}_{[\mu]n} = \sum_{j+k\le n}\left(\int_0^T\left\|\left(-\frac{\partial^2}{\partial t^2}\right)^j(-\triangle_{[\mu]})^k y\right\|^2_{[\mu]} dt\right)^{1/2},$$

where

$$\|y\|_{[0]} = \left(\int_0^1 y^2 x^{3/2}\,dx\right)^{1/2},$$

$$\|y\|_{[1]} = \left(\int_0^1 y^2(1-x)^{N/2-1}\,dx\right)^{1/2}.$$

On the other hand, on $\mathfrak{E}$ we introduce the gradings of norms $(\|\cdot\|^{(\infty)}_n)_n$ and $(\|\cdot\|^{(2)}_n)_n$ by

$$\|y\|^{(\infty)}_n := \sup_{j+k\le n,\mu=0,1}\left\|\left(-\frac{\partial^2}{\partial t^2}\right)^j(-\triangle_{[\mu]})^k y^{[\mu]}\right\|_{L^\infty},$$

$$\|y\|^{(2)}_n := \left(\sum_{j+k\le n,\mu=0,1}\int_0^T\left\|\left(-\frac{\partial^2}{\partial t^2}\right)^j(-\triangle_{[\mu]})^k y^{[\mu]}\right\|^2_{[\mu]} dt\right)^{1/2}.$$

Then it is easy to see that $\mathfrak{E}$ is a tame space as a tame direct summand of the cartesian product $\mathfrak{E}_{[0]}\times\mathfrak{E}_{[1]}$, which is a tame space. (See [7, p. 136, 1.3.3 and 1.3.4]) In fact we consider the linear mappings L: $\mathfrak{E}\to\mathfrak{E}_{[0]}\times\mathfrak{E}_{[1]}$: $h\mapsto(h^{[0]}, h^{[1]})$ and M: $\mathfrak{E}_{[0]}\times\mathfrak{E}_{[1]}$: $(h_0, h_1)\mapsto h_0 + h_1$. It is clear that L is tame and $ML = \mathrm{Id}_{\mathfrak{E}}$. To verify that M is tame, we use the following.

Proposition 7. *If the support of $y(x)$ is included in* $[1/6, 5/6]$, *then*

$$\|\triangle^m_{[\mu]}y\|_{L^\infty} \le C\sum_{0\le k\le m}\|\triangle^k_{[1-\mu]}y\|_{L^\infty}.$$

A proof can be found in Appendix B. Now if $h_\mu\in\mathfrak{E}_{[\mu]}$, then $h = M(h_0, h_1) = h_0 + h_1$, and

$$h^{[0]} = (h_0 + h_1)^{[0]} = \omega h_0 + \omega h_1.$$

Then by [21, Proposition 4] we have

$$\|\triangle_{[0]}^m h^{[0]}\|_{L^\infty} \le C \sum_{k\le m} \|\triangle_{[0]}^k h_0\|_{L^\infty} + \|\triangle_{[0]}^m(\omega h_1)\|_{L^\infty}.$$

Proposition 7 can be applied, since $\mathrm{supp}[\omega h_1] \subset [1/6, 2/3]$, so that

$$\begin{aligned}\text{the second term} &\le C \sum_{k\le m} \|\triangle_{[1]}^k(\omega h_1)\|_{L^\infty} \\ &\le C' \sum_{k\le m} \|\triangle_{[1]}^k h_1\|_{L^\infty}.\end{aligned}$$

Therefore we have

$$\|\triangle_{[0]}^m h^{[0]}\|_{L^\infty} \le C \sum_{k\le m} (\|\triangle_{[0]}^k h_0\|_{L^\infty} + \|\triangle_{[1]}^k h_1\|_{L^\infty}).$$

The same argument gives the estimate of $\|\triangle_{[1]}^m h^{[1]}\|_{L^\infty}$. This implies the tameness of M. Therefore $\mathfrak{E}$ is tame with respect to the grading $(\|\cdot\|_n^{(\infty)})_n$.

By the discussion of [21] it is clear that the mapping $\mathfrak{P}$ is tame. In fact we have

$$\|\mathfrak{P}(w)\|_n^{(\infty)} \le C\|w\|_{n+1}^{(\infty)}.$$

Therefore we can concentrate ourselves to the analysis of the linear equation

$$D\mathfrak{P}(w)h = g \tag{18}$$

when w is chosen from U and g is given in $\mathfrak{E}$. By Proposition 3 and 6 we can write

$$D\mathfrak{P}(w)h = \frac{\partial^2 h}{\partial t^2} - b_2\Lambda h + b_1 x(1-x)\frac{\partial h}{\partial x} + b_0 h, \tag{19}$$

where

$$\Lambda = x(1-x)\frac{\partial^2}{\partial x^2} + \left(\frac{5}{2}(1-x) - \frac{N}{2}x\right)\frac{\partial}{\partial x} \tag{20}$$

and

$$\begin{aligned}&b_2 = 1 + \varepsilon a_1, \quad b_1 = (1+\varepsilon a_1)\frac{L_1}{x(1-x)} + \varepsilon \hat{a}_{21}, \\ &b_0 = (1+\varepsilon a_1)L_0 + \varepsilon a_{20}\end{aligned}$$

are smooth functions of t, x, w, Dw, $D^2 w$, where $D = \partial/\partial x$.

In order to establish the existence and uniqueness of the solution of (18), we introduce the following spaces of functions of $0 \le x \le 1$:

$$\mathfrak{X} = \mathfrak{X}^0 := \left\{ y \,\middle|\, \|y\|_{\mathfrak{X}} := \left(\int_0^1 y^2 x^{3/2}(1-x)^{N/2-1}\,dx \right)^{1/2} < \infty \right\},$$
$$\mathfrak{X}^1 := \left\{ y \in \mathfrak{X} \,\middle|\, \dot{D}y := \sqrt{x(1-x)}\frac{dy}{dx} \in \mathfrak{X} \right\},$$
$$\mathfrak{X}^2 := \{ y \in \mathfrak{X}^1 \mid -\Lambda y \in \mathfrak{X} \}.$$

Then we have

Proposition 8. *Let a be a function in $C^1[0,1]$. If $y \in \mathfrak{X}^2$ and $v \in \mathfrak{X}^1$, then*

$$(-a\Lambda y \mid v)_{\mathfrak{X}} = (a\dot{D}y \mid \dot{D}v)_{\mathfrak{X}} + ((Da)\check{D}y \mid v)_{\mathfrak{X}},$$

where $\check{D} = x(1-x)\,d/dx$. Here, of course,

$$(u \mid v)_{\mathfrak{X}} = \int_0^1 uvx^{3/2}(1-x)^{N/2-1}\,dx.$$

Proof. If $v \in \mathfrak{X}^1$, then

$$v(x) = v\left(\frac{1}{2}\right) + \int_{1/2}^{x} \frac{\dot{D}v(x')}{\sqrt{x'(1-x')}}\,dx'$$

implies

$$|v(x)| \le Cx^{-3/4}(1-x)^{-N/4+1/2},$$

and if $y \in \mathfrak{X}^2$, then

$$x^{5/2}(1-x)^{N/2}\frac{dy}{dx} = x^{5/2}(1-x)^{N/2}\frac{dy}{dx}\bigg|_{x=1/2} - \int_{1/2}^{x} \Lambda y(x')x'^{3/2}(1-x')^{N/2-1}\,dx'$$

implies

$$\left| x^{5/2}(1-x)^{N/2}\frac{dy}{dx} \right| \le Cx^{5/4}(1-x)^{N/4}.$$

(Note that the finite constant

$$x^{5/2}(1-x)^{N/2}\frac{dy}{dx}\bigg|_{x=1/2} + \int_0^{1/2} \Lambda y(x')x'^{3/2}(1-x')^{N/2-1}\,dx'$$

should vanish in order to $\dot{D}y \in \mathfrak{X}$, and so on.) Therefore the boundary terms in the integration by parts vanish as $x \to 0, 1$ and we get the desired equality. $\square$

Using Proposition 8, we can prove the following energy estimate in the same manner as [21, Lemma 3]:

Proposition 9. *Let $g \in C([0,T],\mathfrak{X})$. If $h \in \bigcap_{k=0,1,2} C^{2-k}([0,T],\mathfrak{X}^k)$ satisfies* (18), *then we have, for $0 \le t \le T$,*

$$\|\partial_t h\|_{\mathfrak{X}} + \|h\|_{\mathfrak{X}^1} \le C\left(\|\partial_t h|_{t=0}\|_{\mathfrak{X}} + \|h|_{t=0}\|_{\mathfrak{X}^1} + \int_0^t \|g(t')\|_{\mathfrak{X}}\, dt'\right).$$

Here

$$\|h\|^2_{\mathfrak{X}^1} = \|h\|^2_{\mathfrak{X}} + \|\dot{D}h\|^2_{\mathfrak{X}},$$

and the constant C depends only upon N, T, $\|\partial_t b_2\|_{L^\infty}$, $\|Db_2\|_{L^\infty}$, $\|b_1\|_{L^\infty}$, $\|b_0\|_{L^\infty}$, provided that $|1-b_2| \le 1/2$.

We are considering the initial boundary value problem (IBP):

$$\frac{\partial^2 h}{\partial t^2} + \mathcal{A}h = g(t,x), \quad h(t,\,\cdot\,) \in \mathfrak{X}^1,$$
$$h = \frac{\partial h}{\partial t} = 0 \quad \text{at} \quad t = 0.$$

Here

$$\mathcal{A} = -b_2\Lambda + b_1\check{D} + b_0, \quad \check{D} = x(1-x)\frac{d}{dx}.$$

Note that "$h(t,\cdot) \in \mathfrak{X}^1$" is a Dirichlet boundary condition in some sense. In fact it can be shown that $C_0^\infty(0,1)$ is dense in $\mathfrak{X}^1$.

Anyway, applying Kato's theory developed in [11], we have

Proposition 10. *If $g \in C([0,T],\mathfrak{X}^1) \cup C^1([0,T],\mathfrak{X})$, then there exists a unique solution h of* (*IBP*) *in $\bigcap_{k=0,1,2} C^{2-k}([0,T],\mathfrak{X}^k)$.*

Proof. We write (IBP) as

$$\frac{d}{dt}\begin{pmatrix} h \\ \dot{h} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ \mathcal{A} & 0 \end{pmatrix}\begin{pmatrix} h \\ \dot{h} \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}.$$

Applying the semi-group theory in the space $\mathfrak{H} = \mathfrak{X}^1 \times \mathfrak{X}$ to the family of generators

$$D(\mathfrak{A}(t)) = \mathfrak{X}^2 \times \mathfrak{X}^1,$$
$$\mathfrak{A}(t) = \begin{pmatrix} 0 & -1 \\ \mathcal{A} & 0 \end{pmatrix},$$

we get the result. The proof is same as in the Appendix C of [21]. Note that

$$(\mathcal{A}y \mid v)_{\mathfrak{X}} = (b_2\dot{D}y \mid \dot{D}v)_{\mathfrak{X}} + (((b_1 + Db_2)\check{D} + b_0)y \mid v)_{\mathfrak{X}}$$

for $y \in \mathfrak{X}^2$ and $v \in \mathfrak{X}^1$ thanks to Proposition 8. □

We are going to prove the smoothness of the solution and to get its tame estimates. In order to do it, we use the cut off function ω to separate the singularities at $x = 0$ and $x = 1$, since, although the singularities are of the same type, the calculus structure of Λ^m, $m \in \mathbb{N}$, is little bit complicated.

The equation $\partial^2 h/\partial t^2 + \mathcal{A}h = g$ is split into the following simultaneous system of equations:

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + \mathcal{A}_{[0]}\right)h^{[0]} &= g^{[0]} - (c_1\check{D} + c_0)h^{[1]},\\ \left(\frac{\partial^2}{\partial t^2} + \mathcal{A}_{[1]}\right)h^{[1]} &= g^{[1]} + (c_1\check{D} + c_0)h^{[0]},\end{aligned} \tag{21}$$

where

$$\begin{aligned}&c_1 = (2b_2 - b_1)D\omega, \quad c_0 = b_2(\Lambda\omega),\\ &\mathcal{A}_{[0]} = -b_2\Lambda + (b_1 + c_1)\check{D} + b_0 + c_0,\\ &\mathcal{A}_{[1]} = -b_2\Lambda + (b_1 - c_1)\check{D} + b_0 - c_0.\end{aligned}$$

We can rewrite them as:

$$\begin{aligned}\mathcal{A}_{[0]} &= -b_{[0]2}\triangle_{[0]} + b_{[0]1}x\frac{d}{dx} + b_{[0]0},\\ \mathcal{A}_{[1]} &= -b_{[1]2}\triangle_{[1]} + b_{[1]1}z\frac{d}{dz} + b_{[1]0}, \quad (z = 1 - x),\end{aligned}$$

where

$$\begin{aligned}&b_{[0]2} = b_2(1 - x), \quad b_{[1]2} = b_2x,\\ &b_{[0]1} = \frac{N}{2}b_2 + (b_1 + c_1)(1 - x), \quad b_{[1]1} = \frac{5}{2}b_2 - (b_1 - c_1)x,\\ &b_{[0]0} = b_0 + c_0, \quad b_{[1]0} = b_0 - c_0.\end{aligned}$$

We may assume that $|b_{[\mu]2} - 1| \leq \kappa$ on $x \in I_{[\mu]}$, $\mu = 0, 1$, with a constant κ such that $2/3 < \kappa < 1$, e.g., $\kappa = 5/6$. Here $I_{[0]} = [0, 2/3]$, $I_{[1]} = [1/3, 1]$.

We note that the regularity of the solution h established by Proposition 10 can be reduced to that of $h^{[0]}, h^{[1]}$. In fact, if we know $h^{[0]} \in C^\infty([0, T] \times [0, 2/3])$, then

$h(t,x) = h^{[0]}(t,x)/\omega(x)$ is smooth on $0 \leq x < 2/3$, and the smoothness of $h^{[1]}$ implies that of $h(t,x) = h^{[1]}/(1-\omega(x))$ on $1/3 < x \leq 1$.

But the regularity of the solution of the simultaneous system (21) can be proved by Kato's theory developed in [12], as in Appendix C of [21]. Namely, we consider in the space

$$\begin{aligned}\hat{\mathfrak{H}} &= \mathfrak{H}_{[0]} \times \mathfrak{H}_{[1]} \times \mathbb{R} \\ &= \mathfrak{X}^1_{[0]0} \times \mathfrak{X}_{[0]} \times \mathfrak{X}^1_{[1]0} \times \mathfrak{X}_{[1]} \times \mathbb{R}\end{aligned}$$

the family of generators

$$\begin{aligned}D(\hat{\mathfrak{A}}(t)) &= \hat{\mathfrak{G}} = \mathfrak{G}_{[0]} \times \mathfrak{G}_{[1]} \times \mathbb{R} \\ &= \mathfrak{X}^2_{[0](0)} \times \mathfrak{X}^1_{[0]0} \times \mathfrak{X}^2_{[1](0)} \times \mathfrak{X}^1_{[1]0} \times \mathbb{R}, \\ \hat{\mathfrak{A}}(t) &= \mathfrak{A}_{[0]}(t) \otimes \mathfrak{A}_{[1]}(t) \otimes 0 + B(t), \\ B(t) &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(c_1\check{D}+c_0) & 0 & -g^{[0]} \\ 0 & 0 & 0 & 0 & 0 \\ c_1\check{D}+c_0 & 0 & 0 & 0 & -g^{[1]} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\end{aligned}$$

where

$$\mathfrak{A}_{[\mu]}(t) = \begin{pmatrix} 0 & -1 \\ \mathcal{A}_{[\mu]} & 0 \end{pmatrix}.$$

Here we set

$$\begin{aligned}
\mathfrak{X}_{[0]} &= \left\{ y \;\middle|\; \|y\|_{\mathfrak{X}_{[0]}} := \left(\int_0^{2/3} y(x)^2 x^{3/2}\, dx \right)^{1/2} < \infty \right\}, \\
\mathfrak{X}^1_{[0]} &= \left\{ y \in \mathfrak{X}_{[0]} \;\middle|\; \dot{D}_{[0]} y = \sqrt{x}\frac{dy}{dx} \in \mathfrak{X}_{[0]} \right\}, \\
\mathfrak{X}^1_{[0]0} &= \{ y \in \mathfrak{X}^1_{[0]} \mid y|_{x=2/3} = 0 \}, \\
\mathfrak{X}^2_{[0]} &= \{ y \in \mathfrak{X}^1_{[0]} \mid \triangle_{[0]} y \in \mathfrak{X}_{[0]} \}, \\
\mathfrak{X}^2_{[0](0)} &= \mathfrak{X}^2_{[0]} \cap \mathfrak{X}^1_{[0]0}; \\
\mathfrak{X}_{[1]} &= \left\{ y \;\middle|\; \|y\|_{\mathfrak{X}_{[1]}} := \left(\int_{1/3}^{1} y(x)^2 (1-x)^{N/2-1}\, dx \right)^{1/2} < \infty \right\}, \\
\mathfrak{X}^1_{[1]} &= \left\{ y \in \mathfrak{X}_{[1]} \;\middle|\; \dot{D}_{[1]} y = -\sqrt{1-x}\frac{dy}{dx} \in \mathfrak{X}_{[1]} \right\}, \\
\mathfrak{X}^1_{[1]0} &= \{ y \in \mathfrak{X}^1_{[1]} \mid y|_{x=1/3} = 0 \}, \\
\mathfrak{X}^2_{[1]} &= \{ y \in \mathfrak{X}^1_{[1]} \mid \triangle_{[1]} y \in \mathfrak{X}_{[1]} \}, \\
\mathfrak{X}^2_{[1](0)} &= \mathfrak{X}^2_{[1]} \cap \mathfrak{X}^1_{[1]0}.
\end{aligned}$$

REMARK. 1) It may be difficult to verify that, given a solution (h_0, h_1) of the system (21) such that $h_\mu \in \bigcap_{k=0,1,2} C^{2-k}([0,T], \mathfrak{X}^k_{[\mu]})$, the function h which should be defined by

$$h(t,x) = \begin{cases} h_0(x) & (0 \le x \le 1/3), \\ h_0(x) + h_1(x) & (1/3 < x < 2/3), \\ h_1(x) & (2/3 \le x \le 1) \end{cases}$$

belongs to $C([0,T], \mathfrak{X}^2)$. Therefore we first established the existence of the solution h by Proposition 10. Then, by the uniqueness, we can claim that $h^{[\mu]} = h_\mu$, the solutions of (21).

2) We used

$$\|y\|_{[0]} = \left(\int_0^1 y(x)^2 x^{3/2}\,dx\right)^{1/2}$$

in the definition of the gradings on $\mathfrak{E}_{[0]}$. But $\|y\|_{\mathfrak{X}_{[0]}} = \|y\|_{[0]}$ for $y = h^{[0]}$, since $\mathrm{supp}[h^{[0]}] \subset [0, 2/3]$. So, we can consider $h(t,\,\cdot\,)^{[\mu]} \in \mathfrak{X}^2_{[\mu](0)}$ for the solution h established in Proposition 10.

Then $B(t) \in C([0,T], \mathrm{B}(\hat{\mathfrak{H}}))$ is a smooth bounded perturbation from the stable family $(\mathfrak{A}_{[0]}(t) \otimes \mathfrak{A}_{[1]}(t) \otimes 0)_t$. Hence $(\hat{\mathfrak{A}}(t))_t$ is stable.

In order to consider 'smoothness', 'ellipticity' and compatibility conditions, we introduce the scales of Hilbert spaces

$$\begin{aligned} \hat{\mathfrak{H}}_j &= \mathfrak{X}^{j+1}_{[0](0)} \times \mathfrak{X}^j_{[0]} \times \mathfrak{X}^{j+1}_{[1](0)} \times \mathfrak{X}^j_{[1]} \times \mathbb{R}, \\ \hat{\mathfrak{G}}_j &= \hat{\mathfrak{G}} \cap \hat{\mathfrak{H}}_j = \mathfrak{X}^{j+1}_{[0](0)} \times \mathfrak{X}^j_{[0](0)} \times \mathfrak{X}^{j+1}_{[1](0)} \times \mathfrak{X}^j_{[1](0)} \times \mathbb{R}, \end{aligned}$$

as in Appendix C of [21], where

$$\begin{aligned} \mathfrak{X}^{2m+1}_{[\mu]} &= \{y \in \mathfrak{X}^{2m}_{[\mu]} \mid \dot{D}_{[\mu]} \triangle^m_{[\mu]} y \in \mathfrak{X}_{[\mu]}\}, \\ \mathfrak{X}^{2m+2}_{[\mu]} &= \{y \in \mathfrak{X}^{2m+1}_{[\mu]} \mid \triangle^{m+1}_{[\mu]} y \in \mathfrak{X}_{[\mu]}\}, \\ \mathfrak{X}^j_{[\mu](0)} &= \mathfrak{X}^j_{[\mu]} \cap \mathfrak{X}^1_{[\mu]0}. \end{aligned}$$

The definition of $\|\cdot\|_{\mathfrak{X}^j_{[\mu]}}$ follows that of $\|\cdot\|_j$ in [21], that is:

$$\|y\|_{\mathfrak{X}^j_{[\mu]}} = \left(\sum_{l \le j} (\langle y \rangle_{[\mu]l})^2\right)^{1/2},$$

$$\langle y \rangle_{[\mu]l} = \begin{cases} \|\triangle^m_{[\mu]} y\|_{\mathfrak{X}_{[\mu]}} & \text{as} \quad l = 2m, \\ \|\dot{D}_{[\mu]} \triangle^m_{[\mu]} y\|_{\mathfrak{X}_{[\mu]}} & \text{as} \quad l = 2m+1. \end{cases}$$

In order check the 'smoothness', we note that $c_1 = c_0 = 0$ for $0 \leq x \leq 1/3$ or $2/3 \leq x \leq 1$. This implies that

$$\|(c_1\check{D} + c_0)y^{[1]}\|_{\mathfrak{X}^j_{[0]}} \leq C\|(c_1\check{D} + c_0)y^{[1]}\|_{\mathfrak{X}^j_{[1]}} \leq C'\|y^{[1]}\|_{\mathfrak{X}^{j+1}_{[1]}},$$
$$\|(c_1\check{D} + c_0)y^{[0]}\|_{\mathfrak{X}^j_{[1]}} \leq C\|(c_1\check{D} + c_0)y^{[0]}\|_{\mathfrak{X}^j_{[0]}} \leq C'\|y^{[0]}\|_{\mathfrak{X}^{j+1}_{[0]}}.$$

(See [21, Proposition 6].) Here we have used the following

Proposition 11. *If the support of $y \in C^\infty(0, 1)$ is included in $[1/3, 2/3]$, then*

$$\|y\|_{\mathfrak{X}^j_{[\mu]}} \leq C\|y\|_{\mathfrak{X}^j_{[1-\mu]}},$$

where $\mu = 0, 1$.

A proof can be found in Appendix B.

Then, using this observation, we can reduce the 'ellipticity' of $\hat{\mathfrak{A}}(t)$ to that of $\mathcal{A}_{[\mu]}(t)$, $\mu = 0, 1$.

The compatibility conditions are guaranteed as follows.

We are considering the Cauchy problem

$$\frac{du}{dt} + \hat{\mathfrak{A}}(t)u = 0, \qquad u|_{t=0} = \phi_0,$$

where

$$\hat{\mathfrak{A}}(t) = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ \mathcal{A}_{[0]} & 0 & -\mathcal{C} & 0 & -g^{[0]} \\ 0 & 0 & 0 & -1 & 0 \\ \mathcal{C} & 0 & \mathcal{A}_{[1]} & 0 & -g^{[1]} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\mathcal{C} := c_1\check{D} + c_0,$$
$$\phi_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

As in [11, Section 2], we consider

$$S^0 = I,$$
$$S^{j+1}\phi = -\sum_{k=0}^{j} \binom{j}{k} \left(\frac{d}{dt}\right)^{j-k} \hat{\mathfrak{A}}(0)S^k\phi,$$

$$D_0 = \hat{\mathfrak{H}} = \mathfrak{X}^1_{[0]0} \times \mathfrak{X}_{[0]} \times \mathfrak{X}^1_{[1]0} \times \mathfrak{X}_{[1]} \times \mathbb{R},$$
$$D_{j+1} = \{\phi \in D_j \mid S^k\phi \in \hat{\mathfrak{G}}_{j+1-k},\ 0 \le k \le j\}.$$

We should show that $\phi_0 \in D_n$ for any n. But $g^{[0]}, g^{[1]}$ can be considered as functions in $C^\infty([0, T] \times [0, 1])$ such that, for all positive integer l, $\partial_t^l g^{[0]}(0, x) = 0$ for $2/3 \le x \le 1$ and $\partial_t^l g^{[1]}(0, x) = 0$ for $0 \le x \le 1/3$. We denote

$$\phi_k := S^k\phi_0 = \begin{pmatrix} \phi^k_{[0]0} \\ \phi^k_{[0]1} \\ \phi^k_{[1]0} \\ \phi^k_{[1]1} \\ 0 \end{pmatrix}.$$

Then it is easy to verify by induction that, for $k \ge 1$, the extension $\tilde{\phi}_k = \big(\tilde{\phi}^k_{[0]0}, \tilde{\phi}^k_{[0]1}, \tilde{\phi}^k_{[1]0}, \tilde{\phi}^k_{[1]1}, 0\big)^T$ of ϕ_k defined by

$$\tilde{\phi}^k_{[0]0}(x) = \begin{cases} \phi^k_{[0]0}(x) & (0 \le x \le 2/3), \\ 0 & (2/3 < x \le 1), \end{cases}$$
$$\tilde{\phi}^k_{[0]1}(x) = \begin{cases} \phi^k_{[0]1}(x) & (0 \le x \le 2/3), \\ 0 & (2/3 < x \le 1), \end{cases}$$
$$\tilde{\phi}^k_{[1]0}(x) = \begin{cases} 0 & (0 \le x < 1/3), \\ \phi^k_{[1]0}(x) & (1/3 \le x \le 1), \end{cases}$$
$$\tilde{\phi}^k_{[1]1}(x) = \begin{cases} 0 & (0 \le x < 1/3), \\ \phi^k_{[1]1}(x) & (1/3 \le x \le 1) \end{cases}$$

belongs to $C^\infty([0, 1]; \mathbb{R}^5)$. In other words, the components of ϕ_k satisfy the boundary conditions at $x = 1/3$ and $x = 2/3$ and $\phi_k = S^k\phi_0$ remains in $\hat{\mathfrak{G}}_{k+1}$. It implies that $\phi_0 \in D_n$ for all n.

Summing up, we can claim that $h^{[0]} \in C^\infty([0, T] \times [0, 2/3])$ and $h^{[1]} \in C^\infty([0, T] \times [1/3, 1])$ provided that $g \in C^\infty([0, T] \times [0, 1])$.

Finally, we must deduce the tame estimate of $(w, g) \mapsto h$. We are going to show that

$$\|h\|^{\langle T\rangle}_{n+2} \le C(1 + \|g\|^{\langle T\rangle}_{n+1} + |w|^{\langle T\rangle}_{n+7}).$$

Here

$$\|y\|^{\langle T\rangle}_n := \left(\sum_{j+k \le n,\, \mu=0,1} \int_0^T \|\partial_t^j y^{[\mu]}\|_{\mathfrak{X}^k_{[\mu]}}\, dt \right)^{1/2},$$
$$|y|^{\langle T\rangle}_n := \max_{j+k \le n, \mu=0,1} \|\partial_t^j \dot{D}^k_{[\mu]} y^{[\mu]}\|_{L^\infty([0,T]\times[0,1])}.$$

Let us follow the discussion of [21, Section 5.4]. To do so, we should reconsider the discussion about the single equation, say, we consider a solution H of the boundary value problem

$$\frac{\partial^2 H}{\partial t^2} + \mathcal{A}(\vec{b})H = G(t,x), \quad H|_{x=1} = 0$$

on $0 \le t \le T$. Here $\vec{b}$ stands for the vector (b_0, b_1, b_2). The energy estimate claimed in Proposition 9 should read

$$\|\partial_t H\| + \|H\|_1 \le C\left(\|\partial_t H|_{t=0}\| + \|H|_{t=0}\|_1 + \int_0^t \|G(t')\|\, dt'\right).$$

Even if we consider the $H = h$ which satisfy the initial condition $h|_{t=0} = \partial_t h|_{t=0} = 0$, the higher derivatives $\partial_t^{n+2} h$ may not vanish at $t = 0$. Therefore the estimate of $\|\partial_t^{n+1} h\|_1$ in the proof of [21, Proposition 10] should be replaced by

$$\begin{aligned}\|\partial_t^{n+1} h\|_1 \le C\Bigg(&\|\partial_t^{n+2} h|_{t=0}\| + \|\partial_t^{n+1} h|_{t=0}\|_1 \\ &+ \int_0^t \|\partial_t^{n+1} g\|\, dt' + \int_0^t \|[\partial_t^{n+1}, \mathcal{A}]h\|\, dt'\Bigg).\end{aligned}$$

We claim the estimate

$$\|\partial_t^{n+2} h|_{t=0}\| + \|\partial_t^{n+1} h|_{t=0}\|_1 \le C(1 + W_n(g) + |\vec{b}|_{n+1}^{\langle 0\rangle}), \tag{22}$$

provided that $W_0(g)$, $|\vec{b}|_4^{\langle 0\rangle} \le M_0$. Here

$$W_n(g) := \sum_{j+k\le n} \|\partial_t^j g|_{t=0}\|_k$$

and

$$|y|_n^{\langle 0\rangle} := \max_{j+k\le n} \|\partial_t^j \dot{D}^k y|_{t=0}\|_{L^\infty([0,1])}.$$

To prove (22) it is sufficient to verify the following estimate by induction on n: for all $k \in \mathbb{N}$,

$$\|\partial_t^{n+2} h|_{t=0}\|_k \le C(|\vec{b}|_{n+k+1}^{\langle 0\rangle} W_0(g) + |\vec{b}|_{k+3}^{\langle 0\rangle} W_{n-2}(g) + W_{n+k}(g)).$$

Since the proof of the above inequality by induction on n using the estimate

$$\|\mathcal{A}(\vec{b})y\|_k \le C(\|y\|_{k+2} + |\vec{b}|_{k+3}\|y\|)$$

applied to the relation

$$\partial_t^{n+2} h = -\sum_{j=0}^{n} \binom{n}{j} \mathcal{A}(\partial_t^{n-j}\vec{b})\partial_t^j h + \partial_t^n g$$

is straightforward, we omit it.

Moreover we note that the inequality in the statement of [21, Lemma 4] can be replaced by the stronger one:

$$\|h\|_{n+2}^{\langle t\rangle} \leq C\left(1+\int_0^t \|g\|_{n+1}^{\langle t'\rangle}\,dt' + W_n(g) + \|g\|_n^{\langle T\rangle} + |\vec{b}|_{n+3}^{\langle T\rangle}\right),$$

for $0 \leq t \leq T$, where

$$\|y\|_n^{\langle\tau\rangle} := \left(\sum_{j+k\leq n}\int_0^\tau \|\partial_t^j y\|_k^2\,dt\right)^{1/2},$$

$$|y|_n^{\langle\tau\rangle} := \max_{j+k\leq n}\|\partial_t^j \dot{D}^k y\|_{L^\infty([0,\tau]\times[0,1])}.$$

This can be verified easily by following the discussion in [21, Section 5.4]. Let us omit the detail.

Let us go back to the simultaneous system of equations. Applying the above discussion on a single equation, we have

$$\|h^{[0]}\|_{[0]n+2}^{\langle t\rangle} \leq C\left(1+\int_0^t \|h^{[1]}\|_{[1]n+2}^{\langle t'\rangle}\,dt' + W_n(g) + \|g\|_{n+1}^{\langle T\rangle} + |\vec{b}|_{n+3}^{\langle T\rangle}\right),$$
$$\|h^{[1]}\|_{[1]n+2}^{\langle t\rangle} \leq C\left(1+\int_0^t \|h^{[0]}\|_{[0]n+2}^{\langle t'\rangle}\,dt' + W_n(g) + \|g\|_{n+1}^{\langle T\rangle} + |\vec{b}|_{n+3}^{\langle T\rangle}\right),$$

for $0 \leq t \leq T$, since

$$\|(c_1\check{D} + c_0)h^{[\mu]}\|_{[1-\mu]k} \leq C(1 + \|h^{[\mu]}\|_{[\mu]k+1} + |\vec{b}|_{k+3}^{\langle T\rangle})$$

for $\mu = 0, 1$. Here $\|\cdot\|_{[\mu]k}$ stands for $\|\cdot\|_{\mathfrak{X}^k_{[\mu]}}$. Applying the Gronwall lemma to the quantity

$$U(t) := \|h^{[0]}\|_{[0]n+2}^{\langle t\rangle} + \|h^{[1]}\|_{[1]n+2}^{\langle t\rangle},$$

we get

$$U(t) \leq C(1 + W_n(g) + \|g\|_{n+1}^{\langle T\rangle} + |\vec{b}|_{n+3}^{\langle T\rangle}).$$

This completes the proof, since $W_n(g) \leq C\|g\|_{n+1}^{\langle T\rangle}$ by Sobolev's imbedding.

5. Cauchy problems

We have discussed about the justification of linearized approximations by time-periodic solutions. In this section we want to give a brief mention on the Cauchy problems associated with the equation (6) or (12). We consider the problem (CP):

$$\frac{\partial^2 y}{\partial t^2} + \left(1 + G_{\mathrm{I}}\left(y, r\frac{\partial y}{\partial r}\right)\right)\mathcal{L}y + G_{\mathrm{II}}\left(r, y, r\frac{\partial y}{\partial r}\right) = 0,$$
$$y|_{t=0} = \psi_0(r), \quad \frac{\partial y}{\partial t}\bigg|_{t=0} = \psi_1(r),$$

where the initial data ψ_0, ψ_1 are given functions. We claim

Theorem 2. *Assume that* $6/5 < \gamma \leq 2$ $(\Leftrightarrow 4 \leq N < 12)$ *and that* $\gamma/(\gamma - 1)$ $(= N/2)$ *is an integer, that is,* γ *is either* 2, 3/2, 4/3 *or* 5/4. *Then for any given* $T > 0$ *there exist a sufficiently small positive number* δ *and a sufficiently large integer* $\mathfrak{r}$ *such that if* $\psi_0, \psi_1 \in C^\infty([0, R])$ *satisfy*

$$\max_{j\leq 2(2\mathfrak{r}+1)}\left\{\left\|\left(\frac{d}{dr}\right)^j\psi_0\right\|_{L^\infty(0,R)}, \left\|\left(\frac{d}{dr}\right)^j\psi_1\right\|_{L^\infty(0,R)}\right\} \leq \delta,$$

then there exists a unique solution $y(t, r)$ *of* (CP) *in* $C^\infty([0, T] \times [0, R])$.

A proof of this theorem can be done as follows.

Let us take the function

$$y_1^*(t, r) = \psi_0(r) + t\psi_1(r),$$

which satisfy the initial conditions. Then we should find a solution w introduced by

$$y = y_1^* + w,$$

which should obey the initial conditions

$$w|_{t=0} = \frac{\partial w}{\partial t}\bigg|_{t=0} = 0.$$

The equation which w should satisfies is same as (13), in which the time-periodic function

$$\varepsilon y_1 = \sin(\sqrt{\lambda}t + \theta)\Phi(r)$$

is replaced by

$$y_1^* = \psi_0(r) + t\psi_1(r),$$

and $F_I + F_{II}$ should be replaced by

$$(1 + G_{\mathrm{I}}(y_1^* + w, v_1^* + \Omega))\mathcal{L}y_1^* + G_{\mathrm{II}}(r, y_1^* + w, v_1^* + \Omega).$$

Of course we take $\varepsilon = 1$. Then the mapping $\mathfrak{P}(w)$ and the derivative $D\mathfrak{P}(w)h$ are defined in the same forms as (14) and as (15). Proposition 5 holds valid, since the concrete structure of the function y_1 or y_1^* is not used in the proof; It is sufficient that εy_1 or y_1^* is a small smooth function. Hence Proposition 6 holds valid, when εy_1 is replaced by y_1^*.

Then the proof of Theorem 1 given in Section 4 can be repeated word for word in the present situation. Note that

$$c = -\left(1 + G_{\mathrm{I}}\left(y_1^*, r\frac{\partial y_1^*}{\partial r}\right)\right)\mathcal{L}y_1^* - G_{\mathrm{II}}\left(r, y_1^*, r\frac{\partial y_1^*}{\partial r}\right)$$

and that $\|c\|_n^{(\infty)} \leq C(\|\psi_0\|_{n+1}^{(\infty)} + \|\psi_1\|_{n+1}^{(\infty)})$, provided that $0 \leq t \leq T$. In fact, if we follow the discussion of [7, III.1.], we can show that it is enough to take $\mathfrak{r}$ such that $2\mathfrak{r} > 3/2 + \max\{5, N\}/4$. (But this $\mathfrak{r}$ may not be the best possible.) Anyway this completes the proof of Theorem 2.

REMARK. The corresponding initial data in the Eulerian variables are given by

$$\rho|_{t=0}(r) = \bar{\rho}(\bar{r})\left((1+\psi_0(\bar{r}))^2\left(1+\psi_0(\bar{r}) + \bar{r}\frac{d\psi_0(\bar{r})}{d\bar{r}}\right)\right)^{-1},$$
$$u|_{t=0}(r) = \bar{r}\psi_1(\bar{r})$$

implicitly by $\bar{r} = \bar{r}(m(r))$. Here $m \mapsto \bar{r}(m)$ is the inverse function of

$$\bar{r} \mapsto m = m(\bar{r}) = 4\pi\int_0^{\bar{r}} \bar{\rho}(r)r^2\,dr$$

and $r \mapsto m(r)$ is the inverse function of $m \mapsto r = \bar{r}(m)(1 + \psi_0(\bar{r}(m)))$.

6. Concluding remark

In order that the equilibrium satisfy that $\bar{\rho}^{\gamma-1}$ is analytic at the free boundary $r = R$ and that the eigenfunction y_1 is analytic in r at $r = R$, we have assumed that N is an even integer. But $\gamma = 5/3 (N = 5)$ for mono-atomic gas, and $\gamma = 7/5 (N = 7)$ for the air. Therefore it is desired that the result will be extended to the case when N is an odd integer at least. Moreover for the case when N is not an integer, we might try quite other approach. It seems that these are interesting open problems in view of physical applications.

Appendix A.

Let us consider a solution $\rho = \rho(r)$, $r_0 \leq r < R$, of the Lane–Emden equation

$$-\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = 4\pi g_0\rho, \quad P = A\rho^\gamma.$$

Let $[r_0, R)$ be a right maximal interval of existence of $\rho > 0$, and we assume that $R < +\infty$, $d\rho/dr|_{r=r_0} < 0$. Then there is a positive constant C such that

$$\rho = C(R-r)^{1/(\gamma-1)}\left(1 + \left[\frac{R-r}{R}, C'\left(\frac{R-r}{R}\right)^{\gamma/(\gamma-1)}\right]_1\right)$$

with

$$C' = KR^{\gamma/(\gamma-1)}C^{2-\gamma}, \quad K = \frac{4\pi g_0(\gamma-1)}{A\gamma}.$$

Proof. The variable

$$U := \rho^{\gamma-1}$$

satisfies

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} + KU^m = 0,$$

where $m = 1/(\gamma-1)$. Then

$$v := -\frac{r}{U}\frac{dU}{dr}, \quad w := Kr^2U^{m-1}$$

satisfies the plane autonomous system

$$r\frac{dv}{dr} = -v + v^2 + w,$$
$$r\frac{dw}{dr} = w(2-(m-1)v).$$

The interval $[r_0, R)$ is right maximal. We assumed that $v(r_0) > 0$. We claim that there is $r_1 \in [r_0, R)$ such that $v(r_1) > 1$. Otherwise $0 < v \leq 1$ and $|(r/w)\,dw/dr| \leq m+1$ for $r_0 \leq r < R$. Then it should be $R = +\infty$, a contradiction to the assumption. Hence we can assume that $v(r_0) > 1$. Then $r\,dv/dr \geq v(v-1)$ implies $v \geq 1+\delta$, $dv/dr > 0$ and $r\,dw/dr \leq 2w$. So, it should be that $v(r) \to +\infty$ as $r \to R$, since $R < \infty$. We see $w \leq B$.

Now we introduce the variables

$$x_1 := \frac{1}{v}, \quad x_2 := \frac{w}{v^2},$$
$$t := \exp\left(-\int_{r_0}^{r}\frac{v(r')\,dr'}{r'}\right).$$

Then $(x_1, x_2) \to (0, 0)$, $t \to 0$ as $r \to R$ and $(x_1(t), x_2(t))$, $0 < t \leq 1$, satisfies

$$t\frac{dx_1}{dt} = (1 - x_1 + x_2)x_1,$$
$$t\frac{dx_2}{dt} = (m + 1 - 4x_1 + 2x_2)x_2.$$

As well-known, this Briot–Bouquet system can be reduced to

$$t\frac{dz_1}{dt} = z_1,$$
$$t\frac{dz_2}{dt} = (m+1)z_2$$

by a transformation of the form

$$x_1 = z_1(1 + P_1(z_1, z_2)),$$
$$x_2 = z_2(1 + P_2(z_1, z_2)).$$

Here

$$P_j(z_1, z_2) = [z_1, z_2]_1$$

for $j = 1, 2$. Therefore there are positive constants C_1, C_2 such that

$$x_1 = C_1 t(1 + P_1(C_1 t, C_2 t^{m+1})),$$
$$x_2 = C_2 t^{m+1}(1 + P_2(C_1 t, C_2 t^{m+1})).$$

Since $dr/r = -x_1\, dt/t$, we see

$$\log\frac{R}{r} = \frac{R-r}{R}\left(1 + \left[\frac{R-r}{R}\right]_1\right)$$
$$= C_1 t(1 + [C_1 t, C_2 t^{m+1}]_1),$$

from which

$$C_1 t = \frac{R-r}{R}\left(1 + \left[\frac{R-r}{R}, C'\left(\frac{R-r}{R}\right)^{m+1}\right]_1\right)$$

and

$$x_1 = \frac{R-r}{R}\left(1 + \left[\frac{R-r}{R}, C'\left(\frac{R-r}{R}\right)^{m+1}\right]_1\right),$$

where $C' = C_2/C_1^{m+1}$. Integrating $dU/U = -dr/rx_1$, we have

$$U = C_3\frac{R-r}{R}\left(1 + \left[\frac{R-r}{R}, C'\left(\frac{R-r}{R}\right)^{m+1}\right]_1\right).$$

It is easy to see $C' = KR^2C_3^{m-1}$, and we get the required result. $\square$

Appendix B.

Let us prove Proposition 7, that is,

$$\|\triangle_{[0]}^m y\|_{L^\infty} \leq C\sum_{k\leq m}\|\triangle_{[1]}^k y\|_{L^\infty},$$

provided that $\mathrm{supp}[y] \subset [1/6, 5/6]$.

Note that

$$\triangle_{[0]} = \alpha\triangle_{[1]} + \beta\check{D}_{[1]},$$

where $\check{D}_{[1]} = z\, d/dz = -(1-x)\, d/dx$ and

$$\alpha = \frac{x}{1-x}, \quad \beta = -\frac{1}{1-x}\left(\frac{x}{1-x}\frac{N}{2} + \frac{5}{2}\right)$$

are smooth function on $(0, 1)$. Therefor our task is to estimate

$$\|(\alpha\triangle_{[1]} + \beta\check{D}_{[1]})^m y\|_{L^\infty}.$$

On the other hand, it is easy to verify that there are $\gamma_{\epsilon k}^{(m)} \in C^\infty(0, 1)$ such that

$$(\alpha\triangle_{[1]} + \beta\check{D}_{[1]})^m = \sum_{k\le m}(\gamma_{1k}^{(m)}\check{D}_{[1]}\triangle_{[1]}^k + \gamma_{0k}^{(m)}\triangle_{[1]}^k)$$

with $\gamma_{1m}^{(m)} = 0$. Note that

$$\|\check{D}_{[1]}\triangle_{[1]}^k y\|_{L^\infty} \le \|D\triangle_{[1]}^k\|_{L^\infty} \le \frac{2}{N}\|\triangle_{[1]}^{k+1} y\|_{L^\infty}.$$

(See [21, Proposition 3]). This completes the proof.

Let us prove Proposition 11, that is,

$$\|y\|_{\mathfrak{X}_{[0]}^j} \le C\|y\|_{\mathfrak{X}_{[1]}^j},$$

provided that $\mathrm{supp}[y] \subset [1/3, 2/3]$.

It is clear that

$$\|y\|_{\mathfrak{X}_{[0]}} \le C\|y\|_{\mathfrak{X}_{[1]}},$$

since $x^{3/2} \le 3^{N/2-1}(1-x)^{N/2-1}$ for $1/3 \le x \le 2/3$. Let us estimate $\|\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}}$ and $\|\dot{D}_{[0]}\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}}$, where $\dot{D}_{[0]} = \sqrt{x}d/dx$. As in the above discussion we note that

$$\triangle_{[0]} = \alpha\triangle_{[1]} + \beta\check{D}_{[1]},$$

where $\check{D}_{[1]} = z\, d/dz = -(1-x)\, d/dx$ and

$$\alpha = \frac{x}{1-x}, \quad \beta = -\frac{1}{1-x}\left(\frac{x}{1-x}\frac{N}{2} + \frac{5}{2}\right)$$

are smooth function on (0, 1). Therefor our task is to estimate

$$\|\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}} \leq C\|\triangle_{[0]}^m y\|_{\mathfrak{X}_{[1]}} = C\|(\alpha\triangle_{[1]} + \beta\check{D}_{[1]})^m y\|_{\mathfrak{X}_{[1]}}$$

and

$$\|\dot{D}_{[0]}\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}} \leq C\|\check{D}_{[1]}\triangle_{[0]}^m y\|_{\mathfrak{X}_{[1]}} = C\|\check{D}_{[1]}(\alpha\triangle_{[1]} + \beta\check{D}_{[1]})^m y\|_{\mathfrak{X}_{[1]}}.$$

On the other hand, it is easy to verify that there are $\gamma_{\epsilon k}^{(m)}, \gamma_{\epsilon k}^{(m)\sharp} \in C^\infty(0,1)$ such that

$$(\alpha\triangle + \beta\check{D})^m = \sum_{k\leq m}(\gamma_{1k}^{(m)}\check{D}\triangle^k + \gamma_{0k}^{(m)}\triangle^k),$$

$$\check{D}(\alpha\triangle + \beta\check{D})^m = \sum_{k\leq m}(\gamma_{1k}^{(m)\sharp}\check{D}\triangle^k + \gamma_{0k}^{(m)\sharp}\triangle^k)$$

with $\gamma_{1m}^{(m)} = 0$. Here $\triangle$, $\check{D}$ stand for $\triangle_{[1]}$, $\check{D}_{[1]}$. Hence we have

$$\|\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}} \leq C\|y\|_{\mathfrak{X}_{[1]}^{2m}},$$

$$\|\dot{D}_{[0]}\triangle_{[0]}^m y\|_{\mathfrak{X}_{[0]}} \leq C\|y\|_{\mathfrak{X}_{[1]}^{2m+1}}.$$

This completes the proof.

ACKNOWLEDGMENT. The author would like to express his sincere and deep thanks to the anonymous referee for his/her careful reading of the original manuscript, which contained many careless typos and rude expressions. If this revised manuscript has turned out to be tolerably readable, the author owes it to the kind suggestions by the referee.

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