Stukow, M.
Osaka J. Math.
52 (2015), 495–514

A FINITE PRESENTATION FOR THE HYPERELLIPTIC MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

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(Received May 23, 2013, revised December 25, 2013)

Abstract

We obtain a simple presentation of the hyperelliptic mapping class group $\mathcal{M}^h(N)$ of a nonorientable surface N. As an application we compute the first homology group of $\mathcal{M}^h(N)$ with coefficients in $H_1(N;\mathbb{Z})$.

1. Introduction

Let $N_{g,s}^n$ be a smooth, nonorientable, compact surface of genus g with s boundary components and n punctures. If s and/or n is zero, then we omit it from the notation. If we do not want to emphasise the numbers g, s, n, we simply write N for a surface $N_{g,s}^n$. Recall that N_g is a connected sum of g projective planes, and $N_{g,s}^n$ is obtained from N_g by removing s open disks and specifying the set $\Sigma = \{z_1, \dots, z_n\}$ of n distinguished points in the interior of N_g.

Let $\mathrm{Diff}(N)$ be the group of all diffeomorphisms $h\colon N \to N$ such that h is the identity on each boundary component and $h(\Sigma) = \Sigma$. By $\mathcal{M}(N)$ we denote the quotient group of $\mathrm{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies fix Σ and are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of N.

The mapping class group $\mathcal{M}(S_{g,s}^n)$ of an orientable surface is defined analogously, but we consider only orientation preserving maps. If we include orientation reversing maps, we obtain the so-called *extended mapping class group* $\mathcal{M}^{\pm}(S_{g,s}^n)$.

Suppose that the closed orientable surface S_g is embedded in $\mathbb{R}^3$ as shown in Fig. 1, in such a way that it is invariant under reflections across xy-, yz-, xz-planes. Let $\varrho\colon S_g \to S_g$ be the *hyperelliptic involution*, i.e. the half turn about the y-axis. The *hyperelliptic mapping class group* $\mathcal{M}^h(S_g)$ is defined to be the centraliser of ϱ in $\mathcal{M}(S_g)$. In a similar way we define the *extended hyperelliptic mapping class group* $\mathcal{M}^{h\pm}(S_g)$ to be the centraliser of ϱ in $\mathcal{M}^{\pm}(S_g)$.

2000 Mathematics Subject Classification. Primary 57N05; Secondary 20F38, 57M99.
Supported by MNiSW N201 366436 and NCN 2012/05/B/ST1/02171.

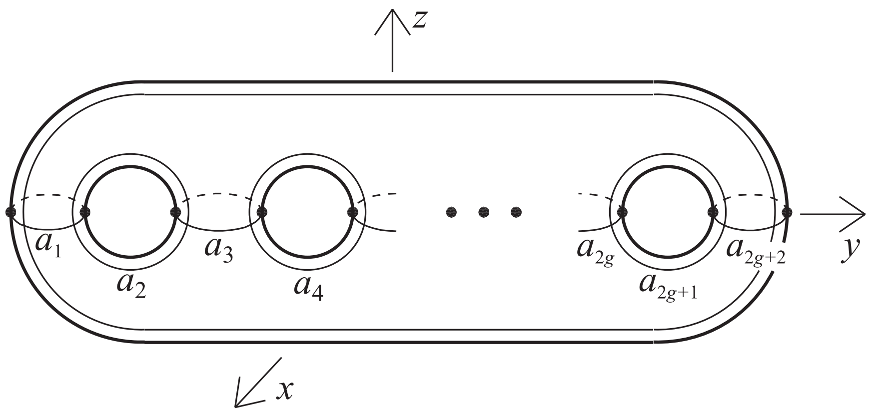

Fig. 1. Surface S_g embedded in $\mathbb{R}^3$.

1.1. Background. The hyperelliptic mapping class group turns out to be a very interesting and important subgroup of the mapping class group. Its algebraic properties have been studied extensively—see [4, 9] and references there. Although $\mathcal{M}^h(S_g)$ is an infinite index subgroup of $\mathcal{M}(S_g)$ for $g \geqslant 3$, it plays surprisingly important role in studying its algebraic properties. For example Wajnryb's simple presentation [18] of the mapping class group $\mathcal{M}(S_g)$ differs from the presentation of the group $\mathcal{M}^h(S_g)$ by adding one generator and a few relations. Another important phenomenon is the fact, that every finite cyclic subgroup of maximal order in $\mathcal{M}(S_g)$ is conjugate to a subgroup of $\mathcal{M}^h(S_g)$ [14].

Homological computations play a prominent role in the theory of mapping class groups. Let us mention that in the case of the hyperelliptic mapping class group, Bödigheimer, Cohen and Peim [5] computed $H^*(\mathcal{M}^h(S_g); \mathbb{K})$ with coefficients in any field $\mathbb{K}$. Kawazumi showed in [9] that if $\mathrm{ch}(\mathbb{K}) \neq 2$ then $H^*(\mathcal{M}^h(S_g); H^1(S_g; \mathbb{K})) = 0$. For the integral coefficients, Tanaka [17] showed that $H_1(\mathcal{M}^h(S_g); H_1(S_g; \mathbb{Z})) \cong \mathbb{Z}_2$. Let us also mention that Morita [11] showed that in the case of the full mapping class group, $H_1(\mathcal{M}(S_g); H_1(S_g, \mathbb{Z})) \cong \mathbb{Z}_{2g-2}$.

1.2. Main results. The purpose of this paper is to extend the notion of the hyperelliptic mapping class group to the nonorientable case. We define this group $\mathcal{M}^h(N)$ in Section 2 and observe that it contains a natural subgroup $\mathcal{M}^{h+}(N)$ of index 2 (Remark 2.3).

Then we obtain simple presentations of these groups (Theorems 4.1 and 4.4). By analogy with the orientable case, these presentations may be thought of as the first approximation of a presentation of the full mapping class group $\mathcal{M}(N)$. In fact, for $g = 3$ the hyperelliptic mapping class group $\mathcal{M}^h(N)$ coincide with the full mapping class group $\mathcal{M}(N)$ (see Corollary 4.3). If $g \geq 4$, then Paris and Szepietowski [12] obtained a simple presentation of $\mathcal{M}(N)$, which can be rewritten (Proposition 3.3 and

Theorem 3.5 of [16]) so that it has the hyperelliptic involution ϱ as one of the generators, and the hyperelliptic relations (Theorem 4.1) appear among defining relations.

As an application of obtained presentations we compute the first homology groups of $\mathcal{M}^h(N)$ and $\mathcal{M}^{h+}(N)$ with coefficients in $H_1(N;\mathbb{Z})$ (Theorems 5.3 and 5.4).

2. Definitions of $\mathcal{M}^h(N_g)$ and $\mathcal{M}^{h+}(N_g)$

Let S_{g-1} be a closed oriented surface of genus $g-1 \geqslant 2$ embedded in $\mathbb{R}^3$ as shown in Fig. 1, in such a way that it is invariant under reflections across xy-, yz-, xz-planes, and let $j\colon S_{g-1} \to S_{g-1}$ be the symmetry defined by $j(x,y,z) = (-x,-y,-z)$. Denote by $C_{\mathcal{M}^{\pm}(S_{g-1})}(j)$ the centraliser of j in $\mathcal{M}^{\pm}(S_{g-1})$. The orbit space $S_{g-1}/\langle j\rangle$ is a nonorientable surface N_g of genus g and it is known (Theorem 1 of [3]) that there is an epimorphism

$$\pi_j\colon C_{\mathcal{M}^{\pm}(S_{g-1})}(j) \to \mathcal{M}(N_g)$$

with kernel $\ker \pi_j = \langle j\rangle$. In particular

$$\mathcal{M}(N_g) \cong C_{\mathcal{M}^{\pm}(S_{g-1})}(j)/\langle j\rangle.$$

Observe that the hyperelliptic involution ϱ is an element of $C_{\mathcal{M}^{\pm}(S_{g-1})}(j)$. Hence the following definition makes sense.

DEFINITION. Define the *hyperelliptic mapping class group* $\mathcal{M}^h(N)$ of a closed nonorientable surface N to be the centraliser of $\pi_j(\varrho)$ in the mapping class group $\mathcal{M}(N)$. We say that $\pi_j(\varrho)$ is the *hyperelliptic involution* of N and by abuse of notation we write ϱ for $\pi_j(\varrho)$.

In order to have a little more straightforward description of ϱ observe, that the orbit space $S_{g-1}/\langle j\rangle$ gives the model of N_g, where N_g is a connected sum of an orientable surface S_r and a projective plane (for g odd) or a Klein bottle (for g even)—see Fig. 2. To be more precise, N_g is the left half of S_{g-1} embedded in $\mathbb{R}^3$ as in Fig. 1 with boundary points identified by the map $(x,y,z) \mapsto (-x,-y,-z)$. Note that $g = 2r+1$ for g odd and $g = 2r+2$ for g even. In such a model, $\varrho\colon N_g \to N_g$ is the map induced by the half turn about the y-axis.

Observe that the set of fixed points of ϱ: $N_g \to N_g$ consists of g points $\{p_1, p_2, \ldots, p_g\}$ and the circle p. Therefore $\mathcal{M}^h(N)$ consists of isotopy classes of maps which must fix the set $\{p_1, p_2, \ldots, p_g\}$ and map the circle p to itself. Moreover, the orbit space $N_g/\langle\varrho\rangle$ is the sphere $S^g_{0,1}$ with one boundary component corresponding to p and g distinguished points corresponding to $\{p_1, p_2, \ldots, p_g\}$. Since elements of $\mathcal{M}^h(N_g)$ may not fix p point–wise, it is more convenient to treat p as the distinguished puncture p_{g+1}, hence we will identify $N_g/\langle\varrho\rangle$ with the sphere $S_0^{g,1}$ with $g+1$ punctures. The notation $S_0^{g,1}$ is meant to indicate that maps of $S_0^{g,1}$ (and their isotopies) could permute the punctures $p_1, \ldots, p_g$, but must fix p_{g+1}.

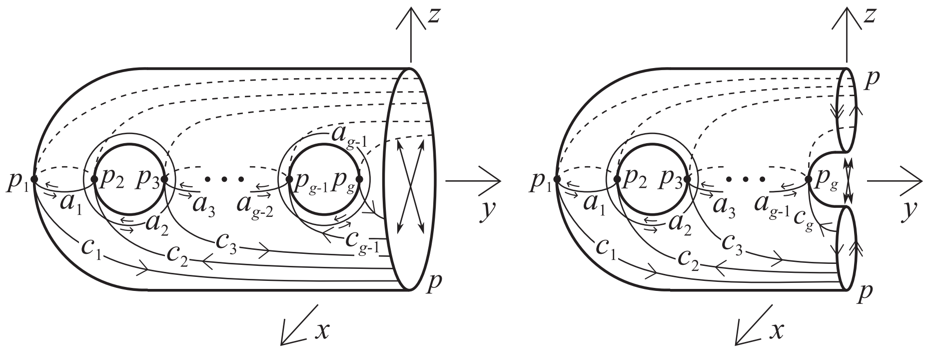

Fig. 2. Nonorientable surface N_g.

The main goal of this section is to prove the following theorem.

Theorem 2.1. *If $g \geqslant 3$ then the projection $N_g \to N_g/\langle \varrho \rangle$ induces an epimorphism*

$$\pi_\varrho \colon \mathcal{M}^h(N_g) \to \mathcal{M}^{\pm}(S_0^{g,1})$$

with $\ker \pi_\varrho = \langle \varrho \rangle$.

Proof. Consider the following diagram

$$\begin{tikzcd}
C_{\mathcal{M}^{\pm}(S_{g-1})}(\langle j, \varrho \rangle) \arrow[r, "\pi_\varrho"] \arrow[d, shift left, "\pi_j"] & C_{\mathcal{M}^{\pm}(S_0^{2g})}(j) \arrow[d, "\pi_j"] \\
\mathcal{M}^h(N_g) \arrow[u, shift left, bend left, "i_j"] \arrow[r, dashed, "\pi_\varrho"] & \mathcal{M}^{\pm}(S_0^{g,1}).
\end{tikzcd}$$

The left vertical map is the restriction of the projection

$$\pi_j \colon C_{\mathcal{M}^{\pm}(S_{g-1})}(j) \to \mathcal{M}(N_g)$$

to the subgroup consisting of elements which centralise ϱ. The nice thing about π_j is that it has a section

$$i_j \colon \mathcal{M}(N_g) \to C_{\mathcal{M}^{\pm}(S_{g-1})}(j).$$

In fact, for any $h \in \mathcal{M}(N_g)$ we can define $i_j(h)$ to be an orientation preserving lift of h.

The upper horizontal map is the restriction of the homomorphism

$$\pi_\varrho \colon \mathcal{M}^{h\pm}(S_{g-1}) \to \mathcal{M}^{\pm}(S_0^{2g})$$

induced by the orbit projection $S_{g-1} \to S_{g-1}/\langle\varrho\rangle$. The fact that this map is a homomorphism was first observed by Birman and Hilden [4]. The kernel of this map is equal to $\langle\varrho\rangle$.

The right vertical map is again the homomorphism induced by the orbit projection $S_0^{2g} \to S_0^{2g}/\langle j\rangle$. However now $j\colon S_0^{2g} \to S_0^{2g}$ is a reflection with a circle of fixed points. The existence of π_j in such a case follows from the work of Zieschang (Proposition 10.3 of [19]).

Hence there is the homomorphism

$$\pi_\varrho\colon \mathcal{M}^h(N_g) \to \mathcal{M}^\pm(S_0^{g,1})$$

defined as the composition

$$\pi_\varrho = \pi_j \circ \pi_\varrho \circ i_j.$$

Moreover,

$$\begin{aligned}\ker \pi_\varrho &= \ker(\pi_j \circ \pi_\varrho \circ i_j) = (\pi_j \circ \pi_\varrho \circ i_j)^{-1}(\mathrm{id})\\ &= i_j^{-1}(\pi_\varrho^{-1}(\pi_j^{-1}(\mathrm{id}))) = i_j^{-1}(\pi_\varrho^{-1}(\langle j\rangle)) = i_j^{-1}(\langle j, \varrho\rangle) = \langle\varrho\rangle. \qquad \square\end{aligned}$$

REMARK 2.2. Theorem 2.1 is not true if $N = N_2$. This corresponds to the fact that the Birman–Hilden theorem does not hold for the closed torus $S = S_1$.

REMARK 2.3. Theorem 2.1 shows that the group $\mathcal{M}^h(N_g)$ contains a very natural subgroup of index 2, namely

$$\mathcal{M}^{h+}(N_g) = \pi_\varrho^{-1}(\mathcal{M}(S_0^{g,1})).$$

Geometrically, the subgroup $\mathcal{M}^{h+}(N_g)$ consists of these elements, which preserve the orientation of the circle p (the circle fixed by ϱ). As we will see later (see Remark 4.6), it seems that the group $\mathcal{M}^{h+}(N)$ corresponds to $\mathcal{M}^h(S)$, whereas $\mathcal{M}^h(N)$ corresponds to $\mathcal{M}^{h\pm}(S)$.

3. Presentations for groups $\mathcal{M}(S_0^{g,1})$ and $\mathcal{M}^\pm(S_0^{g,1})$

Let $w_1, w_2, \ldots, w_g$ be simple arcs connecting punctures $p_1, \ldots, p_{g+1}$ on a sphere S_0^{g+1} as shown in Fig. 3. Recall that to each such arc w_i we can associate the elementary braid σ_i which interchanges punctures p_i and p_{i+1}—see Fig. 3. The following theorem is due to Magnus [10]. It is also proved in Chapter 4 of [2].

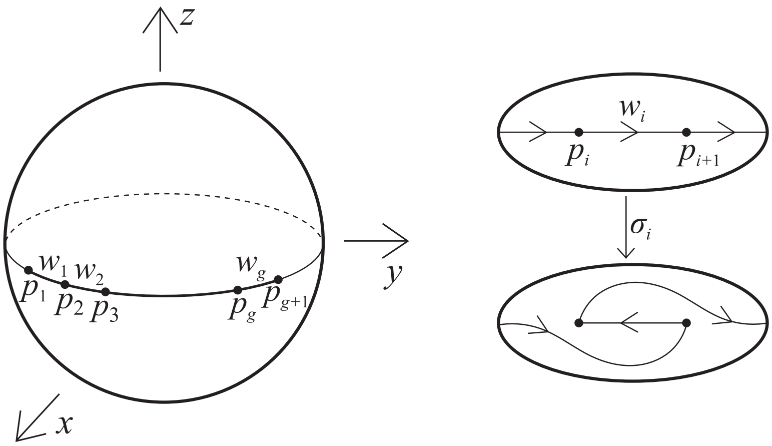

Fig. 3. Sphere S_0^{p+1} and elementary braid σ_i.

Theorem 3.1. *If* $g \geqslant 1$, *then* $\mathcal{M}(S_0^{g+1})$ *has the presentation with generators* $\sigma_1, \dots, \sigma_g$ *and defining relations*:

$$\sigma_k\sigma_j = \sigma_j\sigma_k \quad \textit{for } |k-j| > 1,$$

$$\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1} \quad \textit{for } j = 1, \dots, g-1,$$

$$\sigma_1 \cdots \sigma_{g-1}\sigma_g^2\sigma_{g-1} \cdots \sigma_1 = 1,$$

$$(\sigma_1\sigma_2 \cdots \sigma_g)^{g+1} = 1.$$

In order to avoid unnecessary complications, from now on assume that $g \geqslant 3$. Recall that we denote by $\mathcal{M}(S_0^{g,1})$ the subgroup of $\mathcal{M}(S_0^{g+1})$ consisting of maps which fix p_{g+1}.

Theorem 3.2. *If* $g \geqslant 3$, *then* $\mathcal{M}(S_0^{g,1})$ *has the presentation with generators* $\sigma_1, \dots, \sigma_{g-1}$ *and defining relations*:
(A1) $\sigma_k\sigma_j = \sigma_j\sigma_k$ *for* $|k-j| > 1$ *and* $k, j < g$,
(A2) $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$ *for* $j = 1, 2, \dots, g-2$,
(A3) $(\sigma_1 \cdots \sigma_{g-1})^g = 1$.

Proof. By Lemma 2.2 of [1],

$$\mathcal{M}(S_0^{g,1}) \cong B_g/\langle \Delta^2 \rangle,$$

where $B_g = \mathcal{M}(S_{0,1}^g)$ is the braid group on g strands, and

$$\Delta^2 = (\sigma_1 \cdots \sigma_{g-1})^g$$

is the generator of the center of B_g. Since B_g has the presentation with generators $\sigma_1, \dots, \sigma_{g-1}$ and defining relations (A1), (A2), this completes the proof. $\square$

REMARK 3.3. Theorem 3.2 can be also algebraically deduced from Theorem 3.1. Since $\mathcal{M}(S_0^{g,1})$ is a subgroup of index $g+1$ in $\mathcal{M}(S_0^{g+1})$, for the Schreier transversal we can take

$$(1, \sigma_g, \sigma_g\sigma_{g-1}, \dots, \sigma_g\sigma_{g-1}\cdots\sigma_1).$$

If we now apply Reidemeister–Schreier process, as generators for $\mathcal{M}(S_0^{g,1})$ we get $\sigma_1, \dots, \sigma_{g-1}$ and additionally $\tau_1, \dots, \tau_g$ where

$$\tau_k = \begin{cases} \sigma_g \cdots \sigma_{k+1}\sigma_k^2\sigma_{k+1}^{-1}\cdots\sigma_g^{-1} & \text{for} \quad k = 1, \dots, g-1, \\ \sigma_g^2 & \text{for} \quad k = g. \end{cases}$$

As defining relations we get

$$\begin{aligned}
&\sigma_k\sigma_j = \sigma_j\sigma_k \quad \text{for} \quad |k-j| > 1 \quad \text{and} \quad k, j < g,\\
&\sigma_k\tau_j = \tau_j\sigma_k \quad \text{for} \quad j \neq k, k+1,\\
&\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1} \quad \text{for} \quad j = 1, 2, \dots, g-2,\\
&\sigma_k\tau_{k+1}\sigma_k^{-1} = \tau_{k+1}^{-1}\tau_k\tau_{k+1} \quad \text{for} \quad k = 1, 2, \dots, g-1,\\
&\sigma_k\tau_k\sigma_k^{-1} = \tau_{k+1} \quad \text{for} \quad k = 1, 2, \dots, g-1,\\
&\tau_1\tau_2\cdots\tau_g = 1,\\
&\sigma_{g-1}\cdots\sigma_2\sigma_1\tau_1\sigma_1\sigma_2\cdots\sigma_{g-1} = 1,\\
&(\sigma_{g-1}\sigma_{g-2}\cdots\sigma_1\tau_1)^g = 1.
\end{aligned}$$

If we now remove generators $\tau_1, \dots, \tau_g$ from the above presentation, we obtain the presentation given by Theorem 3.2. The computations are lengthy, but completely straightforward.

Recall that by $\mathcal{M}^{\pm}(S_0^{g,1})$ we denote the extended mapping class group of the sphere $S_0^{g,1}$, that is the extension of degree 2 of $\mathcal{M}(S_0^{g,1})$. Suppose that the sphere $S_0^{g,1}$ is the metric sphere in $\mathbb{R}^3$ with origin $(0,0,0)$ and that punctures $p_1,\dots,p_g$ are contained in the xy-plane. Let $\sigma : S_0^{g,1} \to S_0^{g,1}$ be the map induced by the reflection across the xy-plane. We have the short exact sequence.

$$1 \to \mathcal{M}(S_0^{g,1}) \to \mathcal{M}^{\pm}(S_0^{g,1}) \to \langle\sigma\rangle \to 1.$$

Moreover, $\sigma\sigma_i\sigma^{-1} = \sigma_i^{-1}$ for $i = 1, \dots, g-1$. Therefore Theorem 3.2 implies the following.

Theorem 3.4. *If $g \geqslant 3$, then $\mathcal{M}^{\pm}(S_0^{g,1})$ has the presentation with generators $\sigma_1, \dots, \sigma_{g-1}, \sigma$ and defining relations*:
(B1) $\sigma_k\sigma_j = \sigma_j\sigma_k$ *for* $|k-j| > 1$ *and* $k, j < g$,
(B2) $\sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1}$ *for* $j = 1, 2, \dots, g-2$,
(B3) $(\sigma_1 \cdots \sigma_{g-1})^g = 1$,
(B4) $\sigma^2 = 1$,
(B5) $\sigma\sigma_i\sigma = \sigma_i^{-1}$ *for* $i = 1, 2, \dots, g-1$.

4. Presentations for groups $\mathcal{M}^h(N_g)$ and $\mathcal{M}^{h+}(N_g)$

By Theorem 2.1 there is a short exact sequence.

$$1 \to \langle \varrho \rangle \to \mathcal{M}^h(N_g) \xrightarrow{\pi_\varrho} \mathcal{M}^{\pm}(S_0^{g,1}) \to 1.$$

Moreover, it is known that as lifts of braids $\sigma_1,\dots,\sigma_{g-1} \in \mathcal{M}^{\pm}(S_0^{g,1})$ we can take Dehn twists $t_{a_1}, \dots, t_{a_{g-1}} \in \mathcal{M}^h(N_g)$ about circles $a_1, \dots, a_{g-1}$—cf. Fig. 2 (small arrows in this picture indicate directions of twists). As a lift of σ we take the symmetry s across the xy-plane (the second lift of σ is the symmetry ϱs, that is the symmetry across the yz-plane).

To obtain a presentation for the group $\mathcal{M}^h(N_g)$ we need to lift relations (B1)–(B5) of Theorem 3.4. Each relation of the form

$$w(\sigma_1, \dots, \sigma_{g-1}, \sigma) = 1$$

lifts either to $w(t_{a_1},\dots,t_{a_{g-1}},s) = 1$ or to $w(t_{a_1},\dots,t_{a_{g-1}},s) = \varrho$. In order to determine which of these two cases does occur it is enough to check whether the homeomorphism $w(t_{a_1}, \dots, t_{a_{g-1}}, s)$ changes the orientation of the circle a_1 or not. This can be easily done and as a result we obtain the following theorem.

Theorem 4.1. *If $g \geqslant 3$, then $\mathcal{M}^h(N_g)$ has the presentation with generators $t_{a_1}, \dots, t_{a_{g-1}}, s, \varrho$ and defining relations*:
(C1) $t_{a_k}t_{a_j} = t_{a_j}t_{a_k}$ *for* $|k-j| > 1$ *and* $k, j < g$,
(C2) $t_{a_j}t_{a_{j+1}}t_{a_j} = t_{a_{j+1}}t_{a_j}t_{a_{j+1}}$ *for* $j = 1, 2, \dots, g-2$,
(C3) $(t_{a_1} \cdots t_{a_{g-1}})^g = \begin{cases} 1 & \text{for } g \text{ even,} \\ \varrho & \text{for } g \text{ odd,} \end{cases}$
(C4) $s^2 = 1$,
(C5) $st_{a_j}s = t_{a_j}^{-1}$ *for* $j = 1, 2, \dots, g-1$,
(C6) $\varrho^2 = 1$,
(C7) $\varrho t_{a_j}\varrho = t_{a_j}$ *for* $j = 1, 2, \dots, g-1$,
(C8) $\varrho s \varrho = s$.

Corollary 4.2. *If $g \geqslant 3$, then*

$$H_1(\mathcal{M}^h(N_g)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } g \text{ odd,} \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } g \text{ even.} \end{cases}$$

Proof. Relation (C2) implies that the abelianization of the group $\mathcal{M}^h(N_g)$ is an abelian group generated by t_{a_1}, s, ϱ. Defining relations take form

$$t_{a_1}^{(g-1)g} = \begin{cases} 1 & \text{for } g \text{ even,} \\ \varrho & \text{for } g \text{ odd,} \end{cases}$$
$$s^2 = 1, \quad t_{a_1}^2 = 1, \quad \varrho^2 = 1.$$

Hence $H_1(\mathcal{M}^h(N_g)) = \langle t_{a_1}, s\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for g odd and $H_1(\mathcal{M}^h(N_g)) = \langle t_{a_1}, s, \varrho\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ for g even. $\square$

The main theorem of [7] implies that the group $\mathcal{M}(N_3)$ is generated by a_1, a_2 and a crosscap slide which commutes with ϱ. Hence $\mathcal{M}^h(N_3) = \mathcal{M}(N_3)$ and Theorem 4.1 implies the following.

Corollary 4.3 (Birman–Chillingworth [3])**.** *The group $\mathcal{M}(N_3)$ has the presentation with generators t_{a_1}, t_{a_2}, s and defining relations*:

(D1) $t_{a_1}t_{a_2}t_{a_1} = t_{a_2}t_{a_1}t_{a_2}$,
(D2) $(t_{a_1}t_{a_2}t_{a_1})^4 = 1$,
(D3) $s^2 = 1$,
(D4) $st_{a_j}s = t_{a_j}^{-1}$ *for* $j = 1, 2$.

Proof. By Theorem 4.1, the group $\mathcal{M}(N_3)$ is generated by t_{a_1}, t_{a_2}, ϱ, s with defining relations:
(C2) $t_{a_1}t_{a_2}t_{a_1} = t_{a_2}t_{a_1}t_{a_2}$,
(C3) $(t_{a_1}t_{a_2})^3 = \varrho$,
(C4) $s^2 = 1$,
(C5) $st_{a_j}s = t_{a_j}^{-1}$ for $j = 1, 2$,
(C6) $\varrho^2 = 1$,
(C7) $\varrho t_{a_j}\varrho = t_{a_j}$ for $j = 1, 2$,
(C8) $\varrho s\varrho = s$.
Using (C2), we can rewrite (C3) in the form

$$\varrho = t_{a_1}t_{a_2}t_{a_1}(t_{a_2}t_{a_1}t_{a_2}) = t_{a_1}t_{a_2}t_{a_1}(t_{a_1}t_{a_2}t_{a_1}) = (t_{a_1}t_{a_2}t_{a_1})^2.$$

Hence we can remove ϱ from the generating set and then (C6) will transform into

(D2). It remains to check that relations (C7) and (C8) are superfluous. Let start with (C7).

$$
\begin{aligned}
t_{a_1}\varrho t_{a_1}^{-1} &= t_{a_1}(t_{a_1}t_{a_2}t_{a_1})(t_{a_1}t_{a_2}t_{a_1})t_{a_1}^{-1}\\
&= t_{a_1}(t_{a_2}t_{a_1}t_{a_2})(t_{a_1}t_{a_2}t_{a_1})t_{a_1}^{-1} = (t_{a_1}t_{a_2}t_{a_1})(t_{a_1}t_{a_2}t_{a_1}) = \varrho,\\
t_{a_2}\varrho t_{a_2}^{-1} &= t_{a_2}(t_{a_1}t_{a_2}t_{a_1})(t_{a_1}t_{a_2}t_{a_1})t_{a_2}^{-1}\\
&= t_{a_2}(t_{a_1}t_{a_2}t_{a_1})(t_{a_2}t_{a_1}t_{a_2})t_{a_2}^{-1} = (t_{a_1}t_{a_2}t_{a_1})(t_{a_1}t_{a_2}t_{a_1}) = \varrho.
\end{aligned}
$$

Now we check (C8).

$$
s\varrho s = s(t_{a_1}t_{a_2}t_{a_1})^2 s = (t_{a_1}^{-1}t_{a_2}^{-1}t_{a_1}^{-1})^2 = (t_{a_1}t_{a_2}t_{a_1})^{-2} = (t_{a_1}t_{a_2}t_{a_1})^2 = \varrho. \qquad \square
$$

By restricting homomorphism $\pi_\varrho\colon \mathcal{M}^h(N_g) \to \mathcal{M}^{\pm}(S_0^{g,1})$ to the subgroup $\mathcal{M}^{h+}(N_g)$ we obtain the exact sequence

$$
1 \to \langle\varrho\rangle \to \mathcal{M}^{h+}(N_g) \xrightarrow{\pi_\varrho} \mathcal{M}(S_0^{g,1}) \to 1.
$$

Now if we lift the presentation from Theorem 3.2, we get the following.

Theorem 4.4. *If $g \geqslant 3$, then $\mathcal{M}^{h+}(N_g)$ has the presentation with generators $t_{a_1}, \dots, t_{a_{g-1}}, \varrho$ and defining relations*:
(E1) $t_{a_k}t_{a_j} = t_{a_j}t_{a_k}$ *for* $|k-j| > 1$ *and* $k, j < g$,
(E2) $t_{a_j}t_{a_{j+1}}t_{a_j} = t_{a_{j+1}}t_{a_j}t_{a_{j+1}}$ *for* $j = 1, 2, \dots, g-2$,
(E3) $(t_{a_1}\cdots t_{a_{g-1}})^g = \begin{cases} 1 & \textit{for } g \textit{ even},\\ \varrho & \textit{for } g \textit{ odd},\end{cases}$
(E4) $\varrho^2 = 1$,
(E5) $\varrho t_{a_j}\varrho = t_{a_j}$ *for* $j = 1, 2, \dots, g-1$.

Corollary 4.5. *If $g \geqslant 3$, then*

$$
H_1(\mathcal{M}^{h+}(N_g)) = \begin{cases} \mathbb{Z}_{2(g-1)g} & \textit{for} \quad g \textit{ odd},\\ \mathbb{Z}_{(g-1)g} \oplus \mathbb{Z}_2 & \textit{for} \quad g \textit{ even}.\end{cases}
$$

Proof. Relation (E2) implies that the abelianization of the group $\mathcal{M}^{h+}(N_g)$ is an abelian group generated by t_{a_1}, ϱ. Defining relations take form:

$$
\begin{aligned}
&t_{a_1}^{(g-1)g} = \begin{cases} 1 & \text{for} \quad g \text{ even},\\ \varrho & \text{for} \quad g \text{ odd},\end{cases}\\
&\varrho^2 = 1.
\end{aligned}
$$

Hence $H_1(\mathcal{M}^{h+}(N_g)) = \langle t_{a_1}\rangle \simeq \mathbb{Z}_{2(g-1)g}$ for g odd and $H_1(\mathcal{M}^{h+}(N_g)) = \langle t_{a_1}, \varrho\rangle \simeq \mathbb{Z}_{(g-1)g} \oplus \mathbb{Z}_2$ for g even. $\square$

REMARK 4.6. To put Corollaries 4.2 and 4.5 into perspective, recall that in the oriented case (Theorem 8 of [4]),

$$\begin{aligned}
&\mathcal{M}^h(S_g)\\
&= \langle t_{a_1}, \dots, t_{a_{2g+1}}, \varrho \mid t_{a_k} t_{a_j} = t_{a_j} t_{a_k},\ t_{a_j} t_{a_{j+1}} t_{a_j} = t_{a_{j+1}} t_{a_j} t_{a_{j+1}},\\
&\qquad (t_{a_1} t_{a_2} \cdots t_{a_{2g+1}})^{2g+2} = 1,\ \varrho = t_{a_1} t_{a_2} \cdots t_{a_{2g+1}} t_{a_{2g+1}} \cdots t_{a_2} t_{a_1},\\
&\qquad \varrho^2 = 1, \varrho t_{a_1} \varrho = t_{a_1} \rangle, \quad \text{where} \quad j = 1, 2, \dots, 2g,\ |k - j| > 1.
\end{aligned}$$

The presentation for the group $\mathcal{M}^{h\pm}(S_g)$ is obtained from the above presentation by adding one generator s and three relations:

$$s^2 = 1, \quad s t_{a_1} s = t_{a_1}^{-1}, \quad \varrho s \varrho = s.$$

Consequently, $H_1(\mathcal{M}^{h\pm}(S_g)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and

$$H_1(\mathcal{M}^h(S_g)) = \begin{cases} \mathbb{Z}_{4g+2} & \text{for} \quad g \text{ even},\\ \mathbb{Z}_{8g+4} & \text{for} \quad g \text{ odd}. \end{cases}$$

This suggests that algebraically the group $\mathcal{M}^{h+}(N)$ corresponds to $\mathcal{M}^h(S)$, whereas $\mathcal{M}^h(N)$ corresponds to $\mathcal{M}^{h\pm}(S)$.

5. Computing $H_1(\mathcal{M}^{h+}(N_g); H_1(N_g;\mathbb{Z}))$ and $H_1(\mathcal{M}^h(N_g); H_1(N_g;\mathbb{Z}))$

5.1. Homology of groups. Let us briefly review how to compute the first homology of a group with twisted coefficients. Our exposition follows [6, 17].

For a given group G and G-module M (that is $\mathbb{Z}G$-module) we define the *bar resolution* which is a chain complex $(C_n(G))$ of G-modules, where $C_n(G)$ is the free G-module generated by symbols $[h_1 \mid \cdots \mid h_n]$, $h_i \in G$. For $n = 0$, $C_0(G)$ is the free module generated by the empty bracket $[\,\cdot\,]$. Our interest will restrict to groups $C_2(G), C_1(G), C_0(G)$ for which the boundary operator $\partial_n \colon C_n(G) \to C_{n-1}(G)$ is defined by formulas:

$$\begin{aligned}
&\partial_2([h_1|h_2]) = h_1[h_2] - [h_1h_2] + [h_1],\\
&\partial_1([h]) = h[\,\cdot\,] - [\,\cdot\,].
\end{aligned}$$

The homology of G with coefficients in M is defined as the homology groups of the chain complex $(C_n(G) \otimes M)$, where the chain complexes are tensored over $\mathbb{Z}G$. In particular, $H_1(G; M)$ is the first homology group of the complex

$$C_2(G) \otimes M \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1(G) \otimes M \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0(G) \otimes M.$$

For simplicity, we denote $\partial \otimes \mathrm{id} = \overline{\partial}$ henceforth.

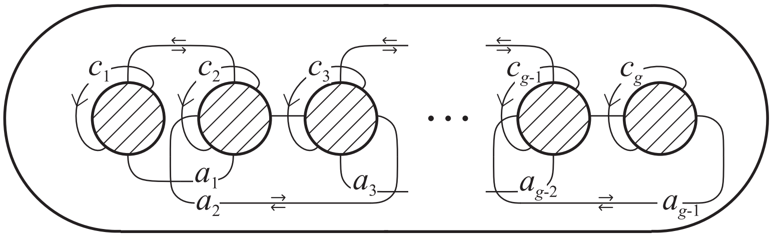

Fig. 4. Surface N_g as a sphere with crosscaps.

If the group G has a presentation $G = \langle X \mid R \rangle$, denote by

$$\langle \overline{X} \rangle = \langle [x] \otimes m \mid x \in X,\ m \in M \rangle \subseteq C_1(G) \otimes M.$$

Then, using the formula for ∂_2, one can show that $H_1(G; M)$ is a quotient of $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$.

The kernel of this quotient corresponds to relations in G (that is elements of R). To be more precise, if $r \in R$ has the form $x_1 \cdots x_k = y_1 \cdots y_n$ and $m \in M$, then r gives the relation (in $H_1(G; M)$)

$$\overline{r} \otimes m\colon\ \sum_{i=1}^{k} x_1 \cdots x_{i-1}[x_i] \otimes m = \sum_{i=1}^{n} y_1 \cdots y_{i-1}[y_i] \otimes m. \tag{5.1}$$

Then

$$H_1(G; M) = \langle \overline{X} \rangle \cap \ker \overline{\partial}_1 / \langle \overline{R} \rangle,$$

where

$$\overline{R} = \{\overline{r} \otimes m \mid r \in R,\ m \in M\}.$$

5.2. Action of $\mathcal{M}^h(N_g)$ on $H_1(N_g; \mathbb{Z})$. Let $c_1, \ldots, c_g$ be one-sided circles indicated in Fig. 4. In this figure surface N_g is represented as the sphere with g crosscaps (the shaded disks represent crosscaps, hence their interiors are to be removed and then the antipodal points on each boundary component are to be identified). The same set of circles is also indicated in Fig. 2—for a method of transferring circles between two models of N_g see Section 3 of [15].

Recall that $H_1(N_g; \mathbb{Z})$ as a $\mathbb{Z}$-module is generated by $\gamma_1 = [c_1], \ldots, \gamma_g = [c_g]$ with respect to the single relation

$$2(\gamma_1 + \gamma_2 + \cdots + \gamma_g) = 0.$$

There is a $\mathbb{Z}_2$-valued intersection paring $\langle\ ,\ \rangle$ on $H_1(N_g; \mathbb{Z})$ defined as the symmetric bilinear form (with values in $\mathbb{Z}_2$) satisfying $\langle \gamma_i, \gamma_j \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant g$. The mapping class group $\mathcal{M}(N_g)$ acts on $H_1(N_g; \mathbb{Z})$ via automorphisms which preserve $\langle\ ,\ \rangle$, hence there is a representation

$$\psi\colon \mathcal{M}(N_g) \to \mathrm{Iso}(H_1(N_g; \mathbb{Z})).$$

In fact it is known that this representation is surjective—see [13, 8].

Since we have very simple geometric definitions of $t_{a_i}, s, \varrho \in \mathcal{M}^h(N_g)$ it is straightforward to check that

$$\psi(t_{a_i}) = I_{i-1} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \oplus I_{g-i-1},$$

$$\psi(t_{a_i}^{-1}) = I_{i-1} \oplus \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \oplus I_{g-i-1},$$

$$\psi(s) = \begin{bmatrix} -1 & 2 & -2 & 2 & \dots & (-1)^g \cdot 2 \\ 0 & 1 & -2 & 2 & \dots & (-1)^g \cdot 2 \\ 0 & 0 & -1 & 2 & \dots & (-1)^g \cdot 2 \\ 0 & 0 & 0 & 1 & \dots & (-1)^g \cdot 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (-1)^g \cdot 1 \end{bmatrix},$$

$$\psi(\varrho) = -I_g,$$

where I_k is the identity matrix of rank k.

The above matrices are written with respect to the generating set $(\gamma_1, \gamma_2, \dots, \gamma_g)$. Note that $H_1(N_g; \mathbb{Z})$ is not free, hence one has to be careful with matrices—two different matrices may represent the same element.

5.3. Computing $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$. Observe that if $G = \mathcal{M}^h(N_g)$, $M = H_1(N_g; \mathbb{Z})$ and $h \in G$ then

$$\overline{\partial}_1([h] \otimes \gamma_j) = (h-1)[\,\cdot\,] \otimes \gamma_j = [\,\cdot\,] \otimes (\psi(h)^{-1} - I_g)\gamma_j.$$

If we identify $C_0(G) \otimes M$ with M by the map $[\,\cdot\,] \otimes m \mapsto m$, this formula takes form

$$\overline{\partial}_1([h] \otimes \gamma_j) = (\psi(h)^{-1} - I_g)\gamma_j.$$

Let us denote $[\varrho] \otimes \gamma_j$, $[s] \otimes \gamma_j$, $[t_{a_i}] \otimes \gamma_j$ respectively by ϱ_j, s_j and $t_{i,j}$. Using the above formula, we obtain

$$\overline{\partial}_1(\varrho_j) = -2\gamma_j,$$

$$\overline{\partial}_1(s_j) = \begin{cases} -2\sum_{k=1}^{j} \gamma_k & \text{for } j \text{ odd}, \\ -\overline{\partial}_1(s_{j-1}) & \text{for } j \text{ even}, \end{cases}$$

$$\overline{\partial}_1(t_{i,j}) = \begin{cases} \gamma_i + \gamma_{i+1} & \text{for } j = i, \\ -\gamma_i - \gamma_{i+1} & \text{for } j = i+1, \\ 0 & \text{otherwise}. \end{cases}$$

Proposition 5.1. *Let $g \geqslant 3$ and $G = \mathcal{M}^{h+}(N_g)$ then $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$ is the abelian group which admits the presentation with generators*:

(F1) $t_{i,j}$, *where* $i = 1, \dots, g-1$ *and* $j = 1, \dots, i-1, i+2, \dots, g$,
(F2) $t_{j,j} + t_{j,j+1}$, *where* $j = 1, \dots, g-1$,
(F3) $2t_{j,j} + \varrho_j + \varrho_{j+1}$, *where* $j = 1, \dots, g-1$,
(F4) $\begin{cases} 2t_{1,1} + 2t_{3,3} + \cdots + 2t_{g-2,g-2} - \varrho_g & \textit{for } g \textit{ odd}, \\ 2t_{1,1} + 2t_{3,3} + \cdots + 2t_{g-1,g-1} & \textit{for } g \textit{ even} \end{cases}$

and relations

$$r_{t_j}\colon 0 = 2t_{j,1} + \cdots + 2(t_{j,j} + t_{j,j+1}) + \cdots + 2t_{j,g} \quad \textit{for } j = 1, \dots, g-1,$$

$$r_\varrho\colon \begin{cases} 2(2t_{1,1} + \varrho_1 + \varrho_2) + \cdots + 2(2t_{g-2,g-2} + \varrho_{g-2} + \varrho_{g-1}) \\ \quad = 2(2t_{1,1} + 2t_{3,3} + \cdots + 2t_{g-2,g-2} - \varrho_g) & \textit{for} \quad g \textit{ odd}, \\ 2(2t_{1,1} + \varrho_1 + \varrho_2) + \cdots + 2(2t_{g-1,g-1} + \varrho_{g-1} + \varrho_g) \\ \quad = 2(2t_{1,1} + 2t_{3,3} + \cdots + 2t_{g-1,g-1}) & \textit{for} \quad g \textit{ even}. \end{cases}$$

Proof. By Theorem 4.4, $\langle \overline{X} \rangle$ is generated by $t_{i,j}$ and ϱ_j. Using formulas for $\overline{\partial}_1(t_{i,j})$ and $\overline{\partial}_1(\varrho_j)$ it is straightforward to check that elements (F1)–(F4) are elements of $\ker \overline{\partial}_1$. Moreover,

$$2t_{j,1} + 2t_{j,2} + \cdots + 2t_{j,g} = [t_{a_j}] \otimes 2(\gamma_1 + \cdots + \gamma_g) = 0,$$

hence r_{t_j} is indeed a relation. Similarly we check that r_ϱ is a relation.

Observe that using relations r_{t_j} and r_ϱ we can substitute for $2t_{j,g}$ and $2\varrho_1$ respectively, hence each element in $\langle \overline{X} \rangle$ can be written as a linear combination of $t_{i,j}$, ϱ_j, where each of $t_{1,g}, t_{2,g}, \dots, t_{g-1,g}, \varrho_1$ has the coefficient 0 or 1. Moreover, for a given $x \in \langle \overline{X} \rangle \subset C_1(G) \otimes H_1(N_g; \mathbb{Z})$ such a combination is unique. Hence for the rest of the proof we assume that linear combinations of $t_{i,j}$, ϱ_j satisfy this condition.

Suppose that $h \in \langle \overline{X} \rangle \cap \ker \overline{\partial}_1$. We will show that h can be uniquely expressed as a linear combination of generators (F1)–(F4).

First observe that $h = h_1 + h_2$, where h_1 is a combination of generators (F1)–(F2), and h_2 does not contain generators of type (F1) nor elements $t_{j,j+1}$. Moreover, h_1 and h_2 are uniquely determined by h.

Next we decompose $h_2 = h_3 + h_4$, where h_3 is a combination of generators (F3) and h_4 does not contain ϱ_j for $j < g$. As before, h_3 and h_4 are uniquely determined by h_2.

Element h_4 has the form

$$h_4 = \sum_{j=1}^{g-1} \alpha_j t_{j,j} + \alpha \varrho_g,$$

for some integers $\alpha, \alpha_1, \dots, \alpha_{g-1}$. Hence

$$0 = \overline{\partial}_1(h_4) = \alpha_1 \gamma_1 + (\alpha_1 + \alpha_2)\gamma_2 + \cdots + (\alpha_{g-2} + \alpha_{g-1})\gamma_{g-1} + (\alpha_{g-1} - 2\alpha)\gamma_g.$$

If g is odd this implies that

$$\alpha_1 = \alpha_3 = \cdots = \alpha_{g-2} = 2k, \quad \alpha_2 = \alpha_4 = \cdots = \alpha_{g-1} = 0, \quad \alpha = -k,$$

for some $k \in \mathbb{Z}$. For g even we get

$$\alpha_1 = \alpha_3 = \cdots = \alpha_{g-1} = 2k, \quad \alpha = \alpha_2 = \alpha_4 = \cdots = \alpha_{g-2} = 0.$$

In each of these cases h_4 is a multiple of the generator (F4). $\square$

By an analogous argument we get

Proposition 5.2. *Let $g \geqslant 3$ and $G = \mathcal{M}^h(N_g)$ then $\langle \overline{X} \rangle \cap \ker \overline{\partial}_1$ is the abelian group which admits the presentation with generators*: (F1)–(F4),
(F5) $s_j + s_{j-1}$, *where j is even,*
(F6) $s_j - \varrho_1 - \varrho_2 - \cdots - \varrho_j$, *where j is odd.*
The defining relations are r_{t_j}, r_ϱ and

$$r_s\colon \begin{cases} 0 = 2(s_2+s_1) + 2(s_4+s_3) + \cdots + 2(s_{g-1}+s_{g-2}) & \\ \qquad + 2(s_g - \varrho_1 - \varrho_2 - \cdots - \varrho_g) & \textit{for} \quad g \textit{ odd}, \\ 0 = 2(s_2+s_1) + 2(s_4+s_3) + \cdots + 2(s_g + s_{g-1}) & \textit{for} \quad g \textit{ even}. \end{cases}$$

5.4. Rewriting relations. Using formula (5.1) we rewrite relations (E1)–(E5) as relations in $H_1(\mathcal{M}^{h+}(N_g); H_1(N_g;\mathbb{Z}))$.

Relation (E1) is symmetric with respect to k and j, hence we can assume that $j+1 < k$. This relation gives

$$\begin{aligned} r^{(E1)}_{k,j:i}\colon 0 &= ([t_{a_k}] + t_{a_k}[t_{a_j}] - [t_{a_j}] - t_{a_j}[t_{a_k}]) \otimes \gamma_i \\ &= t_{k,i} + [t_{a_j}] \otimes \psi(t_{a_k}^{-1})\gamma_i - t_{j,i} - [t_{a_k}] \otimes \psi(t_{a_j}^{-1})\gamma_i \\ &= \pm \begin{cases} 0 & \text{if} \quad i \neq k, k+1, j, j+1, \\ t_{j,k} + t_{j,k+1} & \text{if} \quad i = k \text{ or } i = k+1, \\ t_{k,j} + t_{k,j+1} & \text{if} \quad i = j \text{ or } i = j+1. \end{cases} \end{aligned}$$

Relation (E2) gives

$$\begin{aligned} r^{(E2)}_{j:i}\colon 0 = {} & ([t_{a_j}] + t_{a_j}[t_{a_{j+1}}] + t_{a_j}t_{a_{j+1}}[t_{a_j}] \\ & - [t_{a_{j+1}}] - t_{a_{j+1}}[t_{a_j}] - t_{a_{j+1}}t_{a_j}[t_{a_{j+1}}]) \otimes \gamma_i \\ = {} & \begin{cases} t_{j,i} - t_{j+1,i} & \text{if} \quad i \neq j, j+1, j+2, \\ t_{j,j+2} - t_{j+1,j} & \text{if} \quad i = j+2, \\ (*) + 2(t_{j,j} + t_{j,j+1}) & \text{if} \quad i = j, \\ (*) - (t_{j,j} + t_{j,j+1}) - (t_{j+1,j+1} + t_{j+1,j+2}) & \text{if} \quad i = j+1. \end{cases} \end{aligned}$$

In the above formula $(*)$ denotes some expression homologous to 0 by previously obtained relations. Carefully checking relations $r^{(E1)}_{k,j:i}$ and $r^{(E2)}_{j:i}$ we conclude that generators (F1) generate a cyclic group, and generators (F2) generate a cyclic group of order at most 2.

We next turn to the relation (E5). It gives

$$\begin{aligned} r^{(E5)}_{j:i}\colon 0 &= ([\varrho] + \varrho[t_{a_j}] - [t_{a_j}] - t_{a_j}[\varrho]) \otimes \gamma_i \\ &= \begin{cases} -2t_{j,i} & \text{if} \quad i \neq j,\, j+1, \\ -2t_{j,j} - \varrho_j - \varrho_{j+1} & \text{if} \quad i = j, \\ (\varrho_j + \varrho_{j+1} + 2t_{j,j}) - 2(t_{j,j} + t_{j,j+1}) & \text{if} \quad i = j+1. \end{cases} \end{aligned}$$

These relations imply that generators (F3) are homologically trivial, and generators (F1) generate at most $\mathbb{Z}_2$.

We now turn to the most difficult relation, namely (E3). This relation gives

$$\begin{aligned} r^{(E3)}_i\colon 0 &= \sum_{k=0}^{g-1}\sum_{n=1}^{g-1}(t_{a_1}\cdots t_{a_{g-1}})^k t_{a_1}\cdots t_{a_{n-1}}[t_{a_n}] \otimes \gamma_i - \varepsilon\varrho_i \\ &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes \psi(t_{a_1}\cdots t_{a_{n-1}})^{-1}\sum_{k=0}^{g-1}\psi(t_{a_1}\cdots t_{a_{g-1}})^{-k}\gamma_i - \varepsilon\varrho_i \\ &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n \sum_{k=0}^{g-1} Y_g^k\gamma_i - \varepsilon\varrho_i. \end{aligned}$$

Where $\varepsilon = 0$ for g even, $\varepsilon = 1$ for g odd, and $Y_n = \psi(t_{a_1}\cdots t_{a_{n-1}})^{-1}$. Using the matrix formula for $\psi(t_{a_i}^{-1})$, we obtain

$$Y_n\gamma_i = \begin{cases} -\gamma_{i-1} & \text{if} \quad 2 \leq i \leq n, \\ \gamma_i & \text{if} \quad i > n, \\ 2\gamma_1 + \cdots + 2\gamma_{n-1} + \gamma_n & \text{if} \quad i = 1. \end{cases}$$

In particular

$$Y_g^k\gamma_i = (-1)^k\gamma_{i-k},$$

where we subtract indexes modulo g. Therefore we have

$$r^{(E3)}_i\colon 0 = \sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n \sum_{k=0}^{g-1}(-1)^k\gamma_{i-k} - \varepsilon\varrho_i.$$

In order to simplify computations we replace relations:

$$r^{(E3)}_1, r^{(E3)}_2, \ldots, r^{(E3)}_g$$

with relations:

$$r_1^{(E3)} + r_2^{(E3)}, r_2^{(E3)} + r_3^{(E3)}, \dots, r_{g-1}^{(E3)} + r_g^{(E3)}, r_g^{(E3)}.$$

Let us begin with $r_g^{(E3)}$.

$$\begin{aligned} r_g^{(E3)}\colon 0 &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n \sum_{k=0}^{g-1}(-1)^k \gamma_{g-k} - \varepsilon\varrho_g \\ &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes \Bigg(\sum_{k=0}^{g-n-1}(-1)^k \gamma_{g-k} + \sum_{k=g-n}^{g-2}(-1)^{k+1}\gamma_{g-k-1} \\ &\qquad\qquad + (-1)^{g-1}(2\gamma_1 + \cdots + 2\gamma_{n-1} + \gamma_n)\Bigg) - \varepsilon\varrho_g. \end{aligned}$$

Since all generators of type (F1) are homologous to a single generator, say t, and $2t = 0$, the above relation can be rewritten as

$$r_g^{(E3)}\colon 0 = (g-1)(g-2)t + \sum_{n=1}^{g-1}[t_{a_n}] \otimes ((-1)^{g-n-1}\gamma_{n+1} + (-1)^{g-1}\gamma_n) - \varepsilon\varrho_g.$$

If g is even, this gives the relation

$$\begin{aligned} r_g^{(E3)}\colon 0 &= (-t_{1,1} + t_{1,2}) + (-t_{2,2} - t_{2,3}) + \cdots + (-t_{g-1,g-1} + t_{g-1,g}) \\ &= (t_{1,1} + t_{1,2}) - (t_{2,2} + t_{2,3}) + \cdots + (t_{g-1,g-1} + t_{g-1,g}) \\ &\quad - 2(t_{1,1} + t_{3,3} + \cdots + t_{g-1,g-1}). \end{aligned}$$

If g is odd, we have

$$\begin{aligned} r_g^{(E3)}\colon 0 &= (t_{1,1} - t_{1,2}) + (t_{2,2} + t_{2,3}) + \cdots + (t_{g-1,g-1} + t_{g-1,g}) - \varrho_g \\ &= -(t_{1,1} + t_{1,2}) + (t_{2,2} + t_{2,3}) - \cdots + (t_{g-1,g-1} + t_{g-1,g}) \\ &\quad + 2(t_{1,1} + t_{3,3} + \cdots + t_{g-2,g-2}) - \varrho_g. \end{aligned}$$

In both cases relation $r_g^{(E3)}$ implies that generator (F4) is superfluous.

Now we concentrate on the relation $r_i^{(E3)} + r_{i+1}^{(E3)}$.

$$\begin{aligned} r_i^{(E3)} + r_{i+1}^{(E3)}\colon 0 &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n \sum_{k=0}^{g-1}(-1)^k(\gamma_{i-k} + \gamma_{i+1-k}) - \varepsilon(\varrho_i + \varrho_{i+1}) \\ &= \sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n(\gamma_{i+1} + (-1)^{g-1}\gamma_{i+1}) - \varepsilon(\varrho_i + \varrho_{i+1}). \end{aligned}$$

If g is even, this relation is trivial, and if g is odd it gives

$$\begin{aligned}
r_i^{(E3)} + r_{i+1}^{(E3)}\colon 0 &= 2\sum_{n=1}^{g-1}[t_{a_n}] \otimes Y_n(\gamma_{i+1}) - (\varrho_i + \varrho_{i+1})\\
&= 2(t_{1,i+1} + \cdots + t_{i,i+1} - t_{i+1,i} - \cdots - t_{g-1,i}) - (\varrho_i + \varrho_{i+1})\\
&= (*) + 2(t_{i,i} + t_{i,i+1}) - (2t_{i,i} + \varrho_i + \varrho_{i+1}).
\end{aligned}$$

Hence this relation gives no new information.

Relation (E4) gives no new information, hence we proved the following theorem.

Theorem 5.3. *If $g \geqslant 3$, then*

$$H_1(\mathcal{M}^{h+}(N_g); H_1(N_g;\mathbb{Z})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

5.5. Computing $H_1(\mathcal{M}^h(N_g); H_1(N_g;\mathbb{Z}))$. If $G = \mathcal{M}^h(N_g)$, then by Proposition 5.2 the kernel $\langle\overline{X}\rangle \cap \ker\overline{\partial}_1$ has two more types of generators: (F5), (F6), and by Theorem 4.1 there are three additional relations: (C4), (C5), (C8).

$$\begin{aligned}
r_i^{(\mathrm{C4})}\colon 0 &= [s]\otimes\gamma_i + s[s]\otimes\gamma_i = s_i + [s]\otimes\psi(s)\gamma_i\\
&= 2(-1)^i\left(s_1 + s_2 + \cdots + s_{i-1} + \frac{1+(-1)^i}{2}s_i\right).
\end{aligned}$$

This (inductively) implies that each generator of type (F5) has order at most 2.

$$\begin{aligned}
r_i^{(\mathrm{C8})}\colon 0 &= ([\varrho] + \varrho[s] - [s] - s[\varrho])\otimes\gamma_i = \varrho_i - 2s_i - [\varrho]\otimes\psi(s)\gamma_i\\
&= \varrho_i - 2s_i - (-1)^i(2\varrho_1 + 2\varrho_2 + \cdots + 2\varrho_{i-1} + \varrho_i)\\
&= \begin{cases} -2(s_i - \varrho_1 - \cdots - \varrho_i) & \text{for } i \text{ odd,}\\ -2(s_{i-1} + s_i) + 2(s_{i-1} - \varrho_1 - \cdots - \varrho_{i-1}) & \text{for } i \text{ even.}\end{cases}
\end{aligned}$$

This implies that generator (F6) has also order at most 2.

$$\begin{aligned}
r_i^{(\mathrm{C5})}\colon 0 &= ([t_{a_j}] + t_{a_j}[s] + t_{a_j}s[t_{a_j}] - [s])\otimes\gamma_i\\
&= t_{j,i} + [s]\otimes\psi(t_{a_j}^{-1})\gamma_i + [t_{a_j}]\otimes\psi(s)\psi(t_{a_j}^{-1})\gamma_i - s_i.
\end{aligned}$$

If $i \neq j$ and $i \neq j+1$, then

$$r_i^{(\mathrm{C5})}\colon 0 = (-1)^i(2t_{j,1} + \cdots + 2t_{j,i-1} + (1+(-1)^i)t_{j,i}),$$

which gives no new information. If $i = j$ or $i = j+1$ and j is odd, then

$$r_i^{(\mathrm{C5})}\colon 0 = (*) \pm [(s_j + s_{j+1}) + (t_{j,j} + t_{j,j+1})],$$

where as usual $(*)$ denotes homologically trivial element. This relation implies that generators (F5) are superfluous.

Finally, if $i = j$ or $i = j + 1$ and j is even then

$$r_i^{(C5)}\colon 0 = (*) \pm [(s_{j+1} - \varrho_1 - \cdots - \varrho_{j+1}) - (s_{j-1} - \varrho_1 - \cdots - \varrho_{j-1})].$$

This implies that all generators of type (F6) are homologous, hence we proved the following.

Theorem 5.4. *If $g \geqslant 3$, then*

$$H_1(\mathcal{M}^h(N_g); H_1(N_g; \mathbb{Z})) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

ACKNOWLEDGEMENTS. The author wishes to thank the referee for his/her helpful suggestions.

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