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# *F*-FINITENESS OF HOMOMORPHISMS AND ITS DESCENT

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**Abstract**

Let $p$ be a prime number. We define the notion of $F$-finiteness of homomorphisms of $\mathbb{F}_p$-algebras, and discuss some basic properties. In particular, we prove a sort of descent theorem on $F$-finiteness of homomorphisms of $\mathbb{F}_p$-algebras. As a corollary, we prove the following. Let $g\colon B \to C$ be a homomorphism of Noetherian $\mathbb{F}_p$-algebras. If $g$ is faithfully flat reduced and $C$ is $F$-finite, then $B$ is $F$-finite. This is a generalization of Seydi's result on excellent local rings of characteristic $p$.

## 1. Introduction

Throughout this paper, $p$ denotes a prime number, and $\mathbb{F}_p$ denotes the finite field with $p$ elements.

The notions of Nagata (pseudo-geometric, universally Japanese) and (quasi-)excellent rings give good frameworks to avoid pathologies which appear in the theory of Noetherian rings, see [20], [10], and [18].

In commutative algebra of characteristic $p$, $F$-finiteness of rings is commonly used for a general assumption which guarantees the "tameness" of the theory, as well as Nagata and (quasi-)excellent properties. A commutative ring $R$ of characteristic $p$ is said to be $F$-finite if the Frobenius map $F_R\colon R \to R$ ($F_R(r) = r^p$) is finite (that is, $R$ as the target of $F_R$ is a finite module over $R$ as the source of $F_R$). As the definition suggests, $F$-finiteness is important in studying ring theoretic properties defined via Frobenius maps, such as strong $F$-regularity [14]. Although $F$-finiteness for a Noetherian $\mathbb{F}_p$-algebra is stronger than excellence [16], $F$-finiteness is not so restrictive for practical use. A perfect field is $F$-finite. An algebra essentially of finite type over an $F$-finite ring is $F$-finite. An ideal-adic completion of a Noetherian $F$-finite ring is again $F$-finite. See Examples 3 and 9. It is known that an $F$-finite Noetherian ring is a homomorphic image of an $F$-finite regular ring of finite Krull dimension, and hence it has a dualizing complex [9, Remark 13.6].

In this paper, replacing the absolute Frobenius map by the relative one, we define the $F$-finiteness of homomorphism between rings of characteristic $p$. We say that an $\mathbb{F}_p$-algebra map $A \to B$ is $F$-finite (or $B$ is $F$-finite over $A$) if the relative Frobenius map (Radu–André homomorphism) $\Phi_1(A, B)\colon B^{(1)} \otimes_{A^{(1)}} A \to B$ is finite (Definition 1, see Section 2 for the notation). Thus a ring $B$ of characteristic $p$ is $F$-finite if and

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only if it is $F$-finite over $\mathbb{F}_p$. Replacing absolute Frobenius by relative Frobenius, we get definitions and results on homomorphisms instead of rings. This is a common idea in [23], [2], [3], [4], [5], [7], [12], [6], and [13].

In Section 2, we discuss basic properties of $F$-finiteness of homomorphisms and rings. Some of well-known properties of $F$-finiteness of rings are naturally generalized to those for $F$-finiteness of homomorphisms. $F$-finiteness of homomorphisms has connections with that for rings. For example, if $A \to B$ is $F$-finite and $A$ is $F$-finite, then $B$ is $F$-finite (Lemma 2).

In Section 3, we prove the main theorem (Theorem 21). This is a sort of descent of $F$-finiteness. As a corollary, we prove that for a faithfully flat reduced homomorphism of Noetherian rings $g\colon B \to C$, if $C$ is $F$-finite, then $B$ is $F$-finite. Considering the case that $f$ is a completion of a Noetherian local ring, we recover Seydi's result on excellent local rings of characteristic $p$ [26].

## 2. *F*-finiteness of homomorphisms

Let $k$ be a perfect field of characteristic $p$, and $r \in \mathbb{Z}$. For a $k$-space $V$, the additive group $V$ with the new $k$-space structure $\alpha \cdot v = \alpha^{p^{-r}} v$ is denoted by $V^{(r)}$. An element $v$ of $V$, viewed as an element of $V^{(r)}$ is (sometimes) denoted by $v^{(r)}$. If $A$ is a $k$-algebra, then $A^{(r)}$ is a $k$-algebra with the product $a^{(r)} \cdot b^{(r)} = (ab)^{(r)}$. We denote the Frobenius map $A \to A$ ($a \mapsto a^p$) by $F$ or $F_A$. Note that $F^e\colon A^{(r+e)} \to A^{(r)}$ is a $k$-algebra map. Throughout the article, we regard $A^{(r)}$ as an $A^{(r+e)}$-algebra through $F^e$ ($A$ is viewed as $A^{(0)}$). For an $A$-module $M$, the action $a^{(r)} \cdot m^{(r)} = (am)^{(r)}$ makes $M^{(r)}$ an $A^{(r)}$-module. If $I$ is an ideal of $A$, then $I^{(r)}$ is an ideal of $A^{(r)}$. If $e \geq 0$, then $I^{(e)}A = I^{[p^e]}$, where $I^{[p^e]}$ is the ideal of $A$ generated by $\{a^{p^e} \mid a \in I\}$. In commutative algebra, $A^{(r)}$ is also denoted by ${}^{-r}A$. We employ the notation more consistent with that in representation theory—the $e$-th Frobenius twist of $V$ is denoted by $V^{(e)}$, see [15]. We use this notation for $k = \mathbb{F}_p$.

Let $A \to B$ be an $\mathbb{F}_p$-algebra map, and $e \geq 0$. Then the relative Frobenius map (or Radu–André homomorphism) $\Phi_e(A, B)\colon B^{(e)} \otimes_{A^{(e)}} A \to B$ is defined by $\Phi_e(A, B)(b^{(e)} \otimes a) = b^{p^e} a$.

DEFINITION 1. An $\mathbb{F}_p$-algebra map $A \to B$ is said to be *$F$-finite* if $\Phi_1(A, B)\colon B^{(1)} \otimes_{A^{(1)}} A \to B$ is finite. That is, $B$ is a finitely generated $B^{(1)} \otimes_{A^{(1)}} A$-module through $\Phi_1(A, B)$. We also say that $B$ is $F$-finite over $A$.

**Lemma 2.** *Let $f\colon A \to B$, $g\colon B \to C$, and $h\colon A \to \tilde{A}$ be $\mathbb{F}_p$-algebra maps, and $\tilde{B} := \tilde{A} \otimes_A B$.*

(1) *The following are equivalent.*

  (a) *$f$ is $F$-finite. That is, $\Phi_1(A, B)$ is finite.*

  (b) *For any $e > 0$, $\Phi_e(A, B)$ is finite.*

  (c) *For some $e > 0$, $\Phi_e(A, B)$ is finite.*

(2) *If $f$ and $g$ are $F$-finite, then so is $gf$.*
(3) *If $gf$ is $F$-finite, then so is $g$.*
(4) *The ring $A$ is $F$-finite (that is, the Frobenius map $F_A\colon A^{(1)} \to A$ is finite) if and only if the unique homomorphism $\mathbb{F}_p \to A$ is $F$-finite.*
(5) *If $f\colon A \to B$ is $F$-finite, then the base change $\tilde{f}\colon \tilde{A} \to \tilde{B}$ is $F$-finite.*
(6) *If $B$ is $F$-finite, then $f$ is $F$-finite.*
(7) *If $A$ and $f$ are $F$-finite, then $B$ is $F$-finite.*

Proof. (1) This is immediate, using [12, Lemma 4.1, **2**]. (2) and (3) follow from [12, Lemma 4.1, **1**]. (4) follows from [12, Lemma 4.1, **5**]. (5) follows from [12, Lemma 4.1, **4**]. (6) follows from (3) and (4). (7) follows from (2) and (4). □

EXAMPLE 3. Let $e \geq 1$, and $f\colon A \to B$ be an $\mathbb{F}_p$-algebra map.
(1) If $B = A[x]$ is a polynomial ring, then it is $F$-finite over $A$.
(2) If $B = A_S$ is a localization of $A$ by a multiplicatively closed subset $S$ of $A$, then $\Phi_e(A, B)$ is an isomorphism. In particular, $B$ is $F$-finite over $A$.
(3) If $B = A/I$ with $I$ an ideal of $A$, then

$$B^{(e)} \otimes_{A^{(e)}} A \cong (A^{(e)}/I^{(e)}) \otimes_{A^{(e)}} A \cong A/I^{(e)}A = A/I^{[p^e]}.$$

Under this identification, $\Phi_e(A, B)$ is identified with the projection $A/I^{[p^e]} \to A/I$. In particular, $B$ is $F$-finite over $A$.
(4) If $B$ is essentially of finite type over $A$, then $B$ is $F$-finite over $A$.

Proof. (1) The image of $\Phi_1(A, B)$ is $A[x^p]$, and hence $B$ is generated by $1, x, \ldots, x^{p-1}$ over it. (2) Note that $B^{(e)}$ is identified with $(A^{(e)})_{S^{(e)}}$, where $S^{(e)} = \{s^{(e)} \mid s \in S\}$. So $B^{(e)} \otimes_{A^{(e)}} A$ is identified with $(A^{(e)})_{S^{(e)}} \otimes_{A^{(e)}} A \cong A_{S^{(e)}}$, and $\Phi_e(A, B)$ is identified with the isomorphism $A_{S^{(e)}} \cong A_S$. (3) is obvious. (4) This is a consequence of (1), (2), (3), and Lemma 2 (2). □

**Lemma 4.** *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a sequence of $\mathbb{F}_p$-algebra maps. Then for $e > 0$, the diagram*

$$\begin{array}{ccc} B^{(e)} \otimes_{A^{(e)}} A & \xrightarrow{\Phi_e(A,B)} & B \\ \downarrow{\scriptstyle g^{(e)}\otimes 1} & & \downarrow{\scriptstyle g} \\ C^{(e)} \otimes_{A^{(e)}} A & \xrightarrow{\Phi_e(A,C)} & C \end{array}$$

*is commutative.*

Proof. This is straightforward. □

**Lemma 5.** *Let* $A \xrightarrow{f} B \xrightarrow{g} C$ *be a sequence of* $\mathbb{F}_p$*-algebra maps, and assume that* $C$ *is* $F$*-finite over* $A$*. If* $g$ *is finite and injective, and* $B^{(e)} \otimes_{A^{(e)}} A$ *is Noetherian for some* $e > 0$*, then* $B$ *is* $F$*-finite over* $A$*.*

Proof. By assumption, $C^{(e)} \otimes_{A^{(e)}} A$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$, and $C$ is finite over $C^{(e)} \otimes_{A^{(e)}} A$. So $C$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$. As $B$ is a $B^{(e)} \otimes_{A^{(e)}} A$-submodule of $C$ and $B^{(e)} \otimes_{A^{(e)}} A$ is Noetherian, $B$ is finite over $B^{(e)} \otimes_{A^{(e)}} A$. □

**Lemma 6.** *Let* $A \to B$ *be a ring homomorphism, and* $I$ *a finitely generated nilpotent ideal of* $B$*. If* $B/I$ *is* $A$*-finite, then* $B$ *is* $A$*-finite.*

Proof. As $I^i/I^{i+1}$ is $B/I$-finite for each $i$, it is also $A$-finite. So $B/I^r$ is $A$-finite for each $r$. Taking $r$ large, $B$ is $A$-finite. □

**Lemma 7.** *Let* $f\colon A \to B$ *be an* $\mathbb{F}_p$*-algebra map, and* $I$ *a finitely generated nilpotent ideal of* $B$*. If* $B/I$ *is* $F$*-finite over* $A$*, then* $B$ *is* $F$*-finite over* $A$*.*

Proof. As $B/I$ is $F$-finite over $A$, $B/I$ is $(B^{(1)}/I^{(1)}) \otimes_{A^{(1)}} A$-finite. So $B/I$ is also $B^{(1)} \otimes_{A^{(1)}} A$-finite. By Lemma 6, $B$ is $B^{(1)} \otimes_{A^{(1)}} A$-finite. □

For the absolute $F$-finiteness, we have a better result.

**Lemma 8.** *Let* $B$ *be an* $\mathbb{F}_p$*-algebra, and* $I$ *a finitely generated ideal of* $B$*. If* $B$ *is* $I$*-adically complete and* $B/I$ *is* $F$*-finite, then* $B$ *is* $F$*-finite.*

Proof. $B/I$ is $B^{(1)}/I^{(1)}$-finite. So $B/I^{(1)}B$ is $B^{(1)}$-finite by Lemma 6. As $\bigcap_i I^i = 0$, we have $\bigcap_i (I^{(1)})^i B = 0$. Moreover, $B^{(1)}$ is $I^{(1)}$-adically complete. Hence $B$ is $B^{(1)}$-finite by [19, Theorem 8.4]. □

EXAMPLE 9. Let $A$ be an $\mathbb{F}_p$-algebra.
(1) If $A$ is $F$-finite, then the formal power series ring $A[[x]]$ is so.
(2) Let $J$ be an ideal of $A$. If $A$ is Noetherian and $A/J$ is $F$-finite, then the $J$-adic completion $A^*$ of $A$ is $F$-finite.
(3) If $(A, \mathfrak{m})$ is complete local and $A/\mathfrak{m}$ is $F$-finite, then $A$ is $F$-finite.

Proof. For each of (1)–(3), we use Lemma 8. (1) Set $B = A[[x]]$ and $I = Bx$. Then $B/I \cong A$ is $F$-finite. (2) Set $B = A^*$ and $I = JB$. Then $B/I \cong A/J$ is $F$-finite. (3) is immediate. □

REMARK 10. Let $A$ be a Noetherian ring and $I$ its ideal. If $A$ is $I$-adically complete and $A/I$ is Nagata, then $A$ is Nagata [17]. If $A$ is semi-local, $I$-adically complete, and $A/I$ is quasi-excellent, then $A$ is quasi-excellent [25]. See also [21].

**Lemma 11.** *Let $A$ be an $\mathbb{F}_p$-algebra, and $B$ and $C$ be $A$-algebras. If $B$ and $C$ are $F$-finite over $A$, then*
(1) *$B \otimes_A C$ is $F$-finite over $A$.*
(2) *$B \times C$ is $F$-finite over $A$.*

Proof. (1) $B$ is $F$-finite over $A$, and $B \otimes_A C$ is $F$-finite over $B$ by Lemma 2 (5). By Lemma 2 (2), $B \otimes_A C$ is $F$-finite over $A$. (2) Both $B$ and $C$ are finite over $(B \times C)^{(1)} \otimes_{A^{(1)}} A$, and so is $B \times C$. □

**Lemma 12.** *Let $A \to B$ be an $\mathbb{F}_p$-algebra map, and assume that $B$ and $B^{(e)} \otimes_{A^{(e)}} A$ are Noetherian for some $e > 0$. Then $B$ is $F$-finite over $A$ if and only if $B/P$ is $F$-finite over $A$ for every minimal prime $P$ of $B$.*

Proof. The 'only if' part is obvious by Example 3 (3). We prove the converse. Let $\operatorname{Min} B$ be the set of minimal primes of $B$. Then $\prod_{P \in \operatorname{Min} B} B/P$ is $F$-finite over $A$ by Lemma 11. As $B_{\mathrm{red}} \to \prod_{P \in \operatorname{Min} B} B/P$ is finite injective, and $B_{\mathrm{red}}^{(e)} \otimes_{A^{(e)}} A$ is Noetherian, $B_{\mathrm{red}}$ is $F$-finite over $A$ by Lemma 5. As $B$ is Noetherian, $B$ is $F$-finite over $A$ by Lemma 7. □

REMARK 13. Fogarty asserted that an $\mathbb{F}_p$-algebra map $A \to B$ with $B$ Noetherian is $F$-finite if and only if the module of Kähler differentials $\Omega_{B/A}$ is a finite $B$-module [8, Proposition 1]. The 'only if' part is true and easy. The proof of 'if' part therein has a gap. Although $R_1$ in step (iii) is assumed to be Noetherian, it is not proved that $R'$ in step (iv) is Noetherian. The author does not know if this direction is true or not.

If, moreover, both $A$ and $B$ are Noetherian, the assertion is true. This is an immediate consequence of [1, Proposition 57].

## 3. Descent of $F$-finiteness

In this section, we prove a sort of descent theorem on $F$-finiteness of homomorphisms.

Let $R$ be a commutative ring, and $f\colon M \to N$ an $R$-linear map between $R$-modules. We say that $f$ is pure, if $1_W \otimes f\colon W \otimes_R M \to W \otimes_R N$ is injective for any $R$-module $W$. When we need to clarify the base ring $R$, we also say that $f$ is $R$-pure. A homomorphism of rings $A \to B$ is said to be pure (without mentioning the base ring), if it is $A$-pure (i.e., pure as an $A$-linear map).

**Lemma 14.** *Let $R$ be a commutative ring, $\varphi\colon M \to N$ and $h\colon F \to G$ be $R$-linear maps. If $\varphi$ is $R$-pure and $1_N \otimes h\colon N \otimes F \to N \otimes G$ is surjective, then $1_M \otimes h\colon M \otimes F \to M \otimes G$ is surjective.*

Proof. Let $C := \operatorname{Coker} h$. Then by assumption, $N \otimes C = 0$. By the injectivity of $\varphi \otimes 1_C\colon M \otimes C \to N \otimes C$, we have that $M \otimes C = 0$. □

**Corollary 15.** *Let $A \to B$ be a pure ring homomorphism, and $h\colon F \to G$ an $A$-linear map. If $1_B \otimes h\colon B \otimes_A F \to B \otimes_A G$ is surjective, then $h$ is surjective.*

**Lemma 16.** *Let $A \to B$ be a pure ring homomorphism, and $G$ an $A$-module. If $B \otimes_A G$ is a finitely generated $B$-module, then $G$ is finitely generated as an $A$-module.*

Proof. Let $\theta_1, \dots, \theta_r$ be generators of $B \otimes_A G$. Then we can write $\theta_j = \sum_{i=1}^{s} b_{ij} \otimes g_{ij}$ for some $s > 0$, $b_{ij} \in B$, and $g_{ij} \in G$. Let $F$ be the $A$-free module with the basis $\{ f_{ij} \mid 1 \le i \le s,\ 1 \le j \le r\}$, and $h\colon F \to G$ be the $A$-linear map given by $f_{ij} \mapsto g_{ij}$. Then by construction, $1_B \otimes h$ is surjective. By Corollary 15, $h$ is surjective, and hence $G$ is finitely generated. □

DEFINITION 17 (cf. [13, (2.7)]). Let $e > 0$ be an integer. An $\mathbb{F}_p$-algebra map $A \to B$ is said to be $e$-Dumitrescu if $\Phi_e(A, B)$ is $A$-pure.

**Lemma 18.** *Let $e, e' > 0$. If $A \to B$ is both $e$-Dumitrescu and $e'$-Dumitrescu, then it is $(e + e')$-Dumitrescu. In particular, an $e$-Dumitrescu map is $er$-Dumitrescu map for $r > 0$.*

Proof. This follows from [12, Lemma 4.1, **2**]. □

So a 1-Dumitrescu map is Dumitrescu (that is, $e$-Dumitrescu for all $e > 0$), see [13, Lemma 2.9].

**Lemma 19.** *Let $e > 0$.*
(1) [13, Lemma 2.8],
(2) [13, Lemma 2.12], *and*
(3) [13, Corollary 2.13]
*hold true when we replace all the 'Dumitrescu' therein by '$e$-Dumitrescu'.*

The proof is straightforward, and is left to the reader.

REMARK 20. The precise statement of Lemma 19 for (2) is as follows.

Let $f\colon A \to B$ be a ring homomorphism between rings of characteristic $p$, and $e > 0$ an integer. Assume that $A$ is Noetherian, and the image of the associated map ${}^a f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ contains $\operatorname{Max}(A)$, the set of maximal ideals of $A$. If $f$ is $e$-Dumitrescu, then $f$ is pure.

**Theorem 21.** *Let $f\colon A \to B$ and $g\colon B \to C$ be $\mathbb{F}_p$-algebra maps, and $e > 0$. Assume that $g$ is e-Dumitrescu, and the image of the associated map ${}^a g\colon \operatorname{Spec} C \to \operatorname{Spec} B$ contains the set of maximal ideals $\operatorname{Max} B$ of $B$. If $gf$ is $F$-finite, and $B$ and $C^{(e)} \otimes_{A^{(e)}} A$ are Noetherian, then $f$ is $F$-finite.*

Proof. Note that $\Phi_e(A, C)\colon C^{(e)} \otimes_{A^{(e)}} A \to C$ is a finite map. Note also that $C^{(e)} \otimes_{B^{(e)}} B$ is a $C^{(e)} \otimes_{A^{(e)}} A$-submodule of $C$ through $\Phi_e(B,C)$, since $\Phi_e(B,C)$ is $B$-pure and hence is injective. As $C^{(e)} \otimes_{A^{(e)}} A$ is Noetherian, $C^{(e)} \otimes_{B^{(e)}} B$, which is a submodule of the finite module $C$, is a finite $C^{(e)} \otimes_{A^{(e)}} A$-module. Since $g^{(e)}\colon B^{(e)} \to C^{(e)}$ is pure by Lemma 19 (2) (see Remark 20), $B^{(e)} \otimes_{A^{(e)}} A \to C^{(e)} \otimes_{A^{(e)}} A$ is also pure. Since

$$C^{(e)} \otimes_{B^{(e)}} B \cong (C^{(e)} \otimes_{A^{(e)}} A) \otimes_{B^{(e)} \otimes_{A^{(e)}} A} B$$

is a finite $C^{(e)} \otimes_{A^{(e)}} A$-module, $B$ is a finite $B^{(e)} \otimes_{A^{(e)}} A$-module by Lemma 16. □

A homomorphism $f\colon A \to B$ between Noetherian rings is said to be reduced if $f$ is flat with geometrically reduced fibers.

**Corollary 22.** *Let $g\colon B \to C$ be a faithfully flat reduced homomorphism between Noetherian $\mathbb{F}_p$-algebras. If $C$ is $F$-finite, then $B$ is $F$-finite.*

Proof. By [5, Theorem 3], $g$ is Dumitrescu. As $g$ is faithfully flat, ${}^a g\colon \operatorname{Spec} C \to \operatorname{Spec} B$ is surjective. Letting $A = \mathbb{F}_p$ and $f\colon A \to B$ be the unique map, the assumptions of Theorem 21 are satisfied, and hence $f$ is $F$-finite. That is, $B$ is $F$-finite. □

**Corollary 23** (Seydi [26])**.** *Let $(B, \mathfrak{m})$ be a Nagata local ring with the $F$-finite residue field $k = B/\mathfrak{m}$. Then $B$ is $F$-finite. In particular, $B$ is excellent, and is a homomorphic image of an $F$-finite regular local ring. So $B$ has a dualizing complex.*

Proof. Let $g\colon B \to C = \hat{B}$ be the completion of $B$. Then $C$ is a complete local ring with the residue field $k$. By Example 9 (3), $C$ is $F$-finite. As $g$ is reduced by [10, (7.6.4), (7.7.2)], $B$ is $F$-finite by Corollary 22.

Now $B$ is excellent by [16, Theorem 2.5] and is a homomorphic image of an $F$-finite regular local ring by [9, Remark 13.6]. The last assertion follows from the fact that a homomorphic image of a Gorenstein ring has a dualizing complex if it is of finite Krull dimension. For dualizing complexes, see [11]. □

Even if $A \to B$ is a faithfully flat reduced homomorphism and $B$ is excellent, $A$ need not be quasi-excellent. There is a Nagata local ring $A$ which is not quasi-excellent [24], [22], and its completion $A \to \hat{A} = B$ is an example. On the other hand, if $A \to B$ is a faithfully flat regular homomorphism and $B$ is quasi-excellent, then $A$ is quasi-excellent [19, Theorem 32.2].

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## References

[1] M. André: *Homologie de Frobenius*, Math. Ann. **290** (1991), 129–181.
[2] M. André: *Homomorphismes réguliers en caractéristique p*, C.R. Acad. Sci. Paris Sér. I Math. **316** (1993), no. 7, 643–646.
[3] M. André: *Autre démonstration de théorème liant régularité et platitude en caracteristique p*, Manuscripta Math. **82** (1994), 363–379.
[4] T. Dumitrescu: *On a theorem of N. Radu and M. André*, Stud. Cerc. Mat. **46** (1994), 445–447.
[5] T. Dumitrescu: *Reducedness, formal smoothness and approximation in characteristic p*, Comm. Algebra **23** (1995), 1787–1795.
[6] T. Dumitrescu and C. Ionescu: *Regularity and finite injective dimension in characteristic $p > 0$*, Studia Sci. Math. Hungar. **47** (2010), 108–112.
[7] F. Enescu: *On the behavior of F-rational rings under flat base change*, J. Algebra **233** (2000), 543–566.
[8] J. Fogarty: *Kähler differentials and Hilbert's fourteenth problem for finite groups*, Amer. J. Math. **102** (1980), 1159–1175.
[9] O. Gabber: *Notes on some t-structures*; in Geometric Aspects of Dwork Theory, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, 711–734.
[10] A. Grothendieck: *Éléments de géométrie algébrique* IV, Seconde partie, Publ. Math. IHES **24**, (1965).
[11] R. Hartshorne: Residues and Duality, Lecture Notes in Mathematics **20**, Springer, Berlin, 1966.
[12] M. Hashimoto: *Cohen–Macaulay F-injective homomorphisms*; in Geometric and Combinatorial Aspects of Commutative Algebra (Messina, 1999), Dekker, New York, 2001, 231–244.
[13] M. Hashimoto: *F-pure homomorphisms, strong F-regularity, and F-injectivity*, Comm. Algebra **38** (2010), 4569–4596.
[14] M. Hochster and C. Huneke: *Tight closure and strong F-regularity*, Mém. Soc. Math. France (N.S.) **38** (1989), 119–133.
[15] J.C. Jantzen: Representations of Algebraic Groups, second edition, Amer. Math. Soc., Providence, RI, 2003.
[16] E. Kunz: *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013.
[17] J. Marot: *Sur les anneaux universellement japonais*, Bull. Soc. Math. France **103** (1975), 103–111.
[18] H. Matsumura: Commutative Algebra, second edition, W.A. Benjamin, Inc., New York, 1980.
[19] H. Matsumura: Commutative Ring Theory, first paperback edition, Cambridge Univ. Press, Cambridge, 1989.
[20] M. Nagata: Local Rings, corrected reprint, Robert E. Krieger Publishing Co., Huntington, NY, 1975.
[21] J.-i. Nishimura: *On ideal-adic completion of Noetherian rings*, J. Math. Kyoto Univ. **21** (1981), 153–169.
[22] J.-i. Nishimura: *A few examples of local rings*, I, Kyoto J. Math. **52** (2012), 51–87.
[23] N. Radu: *Une classe d'anneaux noethériens*, Rev. Roumaine Math. Pures Appl. **37** (1992), 79–82.
[24] C. Rotthaus: *Nicht ausgezeichnete, universell japanische Ringe*, Math. Z. **152** (1977), 107–125.
[25] C. Rotthaus: *Komplettierung semilokaler quasiausgezeichneter Ringe*, Nagoya Math. J. **76** (1979), 173–180.

[26] H. Seydi: *Sur la théorie des anneaux excellents en caractéristique p*, Bull. Sci. Math. (2) **96** (1972), 193–198.

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