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GLOBAL DYNAMICS OF A MULTI-GROUP EPIDEMIC MODEL WITH GENERAL EXPOSED DISTRIBUTION AND RELAPSE

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Abstract

In this paper, we investigate a class of multi-group epidemic models with general exposed distribution and relapse. Nonlinear incidence rate is used between compartments. It is showed that global dynamics are completely determined by the threshold parameter R_0 under suitable conditions. More specifically, the disease will die out if $R_0 \leq 1$ and that if $R_0 > 1$, the disease persists in all groups. The approaches used here, are the theory of non-negative matrices, persistence theory in dynamical systems and graph-theoretical approach to the method of Lyapunov functionals. Furthermore, our results demonstrate that heterogeneity and nonlinear incidence rate do not alter the dynamical behavior of the SIR model with general exposed distribution and relapse. On the other hand, our global dynamical results exclude the existence of Hopf bifurcation leading to sustained oscillatory solutions.

1. Introduction

For classical SIR epidemic models, the host population is divided into three disjoint classes called susceptible (S), infective (I) and removed (R). However, it is pointed that many diseases have latency [1]. Susceptible individuals infected with the disease but not yet infective are in the exposed (latent) class. After surviving the latent period, these individuals pass into the infective class, and then recover into the removed class. A fixed latent period can be considered as an approximation of the mean latent period, and this would be appropriate for those diseases whose latent periods vary only relatively slightly. For example, poliomyelitis has a latent period of 1–3 days comparing to its much longer infectious period of 14–20 days (see e.g., Table 3.1 in Anderson and May [1]). However, disease such as tuberculosis, including bovine tuberculosis (a disease spread from animal to animal mainly by direct contact) may take months to develop to the infectious stage, and also can relapse. Since the time it takes from

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the moment of new infection to the moment of becoming infectious may differ from disease to disease; even for the same disease, it differs from individual to individual, it is thus of interest to account for length of the latent period as a random variable. Many relapse phenomenon of disease observed in clinical study is an important feature of some animal and human diseases (see details in [3, 10, 17, 27]), for example, herpes, removed individuals may revert back to the infective class due to reactivation of the latent infection or incomplete treatment [8, 32].

Van den Driessche et al. [31] formulated and studied following more realistic model by considering a general exposed distribution function for the length of the latent period and the possibility of relapse:

$$\begin{cases} \dfrac{dS(t)}{dt} = bN - \beta S(t)\dfrac{I(t)}{N} - bN, \\ E(t) = \displaystyle\int_0^t \beta S(\xi)\frac{I(\xi)}{N} e^{-b(t-\xi)} P(t-\xi)\, d\xi, \\ \dfrac{dR(t)}{dt} = rI(t) - (\alpha + b)R(t), \\ I(t) = N - S(t) - E(t) - R(t), \end{cases} \tag{1.1}$$

where N is the size of the population; $S(t)$, $E(t)$, $I(t)$ and $R(t)$ are the population sizes of susceptible, exposed, infective, and removed classes, respectively; $\alpha > 0$ is a constant rate at which an individual in the recovered class reverts to the infective class; $b > 0$ is the recruitment rate and the removal rate (this guarantees that N can be assumed as a constant populations); $\beta > 0$ denotes the average number of effective contacts of an infective individual per unit time, and $r > 0$ is the rate at which infective individuals recover. $P(t)$ is the probability (without taking death into account) that an exposed individual still remains in the exposed class t time units after entering the exposed class. It is biologically reasonable to assume in [31] that $P(t)$ satisfies the following reasonable properties:

($\mathbf{H}_1$) $P: [0, \infty) \to [0, 1]$ is nonincreasing, piecewise continuous with possibly finitely many jumps and satisfies $P(0^+) = 1$; $\lim_{t\to\infty} P(t) = 0$ with $\int_0^\infty P(t)\, dt$ is positive and finite.

For model (1.1), van den Driessche et al. [31] have shown that the disease-free equilibrium (DFE) is globally asymptotically stable (GAS) if $R_0 < 1$ (see Theorem 3.1 in [31]). One special case with a constant exposed period (the resulting model reduces to a discrete delay differential equation system), they have proved that the system is uniformly persistent and the endemic equilibrium (EE) is locally asymptotically stable (LAS) if $R_0 > 1$ (see Theorems 5.1 and 5.2 in [31]).

Incorporating a nonlinear incidence function into (1.1), Liu et al. [23] studied the following model

$$
(1.2)\qquad
\begin{cases}
\dfrac{dS(t)}{dt} = b - f(S(t))I(t) - bS(t),\\
\dfrac{dE(t)}{dt} = f(S(t))I(t) + \displaystyle\int_0^t f(S(\xi))I(\xi)e^{-b(t-\xi)}\, d_t P(t-\xi)\, d\xi - bE(t),\\
\dfrac{dI(t)}{dt} = -\displaystyle\int_0^t f(S(\xi))I(\xi)e^{-b(t-\xi)}\, d_t P(t-\xi)\, d\xi + \alpha R(t) - (r+b)I(t),\\
\dfrac{dR(t)}{dt} = rI(t) - (\alpha + b)R(t),
\end{cases}
$$

where integrals are in the Riemann–Stieltjes sense and the nonlinear function $f(S(t))$ is assumed to satisfy:

(H$_2$) $f: \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable with $f(0) = 0$, $f'(S) > 0$ for all $S > 0$.

The authors in [23] shown that global threshold dynamics determined in terms of the basic reproduction number R_0 of the model: if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, whereas if $R_0 > 1$, a unique endemic equilibrium exists and is globally asymptotically stable.

In recent years, multi-group epidemic models have been proposed to describe the disease transmission dynamics of many infectious disease in heterogeneous environment, such as measles, mumps, gonorrhea, or to investigate infectious disease with multiple hosts such as West-Nile virus and vector borne diseases such as Malaria [11, 12]. For a heterogeneous host population, the disease can transmit within the same group as well as between groups. Thus host population can be divided into several homogeneous groups in terms of modes of transmission, contact patterns, education levels, ethnic backgrounds, gender, and professions etc. They can also be formed geographically, such as by schools, communities and cities. So that within-group and inter-group interactions could be modeled separately [21, 33, 34]. For more and detailed justifications for multi-group disease models and many different types of heterogeneity epidemic models, see, for example, [7, 11, 12, 14, 19, 21, 28, 33, 34] and the references cited therein.

In the present paper, a general multi-group epidemic model based on (1.2) is proposed to describe the disease spread in a heterogeneous host population with general exposed distribution and relapse. The host population is divided into n homogeneous groups. Let S_i, E_i, I_i and R_i denote the susceptible, infected but non-infectious, infectious, and removed populations in the i-th group, respectively. The disease incidence rate in the i-th group can be calculated as

$$\sum_{j=1}^{n} \beta_{ij} f(S_i(t))I_j(t),$$

where the sum takes into account cross-infections from all groups and β_{ij} represents the transmission coefficient between compartments S_i and I_j. Set parameters as:

b_i the recruitment rate and the removal rate in the i-th group;
α_i constant rate at which an individual in the removed class reverts to the infectious class in the i-th group;
r_i the rate at which infective individuals recover in the i-th group.

All parameter values are assumed to be nonnegative. Thus, based on system (1.2), the new n group model is given by the following nonlinear system of 4n-dimensional differential and integral equations:

$$\begin{cases} \dfrac{dS_i(t)}{dt} = b_i - \displaystyle\sum_{j=1}^{n} \beta_{ij} f(S_i(t))I_j(t) - b_i S_i(t), \\ \dfrac{dE_i(t)}{dt} = \displaystyle\sum_{j=1}^{n} \beta_{ij} f(S_i(t))I_j(t) \\ \qquad + \displaystyle\sum_{j=1}^{n} \int_0^t \beta_{ij} f(S_i(\xi))I_j(\xi)e^{-b_i(t-\xi)}\, d_t P_i(t-\xi)\, d\xi - b_i E_i(t), \\ \dfrac{dI_i(t)}{dt} = -\displaystyle\sum_{j=1}^{n} \int_0^t \beta_{ij} f(S_i(\xi))I_j(\xi)e^{-b_i(t-\xi)}\, d_t P_i(t-\xi)\, d\xi \\ \qquad + \alpha_i R_i(t) - (r_i + b_i)I_i(t), \\ \dfrac{dR_i(t)}{dt} = r_i I_i(t) - (\alpha_i + b_i)R_i(t), \quad i = 1, 2, \ldots, n. \end{cases} \tag{1.3}$$

The first term on the right hand side of second equation in (1.3) is the rate at which new infected individuals come into the exposed class, and the last term explains the natural deaths. The second term accounts for the rate at which the individuals move to the infectious class.

Examples of $f(S_i)I_j$ satisfying ($\mathbf{H}_2$) include common incidence functions such as $f(S_i)I_j = S_i I_j$, see e.g., [11, 13, 18]; $f(S_i)I_j = \eta S_i I_j/(1+\theta S_i)$, see e.g., [2]; $f(S_i)I_j = S_i^q I_j$, see e.g., [36].

Since $E_i(t)$, $i = 1, \ldots, n$, are decoupled from the S_i, I_i and R_i equations, we only need to consider the following sub-system of (1.3) consisting of only the S_i, I_i and R_i equations of (1.3):

$$\begin{cases} \dfrac{dS_i(t)}{dt} = b_i - \displaystyle\sum_{j=1}^{n} \beta_{ij} f(S_i(t))I_j(t) - b_i S_i(t), \\ \dfrac{dI_i(t)}{dt} = -\displaystyle\sum_{j=1}^{n} \int_0^t \beta_{ij} f(S_i(\xi))I_j(\xi)e^{-b_i(t-\xi)}\, d_t P_i(t-\xi)\, d\xi \\ \qquad + \alpha_i R_i(t) - (r_i + b_i)I_i(t), \\ \dfrac{dR_i(t)}{dt} = r_i I_i(t) - (\alpha_i + b_i)R_i(t), \quad i = 1, 2, \ldots, n. \end{cases} \tag{1.4}$$

Assume that each P_i satisfies assumption ($\mathbf{H}_1$) and f satisfies ($\mathbf{H}_2$). The contact matrix $B = (\beta_{ij})_{n\times n}$ encode the patterns of contact and transmission among groups that are built into the model. Associated to B, one can construct a directed graph $L = G(B)$ whose vertex i represents the i-th group, $i = 1, 2, \ldots, n$. A directed edge exists from vertex j to vertex i if and only if $\beta_{ij} > 0$. Throughout the paper, $B = (\beta_{ij})_{n\times n}$ is assumed to be non-negative and irreducible. Biologically, this is the same as assuming that any two groups i and j have a direct or indirect route of transmission. More specifically, individuals in I_j can infect ones in S_i directly or indirectly [11, 12, 21, 33, 34].

The organization of this paper is as follows. In the next section, we give some preliminaries of our main model (1.4). In Section 3, the global asymptotic stability of equilibria for $R_0 \le 1$ and $R_0 > 1$ is investigated, respectively. The proofs of the main results utilize the persistence theory in dynamical systems, Lyapunov functionals and a subtle grouping technique in estimating the derivatives of Lyapunov functionals guided by graph theory, which was recently developed in [11, 12, 21, 22]. For the convenience of the reader, we include in Appendix A results from graph theory that are needed for our proof.

2. Preliminaries

The initial condition of the model (1.3) is assumed to be given as

$$(S_i(0), E_i(0), I_i(0), R_i(0)) \in \mathbb{R}^{4n}_+; \quad S_i(0) + E_i(0) + I_i(0) + R_i(0) \le 1.$$

The Volterra integro-differential equation system (1.3) with properties ($\mathbf{H}_1$) satisfies the hypotheses stated by Miller ([24], p. 338) that are sufficient to ensure the existence, uniqueness and continuity of solutions. Moreover, it can be verified that every solution of (1.3) with nonnegative initial data remain nonnegative. In particular, $S_i(t) > 0$, for $t > 0$. From the first equation of (1.3), it follows that $S_i'(t) \le b_i - b_i S_i(t)$. Hence, $\limsup_{t\to\infty} S_i(t) \le 1$. For each i, adding the four equations in (1.3) gives

$$S_i'(t) + E_i'(t) + I_i'(t) + R_i'(t) = b_i - b_i(S_i(t) + E_i(t) + I_i(t) + R_i(t)),$$

which implies that, for each i, $\limsup_{t\to\infty}(S_i(t) + E_i(t) + I_i(t) + R_i(t)) = 1$. Denote

$$\Gamma = \{(S_i, E_i, I_i, R_i) \in \mathbb{R}^{4n} : \\ S_i, E_i, I_i, R_i > 0,\ S_i \le 1,\ S_i + E_i + I_i + R_i \le 1,\ i = 1, \ldots, n\}.$$

is the feasible region for (1.3), which is positively invariant with respect to model (1.3). All positive semi-orbits in Γ are precompact $\mathbb{R}^{4n}$ (see [4]), and thus have non-empty ω-limit sets. We have the following result.

Lemma 2.1. *All positive semi-orbits in Γ have non-empty ω-limit sets.*

Model (1.4) always admits a disease-free equilibrium (DFE) $P_0 = (S_1^0, 0, 0, 0, \ldots, S_n^0, 0, 0, 0)$ in Γ, where $(S_1^0, \ldots, S_n^0) = (1, \ldots, 1)$. Let

$$q_i = \lim_{t\to\infty} \int_0^t e^{-b_i\xi} P_i(\xi)\, d\xi,$$

which means the average latent period that an individual remains in the exposed class before becoming infective or dying, and we denote

$$Q_i := -\lim_{t\to\infty}\int_0^{\infty} e^{-b_i\xi}\, d_\xi P_i(\xi)\, d\xi. \tag{2.1}$$

Then, $0 < q_i < 1$ and $Q_i = 1 - b_i q_i \in (0, 1)$. Define

$$J_i(t) := -\int_t^{\infty} e^{-b_i\xi}\, d_\xi P_i(\xi)\, d\xi,$$

it follows that $J_i(t) \geq 0$, $\forall t > 0$ and $J_i(0) = Q_i > 0$.

Following the method of Diekmann et al. [9], the basic reproduction number R_0 is defined as the expected number of secondary cases produced in an entirely susceptible population by a typical infected individual during its entire infectious period. Its biological significance is that if $R_0 < 1$ the disease dies out while if $R_0 > 1$ the disease becomes endemic (also see Thieme [29], van den Driessche and Watmough [30]). For model (1.4), we obtain

$$\mathcal{F} = \begin{pmatrix} Q_1 f(S_1^0)\beta_{11} & \cdots & Q_1 f(S_1^0)\beta_{1n} \\ \vdots & \ddots & \vdots \\ Q_n f(S_n^0)\beta_{n1} & \cdots & Q_n f(S_n^0)\beta_{nn} \end{pmatrix}$$

and

$$\mathcal{V} = \operatorname{diag}\left(\frac{b_i(\alpha_i + b_i + r_i)}{\alpha_i + b_i}\right),$$

then the next generation matrix is

$$\mathcal{F}\mathcal{V}^{-1} = \begin{pmatrix} \dfrac{Q_1 f(S_1^0)\beta_{11}(\alpha_1 + b_1)}{b_1(\alpha_1 + b_1 + r_1)} & \cdots & \dfrac{Q_1 f(S_1^0)\beta_{1n}(\alpha_n + b_n)}{b_n(\alpha_n + b_n + r_n)} \\ \vdots & \ddots & \vdots \\ \dfrac{Q_n f(S_n^0)\beta_{n1}(\alpha_1 + b_1)}{b_1(\alpha_1 + b_1 + r_1)} & \cdots & \dfrac{Q_n f(S_n^0)\beta_{nn}(\alpha_n + b_n)}{\alpha_n + b_n + r_n} \end{pmatrix},$$

and hence the basic reproduction number of model (1.4) is calculated by the spectral radius of the next generation matrix

$$\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}).$$

It is well known that $\rho(\mathcal{F}\mathcal{V}^{-1}) = \rho(\mathcal{V}^{-1}\mathcal{F})$. Thus

$$\mathcal{R}_0 = \rho(M^0), \tag{2.2}$$

where

$$M^0 = \mathcal{V}^{-1}\mathcal{F} = \begin{pmatrix} \dfrac{Q_1 f(S_1^0)\beta_{11}(\alpha_1 + b_1)}{b_1(\alpha_1 + b_1 + r_1)} & \cdots & \dfrac{Q_1 f(S_1^0)\beta_{1n}(\alpha_1 + b_1)}{b_1(\alpha_1 + b_1 + r_1)} \\ \vdots & \ddots & \vdots \\ \dfrac{Q_n f(S_n^0)\beta_{n1}(\alpha_n + b_n)}{b_n(\alpha_n + b_n + r_n)} & \cdots & \dfrac{Q_n f(S_n^0)\beta_{nn}(\alpha_n + b_n)}{\alpha_n + b_n + r_n} \end{pmatrix}.$$

Note that (1.4) may not have an endemic equilibrium (EE) for finite time t. According to statements in [24], if (1.4) has an EE, then it must satisfy the limiting system given by

$$\text{(2.3)} \qquad \begin{cases} \dfrac{dS_i(t)}{dt} = b_i - \displaystyle\sum_{j=1}^{n} \beta_{ij} f(S_i(t))I_j(t) - b_i S_i(t), \\ \dfrac{dI_i(t)}{dt} = -\displaystyle\sum_{j=1}^{n} \int_0^{\infty} \beta_{ij} f(S_i(\xi))I_j(\xi)e^{-b_i(t-\xi)}\, d_t P_i(t-\xi)\, d\xi \\ \qquad\qquad + \alpha_i R_i(t) - (r_i + b_i)I_i(t), \\ \dfrac{dR_i(t)}{dt} = r_i I_i(t) - (\alpha_i + b_i)R_i(t), \quad i = 1, 2, \ldots, n. \end{cases}$$

Since the limiting model (2.3) contains an infinite delay, its associated initial condition needs to be restricted in an appropriate fading memory space. For any $\lambda_i \in (0, b_i)$, define the following Banach space of fading memory type (see e.g., [15, 16] and references therein):

$$C_i = \Big\{\phi \in C((-\infty, 0], \mathbb{R}) : \phi(s)e^{\lambda_i s} \text{ is uniformly continuous on } (-\infty, 0], \text{ and } \sup_{s\le 0}|\phi(s)|e^{\lambda_i s} < \infty\Big\},$$

and

$$Y_\Delta = \{\phi_i \in C_i : \phi_i(s) \ge 0 \text{ for all } s \le 0\}$$

with norm $\|\phi\|_i = \sup_{s\le 0}|\phi(s)|e^{\lambda_i s}$. Let $\phi, \varphi \in C_i$ be such that $\phi_t(s) = \phi(t+s)$, $\varphi_t(s) = \varphi(t+s)$, $s \in (-\infty, 0]$. Let ϕ_i, $\varphi_i \in C_i$ and $R_{i,0} \in \mathbb{R}_+$ such that $\phi_i(s) \ge 0$, $\varphi_i(s) \ge 0$, $s \in (-\infty, 0]$. We consider solutions of model (2.3), $(S_{1t}, I_{1t}, R_1(t), \ldots, S_{nt}, I_{nt}, R_n(t))$, with initial conditions

$$\text{(2.4)} \qquad S_{i0} = \phi_i, \quad I_{i0} = \varphi_i, \quad R_i(0) = R_{i,0}, \quad i = 1, 2, \ldots, n.$$

Standard theory of functional differential equations [16] implies $S_{it}, I_{it} \in C_i$ for $t > 0$. We consider model (2.3) in the phase space

$$X = \prod_{i=1}^{n}(C_i \times C_i \times \mathbb{R}).$$

It can be verified that solutions of (2.3) in X with initial conditions (2.4) remain nonnegative.

An equilibrium $P^* = (S_1^*, I_1^*, R_1^*, \ldots, S_n^*, I_n^*, R_n^*)$ in the interior of Γ is called an endemic equilibrium, where $S_i^*, I_i^*, R_i^* > 0$ satisfy the equilibrium equations

$$b_i - \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j^* - b_i S_i^* = 0, \tag{2.5}$$

$$\sum_{j=1}^{n} \beta_{ij} Q_i f(S_i^*) I_j^* + \alpha_i R_i - (r_i + b_i) I_i^* = 0, \tag{2.6}$$

$$r_i I_i^* - (\alpha_i + b_i) R_i^* = 0. \tag{2.7}$$

Next we will give our main results.

3. Main results

Denote

$$H(u) = u - 1 - \ln u, \quad \forall u > 0, \tag{3.1}$$

then we have $H(u) \geq 0$ and $H(u) = 0$ if and only if $u = 1$.

3.1. Global dynamics of disease free equilibrium.

Theorem 3.1. *Assume that each P_i satisfies* ($\mathbf{H}_1$), *f satisfies* ($\mathbf{H}_2$), *and the matrix $B = (\beta_{ij})_{n\times n}$ is irreducible. The following results hold for model* (1.4) *with R_0 given by* (2.2):

(i) *If $R_0 \leq 1$, then the DFE is globally asymptotically stable.*

(ii) *If $R_0 > 1$, then the DFE is unstable.*

Proof. Since $B = (\beta_{ij})_{n\times n}$ is irreducible, the nonnegative matrix

$$M^0 = \left(\frac{Q_i \beta_{ij} f(S_i^0)(\alpha_i + b_i)}{b_i(\alpha_i + r_i + b_i)} \right)_{n\times n}$$

is also irreducible, and M^0 has a positive left eigenvector $(\omega_1, \omega_2, \ldots, \omega_n)$ corresponding to the spectral radius $R_0 = \rho(M^0) \leq 1$. Let

$$c_i = \frac{\omega_i(\alpha_i + b_i)}{b_i(\alpha_i + b_i + r_i)} > 0.$$

Consider a Lyapunov functional

$$L_{\mathrm{DFE}} = \sum_{i=1}^{n} c_i \left[Q_i f(S_i^0) H\left(\frac{f(S_i(t))}{f(S_i^0)} \right) + I_i(t) + \sum_{j=1}^{n} \beta_{ij} U_+ + \frac{\alpha_1}{\alpha_i + b_i} R_i(t) \right],$$

where U^+ is given as $\int_0^t J_i(\xi) f(S_i(t-\xi))I_j(t-\xi)\,d\xi$.

By (3.1) and assumption ($\mathbf{H}_2$), we know that $L_1 \geq 0$ with equality if and only if $S_i(t) = S_i^0$, $I_i(t) = 0$, $R_i(t) = 0$ and $J_i(\xi) f(S_i(t-\xi))I_j(t-\xi) = 0$ for almost all $\xi \geq 0$. Differentiating U_+ along the solution of model (1.4) and using integration by parts, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_0^t J_i(\xi) f(S_i(t-\xi))I_j(t-\xi)\,d\xi\right)\\
&= J_i(t)S_i(0)I_j(0) + \int_0^t J_i(\xi)\frac{\partial}{\partial t}(f(S_i(t-\xi))I_j(t-\xi))\,d\xi\\
&= J_i(t)S_i(0)I_j(0) - \int_0^t J_i(\xi)\frac{\partial}{\partial \xi}(f(S_i(t-\xi))I_j(t-\xi))\,d\xi\\
&= Q_i\, f(S_i(t))I_j(t) + \int_0^t f(S_i(t-\xi))I_j(t-\xi)e^{-b_i\xi}\, d_\xi P_i(\xi)\,d\xi.
\end{aligned}$$

Thus the derivative of L_{DFE} is given as

$$\begin{aligned}
L'_{\mathrm{DFE}}|_{(1.4)} &= \sum_{i=1}^n c_i\Bigg[Q_i\left[\frac{f(S_i(t)) - f(S_i^0)}{f(S_i(t))}\right]\left[b_i - b_i S_i(t) - \sum_{j=1}^n \beta_{ij}\, f(S_i(t))I_j(t)\right]\\
&\quad - \sum_{j=1}^n \int_0^t \beta_{ij}\, f(S_i(t-\xi))I_j(t-\xi)e^{-b_i(\xi)}\, d_\xi P_i(\xi)\,d\xi\\
&\quad + \alpha_i R_i(t) - (r_i + b_i)I_i(t) + \sum_{j=1}^n \beta_{ij} Q_i\, f(S_i(t))I_j(t)\\
&\quad + \sum_{j=1}^n \int_0^t \beta_{ij}\, f(S_i(t-\xi))I_j(t-\xi)e^{-b_i\xi}\, d_\xi P_i(\xi)\,d\xi\\
&\quad + \frac{\alpha_i}{\alpha_i + b_i}(r_i I_i(t) - (\alpha_i + b_i)R_i(t))\Bigg] \qquad (3.2)\\
&= \sum_{i=1}^n c_i Q_i\left[\frac{f(S_i(t)) - f(S_i^0)}{f(S_i(t))}\right]\left[b_i - b_i S_i(t)\right]\\
&\quad + \sum_{i=1}^n \frac{\omega_i(\alpha_i + b_i)}{b_i(\alpha_i + b_i + r_i)}\left(\sum_{j=1}^n \beta_{ij} Q_i\, f(S_i^0)I_j - \frac{b_i(\alpha_i + b_i + r_i)}{(\alpha_i + b_i)} I_i(t)\right)\\
&= \sum_{i=1}^n c_i Q_i\left[\frac{f(S_i(t)) - f(S_i^0)}{f(S_i(t))}\right][b_i - b_i S_i(t)] + (\omega_1, \omega_2, \ldots, \omega_n)(M^0 I - I)\\
&\leq (\rho(M^0) - 1)(\omega_1, \omega_2, \ldots, \omega_n)I \leq 0, \quad \text{if} \quad R_0 \leq 1.
\end{aligned}$$

Here $I(t) = (I_1(t), I_2(t), \ldots, I_n(t))^T$. Let

$$Y = \{(S_1, I_1, R_1, \ldots, S_n, I_n, R_n) \colon L'_{\text{DFE}}|_{(1.4)} = 0\},$$

and Z be the largest compact invariant set in Y. We will show $Z = (S_1^0, 0, 0, \ldots, S_n^0, 0, 0)$. From inequality (3.2) and $c_i > 0$, $L'_{\text{DFE}}|_{(1.4)} = 0$ implies

$$\left[\frac{f(S_i(t)) - f(S_i^0)}{f(S_i(t))}\right][b_i - b_i S_i(t)] = 0,$$

and thus $S_i(t) = S_i^0 = 1$. Hence, from the first equation of (1.4), we obtain

$$\sum_{j=1}^{n} \beta_{ij} f(S_i(t)) I_j(t) = 0,$$

and thus

$$\beta_{ij} f(S_i(t)) I_j(t) = 0,$$

for all $t \geq 0$ and $1 \leq i, j \leq n$. Then, by irreducibility of B, for each j, there exists $i \neq j$ such that $\beta_{ij} \neq 0$, thus $I_j(t) = 0$, $j = 1, \ldots, n$. Therefore $Z = (S_1^0, 0, 0, \ldots, S_n^0, 0, 0)$. Using Lemma 2.1 and the LaSalle–Lyapunov theorem (see Theorem 3.4.7 of [20] or Theorem 5.3.1 of [15]), we conclude that $(S_1^0, 0, 0, \ldots, S_n^0, 0, 0)$ globally attracts all the solutions of model (1.4) if $R_0 \leq 1$.

If $R_0 > 1$ and $I(t) \neq 0$, it follows that $(\rho(M^0) - 1)(\omega_1, \omega_2, \ldots, \omega_n) I > 0$, which implies that, in a sufficiently small enough neighborhood of $(S_1^0, 0, 0, \ldots, S_n^0, 0, 0)$, $L'_{\text{DFE}}|_{(1.4)} > 0$. Therefore, $(S_1^0, 0, 0, \ldots, S_n^0, 0, 0)$ is unstable when $R_0 > 1$. □

3.2. Disease persistence. In this subsection, we obtain some global information about the disease in terms of persistence and show that if $\rho(M^0) > 1$, the disease will persist in all groups. This conclusion together with a well-known result for persistent systems actually implies the existence of an endemic equilibrium for the model (2.3).

For convenience, the positive solution of (2.3) is denoted by

$$\begin{aligned} &S(t, \phi, \varphi, R(0)), I(t, \phi, \varphi, R(0)), R(t, \phi, \varphi, R(0)) \\ &= (S_1(t, \phi, \varphi, R(0)), \ldots, S_n(t, \phi, \varphi, R(0)), \\ &\quad I_1(t, \phi, \varphi, R(0)), \ldots, I_n(t, \phi, \varphi, R(0)), \\ &\quad R_1(t, \phi, \varphi, R(0)), \ldots, R_n(t, \phi, \varphi, R(0))), \end{aligned}$$

whose components are all positive and bounded for $t > 0$.

Theorem 3.2. *Assume that $\rho(M^0) > 1$. Then there exist an $\bar{\varepsilon}$ such that for every $(\phi, \varphi, R(0)) \in Y_\Delta \times Y_\Delta \times \mathbb{R}$ with $\varphi(0) > 0$, the solution $(S(t), I(t), R(t))$ of (2.3) satisfies*

$$\liminf_{t \to \infty} I_i(t, \phi, \varphi, R(0)) \geq \bar{\varepsilon}, \quad i = 1, 2, \ldots, n.$$

Moreover, the model (2.3) *admits at least one positive equilibrium.*

Proof. Define

$$X = \{(\phi, \varphi, R(0)) \in Y_\Delta \times Y_\Delta \times \mathbb{R}\},$$

$$X_0 = \{(\phi, \varphi, R(0)) \in Y_\Delta \times Y_\Delta \times \mathbb{R} \in X : \varphi_i(0) > 0,\ i = 1, 2, \dots, n\},$$

and

$$\partial X_0 = X \backslash X_0.$$

It then suffices to prove that (2.3) is uniformly persistent with respect to $(X_0, \partial X_0)$.

Let $\Phi(t)\colon X \to X$ be the solution semiflow of (2.3), that is,

$$\Phi(t)(\phi, \varphi, R(0)) = (S_t(\phi, \varphi, R(0)), I_t(\phi, \varphi, R(0), R(t, R(0))).$$

It follows from Lemma 2.1 that both X and X_0 are positively invariant for $\Phi(t)$. Clearly, $\partial X_0 = \{(\phi, \varphi, R(0)) \in X : \varphi_i(0) = 0\}$ for at least one $i \in \{1, 2, \dots, n\}$ and it is relatively closed in X. Furthermore, model (2.3) is point dissipative in Γ.

Define

$$\Omega_\partial = \{(\phi, \varphi, R(0)) \in X : (S_t(\phi, \varphi, R(0)), I_t(\phi, \varphi, R(0), R(t, R(0))) \in \partial X_0\}.$$

We next show that

$$\Omega_\partial = \{(\phi, \varphi, R(0)) \in \partial X_0 : I_i(t, \phi, \varphi, R(0)) = 0,\ \forall t \geq 0\}. \tag{3.3}$$

Assume $(\phi, \varphi, R(0)) \in \Omega_\partial$. It suffices to show that $I_i(t, \phi, \varphi, R(0)) = 0,\ \forall t \geq 0$. Suppose this is not true, then there exists an i_0, $0 \leq i_0 \leq n$, and a $t_0 \geq 0$ such that $I_{i_0}(t_0) > 0$. Thus set $\{1, 2, \dots, n\}$ can be departed into Q_1 and Q_2 such that

$$I_i(t_0, \phi, \varphi, R(0)) = 0, \quad \forall i \in Q_1; \quad I_i(t_0, \phi, \varphi, R(0)) > 0, \quad \forall i \in Q_2.$$

Obviously, Q_1 is non-empty due to the definition of Ω_∂ and Q_2 is also non-empty since $I_{i_0}(t_0, \phi, \varphi, R(0)) > 0$. For any $j \in Q_1$, by the irreducibility of the matrix (β_{ij}), there is an $i_1 \in Q_2$ such that

$$\begin{aligned}\frac{dI_j(t)}{dt}\Big|_{t=t_0} = -\sum_{i=1}^{n} \int_0^\infty \beta_{ji}\, f(S_j(t_0 - \xi)) I_i(t_0 - \xi) e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi \\ + \alpha_j R_i(t_0) - (r_i + b_i) I_j(t_0) > 0.\end{aligned}$$

It follows that there is an ϵ_0 such that $I_j(t) > 0$ for $j \in Q_1$ and $t_0 < t < t_0 + \epsilon_0$. Clearly, we can restrict $\epsilon_0 > 0$ small enough such that $I_i(t) > 0$ for $i \in Q_2$ and $t_0 < t < t_0 + \epsilon_0$. This means that $(S_t(\phi, \varphi, R(0)), I_t(\phi, \varphi, R(0), R(t, R(0))) \notin \partial X_0$ for $t_0 < t < t_0 + \epsilon_0$, which contradicts the assumption that $(\phi, \varphi, R(0)) \in \Omega_\partial$. This proves (3.3).

Let us consider the following linear system

$$\frac{dS_i(t)}{dt} = b_i - \sum_{j=1}^{n} \beta_{ij} f(S_i(t))\bar{\varepsilon} - b_i S_i(t), \quad i = 1, 2, \ldots, n. \tag{3.4}$$

Note that for any $\bar{\varepsilon} > 0$ small enough such that (3.4) admits a unique positive equilibrium $(S_1^0(\bar{\varepsilon}), \ldots, S_n^0(\bar{\varepsilon}))$ which is globally asymptotically stable. By the implicit function theorem, it follows that $(S_1^0(\bar{\varepsilon}), \ldots, S_n^0(\bar{\varepsilon}))$ is continuous in ξ_1. Thus, we can further restrict $\bar{\varepsilon}$ small enough such that $(S_1^0(\bar{\varepsilon}), \ldots, S_n^0(\bar{\varepsilon})) > (S_1^0 - \eta, \ldots, S_n^0 - \eta)$.

Next we claim that

$$\limsup_{t\to\infty} \max\{I_i(t, \phi, \varphi, R(0))\} > \bar{\varepsilon}, \quad \text{for all} \quad (\phi, \varphi, R(0)) \in X_0. \tag{3.5}$$

Otherwise, there is a large enough $T_1 > 0$ such that $0 < I_i(t, \phi, \varphi, R(0)) \le \bar{\varepsilon}$, $i = 1, 2, \ldots, n$, for all $t \ge T_1$. Then for $t \ge T_1$, we have

$$\frac{dS_i(t)}{dt} \ge b_i - \sum_{j=1}^{n} \beta_{ij} f(S_i(t))\bar{\varepsilon} - b_i S_i(t), \quad i = 1, 2, \ldots, n. \tag{3.6}$$

Since the equilibrium $(S_1^0(\bar{\varepsilon}), \ldots, S_n^0(\bar{\varepsilon}))$ of (3.6) is globally asymptotically stable and $S^0(\bar{\varepsilon}) > S^0 - \eta$, there is a T_2 such that $S(t) > S^0 - \eta$ for $t \ge T_1 + T_2$. By the continuity of the function f, there exists a positive constant $T_3 > T_1 + T_2$ such that $f(S(t)) > f(S^0) - \eta$ for $t \ge T_3$. Further, we can choose $T_4 > T_3$ large enough such that $-\int_0^t e^{-b_i\xi}\, d_\xi P_i(\xi) > Q_i - \eta > 0$, $\forall t > T_4$. Thus, we can get

$$\rho(M_0(\eta)) = \rho\left(\frac{(Q_i - \eta)\beta_{ij}(f(S_i^0) - \eta)(\alpha_i + b_i)}{b_i(\alpha_i + r_i + b_i)}\right)_{n\times n} > 1, \quad \text{for sufficiently small } \eta.$$

Consequently, for $t \ge T_4$,

$$\begin{cases} \dfrac{dI_i(t)}{dt} \ge -\displaystyle\sum_{j=1}^{n} \int_0^{T_4} \beta_{ij} f(S_i(t-\xi)) I_j(t-\xi) e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi \\ \qquad + \alpha_i R_i(t) - (r_i + b_i) I_i(t), \\ \dfrac{dR_i(t)}{dt} = r_i I_i(t) - (\alpha_i + b_i) R_i(t), \quad i = 1, 2, \ldots, n. \end{cases}$$

Choose sufficiently large $T_5 > T_4$ such that $f(S_i(t - \xi)) > f(S_i^0) - \eta$, $\forall t \ge T_5$ and $\xi \in [0, T_4]$. Hence we have

$$\frac{dI_i(t)}{dt} \ge -(f(S_i^0) - \eta) \sum_{j=1}^{n} \int_0^{T_4} \beta_{ij} I_j(t-\xi) e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi$$
$$+ \alpha_i R_i(t) - (r_i + b_i) I_i(t),$$

for all $t \geq T_5$. By the mean value theorem for integrals we obtain that for any t, there exists a $\xi_t \in [t - T_1, t]$ such that

$$\int_0^{T_4} I_j(t-\xi)e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi = -I_j(\xi_t)\int_0^{T_4} e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi$$
$$> I_j(\xi_t)(Q_i - \varepsilon).$$

Then we get

$$\frac{dI_i(t)}{dt} \geq (f(S_i^0) - \eta)(Q_i - \varepsilon)\sum_{j=1}^{n} \beta_{ij} I_j(\xi_t) + \alpha_i R_i(t) - (r_i + b_i)I_i(t),$$

for all $t \geq T_5$.

Then by a standard comparison argument and the nonnegativity, we know that the assumption $\rho(M_0(\eta)) > 1$ implies that the trivial solution of linear system

$$\begin{cases} \dfrac{dI_i(t)}{dt} \geq (f(S_i^0) - \eta)(Q_i - \varepsilon)\displaystyle\sum_{j=1}^{n} \beta_{ij} I_j(\xi_t) + \alpha_i R_i(t) - (r_i + b_i)I_i(t), \\ \dfrac{dR_i(t)}{dt} = r_i I_i(t) - (\alpha_i + b_i)R_i(t), \quad i = 1, 2, \ldots, n, \quad \text{for all} \quad t \geq T_5 \end{cases}$$

is unstable. This together with (3.7) and the comparison theorem implies that there is at least one $i \in 1, \ldots, n$ such that $I_i(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of solutions. Therefore (3.5) holds. □

Note that $(S_1^0, \ldots, S_n^0)$ is globally asymptotically stable in $\mathbb{R}_+^n/\{0\}$ for system (3.6). By the afore-mentioned claim, it then follows that $(S^0, 0, 0)$ is an isolated invariant set in X, and $W^s(S^0, 0, 0) \cap X_0 = \emptyset$. Clearly, every orbit in Ω_∂ converge to $(S^0, 0, 0)$, and $(S^0, 0, 0)$ is the only invariant set in Ω_∂. By Theorem 4.6 in [29] for a stronger repelling property of ∂X_0, we conclude that system (2.3) is indeed uniformly persistent with respect to $(X_0, \partial X_0)$. Moreover, by theorem 2.4 in [35], system (2.3) has an equilibrium $(S_1^*, \ldots, S_n^*, I_1^*, \ldots, I_n^*, R_1^*, \ldots, R_n^*) \in X_0$.

Let $X(t) = (\phi_{it}, \varphi_{it}, R_i(t))$ be a solution of (2.3). By Lemma 2.1 and Theorem 3.2 and using similar arguments to [26], it follows that the ω-limit set Ω of X is nonempty, compact, and invariant and that Ω is the union of orbits of (2.3). By a similar argument as Lemma 4.1 in [26], we have:

Corollary 3.1. *Suppose that $R_0 > 1$ and $(\phi_{it}, \varphi_{it}, R_i(t))$ be a solution of* (2.3) *that lies in Ω, then there exists a positive constant $\bar{\varepsilon} > 0$ such that $\bar{\varepsilon} < S(t), I(t), R(t) < 1$ for all $t > 0$.*

REMARK 3.1. Uniform persistence of (2.3), together with uniform boundedness of solutions in the interior of Γ, implies the existence of a positive equilibrium of (2.3) (see Theorem 2.8.6 in [5]).

3.3. Global dynamics of endemic equilibrium.

Theorem 3.3. *Consider system* (2.3). *Assume that each* P_i *satisfies* ($\mathbf{H}_1$), f *satisfies* ($\mathbf{H}_2$), *and the matrix* $B = (\beta_{ij})_{n\times n}$ *is irreducible. If* $R_0 > 1$ *and* $(\phi_{it}, \varphi_{it}, R_i(t))$ *is a solution to* (2.3) *that lies in* Γ, *then*

$$\lim_{t\to\infty}(\phi_{it}, \varphi_{it}, R_i(t)) = P^* = (S_1^*, I_1^*, R_1^*, \ldots, S_n^*, I_n^*, R_n^*).$$

Proof. Let $P^* = (S_1^*, I_1^*, R_1^*, \ldots, S_n^*, I_n^*, R_n^*)$ denote the unique endemic equilibrium of model (2.3). Define a Lyapunov functional as

$$L_{EE} = Q_i L_S + L_I + U_- + \frac{\alpha_i}{\alpha_i + b_i} L_R,$$

where

$$L_S = \int_{S_i^*}^{S_i(t)} \frac{f(\lambda) - f(S_i^*)}{f(\lambda)}\, d\lambda, \quad L_I = I_i^* H\left(\frac{I_i}{I_i^*}\right), \quad L_R = R_i^* H\left(\frac{R_i}{R_i^*}\right),$$

and

$$U_- = \sum_{j=1}^{n} \int_0^{\infty} \beta_{ij} f(S_i^*) I_i^* J_i(\xi) H\left(\frac{f(S_i(t-\xi)) I_j(t-\xi)}{f(S_i^*) I_j^*}\right) d\xi.$$

The definition of the fading memory space, Lemma 2.1 and Corollary 3.1 imply L_{EE} is well-defined, that is, L_{EE} is bounded for all $t \geq 0$. It follows from Lemma 2.1 and assumption ($\mathbf{H}_2$) that $L_{EE} \geq 0$ with equality if and only if $S_i(t) = S_i^*$, $I_i(t) = I_i^*$, $R_i(t) = R_i^*$ and $S_i(t-\xi) = S_i^*$, $I_i(t-\xi) = I_i^*$ for almost all $\xi \geq 0$.

Differentiating L_S along the solution of model (2.3) and using equilibrium equations (2.5)–(2.7), we obtain

$$\begin{aligned}
&\frac{dL_S}{dt}\Big|_{(2.3)} \\
&= \frac{f(S_i) - f(S_i^*)}{f(S_i)}\left[b_i S_i^* + \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j^* - \sum_{j=1}^{n} \beta_{ij} f(S_i(t)) I_j(t) - b_i S_i(t)\right] \\
&= \frac{f(S_i) - f(S_i^*)}{f(S_i)}[b_i S_i^* - b_i S_i(t)] + \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j^* - \sum_{j=1}^{n} \beta_{ij} f(S_i(t)) I_j(t) \\
&\quad - \sum_{j=1}^{n} \beta_{ij} \frac{f^2(S_i^*) I_j^*}{f(S_i)} + \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j(t).
\end{aligned} \tag{3.7}$$

Differentiating L_I along the solution of model (2.3), we obtain

$$
\begin{aligned}
\left.\frac{dL_I}{dt}\right|_{(2.3)} &= \frac{I_i - I_i^*}{I_i}\Bigg[-\sum_{j=1}^{n}\int_0^{\infty}\beta_{ij}\, f(S_i(t-\xi))I_j(t-\xi)e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi \\
&\qquad + \alpha_i R_i(t) - \left(\sum_{j=1}^{n}\beta_{ij} Q_i\, f(S_i^*)\frac{I_j^*}{I_i^*} + \frac{\alpha_i R_i^*}{I_i^*}\right) I_i(t)\Bigg] \\
&= \sum_{j=1}^{n}\int_0^{\infty}\beta_{ij}\frac{I_i^* - I_i}{I_i} f(S_i(t-\xi))I_j(t-\xi)e^{-b_i(\xi)}\, d_\xi P_i(\xi)\, d\xi \\
&\quad + \alpha_i R_i(t) - \frac{\alpha_i R_i(t) I_i^*}{I_i} \\
&\quad - \sum_{j=1}^{n}\beta_{ij} Q_i\, f(S_i^*)\frac{I_j^*}{I_i^*} I_i(t) + \sum_{j=1}^{n}\beta_{ij} Q_i\, f(S_i^*) I_j^* - \frac{\alpha_i R_i^*}{I_i^*} I_i(t) + \alpha_i R_i^*.
\end{aligned}
\tag{3.8}
$$

Differentiating L_R along the solution of model (2.3) and using equilibrium equations (2.5)–(2.7), we obtain

$$
\begin{aligned}
\left.\frac{dL_R}{dt}\right|_{(2.3)} &= \frac{R_i - R_i^*}{R_i}\left[\frac{(\alpha_i + b_i)R_i^*}{I_i^*} I_i(t) - (\alpha_i + b_i)R_i(t)\right] \\
&= (\alpha_i + b_i)\left[R_i^* - R_i + \frac{R_i^* I_i}{I_i^*} - \frac{R_i^{*2} I_i(t)}{I_i^* R_i(t)}\right].
\end{aligned}
\tag{3.9}
$$

Differentiating U_- along the solution of model (2.3) and using integration by parts, we obtain

$$
\begin{aligned}
\left.\frac{dU_-}{dt}\right|_{(2.3)} &= \sum_{j=1}^{n}\int_0^{\infty}\beta_{ij}\, f(S_i^*)I_j^* J_i(\xi)\frac{d}{dt} H\left(\frac{f(S_i(t-\xi))I_j(t-\xi)}{f(S_i^*)I_j^*}\right) d\xi \\
&= -\sum_{j=1}^{n}\int_0^{\infty}\beta_{ij}\, f(S_i^*)I_j^* J_i(\xi)\frac{d}{d\xi} H\left(\frac{f(S_i(t-\xi))I_j(t-\xi)}{f(S_i^*)I_j^*}\right) d\xi \\
&= \sum_{j=1}^{n}\beta_{ij}\, f(S_i^*)I_i^* Q_i H\left(\frac{f(S_i(t))I_j(t)}{f(S_i^*)I_j^*}\right) \\
&\quad + \sum_{j=1}^{n}\int_0^{\infty}\beta_{ij}\, f(S_i^*)I_j^* e^{-b_i\xi}\, d_\xi P_i(\xi) H\left(\frac{f(S_i(t-\xi))I_j(t-\xi)}{f(S_i^*)I_j^*}\right) d\xi \\
&= \sum_{j=1}^{n}\beta_{ij} Q_i\, f(S_i(t))I_j(t) \\
&\quad + \sum_{j=1}^{n}\int_0^{\infty}\beta_{ij} e^{-b_i\xi}\, d_\xi P_i(\xi)\Bigg[f(S_i(t-\xi))I_j(t-\xi) \\
&\qquad\qquad - f(S_i^*)I_j^* \ln\frac{f(S_i(t))I_j(t)}{f(S_i(t-\xi))I_j(t-\xi)}\Bigg] d\xi.
\end{aligned}
\tag{3.10}
$$

Combining (3.7)–(3.9) yields

$$
\begin{aligned}
\frac{dL_{EE}}{dt}\bigg|_{(2.3)} =\ & Q_i \frac{f(S_i)-f(S_i^*)}{f(S_i)}[b_i S_i^* - b_i S_i(t)] + \alpha_i R_i^* \left[2 - \frac{R_i^* I_i}{I_i^* R_i} - \frac{R_i I_i^*}{I_i R_i^*}\right] \\
& + \sum_{j=1}^{n} \beta_{ij} Q_i f(S_i^*) I_j^* \left(2 - \frac{f(S_i(t)) I_j(t)}{f(S_i^*) I_j^*} - \frac{f(S_i^*)}{f(S_i)} + \frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*}\right) \\
& + \sum_{j=1}^{n} \int_0^{\infty} \beta_{ij} \frac{I_i^* - I_i}{I_i} f(S_i(t-\xi)) I_j(t-\xi) e^{-b_i(\xi)} d_\xi P_i(\xi)\, d\xi \\
& + \frac{dU_-}{dt}.
\end{aligned} \tag{3.11}
$$

Using (3.10), we rewrite (3.11) as

$$
\begin{aligned}
& \frac{dL_{EE}}{dt}\bigg|_{(2.3)} \\
& = Q_i \frac{f(S_i)-f(S_i^*)}{f(S_i)}[b_i S_i^* - b_i S_i(t)] + \alpha_i R_i^* \left[2 - \frac{R_i^* I_i}{I_i^* R_i} - \frac{R_i I_i^*}{I_i R_i^*}\right] \\
& \quad + \sum_{j=1}^{n} \beta_{ij} Q_i f(S_i^*) I_j^* \left(2 - \frac{f(S_i^*)}{f(S_i)} + \frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*}\right) \\
& \quad + \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j^* \\
& \qquad \times \int_0^{\infty} e^{-b_i \xi} d_\xi P_i(\xi) \left[\frac{f(S_i(t-\xi)) I_j(t-\xi) I_i^*}{f(S_i^*) I_j^* I_i(t)} + \ln \frac{f(S_i(t)) I_j(t)}{f(S_i(t-\xi)) I_j(t-\xi)}\right] d\xi \\
& \le -\sum_{j=1}^{n} \beta_{ij} Q_i f(S_i^*) I_j^* \left(H\left(\frac{S_i^*}{S_i(t)}\right) + \ln \frac{S_i^*}{S_i(t)} + \frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*} - \ln \frac{I_j(t)}{I_j^*} + \ln \frac{I_i(t)}{I_i^*}\right) \\
& \quad + \sum_{j=1}^{n} \beta_{ij} f(S_i^*) I_j^* \int_0^{\infty} e^{-b_i \xi} d_\xi P_i(\xi) H\left(\frac{f(S_i(t-\xi)) I_j(t-\xi) I_i^*}{f(S_i^*) I_j^* I_i(t)}\right) d\xi \\
& \le \sum_{j=1}^{n} \beta_{ij} Q_i f(S_i^*) I_j^* \left(\frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*} - \ln \frac{I_j(t)}{I_j^*} + \ln \frac{I_i(t)}{I_i^*}\right).
\end{aligned}
$$

Here we used the facts that

$$
\frac{f(S_i)-f(S_i^*)}{f(S_i)}[b_i S_i^* - b_i S_i(t)] \le 0, \quad \left[2 - \frac{R_i^* I_i}{I_i^* R_i} - \frac{R_i I_i^*}{I_i R_i^*}\right] \le 0, \tag{3.12}
$$

$$
H\left(\frac{S_i^*}{S_i(t)}\right) \ge 0 \quad \text{and} \quad H\left(\frac{f(S_i(t-\xi)) I_j(t-\xi) I_i^*}{f(S_i^*) I_j^* I_i(t)}\right) \ge 0. \tag{3.13}
$$

Set

$$\bar{\beta}_{ij} = \beta_{ij} Q_i f(S_i^*) I_j^*, \quad 1 \le i,\, j \le n,$$

and

$$\bar{B} = \begin{bmatrix} \sum_{l\neq 1} \bar{\beta}_{1l} & -\bar{\beta}_{21} & \cdots & -\bar{\beta}_{n1} \\ -\bar{\beta}_{12} & \sum_{l\neq 2} \bar{\beta}_{2l} & \cdots & -\bar{\beta}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{\beta}_{1n} & -\bar{\beta}_{2n} & \cdots & \sum_{l\neq n} \bar{\beta}_{nl} \end{bmatrix}.$$

Note that $\bar{B}$ is the Laplacian matrix of the matrix $(\bar{\beta}_{ij})_{n\times n}$ (see Appendix). Since $(\beta_{ij})_{n\times n}$ is irreducible, matrices $(\bar{\beta}_{ij})_{n\times n}$ and $\bar{B}$ are also irreducible. Let C_{ij} denote the cofactor of the $(i,\, j)$ entry of $\bar{B}$. We know that system $\bar{B}v = 0$ has a positive solution $v = (v_1, v_2, \ldots, v_n)$, where $v_i = C_{ii} > 0$ for $i = 1, \ldots, n$.

Set

$$L = \sum_{i=1}^{n} v_i L_{EE},$$

then

$$\begin{aligned} \frac{dL}{dt}\bigg|_{(2.3)} &= \sum_{i=1}^{n} v_i \frac{dL_{EE}}{dt}\Big|_{(2.3)} \\ &\le \sum_{i,j=1}^{n} v_i \beta_{ij} Q_i f(S_i^*) I_j^* \left(\frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*} - \ln \frac{I_j(t)}{I_j^*} + \ln \frac{I_i(t)}{I_i^*} \right) \\ &= \sum_{i,j=1}^{n} v_i \beta_{ij} Q_i f(S_i^*) I_j^* \left(\frac{I_j(t)}{I_j^*} - \frac{I_i(t)}{I_i^*} \right) - \sum_{i,j=1}^{n} v_i \beta_{ij} Q_i f(S_i^*) I_j^* \left(\ln \frac{I_i^* I_j(t)}{I_i(t) I_j^*} \right) \\ &=: \Psi_1 - \Psi_2. \end{aligned}$$

We first show $\Psi_1 \equiv 0$ for all $I_1, I_2, \ldots, I_n > 0$. It follows from $\bar{B}v = 0$ that

$$\sum_{j=1}^{n} \bar{\beta}_{ji} v_j = \sum_{k=1}^{n} \bar{\beta}_{ik} v_i$$

or using $\bar{\beta}_{ji} = \beta_{ji} Q_j f(S_j^*) I_i^*$,

$$\sum_{j=1}^{n} \beta_{ji} Q_i f(S_j^*) I_i^* v_j = \sum_{k=1}^{n} \beta_{ik} Q_k f(S_i^*) I_k^* v_i, \quad i = 1, \ldots, n.$$

This implies that

$$\sum_{i,j=1}^{n} v_i\beta_{ij}Q_jS_i^*I_j = \sum_{i=1}^{n} E_i \sum_{j=1}^{n} \beta_{ji}Q_iS_j^*v_j = \sum_{i=1}^{n} \frac{I_i}{I_i^*} \sum_{k=1}^{n} \beta_{ik}Q_k f(S_i^*)I_k^*v_i$$
$$= \sum_{i,j=1}^{n} v_i\beta_{ij}Q_jS_i^*I_j^*\frac{I_i}{I_i^*},$$

and thus $\Psi_1 \equiv 0$ for all $I_1, I_2, \dots, I_n > 0$.

Next we show $\Psi_2 \equiv 0$ for all $I_1, I_2, \dots, I_n > 0$. Let G denote the directed graph associated with matrix $(\bar{\beta}_{ij})$. G has vertices $1, 2, \dots, n$ with a directed arc (i, j) from k to j if and only if $\bar{\beta}_{ij} \neq 0$. $E(G)$ denotes the set of all directed arcs of G. Using Kirchhoff's Matrix Tree Theorem (see Appendix), we know that $v_i = C_{ii}$ can be interpreted as a sum of weights of all directed spanning subtrees T of G that are rooted at vertex i. Consequently, each term in $v_i\bar{\beta}_{ij}$ is the weight $w(Q)$ of a unicyclic subgraph Q of G, obtained from such a tree T by adding a directed arc (i, j) from the root i to vertex j. Note that the arc (i, j) is part of the unique cycle CQ of Q, and that the same unicyclic graph Q can be formed when each arc of CQ is added to a corresponding rooted tree T. Therefore, the double sum in Ψ_2 can be reorganized as a sum over all unicyclic subgraphs Q containing vertices $1, 2, \dots, n$, that is, $\Psi_2 = \sum_Q H_Q$, where

$$H_Q = w(Q) \cdot \sum_{(i,j)\in E(CQ)} \ln \frac{I_i^*I_j}{I_iI_j^*} = w(Q) \cdot \ln\left(\prod_{(i,j)\in E(CQ)} \frac{I_i^*I_j}{I_iI_j^*} \right).$$

Since $E(CQ)$ is the set of arcs of a cycle CQ, we have

$$\prod_{(i,j)\in E(CQ)} \frac{I_i^*I_j}{I_iI_j^*} \quad \text{and thus} \quad \ln\left(\prod_{(i,j)\in E(CQ)} \frac{I_i^*I_j}{I_iI_j^*} \right) = 0,$$

which implies $H_Q = 0$ for each Q, it follows that $\Psi_2 \equiv 0$, for all $I_1, I_2, \dots, I_n > 0$.

Together with (3.12) and (3.13), we get $(dL/dt)|_{(2.3)} \leq 0$ with equality holds if and only if

$$S_i(t) = S_i(t-\xi) = S_i^*, \quad I_i(t) = I_i(t-\xi) = I_i^*, \quad R_i(t) = R_i^*.$$

Therefore, the only compact invariant subset of the set where $(dL/dt)|_{(2.3)} = 0$ is the singleton $\{P^*\}$. By LaSalles invariance principle, P^* globally attracts in the interior of Γ. That is, $\lim_{t\to\infty}(\phi_{it}, \varphi_{it}, R_i(t)) = P^* = (S_1^*, I_1^*, R_1^*, \dots, S_n^*, I_n^*, R_n^*)$. The proof is complete. □

REMARK 3.2. Compared to results in [11] and [12], the group structure in system (1.4) and (2.3) greatly increases the complexity exhibited in the derivatives of the

Lyapunov functionals. The key to our analysis is a complete description of the patterns exhibited in the derivative of the Lyapunov functionals using graph theory.

Appendix

Given a nonnegative matrix $A = (a_{ij})$, the *directed graph* $G(A)$ associated with $A = (a_{ij})$ has vertices $1, 2, \ldots, n$ with a directed arc (i, j) from i to j iff $a_{ij} = 0$. It is *strongly connected* if any two distinct vertices are joined by an oriented path. Matrix A is irreducible if and only if $G(A)$ is strongly connected [6]. A *tree* is a connected graph with no cycles. A subtree T of a graph G is said to be *spanning* if T contains all the vertices of G. A *directed tree* is a tree in which each edge has been replaced by an arc directed one way or the other. A directed tree is said to be *rooted* at a vertex, called the root, if every arc is oriented in the direction towards to the root. An *oriented cycle* in a directed graph is a simple closed oriented path. A *unicyclic graph* is a directed graph consisting of a collection of disjoint rooted directed trees whose root are on an oriented cycle. We refer the reader to ([25], Theorem 5.5) for more details of these concepts.

For a given nonnegative matrix $A = (a_{ij})$, let

$$L = \begin{bmatrix} \sum_{l\neq 1} \bar{a}_{1l} & -\bar{a}_{21} & \cdots & -\bar{a}_{n1} \\ -\bar{a}_{12} & \sum_{l\neq 2} \bar{a}_{2l} & \cdots & -\bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ -\bar{a}_{1n} & -\bar{a}_{2n} & \cdots & \sum_{l\neq n} \bar{a}_{nl} \end{bmatrix}$$

be the Laplacian matrix of the directed graph $G(A)$ and C_{ij} denote the cofactor of the (i, j) entry of L. For the linear system

$$Lv = 0, \tag{3.14}$$

the following results hold (see details in [12]).

Theorem 3.4 (Kirchhoff's matrix tree theorem). *Assume that* $n \geq 2$ *and that* A *is irreducible. Then following results hold*:
(1) *The solution space of system* (3.14) *has dimension* 1, *with a basis* $(v_1, v_2, \ldots, v_n) = (C_{11}, C_{22}, \ldots, C_{nn})$.
(2) *For* $1 \leq k \leq n$,

$$C_{kk} = \sum_{T\in \mathrm{T}_k} w(T) = \sum_{T\in \mathrm{T}_k} \prod_{(r,m)\in E(T)} a_{rm} > 0,$$

where T_k *is the set of all directed spanning subtrees of* $G(A)$ *that are rooted at vertex* k*,* $w(T)$ *is the weight of a directed tree* T*, and* $E(T)$ *denotes the set of directed arcs in* T*.*

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