Yaguchi, Y.
Osaka J. Math.
52 (2015), 59–70

DETERMINING THE HURWITZ ORBIT OF THE STANDARD GENERATORS OF A BRAID GROUP

YOSHIRO YAGUCHI

(Received January 16, 2012, revised June 18, 2013)

Abstract

The Hurwitz action of the n-braid group B_n on the n-fold product $(B_m)^n$ of the m-braid group B_m is studied. Using a natural action of B_n on trees with n labeled edges and $n+1$ labeled vertices, we determine all elements of the orbit of every n-tuple of the n distinct standard generators of B_{n+1} under the Hurwitz action of B_n.

1. Introduction

Let B_n denote the n-braid group, which has the following presentation [1, 4].

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \ \ (|i-j| = 1), \\ \sigma_i\sigma_j = \sigma_j\sigma_i \ \ (|i-j| > 1) \end{array} \right\rangle,$$

where σ_i is the i-th standard generator represented by a geometric n-braid depicted in Fig. 1.1.

Throughout this paper, n is an integer with $n \geqq 2$. Let G be a group and let G^n be the n-fold direct product of G. For elements g and h of G, let $g * h$ denote $h^{-1}gh$ and let $g \mathbin{\overline{*}} h$ denote $g * (h^{-1}) = hgh^{-1}$.

DEFINITION 1.1. The *Hurwitz action of B_n on G^n* is the right action defined by

$$\begin{aligned} &(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \cdot \sigma_i \\ &= (g_1, \dots, g_{i-1}, g_{i+1}, g_i * g_{i+1}, g_{i+2}, \dots, g_n) \end{aligned}$$

and

$$\begin{aligned} &(g_1, \dots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \dots, g_n) \cdot \sigma_i^{-1} \\ &= (g_1, \dots, g_{i-1}, g_{i+1} \mathbin{\overline{*}} g_i, g_i, g_{i+2}, \dots, g_n), \end{aligned}$$

where $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of B_n.

We call the orbit of $(g_1, \dots, g_n) \in G^n$ under the Hurwitz action of B_n the *Hurwitz orbit* of $(g_1, \dots, g_n)$ and denote it by $(g_1, \dots, g_n) \cdot B_n$. We say two elements of G^n are *Hurwitz equivalent* if they belong to the same Hurwitz orbit.

2010 Mathematics Subject Classification. Primary 20F36; Secondary 20F34.

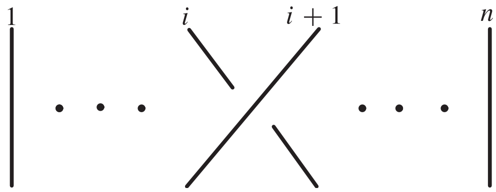

Fig. 1.1.

When G is a braid group B_m and each g_j $(j = 1, \ldots, n)$ is a *positive simple braid* (that means a conjugate of σ_1), the Hurwitz orbit of an n-tuple $(g_1, \ldots, g_n)$ corresponds to an equivalence class of an "algebraic" braided surface in a bidisk $D^2 \times D^2$ with n branch points [11, 13, 14, 15, 17]. Here, we say that two braided surfaces S and S' are *equivalent* if there is a fiber-preserving diffeomorphism $f\colon D^2 \times D^2 \to D^2 \times D^2$ (that is, $f(D^2 \times \{x\}) = D^2 \times \{g(x)\}$ for some diffeomorphism $g\colon D^2 \to D^2$) carrying S to S' rel $D^2 \times \partial D^2$. It is natural to ask if two given braided surfaces are equivalent or not. This question is equivalent to asking if two given n-tuples $(g_1, \ldots, g_n)$ and $(g'_1, \ldots, g'_n)$ of positive simple braids determined by their monodromies are Hurwitz equivalent or not. This is a very hard problem and no algorithm to solve it is known. However, we can determine all elements of the Hurwitz orbits of some n-tuples of positive simple braids.

Throughout this paper, we use the symbol "s_i" to denote the i-th standard generator of B_{n+1}, and "σ_i" to denote that of B_n.

We prove that for every permutation φ of $\{1, \ldots, n\}$, there is a natural bijection from the Hurwitz orbit of $(s_{\varphi(1)}, \ldots, s_{\varphi(n)})$ to a set consisting of all trees satisfying the conditions of Definition 2.2. As a result, we determine all elements in the Hurwitz orbit of $(s_{\varphi(1)}, \ldots, s_{\varphi(n)})$ for every permutation φ of $\{1, \ldots, n\}$ (see Theorem 4.3). We will also prove Theorem 1.2 by using Theorem 4.3.

Let X_{n+1} be the set of the integers $\{2, 3, \ldots, n\}$. For a permutation φ of $\{1, \ldots, n\}$, let $\mathrm{A}^{\varphi} = \{i \in X_{n+1} \mid \varphi^{-1}(i-1) < \varphi^{-1}(i)\}$.

Theorem 1.2. *For permutations φ and ψ of $\{1, \ldots, n\}$, the following conditions are mutually equivalent.*
(1) $(s_{\varphi(1)}, \ldots, s_{\varphi(n)})$ *and* $(s_{\psi(1)}, \ldots, s_{\psi(n)})$ *are Hurwitz equivalent.*
(2) *The products* $s_{\varphi(1)} \cdots s_{\varphi(n)}$ *and* $s_{\psi(1)} \cdots s_{\psi(n)}$ *are equal in* B_{n+1}.
(3) *The sets* A^{φ} *and* A^{ψ} *are equal.*

REMARK 1.3. Let B^+_{n+1} be the semi-group which has the generators $s^*_1, \ldots, s^*_n$ and the relations $s^*_i s^*_j = s^*_j s^*_i$ if $|i - j| > 2$ and $s^*_i s^*_j s^*_i = s^*_j s^*_i s^*_j$ if $|i - j| = 1$. Let $i : B^+_{n+1} \to B_{n+1}$ be the natural semi-group homomorphism, i.e., $i(s^*_i) = s_i$ for each $i \in \{1, \ldots, n\}$. In [7], Garside proved that i is injective. (We call an element of $i(B^+_{n+1})$ a *positive element* of B_{n+1}.) Using the Garside's theorem, T. Ben-Itzhak and M. Teicher showed the equivalence of the conditions (1) and (2) of Theorem 1.2 in [2]. This is explained as follows. If $|i-j|>2$, then (s_i, s_j) and (s_j, s_i) are Hurwitz equivalent. If $|i-j|=1$, then (s_i, s_j, s_i) and (s_j, s_i, s_j) are Hurwitz equivalent. (See [2].) Thus, for permutations φ and ψ of $\{1,\dots,n\}$, $(s_{\varphi(1)},\dots,s_{\varphi(n)})$ and $(s_{\psi(1)},\dots,s_{\psi(n)})$ are Hurwitz equivalent if and only if the products $s^*_{\varphi(1)}\cdots s^*_{\varphi(n)}$ and $s^*_{\psi(1)}\cdots s^*_{\psi(n)}$ in B^+_{n+1} are equal. By the Garside's theorem, this condition is equal to the condition that the products $s_{\varphi(1)}\cdots s_{\varphi(n)}$ and $s_{\psi(1)}\cdots s_{\psi(n)}$ in B_{n+1} are equal. Thus, we have the equivalence of (1) and (2) of Theorem 1.2. In this paper, we will prove the equivalence of (1) and (3) of Theorem 1.2 by using Theorem 4.3.

2. Main result

Let S_{n+1} be the symmetric group of degree $n+1$ and let $\mathcal{T}_{n+1}$ be the set of elements $(\tau_1,\dots,\tau_n)$ of the n-fold direct product $(S_{n+1})^n$ of S_{n+1} such that $\tau_1,\dots,\tau_n$ are transpositions which generate S_{n+1}. For any element $(\tau_1,\dots,\tau_n)$ of $\mathcal{T}_{n+1}$, the Hurwitz orbit $(\tau_1,\dots,\tau_n)\cdot B_n$ is contained in $\mathcal{T}_{n+1}$ [10, 12]. Thus, we obtain an action of B_n on $\mathcal{T}_{n+1}$ as a restriction of the Hurwitz action on $(S_{n+1})^n$. Let $\mathcal{G}_{n+1}$ be the set of trees with n edges labeled $\{e_1,\dots,e_n\}$ and $n+1$ vertices labeled $\{v_1,\dots,v_{n+1}\}$. A natural action of B_n on $\mathcal{G}_{n+1}$ has been observed [5, 6, 8]. It is explained as follows.

Take an element $(\tau_1,\dots,\tau_n)$ of $\mathcal{T}_{n+1}$ and put $\tau_i=(k_i\ l_i)$ $(k_i<l_i)$ for $i\in\{1,\dots,n\}$. We define $\Gamma(\tau_1,\dots,\tau_n)$ as a graph with $n+1$ vertices labeled $\{v_1,\dots,v_{n+1}\}$ and n edges labeled $\{e_1,\dots,e_n\}$ such that the edge labeled e_i connects with two vertices labeled v_{k_i} and v_{l_i} for $i\in\{1,\dots,n\}$. Since $\tau_1,\dots,\tau_n$ generate S_{n+1}, the graph $\Gamma(\tau_1,\dots,\tau_n)$ must be a tree [5, 6]. Hence, $\Gamma(\tau_1,\dots,\tau_n)\in\mathcal{G}_{n+1}$. We call the induced map $\Gamma\colon\mathcal{T}_{n+1}\to\mathcal{G}_{n+1}$ the *graphic map*. The map Γ is bijective [5, 6]. Thus, we obtain the action of B_n on $\mathcal{G}_{n+1}$ defined by $\gamma\odot\beta=\Gamma((\Gamma^{-1}(\gamma))\cdot\beta)$ for $\gamma\in\mathcal{G}_{n+1}$ and $\beta\in B_n$ [5, 6, 8]. We call this action the *Hurwitz action of* B_n *on* $\mathcal{G}_{n+1}$.

For $1\le i\le n$, let t_i be the transposition $(i\ i+1)$ of S_{n+1}. Take a permutation φ of $\{1,\dots,n\}$. Let $p\colon B_{n+1}\to S_{n+1}$ be the canonical projection and let $P=p^n\colon(B_{n+1})^n\to(S_{n+1})^n$ be the map defined by $(b_1,\dots,b_n)\mapsto(p(b_1),\dots,p(b_n))$. Then, for any $\beta\in B_n$, it holds $P((b_1,\dots,b_n)\cdot\beta)=(P(b_1,\dots,b_n))\cdot\beta$. Thus, for every permutation φ of $\{1,\dots,n\}$, we have $P((s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n)=(t_{\varphi(1)},\dots,t_{\varphi(n)})\cdot B_n$. Since $(t_{\varphi(1)},\dots,t_{\varphi(n)})\in\mathcal{T}_{n+1}$ (and hence, $\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)})\in\mathcal{G}_{n+1}$), we have $\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)})\odot B_n\subset\mathcal{G}_{n+1}$. In [16], it is proved that the map $P=P|_{(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n}\colon(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n\to(t_{\varphi(1)},\dots,t_{\varphi(n)})\cdot B_n$ is bijective. Then, $\Gamma\circ P((s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n)=\Gamma((t_{\varphi(1)},\dots t_{\varphi(n)})\cdot B_n)=\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)})\odot B_n$. Since Γ and P are bijective, the map $\Gamma\circ P\colon(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n\to\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)})\odot B_n$ is bijective.

REMARK 2.1. Take an element $(\tau_1,\dots,\tau_n)$ of $\mathcal{T}_{n+1}$. Let $\mathcal{T}_{n+1}(\tau_1,\dots,\tau_n)$ be the subset of $\mathcal{T}_{n+1}$ defined by $\{(\tau'_1,\dots,\tau'_n)\in\mathcal{T}_{n+1}\mid\tau'_1\cdots\tau'_n=\tau_1\cdots\tau_n\}$. In [10], A. Hurwitz proved that there exists $(n+1)^{n-1}$ elements of $\mathcal{T}_{n+1}(\tau_1,\dots,\tau_n)$. In [12], P. Kluitmann proved that the Hurwitz orbit $(\tau_1,\dots,\tau_n)\cdot B_n$ is equal to the set $\mathcal{T}_{n+1}(\tau_1,\dots,\tau_n)$ by an induction on n and using combinatorially calculations of S_{n+1}. Hence, there exists

$(n+1)^{n-1}$ elements in the Hurwitz orbit $(\tau_1,\dots,\tau_n)\cdot B_n$. In [9], S.P. Humphries proved that the Hurwitz orbit $(s_1,\dots,s_n)\cdot B_n$ consists of $(n+1)^{n-1}$ elements. In [16], the author proved that for any permutation φ of $\{1,\dots,n\}$, the Hurwitz orbit $(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n$ consists of $(n+1)^{n-1}$ elements.

Let C be a circle in $\mathbb{R}^2$. Let $P^{n+1}(C) = C\times\cdots\times C$ denote the product space of $n+1$ copies of C. Let $Q^{n+1}(C) = \{(q_1,\dots,q_{n+1}) \in P^{n+1}(C) \mid q_i \neq q_j \text{ for } i \neq j\}$. We call $Q^{n+1}(C)$ the *configuration space of ordered* $n+1$ *distinct points of* C. Fix an element $\mathbf{q} = (q_1,\dots,q_{n+1}) \in Q^{n+1}(C)$ and let $\Gamma^{\mathbf{q}}_{n+1}$ be the set of segments $\{\overline{q_i q_j} \mid 1 \leq i < j \leq n+1\}$ in $\mathbb{R}^2$. Take elements e and e' of $\Gamma^{\mathbf{q}}_{n+1}$. If $\partial e = \{q_i, q_{i'}\}$, $\partial e' = \{q_i, q_{i''}\}$ and $q_{i'} \neq q_{i''}$, i.e., e and e' share a common end point q_i, then we say that e and e' are *adjacent* (at q_i). Moreover, if the end points $q_{i'}$, q_i and $q_{i''}$ appear on C counterclockwise in this order, then we say that e' is a *right adjacent* to e (at q_i).

DEFINITION 2.2. An n-tuple $(e_1,\dots,e_n)$ of elements of $\Gamma^{\mathbf{q}}_{n+1}$ is *good* if it satisfies the conditions (i)–(iii).
(i) If $k \neq l$, then e_k and e_l are disjoint or adjacent.
(ii) If $k < l$ and e_k and e_l are adjacent, then e_l is a right adjacent to e_k.
(iii) The union $e_1 \cup \cdots \cup e_n$ is contractible.

Let $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$ be the set of n-tuples of elements of $\Gamma^{\mathbf{q}}_{n+1}$ which are good. When $(e_1,\dots,e_n)$ is an element of $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$, the union $e_1\cup\cdots\cup e_n$ is regarded as a graph with $n+1$ vertices $q_1,\dots,q_{n+1}$ and n edges $e_1,\dots,e_n$. Note that the graph $e_1\cup\cdots\cup e_n$ is a tree. Putting labels v_i on the points q_i for $1 \leq i \leq n+1$, we regard $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$ as a subset of $\mathcal{G}_{n+1}$. The following is our main theorem.

Theorem 2.3. *For a permutation* φ *of* $\{1,\dots,n\}$, *there exists an element* $\mathbf{q}$ *of* $Q^{n+1}(C)$ *such that the Hurwitz orbit* $\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)}) \odot B_n$ *is equal to the set* $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$ $(\subset \mathcal{G}_{n+1})$. *Hence, the map* $\Gamma\circ P$ *gives a bijection from the Hurwitz orbit* $(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n$ *to the set* $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$.

REMARK 2.4. In [16], the author found an element $\mathbf{q}$ of $Q^{n+1}(C)$ such that the Hurwitz orbit $\Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)}) \odot B_n$ is contained in $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$. Thus, we obtained an action of B_n on $\mathcal{G}(\Gamma^{\mathbf{q}}_{n+1})$ as the restriction of the Hurwitz action of B_n on $\mathcal{G}_{n+1}$. In this paper, we will show the transitively of this action.

3. Some notions

Throughout this section, X_{n+1} is the set of the integers $\{2,3,\dots,n\}$ and A is a fixed subset of X_{n+1}.

For integers i and j with $1 \leqq i < j \leqq n+1$, we define $s^{\mathrm{A}}_{ij} \in B_{n+1}$ and $s^{\mathrm{A}}_{ji} \in B_{n+1}$ by

$$s^{\mathrm{A}}_{ij} = s^{\mathrm{A}}_{ji} = s_i * \left(\prod_{k=i+1}^{j-1} s_k{}^{\epsilon_k} \right),$$

where $\epsilon_k = 1$ if $k \in \mathrm{A}$ and $\epsilon_k = -1$ if $k \notin \mathrm{A}$. We call s^{A}_{ij} a *band generator of* B_{n+1} *associated with* A. Note that a standard generator s_i of B_{n+1} is a band generator $s^{\mathrm{A}}_{i\,i+1}$. Let $\Sigma^{\mathrm{A}}_{n+1}$ be the set of band generators $\{s^{\mathrm{A}}_{ij} \in B_{n+1} \mid 1 \leqq i < j \leqq n+1\}$ associated with A. Let T_{n+1} be the set of transpositions of S_{n+1}. The natural projection $p\colon B_{n+1} \to S_{n+1}$ gives the bijection $p = p|_{\Sigma^{\mathrm{A}}_{n+1}}\colon \Sigma^{\mathrm{A}}_{n+1} \to T_{n+1}$ which satisfies $p(s^{\mathrm{A}}_{ij}) = (i\ \ j)$ for $1 \leqq i < j \leqq n+1$.

Let $\mathrm{P}_k = (k, 0) \in \mathbb{R}^2$ for an integer with $1 \leqq k \leqq n+1$. Let C_1 be the circle in $\mathbb{R}^2$ whose diameter is the segment $\overline{\mathrm{P}_1\mathrm{P}_{n+1}}$. Take the points $\mathrm{Q}_k \in C_1$ for $1 \leqq k \leqq n+1$ such that $\mathrm{Q}_1 = \mathrm{P}_1$, $\mathrm{Q}_{n+1} = \mathrm{P}_{n+1}$ and $\mathrm{Q}_k = (k, y_k)$ for each $2 \leqq k \leqq n$, where $y_k < 0$ if $k \in \mathrm{A}$ and $y_k > 0$ if $k \notin \mathrm{A}$. We call the points $\mathrm{Q}_1, \ldots, \mathrm{Q}_{n+1}$ on the circle C_1 the *points associated with* A. We call the element $\mathbf{Q} = (\mathrm{Q}_1, \ldots, \mathrm{Q}_{n+1})$ of the configuration space $Q^{n+1}(C_1)$ the *ordered* $n+1$ *points associated with* A. Let $\Gamma^{\mathrm{A}}_{n+1}$ denote the set of segments $\Gamma^{\mathrm{A}}_{n+1} = \Gamma^{\mathbf{Q}}_{n+1} = \{\overline{\mathrm{Q}_i\mathrm{Q}_j} \mid 1 \leqq i < j \leq n+1\}$ in $\mathbb{R}^2$. We have a bijection from $\Sigma^{\mathrm{A}}_{n+1}$ to $\Gamma^{\mathrm{A}}_{n+1}$ defined by $s^{\mathrm{A}}_{ij} \mapsto \overline{\mathrm{Q}_i\mathrm{Q}_j}$ for $1 \leqq i < j \leqq n+1$, and we call the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ the *segment corresponding to* s^{A}_{ij}.

REMARK 3.1. In [16], the reason why we call $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ the segment corresponding to s^{A}_{ij} is explained as follows.

Let $\mathrm{P}_0 = \mathrm{Q}_0 = (0,0) \in \mathbb{R}^2$ and $\mathrm{P}_{n+2} = \mathrm{Q}_{n+2} = (n+2, 0) \in \mathbb{R}^2$. Let C_2 be the circle in $\mathbb{R}^2$ whose diameter is the segment $\overline{\mathrm{P}_0\mathrm{P}_{n+2}}$. Let D be the disk in $\mathbb{R}^2$, with $\partial D = C_2$. Take an isotopy $\{h_u\}_{u \in [0,1]}$ of D such that for each $u \in [0,1]$, $h_0 = \mathrm{id}$, $h_u|_{\partial D} = \mathrm{id}$, and for each $u \in [0,1]$ and each $(x, y) \in \bigcup_{i=0}^{n+1} \overline{\mathrm{Q}_i\mathrm{Q}_{i+1}}$, $h_u(x, y) = (x, (1-u)y)$. Then $h_1(\mathrm{Q}_i) = \mathrm{P}_i$ for any i. For $1 \leqq i < j \leqq n+1$, we define α^{A}_{ij} by the arc $h_1(\overline{\mathrm{Q}_i\mathrm{Q}_j})$ in D. Note that $\partial\alpha^{\mathrm{A}}_{ij} = \{\mathrm{P}_i, \mathrm{P}_j\}$, α^{A}_{ij} is upper than P_k if $k \in \mathrm{A}$ and α^{A}_{ij} is lower than P_k if $k \notin \mathrm{A}$. The braid group B_{n+1} is isomorphic to the mapping class group of $(D, \{\mathrm{P}_1, \ldots, \mathrm{P}_{n+1}\})$ relative to the boundary (cf. [3]). The band generator s^{A}_{ij} corresponds to the isotopy class of a homeomorphism from $(D, \{\mathrm{P}_1, \ldots, \mathrm{P}_{n+1}\})$ to itself which twists a sufficiently small disk neighborhood of the arc α^{A}_{ij} by 180°-rotation clockwise using its collar neighborhood. By the homeomorphism $h_1\colon (D, \{\mathrm{Q}_1, \ldots, \mathrm{Q}_{n+1}\}) \to (D, \{\mathrm{P}_1, \ldots, \mathrm{P}_{n+1}\})$, we identify the mapping class group of $(D, \{\mathrm{Q}_1, \ldots, \mathrm{Q}_{n+1}\})$ and that of $(D, \{\mathrm{P}_1, \ldots, \mathrm{P}_{n+1}\})$. Then the band generator s^{A}_{ij} corresponds to the isotopy class of a homeomorphism from $(D, \{\mathrm{Q}_1, \ldots, \mathrm{Q}_{n+1}\})$ to itself which twists a sufficiently small disk neighborhood of the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ by 180°-rotation clockwise. Thus, we say that the segment $\overline{\mathrm{Q}_i\mathrm{Q}_j}$ corresponds to the band generator $s^{\mathrm{A}}_{ij} \in \Sigma^{\mathrm{A}}$.

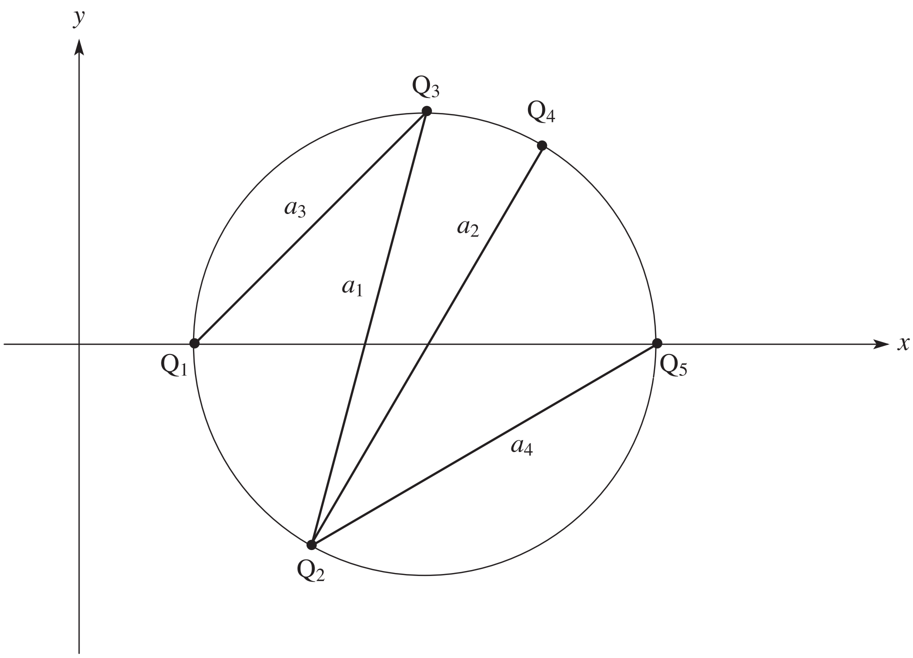

Fig. 3.1.

Let $(g_1, \dots, g_n)$ be an element of the n-fold product $(\Sigma^{\mathrm{A}}_{n+1})^n$ of $\Sigma^{\mathrm{A}}_{n+1}$ and let $(a_1, \dots, a_n)$ be the n-tuple of the segments a_i corresponding to g_i. Then, we call $(a_1, \dots, a_n)$ the *segment system corresponding to* $(g_1, \dots, g_n)$.

DEFINITION 3.2. An element of $(\Gamma^{\mathrm{A}}_{n+1})^n$ *is* A-*good* if it is good. An element of $(\Sigma^{\mathrm{A}}_{n+1})^n$ *is* A-*good* if the corresponding segment system is A-good. We denote the set of elements of $(\Sigma^{\mathrm{A}}_{n+1})^n$ (resp. $(\Gamma^{\mathrm{A}}_{n+1})^n$) which are A-good by $\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$ (resp. $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$).

EXAMPLE 3.3. Let $n = 4$ and $\mathrm{A} = \{2\}$. Then, the element $(s^{\mathrm{A}}_{23}, s^{\mathrm{A}}_{24}, s^{\mathrm{A}}_{13}, s^{\mathrm{A}}_{25})$ of $(\Sigma^{\mathrm{A}}_5)^4$ is an A-good element. The segments $a_1, \dots, a_4$ corresponding to $s^{\mathrm{A}}_{23}, s^{\mathrm{A}}_{24}, s^{\mathrm{A}}_{13}, s^{\mathrm{A}}_{25}$ are depicted in Fig. 3.1.

Let $\Sigma^* = \bigcup_{\mathrm{A}\subset\{2,\dots,n\}}(\Sigma^{\mathrm{A}}_{n+1})^n$ and $\Gamma^* = \bigcup_{\mathrm{A}\subset\{2,\dots,n\}}(\Gamma^{\mathrm{A}}_{n+1})^n$. Let $\Phi\colon \Sigma^* \to \Gamma^*$ be the map which sends an element of $(\Sigma^{\mathrm{A}}_{n+1})^n$ to the segment system corresponding to it for each subset A of $\{2, \dots, n\}$. It is obvious that the map Φ is bijective and $\Phi((\Sigma^{\mathrm{A}}_{n+1})^n) = (\Gamma^{\mathrm{A}}_{n+1})^n$. By Definition 3.2, $\Phi(\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})) = \mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$. For each subset A of $\{2, \dots, n\}$, we simply write Φ for the bijective map $\Phi|_{\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})}\colon \mathcal{G}(\Sigma^{\mathrm{A}}_{n+1}) \to \mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$.

Proposition 3.4. *Let* A *and* B *be subsets of* $\{2, \dots, n\}$. *Then, the conditions* $\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1}) = \mathcal{G}(\Sigma^{\mathrm{B}}_{n+1})$, $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1}) = \mathcal{G}(\Gamma^{\mathrm{B}}_{n+1})$ *and* $\mathrm{A} = \mathrm{B}$ *are mutually equivalent.*

Proof. If $\mathrm{A}=\mathrm{B}$, then it is obvious that the other two conditions hold. By virtue of the bijective map $\Phi\colon \Sigma^* \to \Gamma^*$, the conditions $\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})=\mathcal{G}(\Sigma^{\mathrm{B}}_{n+1})$ and $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})=\mathcal{G}(\Gamma^{\mathrm{B}}_{n+1})$ are equivalent. We prove that the condition $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})=\mathcal{G}(\Gamma^{\mathrm{B}}_{n+1})$ implies $\mathrm{A}=\mathrm{B}$. Let $\mathrm{Q}^{\mathrm{X}}_k$ denote the point (k,y_k), on the circle C_1, associated with a subset $\mathrm{X}\subset\{2,\dots,n\}$. Take elements $(a_1,\dots,a_n)\in\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$ and $(b_1,\dots,b_n)\in\mathcal{G}(\Gamma^{\mathrm{B}}_{n+1})$. Then, by the conditions of Definition 2.2, $\partial a_1\cup\cdots\cup\partial a_n=\{\mathrm{Q}^{\mathrm{A}}_1,\dots,\mathrm{Q}^{\mathrm{A}}_{n+1}\}$ and $\partial b_1\cup\cdots\cup\partial b_n=\{\mathrm{Q}^{\mathrm{B}}_1,\dots,\mathrm{Q}^{\mathrm{B}}_{n+1}\}$. Hence, the condition $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})=\mathcal{G}(\Gamma^{\mathrm{B}}_{n+1})$ implies $\mathrm{Q}^{\mathrm{A}}_k=\mathrm{Q}^{\mathrm{B}}_k$ for $1\le k\le n+1$, and this implies $\mathrm{A}=\mathrm{B}$. This completes the proof of Proposition 3.4. □

Let A be a subset of $\{2,\dots,n\}$ and let $\mathrm{Q}_1,\dots,\mathrm{Q}_{n+1}$ be the points, on the circle C_1, associated with A. We regard $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$ as a subset of $\mathcal{G}_{n+1}$ putting labels $v_1,\dots,v_{n+1}$ on the points $\mathrm{Q}_1,\dots,\mathrm{Q}_{n+1}$, resp. Recall that the Hurwitz action of B_n on $\mathcal{G}_{n+1}$ is defined by $\gamma\odot\beta=\Gamma((\Gamma^{-1}(\gamma))\cdot\beta)$ for $\gamma\in\mathcal{G}_{n+1}$ and $\beta\in B_n$, where Γ is the graphic map $\Gamma\colon\mathcal{T}_{n+1}\to\mathcal{G}_{n+1}$. By Proposition 3.5 (2), we obtain an action of B_n on $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$ as a restriction of the Hurwitz action of B_n on $\mathcal{G}_{n+1}$. We call it the *Hurwitz action of* B_n *on* $\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$.

Proposition 3.5 ([16]). (1) *If* $(g_1,\dots,g_n)\in\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$, *then, for any* $\beta\in B_n$, *we have* $(g_1,\dots,g_n)\cdot\beta\in\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$.
(2) *If* $(a_1,\dots,a_n)\in\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$, *then, for any* $\beta\in B_n$, *we have* $(a_1,\dots,a_n)\odot\beta\in\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$.

Take an element $(g_1,\dots,g_n)=(s^{\mathrm{A}}_{i_1j_1},\dots,s^{\mathrm{A}}_{i_nj_n})\in\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$. Then, $\Phi(g_1,\dots,g_n)=(\overline{\mathrm{Q}_{i_1}\mathrm{Q}_{j_1}},\dots,\overline{\mathrm{Q}_{i_n}\mathrm{Q}_{j_n}})=\Gamma((i_1\ j_1),\dots,(i_n\ j_n))=\Gamma\circ P(g_1,\dots,g_n)$ $(\in\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1}))$. Moreover, for any element $\beta\in B_n$, $(g_1,\dots,g_n)\cdot\beta\in\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$ by Proposition 3.5 (1), and we have $\Phi((g_1,\dots,g_n)\cdot\beta)=\Gamma\circ P((g_1,\dots,g_n)\cdot\beta)=\Gamma(P(g_1,\dots,g_n)\cdot\beta)=\Gamma(((i_1\ j_1),\dots,(i_n\ j_n))\cdot\beta)=\Gamma((\Gamma^{-1}(\overline{\mathrm{Q}_{i_1}\mathrm{Q}_{j_1}},\dots,\overline{\mathrm{Q}_{i_n}\mathrm{Q}_{j_n}}))\cdot\beta)=(\overline{\mathrm{Q}_{i_1}\mathrm{Q}_{j_1}},\dots,\overline{\mathrm{Q}_{i_n}\mathrm{Q}_{j_n}})\odot\beta=\Phi(g_1,\dots,g_n)\odot\beta$. Thus, we have the following proposition.

Proposition 3.6. *For an element* $(g_1,\dots,g_n)$ *of* $\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})$, *we have*
(1) $\Phi(g_1,\dots,g_n)=\Gamma\circ P(g_1,\dots,g_n)$ $(\in\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1}))$ *and*
(2) *for any element* $\beta\in B_n$, $\Phi((g_1,\dots,g_n)\cdot\beta)=\Phi(g_1,\dots,g_n)\odot\beta$.

By Proposition 3.6 (2), the bijective map $\Phi\colon\mathcal{G}(\Sigma^{\mathrm{A}}_{n+1})\to\mathcal{G}(\Gamma^{\mathrm{A}}_{n+1})$ implies that two statements of (1) and (2) of Proposition 3.5 are equivalent.

4. Proof of Theorem 1.2 and Theorem 2.3.

Throughout this section, n is a fixed integer n with $n\ge 2$ and X_{n+1} is the set of the integers $\{2,\dots,n\}$.

For a permutation φ of $\{1,\dots,n\}$, let $\mathrm{A}^{\varphi}=\{i\in X_{n+1}\mid\varphi^{-1}(i-1)<\varphi^{-1}(i)\}$.

For the first step of the proof of Theorem 1.2 and Theorem 2.3, we prepare the following lemma which is proved in [16]:

Lemma 4.1 ([16]). *$(s_{\varphi(1)}, \dots, s_{\varphi(n)})$ is an element of $\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}^\varphi})$.*

Let A be a subset of X_{n+1}. By virtue of the bijective map $\Phi\colon \mathcal{G}(\Sigma_{n+1}^{\mathrm{A}}) \to \mathcal{G}(\Gamma_{n+1}^{\mathrm{A}})$, two statements of (1) and (2) of Theorem 4.2 are equivalent. Theorem 4.2 is also our main result in this paper.

Theorem 4.2. (1) *For each element $(a_1, \dots, a_n)$ of $\mathcal{G}(\Gamma_{n+1}^{\mathrm{A}})$, the Hurwitz orbit $(a_1, \dots, a_n) \odot B_n$ is equal to the set $\mathcal{G}(\Gamma_{n+1}^{\mathrm{A}})$.*
(2) *For each element $(g_1, \dots, g_n)$ of $\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}})$, the Hurwitz orbit $(g_1, \dots, g_n) \cdot B_n$ is equal to the set $\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}})$.*

The following theorem is directly obtained from Lemma 4.1 and Theorem 4.2 (2).

Theorem 4.3. *For a permutation φ of $\{1, \dots, n\}$, the Hurwitz orbit $(s_{\varphi(1)}, \dots, s_{\varphi(n)}) \cdot B_n$ is equal to the set $\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}^\varphi})$.*

We can prove Theorem 1.2 and Theorem 2.3 by using Theorem 4.3.

Proof of Theorem 1.2. We prove that the conditions (1) and (3) are equivalent. Take permutations φ and ψ of $\{1, \dots, n\}$. The condition (1) is equivalent to $(s_{\varphi(1)}, \dots, s_{\varphi(n)}) \cdot B_n = (s_{\psi(1)}, \dots, s_{\psi(n)}) \cdot B_n$. By Theorem 4.3, this is equivalent to $\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}^\varphi}) = \mathcal{G}(\Sigma_{n+1}^{\mathrm{A}^\psi})$. By Proposition 3.4, this is equivalent to $\mathrm{A}^\varphi = \mathrm{A}^\psi$, and we have the result. □

Proof of Theorem 2.3. Fix a permutation φ of $\{1,\dots,n\}$ and let $\mathrm{A} = \mathrm{A}^\varphi$. By Theorem 4.3, $(s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n = \mathcal{G}(\Sigma_{n+1}^{\mathrm{A}})$. Then, $\Phi((s_{\varphi(1)},\dots,s_{\varphi(n)})\cdot B_n) = \Phi(\mathcal{G}(\Sigma_{n+1}^{\mathrm{A}})) = \mathcal{G}(\Gamma_{n+1}^{\mathrm{A}})$. Let $\mathbf{Q} = (\mathrm{Q}_1,\dots,\mathrm{Q}_{n+1})$ be the ordered $n+1$ points of associated with A, that is an element of the configuration space $Q^{n+1}(C_1)$ of the circle C_1. Then, $\mathcal{G}(\Gamma_{n+1}^{\mathrm{A}}) = \mathcal{G}(\Gamma_{n+1}^{\mathbf{Q}})$ and we regard $\mathcal{G}(\Gamma_{n+1}^{\mathbf{Q}})$ as a subset of $\mathcal{G}_{n+1}$ putting labels $v_1, \dots, v_{n+1}$ on the points $\mathrm{Q}_1, \dots, \mathrm{Q}_{n+1}$, resp. By Proposition 3.6 (2), $\Phi((s_{\varphi(1)}, \dots, s_{\varphi(n)}) \cdot B_n) = \Phi(s_{\varphi(1)},\dots,s_{\varphi(n)}) \odot B_n$. By Proposition 3.6 (1), $\Phi(s_{\varphi(1)},\dots,s_{\varphi(n)}) \odot B_n = \Gamma \circ P((s_{\varphi(1)},\dots, s_{\varphi(n)})\cdot B_n) = \Gamma((P(s_{\varphi(1)},\dots,s_{\varphi(n)}))\cdot B_n) = \Gamma((t_{\varphi(1)},\dots,t_{\varphi(n)})\cdot B_n) = \Gamma(t_{\varphi(1)},\dots,t_{\varphi(n)}) \odot B_n$, where $\Gamma\colon \mathcal{T}_{n+1} \to \mathcal{G}_{n+1}$ is the graphic map. Thus, $\Gamma(t_{\varphi(1)}, \dots, t_{\varphi(n)}) \odot B_n = \mathcal{G}(\Gamma_{n+1}^{\mathbf{Q}})$, and we have the result. □

Let C be a circle in $\mathbb{R}^2$. Fix an element $\mathbf{q} = (q_1, \dots, q_{n+1})$ of the configuration space $Q^{n+1}(C)$. Let Γ_{n+1} denote the set of segments $\Gamma_{n+1}^{\mathbf{q}} = \{\overline{q_i q_j} \mid 1 \leqq i < j \leqq n+1\}$ in $\mathbb{R}^2$. Let $\mathcal{G}(\Gamma_{n+1})$ be the set of good elements of $(\Gamma_{n+1})^n$. Let $r_1, \dots, r_{n+1}$ be $n+1$ points with $\{r_1, \dots, r_{n+1}\} = \{q_1, \dots, q_{n+1}\}$ such that $r_1, \dots, r_{n+1}$ stand on C counterclockwise in this order.

Let A be an subset of X_{n+1} and let $\mathrm{Q}_1,\dots,\mathrm{Q}_{n+1}$ be the $n+1$ points, on the circle the C_1, associated with A. Let $\mathrm{R}_1, \dots, \mathrm{R}_{n+1}$ be $n+1$ points with $\{\mathrm{R}_1, \dots, \mathrm{R}_{n+1}\} =$

$\{Q_1, \dots, Q_{n+1}\}$ such that $R_1, \dots, R_{n+1}$ stand on the circle C_1 counterclockwise in this order. Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a self-homeomorphism of $\mathbb{R}^2$ such that $h(C_1) = C$, $h(R_i) = r_i$ for $1 \le i \le n+1$ and $h(\overline{R_i R_j}) = \overline{r_i r_j}$ for $1 \le i < j \le n+1$. Then, h induces the bijection $h\colon \Gamma^{A}_{n+1} \to \Gamma_{n+1}$. Let $f\colon (\Gamma^{A}_{n+1})^n \to (\Gamma_{n+1})^n$ be the n-fold product of h, i.e., for each element $(a_1,\dots,a_n)$ of $(\Gamma^{A}_{n+1})^n$, $f(a_1,\dots,a_n) = (h(a_1),\dots,h(a_n))$. Then, it is obvious that $f(\mathcal{G}(\Gamma^{A}_{n+1})) = \mathcal{G}(\Gamma_{n+1})$ and the map $F = f|_{\mathcal{G}(\Gamma^{A}_{n+1})}\colon \mathcal{G}(\Gamma^{A}_{n+1}) \to \mathcal{G}(\Gamma_{n+1})$ is bijective. If $(e_1,\dots,e_n) \in \mathcal{G}(\Gamma_{n+1})$, then, for any $\beta \in B_n$, we easily see that $(e_1,\dots,e_n) \odot \beta = F((F^{-1}(e_1, \dots, e_n)) \odot \beta)$. By virtue of the map F and the Hurwitz action of B_n on $\mathcal{G}(\Gamma_{n+1})$, Proposition 3.5 is equivalent to Proposition 4.4, and Theorem 4.2 is equivalent to Theorem 4.5. The rest of this section is devoted to proving Theorem 4.5 (and hence Theorem 4.2).

Proposition 4.4. *If $(e_1, \dots, e_n) \in \mathcal{G}(\Gamma_{n+1})$, then, for any $k \in \{1, \dots, n-1\}$ and any $\epsilon \in \{-1, 1\}$, we have $(e_1, \dots, e_n) \odot \sigma_k^{\epsilon} \in \mathcal{G}(\Gamma_{n+1})$.*

Theorem 4.5. *For each element $(e_1, \dots, e_n)$ of $\mathcal{G}(\Gamma_{n+1})$, the Hurwitz orbit $(e_1, \dots, e_n) \odot B_n$ is equal to the set $\mathcal{G}(\Gamma_{n+1})$.*

For elements a and b of Γ_{n+1} satisfying that a and b are disjoint or b is a right adjacent to a, we define the elements $a * b$ and $b \mathbin{\overline{*}} a$ of Γ_{n+1} as follows.

(1) If a and b are disjoint, then, let $a * b = a$ and let $b \mathbin{\overline{*}} a = b$.

(2) If b is a right adjacent to a and $a = \overline{q_x q_y}$, $b = \overline{q_y q_z}$, then, let $a * b = b \mathbin{\overline{*}} a = \overline{q_x q_z}$.

Let $(e_1, \dots, e_n)$ be an element of $\mathcal{G}(\Gamma_{n+1})$. By the conditions (i) and (ii) of Definition 2.2, for any $k \in \{1, \dots, n-1\}$, e_k and e_{k+1} are disjoint or e_{k+1} is a right adjacent to e_k. This implies that the elements $e_k * e_{k+1}$ and $e_{k+1} \mathbin{\overline{*}} e_k$ of Γ_{n+1} are defined. Then, we have the following proposition.

Proposition 4.6 ([16]). *The Hurwitz action of B_n on $\mathcal{G}(\Gamma_{n+1})$ is given by*

$$\begin{aligned}(e_1, \dots, e_{k-1}, e_k, e_{k+1}, e_{k+2}, \dots, e_n) \odot \sigma_k \\ = (e_1, \dots, e_{k-1}, e_{k+1}, e_k * e_{k+1}, e_{k+2}, \dots, e_n)\end{aligned}$$

and

$$\begin{aligned}(e_1, \dots, e_{k-1}, e_k, e_{k+1}, e_{k+2}, \dots, e_n) \odot \sigma_k^{-1} \\ = (e_1, \dots, e_{k-1}, e_{k+1} \mathbin{\overline{*}} e_k, e_k, e_{k+2}, \dots, e_n).\end{aligned}$$

For a power $\alpha = (1\ 2\ \cdots\ n+1)^t$ $(t \in \mathbb{Z})$ of the $n+1$-cyclic permutation $(1\ 2\ \cdots\ n+1)$, let $c_k = \overline{r_{\alpha(k)} r_{\alpha(k+1)}}$ for $1 \le k \le n$. Then, we see that $(c_1, \dots, c_n)$ is an element of $\mathcal{G}(\Gamma_{n+1})$ and we call it *cyclic*.

Lemma 4.7. *Any two cyclic elements of $\mathcal{G}(\Gamma_{n+1})$ are Hurwitz equivalent.*

Proof. Let $\overrightarrow{c_0} = (\overline{r_1r_2},\ldots,\overline{r_nr_{n+1}})$, that is a cyclic element of $\mathcal{G}(\Gamma_{n+1})$. It is enough to prove that any cyclic element $\overrightarrow{c}$ is Hurwitz equivalent to $\overrightarrow{c_0}$. Put $\overrightarrow{c} = (c_1,\ldots,c_n) = (\overline{r_{\alpha(1)}r_{\alpha(2)}},\ldots,\overline{r_{\alpha(n)}r_{\alpha(n+1)}})$ for a power α of the cyclic permutation $(1\ 2\ \cdots\ n+1)$. By Proposition 4.6, $\overrightarrow{c} \odot (\sigma_1\sigma_2\cdots\sigma_{n-1}) = (c_2,c_3,\ldots,c_n,(\cdots((c_1 * c_2) * c_3) * \cdots) * c_n)$. Since the points $r_1,\ldots,r_{n+1}$ stand on C_1 counterclockwise in this order, so do the points $r_{\alpha(1)}$, $r_{\alpha(i)}$ and $r_{\alpha(i+1)}$ for $2 \le i \le n$. Hence, $\overline{r_{\alpha(1)}r_{\alpha(i)}}$ is a right adjacent to $c_i = \overline{r_{\alpha(i)}r_{\alpha(i+1)}}$. Since $c_1 = \overline{r_{\alpha(1)}r_{\alpha(2)}}$ and $\overline{r_{\alpha(1)}r_{\alpha(i)}} * c_i = \overline{r_{\alpha(1)}r_{\alpha(i+1)}}$, we have $(\cdots((c_1 * c_2) * c_3) * \cdots) * c_n = \overline{r_{\alpha(1)}r_{\alpha(n+1)}}$. Thus, $\overrightarrow{c} \odot (\sigma_1\cdots\sigma_{n-1}) = (\overline{r_{\alpha(2)}r_{\alpha(3)}},\ldots,\overline{r_{\alpha(n)}r_{\alpha(n+1)}},\overline{r_{\alpha(n+1)}r_{\alpha(1)}})$. If $\alpha(t) = n+1$, then $\overrightarrow{c} \odot (\sigma_1\cdots\sigma_{n-1})^t = \overrightarrow{c_0}$ and we have the result. □

Let $C\Gamma_{n+1} = \{\overline{r_1r_2},\ldots,\overline{r_nr_{n+1}},\overline{r_{n+1}r_1}\}$. Then, the following lemma holds.

Lemma 4.8. *For any element $(e_1,\ldots,e_n)$ of $\mathcal{G}(\Gamma_{n+1})$, there exists an element $k \in \{1,\ldots,n\}$ such that $e_k \in C\Gamma_{n+1}$.*

Proof. Suppose that there exists an element $(e_1,\ldots,e_n)$ of $\mathcal{G}(\Gamma_{n+1})$ such that $e_k \in \Gamma_{n+1} \setminus C\Gamma_{n+1}$ for each $k \in \{1,\ldots,n\}$. Since the $n+1$-gon $|r_1r_2\cdots r_{n+1}|$ is convex, there exist elements $i, j \in \{1,\ldots,n\}$ $(i \ne j)$ such that e_i and e_j intersect in their interior. This contradicts the condition (i) of Definition 2.2. Thus, we have the result. □

For a power α of the $n+1$-cyclic permutation $(1\ 2\ \cdots\ n+1)$, let $\mathcal{G}_\alpha(\Gamma_{n+1})$ be the set defined by $\{(e_1,\ldots,e_n) \in \mathcal{G}(\Gamma_{n+1}) \mid e_n = \overline{r_{\alpha(n)}r_{\alpha(n+1)}}\}$. Let $\Gamma_{n,\alpha}$ be the subset of Γ_{n+1} defined by $\{\overline{r_ir_j} \in \Gamma_{n+1} \mid i, j \ne \alpha(n+1)\}$ and let $\mathcal{G}(\Gamma_{n,\alpha})$ be the set of good elements of $(\Gamma_{n,\alpha})^{n-1}$. Then, we have the following.

Lemma 4.9. *Let $(e_1,\ldots,e_n)$ be an element of $\mathcal{G}_\alpha(\Gamma_{n+1})$ for a power α of the $(n+1)$-cyclic permutation $(1\ 2\ \cdots\ n+1)$. Then, we have:*
(1) *the degree of the vertex $r_{\alpha(n+1)}$ of the graph $e_1 \cup \cdots \cup e_n$ is 1, and*
(2) *$(e_1,\ldots,e_{n-1})$ is an element of $\mathcal{G}(\Gamma_{n,\alpha})$.*

Proof. First, we prove (1). Suppose that the degree of the vertex $r_{\alpha(n+1)}$ is greater than 1. Then, there exists $\overline{r_{\alpha(n+1)}r_x} \in \{e_1,\ldots,e_{n-1}\}$ for some $x \in \{1,\ldots,n+1\} \setminus \{\alpha(n),\alpha(n+1)\}$. Since the points $r_{\alpha(1)},\ldots,r_{\alpha(n)}$ and $r_{\alpha(n+1)}$ stand on C counterclockwise in this order, so do the points $r_{\alpha(n)}$, $r_{\alpha(n+1)}$ and r_x in this order. Thus, $\overline{r_{\alpha(n+1)}r_x}$ is right adjacent to $\overline{r_{\alpha(n)}r_{\alpha(n+1)}} = e_n$. This contradicts the condition (ii) of Definition 2.2. Thus, the degree of the vertex $r_{\alpha(n+1)}$ must be 1.

By (1), the union $e_1 \cup \cdots \cup e_{n-1}$ is contractible, hence $(e_1,\ldots,e_{n-1})$ satisfies the condition (iii) of Definition 2.2. It is obvious that $(e_1,\ldots,e_{n-1})$ satisfies (i) and (ii) of Definition 2.2. Thus, we obtain (2). □

Theorem 4.5 is proved by Proposition 4.4, Lemma 4.7 and the following lemma.

Lemma 4.10. *Any element of* $\mathcal{G}(\Gamma_{n+1})$ *is Hurwitz equivalent to a cyclic element of* $\mathcal{G}(\Gamma_{n+1})$.

Proof. Let $(e_1, \dots, e_n)$ be an element of $\mathcal{G}(\Gamma_{n+1})$. We prove this by induction on n.

First, consider a case where $n = 2$. Then, $(e_1, e_2) = (\overline{r_1 r_2}, \overline{r_2 r_3})$, $(\overline{r_2 r_3}, \overline{r_3 r_1})$ or $(\overline{r_3 r_1}, \overline{r_1 r_2})$. They are cyclic and we have the result when $n = 2$.

Next, consider a case where $n > 2$. By Lemma 4.8, we can take an element $k \in \{1, \dots, n\}$ such that $e_k \in C\Gamma_{n+1}$. Let

$$(e'_1, \dots, e'_n) = (e_1, \dots, e_n) \odot (\sigma_{n-1}\sigma_{n-2}\dots\sigma_k)^{-1}.$$

Using Proposition 4.6, by direct calculations

$$(e'_1, \dots, e'_n) = (e_1, e_2, \dots, e_{k-1}, e_{k+1} \mathbin{\overline{*}} e_k, e_{k+2} \mathbin{\overline{*}} e_k, \dots, e_n \mathbin{\overline{*}} e_k, e_k).$$

By Proposition 4.4, $(e'_1, \dots, e'_n)$ is an element of $\mathcal{G}(\Gamma_{n+1})$. Since $e'_n = e_k \in C\Gamma_{n+1}$, $e'_n = \overline{r_{\alpha(n)} r_{\alpha(n+1)}}$ for a power α of the $n+1$-cyclic permutation $(1\ 2\ \cdots\ n+1)$. Then, $(e'_1, \dots, e'_n)$ is an element of $\mathcal{G}_\alpha(\Gamma_{n+1})$. By Lemma 4.9 (2), $(e'_1, \dots, e'_{n-1})$ is an element of $\mathcal{G}(\Gamma_{n,\alpha})$. By the assumption of the induction, $(e'_1, \dots, e'_{n-1})$ is Hurwitz equivalent to a cyclic element $(\overline{r_{\beta(\alpha(1))} r_{\beta(\alpha(2))}}, \dots, \overline{r_{\beta(\alpha(n-1))} r_{\beta(\alpha(n))}})$ of $\mathcal{G}(\Gamma_{n,\alpha})$, where β is a power of the n-cyclic permutation $(\alpha(1)\ \alpha(2)\ \cdots\ \alpha(n))$. By Lemma 4.7, it is Hurwitz equivalent to $(\overline{r_{\alpha(1)} r_{\alpha(2)}}, \dots, \overline{r_{\alpha(n-1)} r_{\alpha(n)}})$. Then, $(e'_1, \dots, e'_{n-1}, e'_n)$ is Hurwitz equivalent to

$$(\overline{r_{\alpha(1)} r_{\alpha(2)}}, \dots, \overline{r_{\alpha(n-1)} r_{\alpha(n)}}, e'_n) = (\overline{r_{\alpha(1)} r_{\alpha(2)}}, \dots, \overline{r_{\alpha(n)} r_{\alpha(n+1)}}),$$

that is a cyclic element of $\mathcal{G}(\Gamma_{n+1})$, and we have the result. □

Proof of Theorem 4.5. By Proposition 4.4, it is sufficient to prove that any two elements of $\mathcal{G}(\Gamma_{n+1})$ are Hurwitz equivalent. Take an element $\vec{g}$ of $\mathcal{G}(\Gamma_{n+1})$. By Lemma 4.10, $\vec{g}$ is Hurwitz equivalent to a cyclic element $\vec{c}$ of $\mathcal{G}(\Gamma_{n+1})$. By Lemma 4.7, $\vec{c}$ is Hurwitz equivalent to a special cyclic element $\vec{c_0}$. Hence, any element of $\mathcal{G}(\Gamma_{n+1})$ is Hurwitz equivalent to $\vec{c_0}$. This completes the proof of Theorem 4.5. □

ACKNOWLEDGMENTS. The author would like to thank Professor Takao Matumoto, Professor Seiichi Kamada and Professor Makoto Sakuma for their help. This work was supported by JSPS KAKENHI Grant Number 23840026.

References

[1] E. Artin: *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.

[2] T. Ben-Itzhak and M. Teicher: *Properties of Hurwitz equivalence in the braid group of order n*, J. Algebra **264** (2003), 15–25.

[3] J.S. Birman: Braids, Links, and Mapping Class Groups, Ann. of Math. Studies **82**, Princeton Univ. Press, Princeton, NJ, 1974.

[4] F. Bohnenblust: *The algebraical braid group*, Ann. of Math. (2) **48** (1947), 127–136.

[5] F. Catanese and M. Paluszny: *Polynomial-lemniscates, trees and braids*, Topology **30** (1991), 623–640.

[6] F. Catanese and B. Wajnryb: *The fundamental group of generic polynomials*, Topology **30** (1991), 641–651.

[7] F.A. Garside: *The braid group and other groups*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 235–254.

[8] S.P. Humphries: *Finite Hurwitz braid group actions on sequences of Euclidean reflections*, J. Algebra **269** (2003), 556–588.

[9] S.P. Humphries: *Finite Hurwitz braid group actions for Artin groups*, Israel J. Math. **143** (2004), 189–222.

[10] A. Hurwitz: *Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. **55** (1901), 53–66.

[11] S. Kamada: Braid and Knot Theory in Dimension Four, Mathematical Surveys and Monographs **95**, Amer. Math. Soc., Providence, RI, 2002.

[12] P. Kluitmann: *Hurwitz action and finite quotients of braid groups*; in Braids (Santa Cruz, CA, 1986), Contemp. Math. **78**, Amer. Math. Soc., Providence, RI, 1988, 299–325.

[13] B.G. Moishezon: *Stable branch curves and braid monodromies*; in Algebraic Geometry (Chicago, Ill., 1980), Lecture Notes in Math. **862**, Springer, Berlin, 1981, 107–192.

[14] L. Rudolph: *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37.

[15] Y. Yaguchi: *Isotropy subgroup of Hurwitz action of the* 3*-braid group on the braid systems*, J. Knot Theory Ramifications **18** (2009), 1021–1030.

[16] Y. Yaguchi: *The orbits of the Hurwitz action of the braid groups on the standard generators*, Fund. Math. **210** (2010), 63–71.

[17] Y. Yaguchi: *Isotropy subgroup of the Hurwitz action of the* 4*-braid group on braid systems*, J. Gökova Geom. Topol. **4** (2010), 82–95.

Department of Mathematics
Hiroshima University
Higashi-Hiroshima
Hiroshima, 739-8526
Japan

Current address:
School of Science
Gunma National College of Technology
Maebashi, Gunma, 371-8530
Japan
e-mail: yaguchi1984@gmail.com