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GENERALIZED ALMOST CONTACT STRUCTURES AND GENERALIZED SASAKIAN STRUCTURES

KEN'ICH SEKIYA

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Abstract

We introduce generalized almost contact structures which admit the B-field transformations on odd dimensional manifolds. We provide the notion of generalized Sasakian structures from the view point of the generalized almost contact structures. We obtain a generalized Sasakian structure on a non-compact manifold which does not arise as a pair of ordinary Sasakian structures. However we show that a generalized Sasakian structure on a compact 3-dimensional manifold is equivalent to a pair of Sasakian structures with the same metric.

1. Introduction

Both generalized complex structures and generalized Kähler structures are geometric structures on even dimensional manifolds which have been extensively studied in differential geometry and mathematical physics [5, 7]. It is natural to ask what is an analog of generalized geometry on odd dimensional manifolds. Vaisman introduced generalized F-structures and generalized almost contact structures [9, 10]. He also defined generalized Sasakian structures from the view point of generalized Kähler structures. Poon and Wade studied integrability conditions of generalized almost contact structures and gave nontrivial examples on the three-dimensional Heisenberg group and its cocompact quotients [8]. Vaisman showed that a generalized Sasakian structure appears as a pair of almost contact structures [9]. However, examples of generalized Sasakian structures which do not arise as a pair of Sasakian structures were not known.

The purpose of this paper is to investigate generalized geometry on odd dimensional manifolds. We introduce the new notion of generalized almost contact structures which includes the one in [9], [8] as special cases. We use two sections E_+ and E_- of $TM \oplus T^*M$ to define generalized almost contact structures which admit B-field transformations naturally. An almost contact structure is a triple (φ, ξ, η), where φ is an endomorphism of TM, $\xi \in TM$ and $\eta \in T^*M$ which satisfies

$$\eta(\xi) = 1, \quad \varphi \circ \varphi = -id + \eta \otimes \xi,$$

where *id* denotes the identity map of TM. An almost contact structure gives rises to

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an almost complex structure I on the cone $C(M) = M \times \mathbb{R}_{>0}$,

$$I = \varphi + \eta \otimes r\frac{\partial}{\partial r} - \frac{1}{r}\,dr \otimes \xi,$$

where r denotes the coordinate on $\mathbb{R}_{>0}$. We define a generalized almost contact structure to be a triple (Φ, E_+, E_-) by replacing φ with an endomorphism Φ of $TM \oplus T^*M$ and ξ, η with sections E_+, E_- of $TM \oplus T^*M$, respectively which satisfy

$$\begin{aligned}
&\Phi + \Phi^* = 0,\\
&2\langle E_+, E_-\rangle = 1, \quad \langle E_\pm, E_\pm\rangle = 0,\\
&\Phi \circ \Phi = -id + E_+ \otimes E_- + E_- \otimes E_+
\end{aligned}$$

(see Definition 3.1 for more detail). By an analogue to the case of almost contact structures, we define bundle endomorphisms to construct generalized complex structures on the cone $C(M)$. We define a bundle endomorphism $\Psi(E_+, E_-)$ of $TC(M) \oplus T^*C(M)$ by

$$\Psi(E_+, E_-) = E_- \otimes r\frac{\partial}{\partial r} - r\frac{\partial}{\partial r} \otimes E_- + E_+ \otimes \frac{1}{r}\,dr - \frac{1}{r}\,dr \otimes E_+,$$

then it follows that

$$\Phi + \Psi(E_+, E_-)$$

is a generalized almost complex structure on $C(M)$. In Sasakian geometry, the Riemannian cone metric $\tilde{g} = dr^2 + r^2 g$ on $C(M)$ is, by definition, a Kähler metric. This suggests that

$$R(\Phi + \Psi(E_+, E_-))R^{-1}$$

is more important generalized almost complex structures rather than $\Phi + \Psi(E_+, E_-)$ when we pursue an analogy of Sasakian geometry, where R denotes an element of the special orthogonal group $\mathrm{SO}(TM \oplus T^*M)$ given by

$$R(X + \alpha) = r^{-1}X + r\alpha, \quad X \in TM,\ \alpha \in T^*M.$$

From the view point of generalized almost contact structures, we define a generalized Sasakian structure. We show that on a compact connected 3-dimensional manifold a generalized Sasakian structure is equivalent to a pair of Sasakian structures with the same metric (Theorem 4.6). We obtain a non-compact example of a generalized Sasakian structure which does not arise as a pair of Sasakian structures (Theorem 4.2).

2. Generalized complex structures

In this section we give a brief explanation of generalized complex structures. Let M be an even dimensional smooth manifold. The space of sections of the vector bundle

$TM \oplus T^*M \to M$ is endowed with the following $\mathbb{R}$-bilinear operations.

- A symmetric bilinear form $\langle -, - \rangle$ is defined by

$$\langle X+\alpha, Y+\beta \rangle = \frac{1}{2}(\iota_X\beta + \iota_Y\alpha).$$

- The Courant bracket $[\![-,-]\!]$ is a skew-symmetric bracket,

$$[\![X+\alpha, Y+\beta]\!] = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha),$$

where $X, Y \in TM$ and $\alpha, \beta \in T^*M$.

A subbundle is Courant involutive if the space of sections of the subbundle is closed under the Courant bracket.

DEFINITION 2.1 ([5]). A generalized almost complex structure on M is an endomorphism of the direct sum $TM \oplus T^*M$ which satisfies two conditions,

$$\mathcal{J} + \mathcal{J}^* = 0, \quad \mathcal{J}^2 = -id,$$

where $\mathcal{J}^*$ is defined by $\langle \mathcal{J}A, B \rangle = \langle A, \mathcal{J}^*B \rangle$ for any $A, B \in \Gamma(TM \oplus T^*M)$. Let L be the $+\sqrt{-1}$-eigenspace of $\mathcal{J}$ in $TM \oplus T^*M$. If L is Courant involutive, then $\mathcal{J}$ is called a generalized complex structure.

The following are well known.

Lemma 2.1 ([5]). *L is a maximal isotropic subspace.*

Proposition 2.2 ([5]). *Let L be a maximal isotropic subbundle of $TM \oplus T^*M$. Then the following three conditions are equivalent*:

- *L is Courant involutive,*
- $\mathrm{Nij}|_L = 0$,
- $\mathrm{Jac}|_L = 0$,

where Nij *and* Jac *are given by*

$$\mathrm{Nij}(A,B,C) = \frac{1}{3}(\langle [\![A,B]\!], C\rangle + \langle [\![B,C]\!], A\rangle + \langle [\![C,A]\!], B\rangle),$$
$$\mathrm{Jac}(A,B,C) = [\![[\![A,B]\!], C]\!] + [\![[\![B,C]\!], A]\!] + [\![[\![C,A]\!], B]\!],$$

for any $A, B, C \in \Gamma(TM \oplus T^*M)$.

Let B be a smooth 2-form. Then the invertible bundle map given by,

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : X + \alpha \mapsto X + \alpha + \iota_X B$$

is orthogonal.

Lemma 2.3 ([5]). *A map e^B is an automorphism of the Courant bracket if and only if B is closed, i.e. $dB = 0$.*

DEFINITION 2.2 ([5]). A generalized Kähler structure is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures such that $G = -\mathcal{J}_1\mathcal{J}_2$ gives a positive definite metric on $TM \oplus T^*M$.

Lemma 2.4 ([5]). *A generalized Kähler metric is uniquely determined by a Riemannian metric g together with a 2-form b as follows,*

$$G(g, b) = \begin{pmatrix} -g^{-1}b & g^{-1} \\ g - bg^{-1}b & bg^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

Let C_+ be a positive definite subbundle of $TM \oplus T^*M$ and C_- a negative definite subbundle with respect to the inner product which are given by

$$C_\pm = \{X \pm g(X, \cdot\,) + b(X, \cdot\,);\ X \in TM\}.$$

By the projection from $C_\pm$ to TM, $\mathcal{J}_1$ induces two almost complex structures $J_\pm$ on TM. If both (g, J_+) and (g, J_-) are Hermitian structures, $(g, J_\pm)$ is called a bi-Hermitian structure.

Theorem 2.5 ([5]). *A generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is equivalent to a bi-Hermitian structure $(g, b, J_\pm)$ which satisfies the following condition.*

- *For all vector fields X, Y, Z,*

$$db(X, Y, Z) = d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z),$$

where $\omega_\pm(X, Y) = g(X, J_\pm Y)$.

3. Generalized almost contact structures

An almost contact structure on an odd dimensional manifold M is a triple (φ, ξ, η), where φ is an endomorphism of TM, ξ is a vector field and η is a 1-form which satisfies

$$\eta(\xi) = 1, \quad \varphi \circ \varphi = -id + \eta \otimes \xi.$$

We replace φ by an endomorphism Φ of $TM \oplus T^*M$ and ξ, η by sections $E_\pm$ of $TM \oplus T^*M$ respectively. We define a generalized almost contact structure:

DEFINITION 3.1. A generalized almost contact structure on a smooth manifold M is a triple (Φ, E_+, E_-), where Φ is an endomorphism of $TM \oplus T^*M$ and $E_\pm$ are sections of $TM \oplus T^*M$ which satisfy

$$\Phi + \Phi^* = 0,$$
$$2\langle E_+, E_- \rangle = 1, \quad \langle E_\pm, E_\pm \rangle = 0,$$
$$\Phi \circ \Phi = -id + E_+ \otimes E_- + E_- \otimes E_+.$$

Let $E_\pm = \xi_\pm + \eta_\pm$ where $\xi_\pm$ are vector fields and $\eta_\pm$ are 1-forms. Then we have

$$\Phi \circ \Phi = -id + \begin{pmatrix} \eta_+ \otimes \xi_- + \eta_- \otimes \xi_+ & \xi_+ \otimes \xi_- + \xi_- \otimes \xi_+ \\ \eta_+ \otimes \eta_- + \eta_- \otimes \eta_+ & \xi_+ \otimes \eta_- + \xi_- \otimes \eta_+ \end{pmatrix}.$$

REMARK 3.1. Vaisman, Poon and Wade discussed the restrictive case of $\xi_- = \eta_+ = 0$ [8, 9, 10]. However, their definition is not compatible with the B-field transformations. Note that a generalized almost contact structure of Definition 3.1 satisfies the condition of generalized F-structure [10].

EXAMPLE 3.1 ([8]). Let (φ, ξ, η) be an almost contact structure. Then we have a generalized almost contact structure by setting

$$\Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}, \quad E_+ = \xi, \quad E_- = \eta,$$

where $(\varphi^*\alpha)(X) = \alpha(\varphi X)$, $X \in TM$, $\alpha \in T^*M$.

EXAMPLE 3.2 ([8]). A $(2n+1)$-dimensional manifold M is a contact manifold if there exists a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. A 1-form η is called a contact 1-form. Then there is a unique vector field ξ satisfying the two conditions

$$\iota_\xi d\eta = 0, \quad \eta(\xi) = 1.$$

This vector field is called the Reeb field of the contact form η. Since η is a contact 1-form, the map

$$\rho(X) := \iota_X d\eta - \eta(X)\eta$$

is an isomorphism from the tangent bundle to the cotangent bundle. We define a bivector field π by

$$\pi(\alpha, \beta) := d\eta(\rho^{-1}(\alpha), \rho^{-1}(\beta)).$$

Then we have a generalized almost contact structure by setting

$$\Phi = \begin{pmatrix} 0 & \pi \\ d\eta & 0 \end{pmatrix}, \quad E_+ = \eta, \quad E_- = \xi.$$

Lemma 3.1. *Let* $(\Phi, E_\pm)$ *be a generalized almost contact structure. Then we have the following identities,*

$$\Phi(E_\pm) = 0.$$

Proof. Since $\Phi + \Phi^* = 0$, we have

$$\langle \Phi E_+, E_+ \rangle = \langle E_+, -\Phi E_+ \rangle = -\langle \Phi E_+, E_+ \rangle.$$

Thus it follows that we have

$$\langle \Phi E_+, E_+ \rangle = 0.$$

From $(\Phi \circ \Phi)(E_+) = 0$, we obtain

$$\begin{aligned} 0 &= \Phi \circ (\Phi \circ \Phi)(E_+) = (\Phi \circ \Phi) \circ \Phi(E_+) \\ &= -\Phi E_+ + 2\langle E_+, \Phi E_+ \rangle E_- + 2\langle E_-, \Phi E_+ \rangle E_+ \\ &= -\Phi E_+ + 2\langle E_-, \Phi E_+ \rangle E_+. \end{aligned} \tag{3.1}$$

We also obtain

$$0 = \Phi \circ (\Phi \circ \Phi) \circ \Phi(E_+) = 2\langle E_-, \Phi E_+ \rangle \Phi E_+. \tag{3.2}$$

From (3.1) and (3.2), we have

$$\Phi E_+ = 0.$$

Similarly, we have

$$\Phi E_- = 0. \qquad \square$$

By a simple calculation, we have

Lemma 3.2. *Let* $(\Phi, E_\pm)$ *be a generalized almost contact structure and* B *a smooth* 2*-form. Then* $(e^B \Phi e^{-B}, e^B E_\pm)$ *is a generalized almost contact structure.*

By Definition 3.1, we have

$$\Phi^3 + \Phi = 0.$$

Thus Φ has three eigenvalues, namely 0, $+\sqrt{-1}$, $-\sqrt{-1}$. The kernel of Φ is given by

$$L_{E_+} \oplus L_{E_-},$$

where $L_{E_\pm}$ are line bundles generated by $E_\pm = \xi_\pm + \eta_\pm$, respectively. We define

$$E^{(1,0)} = \{X + \alpha - \sqrt{-1}\Phi(X + \alpha);\ X \in TM,\ \alpha \in T^*M,\ \langle X + \alpha, E_\pm\rangle = 0\},$$
$$E^{(0,1)} = \{X + \alpha + \sqrt{-1}\Phi(X + \alpha);\ X \in TM,\ \alpha \in T^*M,\ \langle X + \alpha, E_\pm\rangle = 0\}.$$

Then $E^{(1,0)}$ is $+\sqrt{-1}$-eigenbundle and $E^{(0,1)}$ is $-\sqrt{-1}$-eigenbundle. We consider the following four different complex vector bundles,

$$\begin{aligned} L^+ &= L_{E_+} \oplus E^{(1,0)}, \quad \overline{L^+} = L_{E_+} \oplus E^{(0,1)}, \\ L^- &= L_{E_-} \oplus E^{(1,0)}, \quad \overline{L^-} = L_{E_-} \oplus E^{(0,1)}. \end{aligned} \tag{3.3}$$

Lemma 3.3. *Bundles $E^{(1,0)}$, $E^{(0,1)}$, $L^\pm$, $\overline{L^\pm}$ are isotropic.*

Proof. Let A, B are sections of $E^{(1,0)}$. By our definition, we have $\langle A, E_\pm\rangle = 0$. It follows from $\Phi + \Phi^* = 0$ that

$$\langle \Phi A, \Phi B\rangle = \langle \sqrt{-1}A, \sqrt{-1}B\rangle = -\langle A, B\rangle,$$
$$\langle \Phi A, \Phi B\rangle = \langle A, -\Phi^2 B\rangle = \langle A, B\rangle.$$

Therefore $E^{(1,0)}$ is isotropic. Similarly, $E^{(0,1)}$, $L^\pm$, $\overline{L^\pm}$ are isotropic since $\langle E_\pm, E_\pm\rangle = 0$. □

According to [8], we define

DEFINITION 3.2. Let $(\Phi, E_\pm)$ be a generalized almost contact structure. If either of $L^\pm$ is Courant involutive, it is called a generalized contact structure. If both $L^\pm$ are Courant involutive, it is called a strong generalized contact structure.

An almost contact metric structure on M is (g, φ, ξ, η), where (φ, ξ, η) is an almost contact structure and g is a Riemannian metric which satisfies

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in TM.$$

We define a generalized almost contact metric structure:

DEFINITION 3.3. Let $(\Phi, E_\pm)$ be a generalized almost contact structure. If $G\colon TM \oplus T^*M \to TM \oplus T^*M$ is a generalized Riemannian metric which satisfies

$$-\Phi G \Phi = G - E_+ \otimes E_+ - E_- \otimes E_-,$$

then $(G, \Phi, E_\pm)$ is a generalized almost contact metric structure.

This definition satisfies the condition of generalized metric F-structure without a signature [10].

From Definition 3.3, it follows that $G\Phi = \Phi G$, $G(E_\pm) = E_\mp$ and we have that $(G, G\Phi = \Phi G, GE_\pm = E_\mp)$ is also a generalized almost contact metric structure.

4. Generalized Sasakian structure

There is the intriguing correspondence between the geometry on the cone $C(M) = M \times \mathbb{R}_{>0}$ and the geometry on M [3]. In fact, an almost contact structure (φ, ξ, η) gives rises to an almost complex structure I on $C(M)$;

$$I = \varphi + \eta \otimes \frac{\partial}{\partial t} - dt \otimes \xi,$$

where $e^t = r$ denotes the coordinate on $\mathbb{R}_{>0}$. If I is integrable, an almost contact structure is called a normal almost contact structure. Let $(\Phi, E_\pm = \xi_\pm + \eta_\pm)$ be a generalized almost contact structure on M. we recall a bundle map $\Psi\colon TC(M) \oplus T^*C(M) \to TC(M) \oplus T^*C(M)$ by

$$\begin{aligned}\Psi(E_+, E_-) &= E_- \otimes \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \otimes E_- + E_+ \otimes dt - dt \otimes E_+ \\ &= \begin{pmatrix} \eta_- \otimes \frac{\partial}{\partial t} - dt \otimes \xi_+ & \xi_- \otimes \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \otimes \xi_- \\ \eta_+ \otimes dt - dt \otimes \eta_+ & \xi_+ \otimes dt - \frac{\partial}{\partial t} \otimes \eta_- \end{pmatrix}.\end{aligned}$$

Then it follows that

$$\Phi + \Psi(E_+, E_-)$$

is a generalized almost complex structures on $C(M)$.

Proposition 4.1. *There is a one-to-one correspondence between generalized almost contact structures $(\Phi, E_\pm)$ on M and generalized almost complex structures $\mathcal{J}$ on $C(M)$ such that*

$$\mathcal{L}_{\partial/\partial t}\mathcal{J} = 0,$$
$$\mathcal{J}\frac{\partial}{\partial t} \in TM \oplus T^*M, \quad \mathcal{J}dt \in TM \oplus T^*M.$$

Proof. Let $\mathcal{J}$ be a generalized almost complex structure which satisfies above conditions. Since $\mathcal{J} = -\mathcal{J}^*$, if $\mathcal{L}_{\partial/\partial t}\mathcal{J} = 0$ then we can write

$$\mathcal{J} = \mathcal{J}_M + A \otimes \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \otimes A + B \otimes dt - dt \otimes B + h\frac{\partial}{\partial t} \otimes dt - h\, dt \otimes \frac{\partial}{\partial t}$$

where $\mathcal{J}_M : TM \oplus T^*M \to TM \oplus T^*M$, $A, B \in TM \oplus T^*M$ and $h \in C^\infty(M)$. From $\mathcal{J} dt \in TM \oplus T^*M$, we have $h = 0$. $\mathcal{J}^2 = -id$ implies that $(\mathcal{J}_M, B, A)$ is a generalized almost contact structure.

$\Phi + \Psi(E_+, E_-)$ is clearly a generalized almost complex structure which satisfies above conditions. □

The integrability condition of $\Phi + \Psi(E_+, E_-)$ is given by the following proposition.

Proposition 4.2. *A generalized almost complex structure $\Phi + \Psi(E_+, E_-)$ on $C(M)$ is integrable if and only if a generalized almost contact structure is a strong generalized almost contact structure and $[\![E_+, E_-]\!] = 0$.*

Proof. Since $E^{(1,0)}$ is $+\sqrt{-1}$-eigenbundle of Φ, $+\sqrt{-1}$-eigenbundle of $\Phi + \Psi(E_+, E_-)$ is generated by

$$E^{(1,0)}, \quad E_+ - \sqrt{-1}\frac{\partial}{\partial t}, \quad E_- - \sqrt{-1}\, dt.$$

By simple calculations, we have

$$\begin{aligned} \left[\!\!\left[X + \alpha,\, E_+ - \sqrt{-1}\frac{\partial}{\partial t} \right]\!\!\right] &= [\![X + \alpha,\, E_+]\!], \\ \left[\!\!\left[X + \alpha,\, E_- - \sqrt{-1}\, dt \right]\!\!\right] &= [\![X + \alpha,\, E_-]\!], \\ \left[\!\!\left[E_+ - \sqrt{-1}\frac{\partial}{\partial t},\, E_- - \sqrt{-1}\, dt \right]\!\!\right] &= [\![E_+,\, E_-]\!], \end{aligned}$$

where $X + \alpha \in \Gamma(E^{(1,0)})$. Since $[\![E_+, E_-]\!]$ is a real section, $+\sqrt{-1}$-eigenbundle of $\Phi + \Psi(E_+, E_-)$ is Courant involutive if and only if both $L^\pm$ are Courant involutive and $[\![E_+, E_-]\!] = 0$. □

Let R be an endomorphism of $TM \oplus T^*M$ given by

$$R = \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix}.$$

Then the adjoints

$$R(\Phi + \Psi(E_+, E_-))R^{-1}$$

is also a generalized almost complex structures on $C(M)$. Let g be a Riemannian metric on M. In Sasakian geometry, the Riemannian cone metric on $C(M)$ is

$$\tilde{g} = dr^2 + r^2 g.$$

Since $R(\Phi + \Psi(E_+, E_-))R^{-1}$ correspond to the cone metric, $R(\Phi + \Psi(E_+, E_-))R^{-1}$ is more important than $\Phi + \Psi(E_+, E_-)$ when we consider about Sasakian structures.

The integrability condition of $R(\Phi + \Psi(E_+, E_-))R^{-1}$ is given by the following theorem.

Theorem 4.3. *A generalized almost complex structure $R(\Phi + \Psi(E_+, E_-))R^{-1}$ on $C(M)$ is integrable if and only if the Nijenhuis operator on M satisfies*

$$\begin{aligned}&\mathrm{Nij}_M(A, B, C)\\&= 2\sqrt{-1}(\langle E_-, A\rangle\langle B, C\rangle_- + \langle E_-, B\rangle\langle C, A\rangle_- + \langle E_-, C\rangle\langle A, B\rangle_-)\end{aligned}$$

for any $A, B, C \in \Gamma(E^{(1,0)} \oplus L_{E_+} \oplus L_{E_-})$, where

$$\langle X + \alpha, Y + \beta\rangle_- = \frac{1}{2}(\alpha(Y) - \beta(X)).$$

Proof. Let L be $+\sqrt{-1}$-eigenbundle of $R(\Phi + \Psi(E_+, E_-))R^{-1}$. $R(\Phi + \Psi(E_+,E_-)R^{-1}$ is integrable if and only if $\mathrm{Nij}_{C(M)}|_L = 0$. Since the $+\sqrt{-1}$-eigenbundle L is isotropic, $\mathrm{Nij}_{C(M)}|_L$ is a trilinear operator. Thus we only need to consider elements in $E^{(1,0)}$, E_+ and E_-. Let $X + \alpha, Y + \beta, Z + \gamma$ be elements of $E^{(1,0)}$. Then we have from Definition 3.1

$$\begin{aligned}&[\![R(X + \alpha), R(Y + \beta)]\!]\\&= e^{-t}R[\![X + \alpha, Y + \beta]\!] + (\alpha(Y) - \beta(X))\,dt.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\left[\!\!\left[R(X + \alpha), R\left(E_+ - \sqrt{-1}\frac{\partial}{\partial t}\right)\right]\!\!\right]\\&= e^{-t}R[\![X + \alpha, E_+]\!] - \sqrt{-1}e^{-2t}X + \sqrt{-1}\alpha + (\alpha(\xi_+) - \eta_+(X))\,dt,\\&[\![R(X + \alpha), R(E_- - \sqrt{-1}dt)]\!]\\&= e^{-t}R[\![X + \alpha, E_-]\!] + (\alpha(\xi_-) - \eta_-(X))\,dt,\\&\left[\!\!\left[R\left(E_+ - \sqrt{-1}\frac{\partial}{\partial t}\right), R(E_- - \sqrt{-1}\,dt)\right]\!\!\right]\\&= e^{-t}R[\![E_+, E_-]\!] + \sqrt{-1}e^{-2t}\xi_- - \sqrt{-1}\eta_- + (\eta_+(\xi_-) - \eta_-(\xi_+))\,dt.\end{aligned}$$

Then it follows that

$$\begin{aligned}&\mathrm{Nij}_{C(M)}(R(X + \alpha), R(Y + \beta), R(Z + \gamma))\\&= e^{-t}\mathrm{Nij}_M(X + \alpha, Y + \beta, Z + \gamma).\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Nij}_{C(M)}\left(R(X+\alpha),\, R(Y+\beta),\, R\left(E_+ - \sqrt{-1}\frac{\partial}{\partial t}\right)\right)\\
&= e^{-t}\mathrm{Nij}_M(X+\alpha,\, Y+\beta,\, E_+) + \frac{1}{2}\sqrt{-1}e^{-t}(\beta(X)-\alpha(Y)),\\
&\mathrm{Nij}_{C(M)}\left(R(X+\alpha),\, R(Y+\beta),\, R(E_- - \sqrt{-1}dt)\right)\\
&= e^{-t}\mathrm{Nij}_M(X+\alpha,\, Y+\beta,\, E_-),\\
&\mathrm{Nij}_{C(M)}\left(R(X+\alpha),\, R\left(E_+ - \sqrt{-1}\frac{\partial}{\partial t}\right),\, R(E_- - \sqrt{-1}dt)\right)\\
&= e^{-t}\mathrm{Nij}_M(X+\alpha,\, E_+,\, E_-) - \frac{1}{2}\sqrt{-1}e^{-t}(\eta_-(X)-\alpha(\xi_-)).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\mathrm{Nij}_{C(M)}(A,\, B,\, C)\\
&= e^{-t}\mathrm{Nij}_M(A,\, B,\, C) - 2\sqrt{-1}e^{-t}\langle E_-,\, A\rangle\langle B,\, C\rangle_-\\
&\quad - 2\sqrt{-1}e^{-t}\langle E_-,\, B\rangle\langle C,\, A\rangle_- - 2\sqrt{-1}e^{-t}\langle E_-,\, C\rangle\langle A,\, B\rangle_-
\end{aligned}$$

for any $A, B, C \in \Gamma(E^{(1,0)} \oplus L_{E_+} \oplus L_{E_-})$. Therefore the integrability condition is given by

$$\begin{aligned}
&\mathrm{Nij}_M(A,\, B,\, C)\\
&= 2\sqrt{-1}(\langle E_-,\, A\rangle\langle B,\, C\rangle_- + \langle E_-,\, B\rangle\langle C,\, A\rangle_- + \langle E_-,\, C\rangle\langle A,\, B\rangle_-)
\end{aligned}$$

for any $A, B, C \in \Gamma(E^{(1,0)} \oplus L_{E_+} \oplus L_{E_-})$. □

An immediate corollary of Theorem 4.3 is

Corollary 4.4. *Let $(\Phi,\, E_\pm)$ be a generalized almost contact structure. If $R(\Phi + \Psi(E_+,\, E_-))R^{-1}$ is a generalized complex structure on $C(M)$, then $E^{(1,0)} \oplus L_{E_-}$ is Courant involutive. Therefore $(\Phi,\, E_\pm)$ is a generalized contact structure.*

Proof. It follows from Theorem 4.3 that

$$\mathrm{Nij}_M(A,\, B,\, C) = 0, \quad A,\, B,\, C \in E^{(1,0)} \oplus L_{E_-}.$$

Therefore $E^{(1,0)} \oplus L_{E_-}$ is Courant involutive. □

DEFINITION 4.1. Let $(\Phi, E_\pm)$ be a generalized almost contact structure. If a generalized almost complex structure $R(\Phi + \Psi(E_+,\, E_-))R^{-1}$ is integrable, a generalized almost contact structure is a called normal generalized almost contact structure.

Note that this definition differs from a Vaisman's definition [9]. We define a generalized Sasakian structure in terms of a generalized almost contact metric structure.

DEFINITION 4.2. A generalized Sasakian structure on M is a generalized almost contact metric structure $(G, \Phi, E_\pm)$ such that $R(\Phi + \Psi(E_+, E_-))R^{-1}$ and $R(G\Phi + \Psi(GE_+, GE_-))R^{-1}$ are generalized complex structures on $C(M)$.

A generalized Sasakian structure $(G, \Phi, E_\pm)$ on M induces a generalized Kähler structure $(R(\Phi + \Psi(E_+, E_-))R^{-1}, R(G\Phi + \Psi(GE_+, GE_-)R^{-1})$ on $C(M)$.

REMARK 4.1. Definition 4.2 coincides with Vaisman's definition in the case of $\kappa = 0$ under a modification of degree r [9, 10] (also see Proposition 4.1). The Sasakian structure due to Vaisman allows transformations by 2-forms $2r\, dr \wedge \kappa$ ($\kappa \in T^*M$), however the one by our definition does not admit such a B-field transformation. Generalized almost contact structures admit B-field transformations by 2-forms on M. However, Lemma 2.3 and

$$d(r^2\alpha) \neq 0, \quad \forall \alpha \in \Lambda^2 T^*M$$

show that our definition of generalized Sasakian structures does not admit any B-field transformation. If $\langle \kappa, E_\pm \rangle = 0$, there exists a generalized almost contact structure $(\Phi^\kappa, E^\kappa_\pm)$ such that

$$\begin{pmatrix} 1 & 0 \\ \dfrac{2}{r}\, dr \wedge \kappa & 1 \end{pmatrix} (\Phi + \Psi(E_+, E_-)) \begin{pmatrix} 1 & 0 \\ -\dfrac{2}{r}\, dr \wedge \kappa & 1 \end{pmatrix} = \Phi^\kappa + \Psi(E^\kappa_+, E^\kappa_-).$$

However $(G, \Phi^\kappa, E^\kappa_\pm)$ is not a generalized almost contact metric structure.

EXAMPLE 4.1. Let (g, φ, ξ, η) be a Sasakian structure. If we set

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^* \end{pmatrix}, \quad E_+ = \xi, \quad E_- = \eta$$

then $(G, \Phi, E_\pm)$ becomes a generalized Sasakian structure.

The next theorem corresponds to Theorem 2.5.

Theorem 4.5 ([9]). *A generalized Sasakian structure on a manifold M is equivalent to a pair $(\varphi_\pm, \xi_\pm, \eta_\pm, g)$ of normal almost contact metric structures with the same* *metric g which satisfies the following conditions*

$$\mathcal{L}_{\xi_+}\theta_+ = -\mathcal{L}_{\xi_-}\theta_-, \tag{4.1}$$

$$\theta_\pm - d\eta_\pm + \frac{1}{4}\mathcal{L}_{\xi_\pm}\mathcal{L}_{\xi_\pm}\theta_\pm = 0, \tag{4.2}$$

$$d\theta_\pm - \eta_\pm \wedge \mathcal{L}_{\xi_\pm}\theta_\pm - \frac{1}{2}(d\mathcal{L}_{\xi_\pm}\theta_\pm)^{c_\pm} = 0, \tag{4.3}$$

where $\theta = g(\,\cdot\,, \varphi)$ and the upper indices $c_\pm$ denote

$$\alpha^{c_\pm}(X_1, \ldots, X_k) = \alpha(\varphi_\pm X_1, \ldots, \varphi_\pm X_k), \quad \forall \alpha \in \Omega^k(M).$$

Note that a pair of Sasakian structures with the same metric satisfies these conditions. In the case of a compact connected 3-dimensional manifold, a generalized Sasakian structure is equivalent to a pair of Sasakian structures with the same metric. In fact, we have

Theorem 4.6. *Let M be a compact connected 3-dimensional manifold. Then a pair $(\varphi_\pm, \xi_\pm, \eta_\pm, g)$ of normal almost contact metric structures corresponds to a generalized Sasakian structure if and only if both structures are Sasakian.*

Proof. A normal almost contact metric structure (φ, ξ, η, g) is a Sasakian structure if and only if $\theta = d\eta$, where $\theta = g(\,\cdot\,, \varphi)$ (cf. Definition 6.4.4 and Definition 6.5.13 in [3]). Thus it is sufficient to show that $\theta_\pm = d\eta_\pm$. Since M is 3-dimensional, we have

$$\eta_+ \wedge d\mathcal{L}_{\xi_+}\theta_+ = 0.$$

The inner product by ξ_+ yields

$$\eta_+ \wedge \mathcal{L}_{\xi_+}\mathcal{L}_{\xi_+}\theta_+ = d\mathcal{L}_{\xi_+}\theta_+.$$

From (4.2) and Stokes' theorem, we have

$$0 \neq \int \eta_+ \wedge \theta_+ = \int \eta_+ \wedge \left(d\eta_+ - \frac{1}{4}\mathcal{L}_{\xi_+}\mathcal{L}_{\xi_+}\theta_+ \right) = \int \eta_+ \wedge d\eta_+.$$

Let U be the open set given by

$$U = \{x \in M;\, (\eta_+ \wedge d\eta_+)_x \neq 0\}.$$

Then U is not empty. It follows from Darboux's theorem that we have local coordinates (x, y, z) such that

$$\eta_+ = dz - y\,dx, \quad \xi_+ = \frac{\partial}{\partial z}.$$

Since $\iota_{\xi_+}\theta_+ = 0$, there exits a function $f \neq 0$ such that

$$\theta_+ = f\,dx \wedge dy = f\,d\eta_+.$$

From (4.1), we have

$$0 = -\iota_{\xi_-}\mathcal{L}_{\xi_-}\theta_- = \iota_{\xi_-}\mathcal{L}_{\xi_+}\theta_+ = \iota_{\xi_-}\left(\frac{\partial f}{\partial z}\,dx \wedge dy\right).$$

Let V be the open set given by

$$V = \left\{x \in U;\ \frac{\partial f}{\partial z} \neq 0\right\}$$

We assume that V is not empty. Then we have $\xi_+ = \pm\xi_-$ on V. Since $\iota_{\xi_-}\theta_- = 0$, we obtain

$$\theta_- = h\,d\eta_- = \pm h\,d\eta_+,$$

where h is a function. From (4.1), we have

$$\frac{\partial f}{\partial z}\,d\eta_+ = -\frac{\partial h}{\partial z}\,d\eta_+.$$

Then, from (4.2), we have

$$\left(f - 1 + \frac{1}{4}\frac{\partial^2 f}{\partial z^2}\right)d\eta_+ = 0,$$
$$\pm\left(h - 1 - \frac{1}{4}\frac{\partial^2 f}{\partial z^2}\right)d\eta_+ = 0.$$

Thus it follows that

$$f - 1 = -(h - 1).$$

Thus, for $X, Y \in TM$, we obtain

$$g(Y, \varphi_- X) = \theta_-(Y, X) = \pm\left(\frac{2}{f} - 1\right)\theta_+(Y, X) = g\left(Y, \pm\left(\frac{2}{f} - 1\right)\varphi_+ X\right).$$

Thus it follows that

$$\varphi_- = \pm\left(\frac{2}{f} - 1\right)\varphi_+.$$

Since $\varphi_\pm^2 = -id + \eta_\pm \otimes \xi_\pm$, we have $f = 1$. However this is a contradiction because $\partial f/\partial z \neq 0$. Therefore $\partial f/\partial z = 0$ on U, we have $L_{\xi_+}\theta_+ = 0$ and $\theta_\pm = d\eta_\pm$ on U. Since $\eta_+ \wedge \theta_+ \neq 0$ on M, we have $\overline{U} \subset U$. Since M is connected and U is not empty, we have $U = M$ and $\theta_\pm = d\eta_\pm$ on M. $\square$

On a compact 3-dimensional manifold, a generalized Sasakian structure is equivalent to a pair of Sasakian structures. However, there exists a non-compact example which is not a pair of Sasakian structures.

EXAMPLE 4.2. Let (M', g', J', ω') be a Kähler manifold and $M = M' \times (0, \pi/2)$. To construct normal almost contact metric structures, we define

$$\begin{aligned} &\varphi = J', \quad \xi = \frac{\partial}{\partial z}, \quad \eta = dz, \\ &g = \sin(2z) g' + dz \otimes dz, \end{aligned}$$

where z denotes the coordinate on $(0, \pi/2)$. Then $(g, \pm\varphi, \xi, \eta)$ are normal almost contact metric structures but not Sasakian structures.

On $C(M) = M' \times (0, \pi/2) \times \mathbb{R}_{>0}$, we define complex structures and a metric by

$$\begin{aligned} &J_\pm = \pm\varphi - \frac{1}{r}\, dr \otimes \frac{\partial}{\partial z} + dz \otimes r \frac{\partial}{\partial r}, \\ &\tilde{g} = r^2 g + dr \otimes dr. \end{aligned}$$

Then $(\tilde{g}, J_\pm)$ is a bi-Hermitian structure and

$$\begin{aligned} &\omega_\pm = \tilde{g}(\,\cdot\,, J_\pm \,\cdot\,) = \pm r^2 \sin(2z)\omega' + 2r\, dr \wedge dz, \\ &d\omega_\pm = \pm 2r \sin(2z)\, dr \wedge \omega' \pm 2r^2 \cos(2z)\, dz \wedge \omega'. \end{aligned}$$

Thus

$$\begin{aligned} &d\omega_\pm(J_\pm \,\cdot\,, J_\pm \,\cdot\,, J_\pm \,\cdot\,) \\ &= \pm 2r \sin(2z)(r\, dz) \wedge \omega' \pm 2r^2 \cos(2z)\left(-\frac{1}{r}\, dr\right) \wedge \omega' \\ &= \pm 2r^2 \sin(2z)\, dz \wedge \omega' \mp 2r \cos(2z)\, dr \wedge \omega' \\ &= \pm d(-r^2 \cos(2z)\omega'). \end{aligned}$$

Therefore $(\tilde{g}, -r^2\cos(2z)\omega', J_\pm)$ is a generalized Kähler structure and induces a generalized Sasakian structure. If we set $\rho = (\omega')^{-1}$ on M', we have

$$\begin{aligned} G &= \begin{pmatrix} 1 & 0 \\ -\cos(2z)\omega' & 1 \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \cos(2z)\omega' & 1 \end{pmatrix}, \\ \Phi &= \begin{pmatrix} 1 & 0 \\ -\cos(2z)\omega' & 1 \end{pmatrix} \begin{pmatrix} 0 & \dfrac{1}{\sin(2z)}\rho \\ -\sin(2z)\omega' & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \cos(2z)\omega' & 1 \end{pmatrix}, \\ E_+ &= \begin{pmatrix} 1 & 0 \\ -\cos(2z)\omega' & 1 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial z} \\ 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 1 & 0 \\ -\cos(2z)\omega' & 1 \end{pmatrix} \begin{pmatrix} 0 \\ dz \end{pmatrix}. \end{aligned}$$

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2-27-3-103, kitamikata, takatsu-ku
kawasaki city Kanagawa 213-0005
Japan
e-mail: kenichisekiya.g@gmail.com