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WEYL GROUP SYMMETRY ON THE GKM GRAPH OF A GKM MANIFOLD WITH AN EXTENDED LIE GROUP ACTION

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Abstract

We consider the class of manifolds with compact Lie group actions which restrict to GKM-actions on the maximal torus. First, we see their GKM-graphs admit symmetry of the Weyl groups. And then, we study its combinatorial abstraction; starting with abstract GKM-graphs with symmetry, we derive certain properties which reflect topology in a purely combinatorial way.

1. Introduction

Let T be a compact n-dimensional torus acting on a closed oriented m-manifold M with finite fixed points. If M satisfies certain conditions (see §2), it is called a *GKM-manifold*. Goresky, Kottwitz and MacPherson ([4]) developed a powerful method to compute the torus equivariant cohomology $H_T^*(M;\mathbb{R})$ for a GKM-manifold M, by associating it a combinatorial data called the *GKM-graph*.

What we focus on in this note are the *GKM-manifolds with extended Lie group actions*. Let G be a compact, simple, simply-connected Lie group with the maximal torus T. If M admits a G-action whose restriction to T equips M a GKM-manifold structure, we call M a *GKM-manifold with an extended G-action* (see [9]). This class of manifolds includes interesting examples of flag varieties, and more generally, maximal rank homogeneous spaces. Our goal is to see what additional structures are imposed on the GKM-graph, and to what extent the topology of M is captured combinatorially.

The organization of this paper is as follows: After briefly recalling the GKM-theory in §2, we see in §3 that the GKM-graph of a GKM-manifold with an extended G-action has a symmetry of the Weyl group W of G, and that the localization map is compatible with it. In §4, we start with an abstract GKM-graph Γ with the symmetry of a finite Coxeter group W. We discuss the cohomology ring $H^*(\Gamma;\mathbb{R})$ of the graph and derive its properties combinatorially. In particular, we see a necessary condition for Γ to admit the symmetry of W. Finally in §5, we see that with our definition of the W-symmetry, a series of operators on $H^*(\Gamma;\mathbb{R})$ indexed by W are defined.

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They correspond topologically to the *left divided difference operators*, and $H^*(\Gamma;\mathbb{R})$ is equipped with an action of the *nil-Hecke ring* ([8]). As an application of this fact, we obtain a retraction $H^*(\Gamma;\mathbb{R}) \to H^*(\Gamma;\mathbb{R})^W$, which is reminiscent of the Becker–Gottlieb transfer ([1]) $\tau\colon H_T^*(M;\mathbb{R}) \to H_G^*(M;\mathbb{R})$.

We assume that the coefficients for the cohomology ring is $\mathbb{R}$ unless otherwise stated.

2. GKM theory

For a closed oriented manifold M with a T-action, we consider its T-equivariant cohomology $H_T^*(M) := H^*(ET \times_T M)$. Here $ET \times_T M$ is the Borel construction, which is the quotient space of $ET \times M$ by the equivalence relation $(u,m) \sim (ut^{-1}, tm)$, where $u \in ET$, $m \in M$, and $t \in T$. When M is a point, $H_T^*(pt) = H^*(BT)$ is the symmetric algebra over the dual Lie algebra $\mathfrak{t}^*$ of T.

M is said to be a GKM-manifold if it satisfies the following conditions:

- it admits a T-invariant almost complex structure,
- it is *equivariant formal*, i.e., $H_T^*(M)$ is a free $H_T^*(pt)$-module,
- the fixed points set M^T is finite,
- the weights of the isotropic T-action on the tangent space $\mathrm{T}_p(M)$ are pairwise linearly independent for $\forall\, p \in M^T$.

It follows (see [5]) that the set of the one-dimensional orbits $\{x \in M \mid \dim(Tx) = 1\}$ is the disjoint union $\bigsqcup_{\hat{e}\in\hat{E}} X_{\hat{e}}^o$, where $\hat{E}$ is a finite index set and the closure $X_{\hat{e}}$ of each component $X_{\hat{e}}^o$ is diffeomorphic to S^2 containing exactly two fixed points e_p and e_q at the north and the south poles. Then, the *GKM-graph* $\Gamma(M) = (V, E, \alpha, \theta)$ for M is constructed as follows:

- The vertex set V is the fixed points set M^T.
- The edge set E; for each two sphere $X_{\hat{e}}$ ($\hat{e} \in \hat{E}$), we draw an edge $e_p \to e_q$.
- The *axial function* α: $E \to \mathfrak{t}^*$ is defined by assigning to $e_p \to e_q \in E$ the weight of the T-action on $T_{e_p}X_{\hat{e}}$. Note that α_e annihilates the Lie algebra of the stabilizer group of $X_{\hat{e}}$. We graphically denote an edge with the value of axial function by $p \xrightarrow{\alpha(e)} q$.
- Let $\mathrm{out}(p)$ be the set of the outgoing edges from $p \in V$. The restriction of TM around p splits into the sum of the plane bundles $\bigoplus_{e'\in\mathrm{out}(e_p)} L_{e'}$, where the restriction of $L_{e'}$ at e_p is isomorphic to $T_{e_p}X_{\hat{e}'}$. For each edge $e \in E$, the *connection* θ_e: $\mathrm{out}(e_p) \to \mathrm{out}(e_q)$ is the bijection which assigns to $e' \in \mathrm{out}(e_p)$ the edge corresponding to $L_{e'}|_{e_q}$.

The relationship between the topology of a GKM-manifold and the combinatorics of its GKM-graph is bridged by the *localization map*. The inclusion map $i\colon M^T \hookrightarrow M$ is T-equivariant, and hence it induces a map on the equivariant cohomology. We call the induced map

$$i^* = \bigoplus_{p\in M^T} i_p^*\colon H_T^*(M) \to \bigoplus_{p\in M^T} H_T^*(p) \cong \bigoplus_{p\in M^T} H^*(BT)$$

the localization map. We often denote $i_p^*(h)$ by h_p for $h \in H_T^*(M)$.

The main result in [4] states that the GKM-graph determines the image of the localization map, and thus encodes the T-equivariant cohomology:

Theorem 2.1 ([4]). *The localization map gives the following isomorphism*:

$$H_T^*(M) \cong \left\{ \bigoplus_{p \in M^T} h_p \;\middle|\; h_p \in H^*(BT),\ h_p - h_q \in (\alpha(e)) \text{ if } p \xrightarrow{\alpha(e)} q \right\},$$

where $\mathfrak{t}^*$ is identified with $H^2(BT)$. Note that the right hand side is determined purely combinatorially, and is often called the GKM-description of the equivariant cohomology.

3. Weyl group action

Let M be a GKM-manifold with an extended G action, where G is a compact, simple, simply-connected Lie group with the maximal torus T. The Weyl group W of G is by definition $N(T)/T$, where $N(T)$ is the normalizer of the maximal torus in G. We denote the positive roots by Π^+, and the reflection corresponding to $\alpha \in \Pi^+$ by $s_\alpha \in \mathrm{Aut}(\mathfrak{t}^*)$.

We define a right action of W on $ET \times_T M$ by $w(u, x) = (uw, w^{-1}x)$, which in turn induces a left action on $H_T^*(M)$. In particular, W acts on $H_T^*(pt) = H^*(BT)$. If we regard $H^*(BT)$ as the symmetric algebra over $\mathfrak{t}^* = \langle t_1, \dots, t_n \rangle$, the action is nothing but the standard action on the variables:

$$w(h)(t_1, \dots, t_n) = h(wt_1, \dots, wt_n), \quad w \in W,\ h \in H^*(BT) = \mathbb{R}[t_1, \dots, t_n].$$

Note that the W-action is trivial on $EG \times_G M$, and so is on $H_G^*(M)$.

REMARK 3.1. The W action is not T-equivariant, and hence, it is well-defined only on $ET \times_T M$ but not on M.

We first investigate the W-action on $H_T^*(M)$ in terms of the GKM-description.

Lemma 3.2. *For $w \in W$, $p \in M^T$, and $h \in H_T^*(M)$, we have*

$$w(h)_p = w(h_{w^{-1}p}) \in H^*(BT).$$

Proof. By the following commutative diagram

$$\begin{array}{ccc} ET \times_T \{p\} & \xrightarrow{w} & ET \times_T \{w^{-1}p\} \\ \big\downarrow{\scriptstyle i_p} & & \big\downarrow{\scriptstyle i_{w^{-1}p}} \\ ET \times_T M & \xrightarrow{w} & ET \times_T M, \end{array}$$

we obtain $i_p^* \circ w = w \circ i_{w^{-1}p}^*$. □

The GKM-graph $\Gamma(M)$ admits the following symmetry of W.

Theorem 3.3. (1) *If there is an edge* $p \xrightarrow{\alpha(e)} q$, *then so is* $wp \xrightarrow{w(\alpha(e))} wq$.
(2) *If* $q = s_\alpha p$ *for some* $\alpha \in \Pi^+$, *there is an edge* $p \xrightarrow{\alpha} q$.
(3) $w(\theta_e(e')) = \theta_{w(e)}(w(e'))$ *for all* $e, e' \in E$ *and* $w \in W$.

Proof. (1) Assume that $T_p(M) = \bigoplus_{e \in \text{out } p} \mathbb{C}_{\alpha(e)}$, where $\mathbb{C}_\alpha$ is the complex one-dimensional T-module with the weight $\alpha(e)$. Then the tangent space at wp decomposes to $T_{wp}(M) = \bigoplus_{e \in \text{out } p} \mathbb{C}_{w\alpha(e)}$.

(2) Let $P_\alpha \subset G$ be the subgroup corresponding to the root α. The P_α-orbit of p is $P_\alpha/T \cong S^2$ with p and q as its north and south poles. Since, $T_p(P_\alpha/T) \cong \mathbb{C}_\alpha$ by definition, the assertion follows.

(3) The decomposition $\bigoplus_{e' \in \text{out}(p)} L_{e'}$ in the definition of θ is mapped by w to $\bigoplus_{e' \in \text{out}(p)} wL_{e'}$. $\square$

The relationship between the G-equivariant cohomology and the T-equivariant cohomology is summarized as follows.

Proposition 3.4. (1) $H_T^*(M) \cong H^*(BT) \otimes H^*(M)$ *as* $H^*(BT)$*-modules.*
(2) $H_G^*(M) \cong H^*(BG) \otimes H^*(M)$ *as* $H^*(BG)$*-modules.*
(3) $H_T^*(M) \cong H^*(BT) \otimes_{H^*(BG)} H_G^*(M)$ *as* $H^*(BT)$*-algebras.*
(4) $H_G^*(M) \cong H_T^*(M)^W$ *as* $H^*(BG)$*-algebras.*

Proof. (1) Consider the fibration $M \hookrightarrow ET \times_T M \to BT$. Since $H_T^*(M)$ is a free $H^*(BT)$-module by assumption, we have $H_T^*(M) \cong H^*(BT) \otimes H^*(M)$ by the Serre spectral sequence.

(2) By Borel's localization theorem, $H^*(M)$ should be concentrated in even degrees. Then by the Serre spectral sequence for the fibration $M \hookrightarrow EG \times_G M \to BG$, we have $H_G^*(M) \cong H^*(BG) \otimes H^*(M)$.

(3) Consider the following pullback diagram:

$$\begin{array}{ccccc}
 & & G/T & = & G/T \\
 & & \downarrow & & \downarrow \\
M & \hookrightarrow & ET \times_T M & \longrightarrow & BT \\
\| & & \downarrow & & \downarrow \\
M & \hookrightarrow & EG \times_G M & \longrightarrow & BG
\end{array}$$

The Eilenberg–Moore spectral sequence collapses at E_2-term and we have $H_T^*(M) \cong H^*(BT) \otimes_{H^*(BG)} H_G^*(M)$ as $H^*(BT)$-algebras. Note that there are no extension problem in this case.

(4) Since W acts on $H_G^*(M)$ trivially, the assertion follows from the previous isomorphism. □

EXAMPLE 3.5. We recover the well-known specialization formula for the double Schubert polynomials. Let M be the flag variety G/T, with the G-action induced by the left multiplication. The fixed point set of the action restricted to the maximal torus T is $\{wT/T \mid w \in W\}$. Since $H_G^*(G/T) = H^*(BT)$, we have $H_T^*(G/T) = H^*(BT) \otimes_{H^*(BG)} H^*(BT)$ by Proposition 3.4. It is well-known that $H_T^*(G/T)$ admits a free $H^*(BT)$-module basis indexed by W called the *Schubert classes* (see, for example, [6]). Let $\mathcal{S}_w(t; x) \in \mathbb{R}[t_1, \dots, t_n, x_1, \dots, x_n]$ be a polynomial representing a Schubert class $\mathcal{S}_w \in H_T^*(G/T)$, where t_i's and x_i's are the generators for the left and the right factors of $H^*(BT) \otimes H^*(BT)$. Then, the localization at a fixed point $v \in W$ is the specialization at $x_i = v^{-1}t_i$:

$$i_v^*(\mathcal{S}_w) = \mathcal{S}_w(t_1, \dots, t_n; vt_1, \dots, vt_n).$$

Proof. Since $EG \times_T \{*\} \to EG \times_G G/T$ induces the isomorphism $H^*(BT) \cong H_G^*(G/T)$, the localization map at the identity element i_e^* is the identity map on the right factor. Hence, by Lemma 3.2,

$$\begin{aligned} i_v^*(\mathcal{S}_w) &= v \circ i_e^* \circ v^{-1}(\mathcal{S}_w) \\ &= v \circ i_e^* \mathcal{S}_w(v^{-1}t_1, \dots, v^{-1}t_n; x_1, \dots, x_n) \\ &= v\mathcal{S}_w(v^{-1}t_1, \dots, v^{-1}t_n; t_1, \dots, t_n) \\ &= \mathcal{S}_w(t_1, \dots, t_n; vt_1, \dots, vt_n). \end{aligned}$$ □

4. Abstract GKM-graph with symmetry

Modeled after the topological setting we have just seen, we consider an abstract GKM-graph with the symmetry of a finite Coxeter group.

DEFINITION 4.1. Let $\mathfrak{t}^*$ be the vector space generated on a basis $t_1, \dots, t_n$. A connected, finite, m-valent graph $\Gamma = (V, E, \alpha, \theta)$ with the following additional structures is said to be a GKM-graph:

(1) The axial function $\alpha \colon E \to \mathfrak{t}^*$ satisfies that $\{\alpha(e) \mid e \in \mathrm{out}(p)\}$ are pairwise linearly independent for all $p \in V$.

(2) The connection θ assigns for each edge $e \in E$ a bijection between out(e_p) and out(e_q), satisfying that $\alpha(\theta_e(e')) - \alpha(e')$ is a scalar multiple of $\alpha(e)$ for any adjacent edges e and e'.

We say that Γ has the symmetry of a rank n finite Coxeter group W if W acts as an automorphism on the vertices and edges and further it satisfies

(1) $\alpha(w(e)) = w(\alpha(e))$ for $\forall e \in E$.

(2) If $q = s_\alpha p$ and $q \neq p$ for some $\alpha \in \Pi^+$, there is an edge $p \xrightarrow{\alpha} q$.

(3) $w(\theta_e(e')) = \theta_{w(e)}(w(e'))$ for all $e, e' \in E$ and $w \in W$.
In the above, W action on $\mathfrak{t}^*$ is given by the standard representation.

We see in §3, for a GKM manifold M with an extended action of a Lie group G, the GKM-graph $\Gamma(M)$ admits the symmetry of the Weyl group of G.

The cohomology of Γ is defined following Theorem 2.1:

DEFINITION 4.2. Let $\mathbb{R}[t_1, \ldots, t_n]$ be the symmetric algebra over $\mathfrak{t}^*$, where the generators t_i's are considered to be of degree two.

$$H^*(\Gamma) = \left\{ \bigoplus_{p \in V} h_p \;\middle|\; h_p \in \mathbb{R}[t_1, \ldots, t_n],\ h_p - h_q \in (\alpha(e)) \text{ if } p \xrightarrow{\alpha(e)} q \right\}.$$

We consider $H^*(\Gamma)$ as a $\mathbb{R}[t_1, \ldots, t_n]$-submodule of $\bigoplus_{p \in V} \mathbb{R}[t_1, \ldots, t_n]$, where the module structure is diagonal.

Note that $\mathbf{1} = \bigoplus_{p \in V} 1$ is the generator for $H^0(\Gamma) \cong \mathbb{R}$.
If Γ has the symmetry of W, then W acts on the cohomology.

Lemma 4.3. *For $h \in H^*(\Gamma)$, the element $w(h) \in \bigoplus_{p \in V} \mathbb{R}[t_1, \ldots, t_n]$ defined by*

$$w(h)_p = w(h_{w^{-1}p}) \tag{4.1}$$

belongs to $H^*(\Gamma)$.

Proof. Suppose that there is an edge $p \xrightarrow{\alpha} q$. Then there should be an edge $w^{-1}p \xrightarrow{w^{-1}\alpha} w^{-1}q$. $w(h)_p - w(h)_q = w(h_{w^{-1}p} - h_{w^{-1}q})$ is divisible by $w \circ w^{-1}\alpha = \alpha$. □

The action reduces trivially on the "ordinary cohomology."

Lemma 4.4. *For $h \in H^*(\Gamma)$ and $\alpha \in \Pi^+$, we have*

$$h - s_\alpha(h) \in (\alpha \cdot \mathbf{1}).$$

Proof. Since

$$(h - s_\alpha(h))_p = h_p - s_\alpha(h_{s_\alpha p}) = (h_p - h_{s_\alpha p}) + (h_{s_\alpha p} - s_\alpha(h_{s_\alpha p}))$$

is divisible by α, the assertion follows. □

Since $w_1 w_2(h) - h = (w_1 w_2(h) - w_2(h)) + (w_2(h) - h)$ for $w_1, w_2 \in W$, it follows that $w(h) - h \in (\mathbb{R}^+[t_1, \ldots, t_n] \cdot \mathbf{1})$ for any $w \in W$, that is, W acts trivially on $H^*(\Gamma)/(t_1, \ldots, t_n)$.

REMARK 4.5. Note that if Γ is a GKM-graph of some GKM G-manifold, it is an easy topological consequence; The fiber inclusions $\iota\colon M \to ET \times_T M$ and $w \circ \iota$ is homotopic since G is connected and there exists a path from (a representative of) w to the identity element. Therefore, they induces the same map $H_T^*(M) \to H^*(M) \cong H_T^*(M)/(H^+(BT))$.

From this lemma, we deduce a necessary condition for Γ to admit the symmetry of W. For this purpose, we define a graph theoretical analogy for torus invariant submanifolds. A connected subgraph Λ of Γ is said to be *closed under the connection* θ when $\theta_e(e') \in \Lambda$ for $\forall e, e' \in \Lambda$. Since θ_e is bijection for all $e \in E$, Λ is always a regular graph. Let Γ_k be the set of all the k-valent subgraphs which are closed under the connection. For $\Lambda \in \Gamma_k$, we obtain the class $\lambda \in H^{2m-2k}(\Gamma)$ which is supported on Λ by

$$\lambda_p = \begin{cases} \prod_{e \in \text{out } p \setminus \Lambda} \alpha(e) & (p \in \Lambda), \\ 0 & (p \notin \Lambda). \end{cases}$$

Note that $\lambda \in H^*(\Gamma)$ is clear from the definition of θ. Suppose that another class $\lambda' \in H^{2m-2k}(\Gamma)$ has the same support as λ, that is, $\lambda'_p \neq 0$ iff $p \in \Lambda$. Then it must be a scalar multiple of λ. To see this, observe that there exists $c_p \in \mathbb{R}$ for each $p \in \Lambda$ such that $\lambda'_p = c_p \lambda_p$. Take $q \in \Lambda$ which is adjacent to p by an edge e. Then $(\lambda' - c_p\lambda)_q$ must be divisible by $\alpha(e)\prod_{e' \in \text{out } q \setminus \Lambda} \alpha(e')$ and so must be 0 by degree reason. It follows that $c_q = c_p$, and by the connectivity of Λ all the c_p's for $p \in \Lambda$ must be equal. Since W permutes Γ_k, we have

Proposition 4.6. *W acts on the set $\{\pm\lambda \in H^{2m-2k}(\Gamma) \mid \Lambda \in \Gamma_k\}$ as a signed permutation.*

In some cases, this imposes a cohomological restriction on what kind of W can act on Γ.

EXAMPLE 4.7. Assume that the classes corresponding to Γ_{m-1} gives a free basis for $H^2(\Gamma)$. Since $H^2(\Gamma)$ contains $\mathfrak{t}^* \cdot \mathbf{1}$, the action is faithful. Furthermore, each reflection $s_\alpha \in W$ fixes all but one or two elements. To see this, first observe that the trace of the action of s_α on $\mathfrak{t}^*$ is $n-2$ since it is a reflection. In addition, by the previous Lemma, s_α acts trivially on $H^2(\Gamma)/\mathfrak{t}^*$. Therefore, the trace of the action of s_α on $H^2(\Gamma)$ must be two less than the rank of $H^2(\Gamma)$. It follows that we can choose a subset of $\{\pm\lambda \in H^2(\Gamma)\}$ of cardinality equal to or less than $2(n+1)$, on which W acts faithfully as a signed permutation. In particular, such W is restricted to of classical types. (Readers also refer to [12] and [10]).

5. Divided difference operators

In this section, we construct a series of operators on $H^*(\Gamma)$. Topologically, they coincide with the left divided difference operators.

First, we need the following lemma:

Lemma 5.1. *$(h - s_\alpha(h))/\alpha$ is in $H^*(\Gamma)$.*

Proof. First see

$$(h - s_\alpha(h))_p - (h - s_\alpha(h))_q = h_p - h_q + s_\alpha(h_{s_\alpha q} - h_{s_\alpha p}).$$

If $q \neq s_\alpha p$ and $p \xrightarrow{\beta} q$, it follows $s_\alpha p \xrightarrow{s_\alpha(\beta)} s_\alpha q$. Hence, both $h_p - h_q$ and $s_\alpha(h_{s_\alpha q} - h_{s_\alpha p})$ is divisible by β and since the labels are pairwise linearly independent, $(h - s_\alpha(h))_p/\alpha - (h - s_\alpha(h))_q/\alpha$ is divisible by β. If $q = s_\alpha p$ and $p \xrightarrow{\alpha} q$, we put $h_p - h_{s_\alpha p} = \alpha \cdot f$ with $f \in \mathbb{R}[t_1, \ldots, t_n]$. Then we see

$$h_p - h_q + s_\alpha(h_{s_\alpha q} - h_{s_\alpha p}) = \alpha \cdot f + s_\alpha(\alpha \cdot f) = \alpha(f - s_\alpha(f))$$

is divisible by α^2. □

Now, we can define the operators.

DEFINITION 5.2. The *left divided difference operator* associated to $\alpha \in \Pi^+$ is defined to be

$$\partial_\alpha \colon H^*(\Gamma) \to H^{*-2}(\Gamma), \qquad h \mapsto \frac{h - s_\alpha(h)}{\alpha},$$

or equivalently,

$$\partial_\alpha(h)_p = \frac{1}{\alpha}(h_p - s_\alpha h_{s_\alpha p}).$$

It is well-known (see [8], for example) that they satisfy the braid relation for W, and hence, we can define ∂_w for $w \in W$ as the composition $\partial_{\alpha_1}\partial_{\alpha_2}\cdots\partial_{\alpha_l}$, where $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_l}$ is a reduced decomposition for w.

Let us recall the definition of the *nil-Hecke ring* from [8]: Let $S = \mathbb{R}[t_1, \ldots, t_n]$ be the symmetric algebra over $\mathfrak{t}^*$ and Q be its quotient ring. We denote the smash product of the group ring $\mathbb{R}[W]$ and Q by Q_W. Then the nil-Hecke ring R is the sub ring of Q_W generated freely as a left S-module by ∂_w's for $\forall w \in W$.

Theorem 5.3. *Let Γ be a GKM-graph with the symmetry of a finite Coxeter group W* (Definition 4.1). *Then, $H^*(\Gamma)$ is a module over the nil-Hecke ring R.*

EXAMPLE 5.4. When $M = G/T$, the left divided difference operator ∂_w exhibits the hierarchy of the Schubert classes: (We use the same notation as in Example 3.5.)

$$\partial_v(\mathcal{S}_w) = \begin{cases} \mathcal{S}_{vw} & (v \leq w), \\ 0 & (\text{otherwise}). \end{cases}$$

This is easily seen by comparing the support, since the Schubert classes are known to be characterized by the upper-triangularity of the support (see, for example, [6]). Note that the operators here are so-called the *left* divided difference, and in [8] they consider the *right* divided difference.

Now we consider the W-invariant subalgebra of $H^*(\Gamma)$.

DEFINITION 5.5. Let $H_W^*(\Gamma) = H^*(\Gamma)^W$. More explicitly, it is described as follows: Fix a set of representatives $p_1, \dots, p_k$ for the right W-cosets $W\backslash V$ of $W \curvearrowright V$ and denote by $W^{p_i} = \{w \in W \mid wp_i = p_i\}$ the isotropy subgroup at p_i. Since, by (4.1), $h_{wp} = w(h_p)$ for $h \in H^*(\Gamma)^W$, we have

$$H_W^*(\Gamma) = \left\{ \bigoplus_{i=1}^{k} h_{p_i} \;\middle|\; h_{p_i} \in \mathbb{R}[t_1, \dots, t_n]^{W^{p_i}},\ h_{p_i} - wh_{p_j} \in (\alpha(e)) \right.$$
$$\left. \text{if } \exists w \in W/W^{p_j} \text{ s.t. } p_i \xrightarrow{\alpha(e)} wp_j \right\}.$$

REMARK 5.6. Since $H_G^*(M) \cong H_T^*(M)^W$ by Proposition 3.4, we have $H_W^*(\Gamma(M)) \cong H_G^*(M)$. The above definition gives a combinatorial description for $H_G^*(M)$.

As an application of the left divided difference operators, we can define the *Becker–Gottlieb transfer*. Let $w_0 \in W$ be the longest element.

DEFINITION 5.7. Define

$$\tau \colon H^*(\Gamma) \to H_W^*(\Gamma)$$

by

$$\tau(h) = \partial_{w_0}(d\mathbf{1} \cdot h),$$

where $d = \prod_{\alpha \in \Pi^+} \alpha$.

Then, we recover the famous theorem by Brumfiel and Madsen ([2]).

Proposition 5.8. *τ is given by*

$$\tau(h) = \sum_{w \in W} w(h).$$

In particular, $\tau/|W|$ is a left inverse to the inclusion $\iota\colon H^*_W(\Gamma) \hookrightarrow H^*(\Gamma)$, that is, $\tau \circ \iota/|W|$ is the identity map.

Proof. Fix a reduced expression $w_0 = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{l(w_0)}}$. It is well-known (see [3], for example) that

$$\partial_{w_0} = \alpha_1^{-1}(1 - s_{\alpha_1})\alpha_2^{-1}(1 - s_{\alpha_2}) \cdots \alpha_{l(w_0)}^{-1}(1 - s_{\alpha_{l(w_0)}}) = \frac{1}{d} \sum_{w \in W} (-1)^{l(w)} w.$$

Since $w(d) = (-1)^{l(w)} d$, we have the assertion. □

EXAMPLE 5.9. In [7, §7], it is shown that

$$\frac{1}{|W|} \tau(\mathcal{S}_w) = (-1)^{l(w)} s_{w^{-1}},$$

where $\mathcal{S}_w$ is the equivariant Schubert class associated to $w \in W$ and $s_{w^{-1}}$ is the ordinary Schubert class associated to $w^{-1} \in W$ as in Example 3.5.

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