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THE HOMOTOPY TYPES OF *SU*(5)-GAUGE GROUPS

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Abstract

Let $\mathcal{G}_k$ be the gauge group of the principal $SU(5)$-bundle over S^4 with second Chern class k. We show that there is a p-local homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ for any prime p if and only if $(120, k) = (120, k')$.

1. Introduction

Let G be a simply-connected, simple compact Lie group. Principal G-bundles over S^4 are classified by the value of the second Chern class, which can take any integer value. Let $P_k \to S^4$ represent the equivalence class of principal G-bundle whose second Chern class is k. Let $\mathcal{G}_k$ be the *gauge group* of this principal G-bundle, which is the group of G-equivariant automorphisms of P_k which fix S^4.

While there are countably many inequivalent principal G-bundles, Crabb and Sutherland [5] showed that the gauge groups $\{\mathcal{G}_k\}_{k=1}^{\infty}$ have only finitely many distinct homotopy types. There has been a great deal of interest recently in determining the precise number of possible homotopy types. The following classifications are known. For two integers a, b, let (a, b) be their greatest common divisor. If $G = SU(2)$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(12, k) = (12, k')$ [11]; if $G = SU(3)$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(24, k) = (24, k')$ [7]; if $G = Sp(2)$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ when localized at any prime p or rationally if and only if $(40, k) = (40, k')$ [17]; and in a non-simply-connected case, if $G = SO(3)$ then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(24, k) = (24, k')$ [10].

In this paper we will classify the p-local homotopy types of gauge groups of principal $SU(5)$-bundles. It should be emphasized that in all the previous cases, the classification proofs relied heavily on the fact that as a CW-complex G has very few cells (at most 3). This is not the case for $SU(5)$, which has 15 cells. To deal with this previously inaccessible case, we make use of some new results in [9]. We prove the following.

Theorem 1.1. *For $G = SU(5)$, there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ when localized at any prime p or rationally if and only if $(120, k) = (120, k')$.*

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One motivating reason for studying $SU(5)$-gauge groups is that they are of interest to physics. The standard model, which accurately describes the behavior of elementary particles subject to the strong and weak nuclear forces and electromagnetism, is based on $SU(3) \times SU(2) \times U(1)$-gauge groups. Since the early 1970s, physicists have sought for a grand unified model which merges the gauge theory from the three Lie groups in the standard model into that from a single Lie group. One candidate for such a Lie group is $SU(5)$, others include $SO(10)$ and E_6. The corresponding models in each case have been heavily studied.

2. Determining homotopy types of gauge groups

We begin by collecting some preliminary information. First, we establish a context in which homotopy theory can be applied to study gauge groups. This works for any simply-connected, simple compact Lie group G. Let BG and $B\mathcal{G}_k$ be the classifying spaces of G and $\mathcal{G}_k$ respectively. Let $\mathrm{Map}(S^4, BG)$ and $\mathrm{Map}^*(S^4, BG)$ respectively be the spaces of freely continuous and pointed continuous maps between S^4 and BG. The components of each space are in one-to-one correspondence with the integers, where the integer is determined by the degree of a map $S^4 \to BG$. By [6] or [1], there is a homotopy equivalence $B\mathcal{G}_k \simeq \mathrm{Map}(S^4, BG)$ between $B\mathcal{G}_k$ and the component of $\mathrm{Map}(S^4, BG)$ consisting of maps of degree k. Evaluating a map at the basepoint of S^4, we obtain a map $ev\colon B\mathcal{G}_k \to BG$ whose fibre is homotopy equivalent to $\mathrm{Map}_k^*(S^4, BG)$. It is well known that each component of $\mathrm{Map}^*(S^4, BG)$ is homotopy equivalent to $\Omega_0^3 G$, the component of $\Omega^3 G$ containing the basepoint. Putting all this together, for each $k \in \mathbb{Z}$, there is a homotopy fibration sequence

$$G \xrightarrow{\partial_k} \Omega_0^3 G \to B\mathcal{G}_k \xrightarrow{ev} BG \tag{1}$$

where ∂_k is the fibration connecting map.

Since $\mathcal{G}_k$ is the homotopy fibre of ∂_k, its topology is governed to a great extent by properties of ∂_k. We mention two for now. First, by [12], the triple adjoint $S^3 \wedge G \to G$ of ∂_k is homotopic to the Samelson product $\langle k \cdot i, 1\rangle$, where i is the inclusion of S^3 into G and 1 is the identity map on G. The linearity of the Samelson product therefore implies that $\partial_k \simeq k \circ \partial_1$, where k is the k^{th}-power map on $\Omega_0^3 G$. Second, the order of ∂_k is finite. For, rationally, G is homotopy equivalent to a product of Eilenberg–MacLane spaces, and the homotopy equivalence can be chosen to be one of H-maps. Since Eilenberg–MacLane spaces are homotopy commutative, any Samelson product into such a space is null homotopic. Thus, rationally, the adjoint of ∂_k is null homotopic, implying that ∂_k is rationally null homotopic. Therefore, the order of ∂_k is finite.

In determining the homotopy types of $\mathcal{G}_k$, the order of ∂_1 plays a prominent role. We have just seen that the order of ∂_1 is finite, and since $\partial_k \simeq k \circ \partial_1$, once the order of ∂_1 is known so is that of ∂_k. When $G = SU(n)$, Hamanaka and Kono [7] gave a lower bound on the order of ∂_1 and the number of homotopy types of $\mathcal{G}_k$.

Lemma 2.1. *Let $G = SU(n)$. Then the following hold*:
(a) *the order of ∂_1 is divided by $n(n^2-1)$;*
(b) *if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(n(n^2-1), k) = (n(n^2-1), k')$.*

In particular, if $G = SU(5)$ then 120 divides the order of ∂_1 and a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ implies that $(120, k) = (120, k')$. In Section 6 we will prove an upper bound on the order of ∂_1 that matches the lower bound.

Theorem 2.2. *The map $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ has order* 120.

Granting Theorem 2.2 for now, we can prove Theorem 1.1 by using the following general result, proved in [17]. Let Y be an H-space with a homotopy inverse, and let $k\colon Y \to Y$ be the k^{th}-power map.

Lemma 2.3. *Let X be a space and Y be an H-space with a homotopy inverse. Suppose there is a map $X \xrightarrow{f} Y$ of order m, where m is finite. Let F_k be the homotopy fibre of $k \circ f$. If $(m, k) = (m, k')$ then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.*

Proof of Theorem 1.1. By Theorem 2.2, the map $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ has order 120. So Lemma 2.3 implies that if $(120, k) = (120, k')$, then $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ when localized at any prime p or rationally. On the other hand, by Lemma 2.1, if $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ then $(120, k) = (120, k')$. Thus there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ at each prime p and rationally if and only if $(120, k) = (120, k')$. □

It remains to prove Theorem 2.2. In general, it is difficult to determine the order of ∂_1 precisely. The following Lemma proved in [9] helps gives some crude information on the order when $G = SU(n)$.

Lemma 2.4. *For $G = SU(n)$, there is a homotopy commutative square*

$$\begin{array}{ccc} SU(n) & \xrightarrow{\partial_1} & \Omega^3_0 SU(n) \\ \downarrow{\scriptstyle \pi} & & \| \\ SU(n)/SU(n-2) & \xrightarrow{f} & \Omega^3_0 SU(n) \end{array}$$

for some map f, where π is the standard quotient map.

Another useful fact from [9] is the following. Let $SU(n) \to SU(n+1)$ be the standard group inclusion, and let $SU(n)/SU(n-2) \to SU(n)/SU(n-1) = S^{2n-1}$ be the usual quotient map.

Lemma 2.5. *There is a homotopy commutative square*

$$\begin{array}{ccc} SU(n)/SU(n-2) & \xrightarrow{f} & \Omega^3_0 SU(n) \\ \downarrow & & \downarrow \\ S^{2n-1} & \xrightarrow{f'} & \Omega^3_0 SU(n+1). \end{array}$$

3. An initial upper bound on the order of ∂_1

By Lemma 2.4, there is a homotopy commutative square

$$\begin{array}{ccc} SU(5) & \xrightarrow{\partial_1} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle \pi} & & \| \\ SU(5)/SU(3) & \xrightarrow{f} & \Omega^3_0 SU(5) \end{array}$$

for some map f. We begin by discussing some properties of f. Let

$$\overline{f}\colon \Sigma SU(5)/SU(3) \to \Omega^2 SU(5)$$

be the adjoint of f. By [8], there is a homotopy equivalence $\Sigma SU(5)/SU(3) \simeq \Sigma^6 \mathbb{C}P^2 \vee S^{17}$. So we regard $\overline{f}$ as a map $\Sigma^6 \mathbb{C}P^2 \vee S^{17} \to \Omega^2 SU(5)$. Let $\overline{f}_1$ and $\overline{f}_2$ be the restrictions of $\overline{f}$ to $\Sigma^6 \mathbb{C}P^2$ and S^{17} respectively. By [7], there is an isomorphism $[\Sigma^8 \mathbb{C}P^2, SU(5)] \cong \mathbb{Z}/360\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$. So we immediately have the following.

Lemma 3.1. *The maps $\overline{f}_1$ and $\overline{f}_2$ represent homotopy classes in $[\Sigma^8 \mathbb{C}P^2, SU(5)] \cong \mathbb{Z}/360\mathbb{Z} \oplus \mathbb{Z}/120\mathbb{Z}$ and $\pi_{19}(SU(5))$ respectively.*

It is not clear what the group $\pi_{19}(SU(5))$ is, although in Section 5 we will see that its 3-component is 0. For the moment, however, we are concerned with the p-component for every prime p.

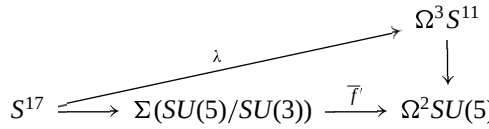
Lemma 3.2. *The order of $\overline{f}_2$ divides* 2.

Proof. By taking adjoints in Lemma 2.5, there is a homotopy square

$$\begin{array}{ccc} \Sigma(SU(5)/SU(3)) & \xrightarrow{\overline{f}} & \Omega^2 SU(5) \\ \downarrow & & \downarrow \\ \Sigma S^9 & \xrightarrow{\overline{f}'} & \Omega^2 SU(6) \end{array}$$

where $\overline{f}'$ is the adjoint of f'. Thus if we restrict $\overline{f}$ to $S^{17} \to \Sigma(SU(5)/SU(3))$ and compose into $\Omega^2 SU(6)$, the result is null homotopic. Therefore, since the homotopy

fibre of the inclusion $SU(5) \to SU(6)$ is ΩS^{11}, we obtain a lift



for some map λ. In particular, λ represents a homotopy class in $\pi_{20}(S^{11})$, which by [18], is isomorphic to the direct sum of three copies of $\mathbb{Z}/2\mathbb{Z}$. Thus the order of λ divides 2, implying that the order of f_2 also divides 2. □

The information on f gives an upper bound on the order of the map $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$.

Lemma 3.3. *The order of the map* $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ *divides* 360.

Proof. By Lemma 2.4, ∂_1 factors as the composite $SU(5) \xrightarrow{\pi} SU(5)/SU(3) \xrightarrow{f} \Omega^3_0 SU(5)$. By Lemmas 3.1 and 3.2, the order of f divides 360. Hence the order of ∂_1 also divides 360. □

By Lemma 2.1, the order of $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ is a multiple of 120, while by Lemma 3.3 the order divides 360. We wish to show that the order of ∂_1 is exactly 120, so we need to improve the upper bound by a factor of 3. To do so it suffices to localize at 3 and show the 3-local order of ∂_1 is 3 rather than 9.

It will help to first investigate a 3-local property of f. Localized at 3, by [8] there is a homotopy equivalence $SU(5)/SU(3) \simeq S^7 \times S^9$. Let h be the composite $h\colon S^9 \xrightarrow{i_2} S^7 \times S^9 \xrightarrow{f} \Omega^3_0 SU(5)$, where i_2 is the inclusion of the second factor. In the following lemma we show that h has order 9, implying that the 3-component of the order of f is at least 9. In particular, in trying to improve the upper bound on the 3-component of the order of ∂_1 from 9 to 3, we need to study the map $SU(5) \xrightarrow{\pi} SU(5)/SU(3)$ as well as the map $SU(5)/SU(3) \xrightarrow{f} \Omega^3_0 SU(5)$.

Lemma 3.4. *Localized at* 3, *the map* $S^9 \xrightarrow{h} \Omega^3_0 SU(5)$ *has order* 9.

Proof. Let $c\colon S^9 \to SU(5)$ be the characteristic map. It is well known that this has the property that the composite $S^9 \xrightarrow{c} SU(5) \xrightarrow{q} S^9$ has degree 4!. In particular, localized at 3, we may regard $q \circ c$ as having degree 3, up to multiplication by a unit in the 3-local integers. Also, by [12], the triple adjoint of the map $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$

is homotopic to the Samelson product $S^3 \wedge SU(5) \xrightarrow{\langle i,1\rangle} SU(5)$ where i is the inclusion of the bottom cell and 1 is the identity map. By [4], the composite $S^3 \wedge S^9 \xrightarrow{1\wedge c} S^3 \wedge SU(5) \xrightarrow{\langle i,1\rangle} SU(5)$ has order dividing $6!/(1!\cdot 4!) = 30$. In particular, adjointing back and localizing at 3, the order of $\partial_1 \circ c$ divides 3.

Now consider the diagram

$$\begin{array}{ccccc} S^9 & \xrightarrow{c} & SU(5) & \xrightarrow{\partial_1} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle \pi} & & \| \\ S^9 & \xrightarrow{i_2} & S^7 \times S^9 & \xrightarrow{f} & \Omega^3_0 SU(5). \end{array}$$

The left square homotopy commutes since $\pi_9(S^7) = 0$ at 3 and since $q \circ c$ has degree 3. The right square homotopy commutes by Lemma 2.4. From the first paragraph, the upper row has order dividing 3. Hence the lower direction around the diagram is nontrivial. That is, as $h = f \circ i_2$, we have $3 \cdot h$ nontrivial, implying that the order of h divides 9. On the other hand, by [2] and [3], the three-component of $\pi_9(\Omega^3_0 SU(5))$ is $\mathbb{Z}/9\mathbb{Z}$. Hence h has order at most 9. Therefore, h represents a generator of $\pi_9(\Omega^3_0 SU(5))$ and has order 9. □

4. Some decompositions

For the remainder of the paper, assume that all spaces and maps have been localized at 3, and homology is taken with mod-3 coefficients. By [13], there are homotopy equivalences $SU(4) \simeq B(3,7) \times S^5$ and $SU(5) \simeq B(3,7) \times B(5,9)$, where $H^*(B(3,7)) \cong \Lambda(x_3, \mathcal{P}^1(x_3))$, $H^*(B(5,9)) \cong \Lambda(x_5, \mathcal{P}^1(x_5))$, and there are homotopy fibrations $S^3 \to B(3,7) \xrightarrow{r} S^7$ and $S^5 \to B(5,9) \xrightarrow{s} S^9$.

In what follows, it is helpful to define some maps. Let $\pi\colon SU(5) \to SU(5)/SU(3)$ and $q\colon SU(5) \to S^9$ be the standard quotient maps. Let $e\colon B(3,7) \times B(5,9) \to SU(5)$ be the homotopy equivalence from [13], and let e_1, e_2 be the restrictions of e to $B(3,7)$ and $B(5,9)$ respectively. Observe that e_1 can be chosen to be the composite $B(3,7) \to SU(4) \to SU(5)$, implying that $q \circ e_1$ is null homotopic. Also observe that s can be chosen to be the composite $B(5,9) \xrightarrow{e_2} SU(5) \xrightarrow{q} S^9$.

As mentioned in Section 3, there is a homotopy equivalence $SU(5)/SU(3) \simeq S^7 \times S^9$. However, the decompositions of $SU(5)$ and $SU(5)/SU(3)$ may not be compatible, in the sense that there *may not* be a homotopy commutative diagram

$$\begin{array}{ccc} B(3,7) \times B(5,9) & \xrightarrow{e} & SU(5) \\ \downarrow{\scriptstyle r\times s} & & \downarrow{\scriptstyle \pi} \\ S^7 \times S^9 & = & S^7 \times S^9. \end{array}$$

However, we will show that an appropriate diagram does exist if we weaken to consider only the map $SU(5) \xrightarrow{q} S^9$.

Lemma 4.1. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} B(3,7) \times B(5,9) & \xrightarrow{e} & SU(5) \\ \downarrow{\scriptstyle s\circ\pi_2} & & \downarrow{\scriptstyle q} \\ S^9 & = & S^9 \end{array}$$

where π_2 is the projection onto the second factor.

Proof. In general, for a homotopy fibration $F \to E \to B$ the connecting map $\delta\colon \Omega B \to F$ induces a homotopy action $\theta\colon F \times \Omega B \to F$ with the property that there is a homotopy commutative square

$$\begin{array}{ccc} \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\ \downarrow{\scriptstyle \delta\times 1} & & \downarrow{\scriptstyle \delta} \\ F \times \Omega B & \xrightarrow{\theta} & F \end{array}$$

where μ is the loop multiplication.

In our case, there is a homotopy fibration sequence $SU(5) \xrightarrow{q} S^9 \to BSU(4) \to BSU(5)$. Now consider the diagram

$$\begin{array}{ccccc} B(3,7) \times B(5,9) & \xrightarrow{e_1\times e_2} & SU(5) \times SU(5) & \xrightarrow{\mu} & SU(5) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle q\times 1} & & \downarrow{\scriptstyle q} \\ B(5,9) & \xrightarrow{*\times e_2} & S^9 \times SU(5) & \xrightarrow{\theta} & S^9. \end{array}$$

The right square homotopy commutes since $q \circ e_1$ is null homotopic. The left square homotopy commutes by the homotopy action induced by the fibration connecting map q. The top row is homotopic to e. As the restriction of θ to $SU(5)$ is q, the bottom row is homotopic to $q \circ e_2 \simeq s$. Thus the diagram as a whole shows that $q \circ e \simeq s \circ \pi_2$, which proves the lemma. □

Define the space C and the map δ by the homotopy cofibration

$$SU(5) \xrightarrow{q} S^9 \xrightarrow{\delta} C.$$

In the next two propositions we examine properties of this cofibration. First we decompose the space C and then factor the map δ.

Proposition 4.2. *There is a homotopy equivalence*

$$C \simeq S^6 \vee S^{15} \vee (\Sigma B(3,7)) \vee (\Sigma B(3,7) \wedge B(5,9)).$$

Proof. The homotopy commutative diagram in the statement of Lemma 4.1 implies that there is a homotopy cofibration diagram

$$\begin{array}{ccccc} B(3,7)\times B(5,9) & \xrightarrow{s\circ\pi_2} & S^9 & \longrightarrow & D \\ \downarrow{\scriptstyle e} & & \| & & \downarrow{\scriptstyle \gamma} \\ SU(5) & \xrightarrow{q} & S^9 & \longrightarrow & C \end{array}$$

where D is the homotopy cofibre of $s \circ \pi_2$ and γ is some induced map of cofibres. The cofibrations in the top and bottom rows induce long exact sequences of homology groups, and the cofibration diagram induces a map of these long exact sequences. Since e is a homotopy equivalence and the map between S^9's is the identity, the five-lemma implies that γ_* is an isomorphism in every degree. Hence γ is a homotopy equivalence. So to prove the lemma it is equivalent to decompose D.

In general, there is a homotopy decomposition $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. So the homotopy cofibre of the projection $A \times B \xrightarrow{\pi_2} B$ is homotopy equivalent to $\Sigma A \vee (\Sigma A \wedge B)$. If we compose the projection with a map $h\colon B \to X$, then we obtain a homotopy cofibration diagram

$$\begin{array}{ccccccc} A\times B & \xrightarrow{\pi_2} & B & \longrightarrow & \Sigma A \vee (\Sigma A \wedge B) & \longrightarrow & \Sigma(A\times B) \\ \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle j} & & \| \\ A\times B & \xrightarrow{h\circ\pi_2} & X & \longrightarrow & Y & \xrightarrow{k} & \Sigma(A\times B) \\ & & \downarrow & & \downarrow & & \\ & & Z & = & Z & & \end{array}$$

which defines the spaces Y and Z and the maps j and k. The homotopy commutativity of the upper right square implies that the composite $Y \xrightarrow{k} \Sigma(A \times B) \to \Sigma A \vee (\Sigma A \wedge B)$ is a left homotopy inverse of j. Since k induces a homotopy coaction $\psi\colon Y \to Y \vee \Sigma(A \times B)$, the composite $Y \xrightarrow{\psi} Y \vee \Sigma(A \times B \to Z \vee (\Sigma A \vee (\Sigma A \wedge B))$ is a homotopy equivalence. Therefore, in our case of the homotopy cofibration $B(3,7) \times B(5,9) \xrightarrow{s\circ\pi_2} S^9 \to D$, we have $D \simeq Z \vee \Sigma B(3,7) \vee (\Sigma B(3,7) \wedge B(5,9))$, where Z is the homotopy cofibre of s.

To complete the proof of the decomposition of C, it remains to show that $Z \simeq S^6 \vee S^{15}$. By definition, Z is the homotopy cofibre of the map $B(5,9) \xrightarrow{s} S^9$. Since $H_*(B(5,9)) \cong \Lambda(x_5, x_9)$ and s_* is a projection, there is a vector space isomorphism $H_*(Z) \cong \{x_6, x_{15}\}$. Thus Z is a two-cell complex, so there is a homotopy cofibration

$S^{14} \xrightarrow{l} S^6 \to Z$ where l attaches the top cell to Z. By [18], the 3-component of $\pi_{14}(S^6)$ is 0. Thus l is null homotopic, implying that $Z \simeq S^6 \vee S^{15}$. □

Let $C' = S^{15} \vee (\Sigma B(3,7)) \vee (\Sigma B(3,7) \wedge B(5,9))$, so $C \simeq S^6 \vee C'$. As is usual, for $n \geq 3$, let $\alpha_1 \colon S^{n+3} \to S^n$ be a map representing the generator of the stable 3-stem (localized at 3).

Proposition 4.3. *The map $S^9 \xrightarrow{\delta} C$ factors as the composite $S^9 \xrightarrow{\alpha_1} S^6 \xrightarrow{i_1} S^6 \vee C' \xrightarrow{\simeq} C$, where i_1 is the inclusion of the first wedge summand.*

Proof. By connectivity, the map $S^9 \xrightarrow{\delta} C$ factors through the 9-skeleton of C. From the decomposition of C in Proposition 4.2, we see that its 9-skeleton is homotopy equivalent to $S^6 \vee \Sigma A \vee S^9$, where A is the 7-skeleton of $B(3,7)$ and S^9 is the bottom cell of $\Sigma B(3,7) \wedge B(5,9)$. So δ factors through a map $\delta' \colon S^9 \to S^6 \vee \Sigma A \vee S^9$.

Consider the homotopy fibration $F \to S^6 \vee \Sigma A \vee S^9 \to S^6 \times \Sigma A \times S^9$, where the right map is the inclusion of the wedge into the product and the fibration defines the space F. The Hilton-Milnor Theorem implies that F is 8-connected and has a single cell in dimension 9. Further, the map $S^9 \hookrightarrow F \to S^6 \vee \Sigma A \vee S^9$ is homotopic to the composite $\omega \colon S^9 \xrightarrow{w} S^6 \vee S^4 \hookrightarrow S^6 \vee \Sigma A \hookrightarrow S^6 \vee \Sigma A \vee S^9$, where w is the Whitehead product of the identity maps on S^6 and S^4. Thus δ' is homotopic to $p_1 \circ \delta' + p_2 \circ \delta' + p_3 \circ \delta' + t \cdot w$, where p_i is the pinch map from $S^6 \vee \Sigma A \vee S^9$ to the respective wedge summands S^6, ΣA and S^9 and t is some element of $\mathbb{Z}_{(3)}$. We aim to show that $p_1 \circ \delta' \simeq \alpha_1$, $p_2 \circ \delta' \simeq *$, $p_3 \circ \delta' \simeq *$ and $t = 0$. If so, then δ' factors through α_1, implying that the same is true of δ, completing the proof.

Now consider the homotopy cofibration sequence $SU(5) \xrightarrow{q} S^9 \xrightarrow{\delta} C \to \Sigma SU(5)$. First, since $H^*(SU(5);\mathbb{Z})$ is torsion free, the composite $S^9 \xrightarrow{\delta'} S^6 \vee \Sigma A \vee S^9 \xrightarrow{p_3} S^9$ must be of degree zero, for otherwise δ' and hence δ is degree $d \neq 0$ in H^9, implying that there is d-torsion in $H^*(SU(5);\mathbb{Z})$, a contradiction. Thus $p_3 \circ \delta' \simeq *$. Second, observe that the degree 5, 9 generators in $H^*(SU(5))$ are connected by the Steenrod operation $\mathcal{P}^1$, an operation which detects α_1. Since q^* is an inclusion of the degree 9 generator, we must have δ detecting α_1. That is, $p_1 \circ \delta' \simeq \alpha_1$. Third, consider the homotopy cofibration $S^4 \to \Sigma A \to S^8$ which includes the bottom cell into ΣA and pinches out to the top cell. The Serre exact sequence implies that this homotopy cofibration is also a homotopy fibration in dimensions < 10. In particular, there is an exact sequence $\pi_9(S^4) \to \pi_9(\Sigma A) \to \pi_9(S^8)$. By [18], the three-components of $\pi_9(S^4)$ and $\pi_9(S^8)$ are both 0, so the three-component of $\pi_9(\Sigma A)$ is 0. Hence $p_2 \circ \delta' \simeq *$. Finally, suppose that $t \cdot w \neq *$. Then as w factors through the Whitehead product ω and ω detects a nontrivial cup-product, the map $\delta \simeq \alpha_1 + t \cdot w$ would detect a nontrivial cup-product in the integral cohomology its cofibre $\Sigma SU(5)$. But all cup-products in $H^*(\Sigma SU(5);\mathbb{Z})$ vanish. Hence $t \cdot w$ must be trivial, implying that $t = 0$. □

5. Some properties of $B(3,7)$ and $B(5,9)$

We will need to know some information about certain homotopy groups of $SU(5)$. This makes use of the homotopy equivalence $SU(5) \simeq B(3,7) \times B(5,9)$ and calculations of the low dimensional homotopy groups of $B(3,7)$ by Toda [19] and $B(5,9)$ by Oka [15].

(2)

	$B(3,7)$	$B(5,9)$	$SU(5)$
π_{10}	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$
π_{12}	0	$\mathbb{Z}/9\mathbb{Z}$	$\mathbb{Z}/9\mathbb{Z}$
π_{19}	0	0	0

As well, let $\mathbb{Z}_{(3)}$ be the 3-local integers. Then $\pi_7(B(3,7)) \cong \mathbb{Z}_{(3)}$ and if $c\colon S^7 \to B(3,7)$ represents the generator, then the composite $S^7 \xrightarrow{c} B(3,7) \xrightarrow{r} S^7$ is of degree 3. We now prove a lemma which gives two ways of describing the generator of $\pi_{10}(B(3,7))$. For $n \geq 3$, let $\alpha_2\colon S^{n+7} \to S^7$ represent the generator of the stable 7-stem (again, at 3).

Lemma 5.1. *There is a homotopy commutative square*

$$\begin{array}{ccc} S^{10} & \xrightarrow{\alpha_1} & S^7 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle c} \\ S^3 & \longrightarrow & B(3,7). \end{array}$$

Proof. By (2), $\pi_{10}(B(3,7)) \cong \mathbb{Z}/3\mathbb{Z}$ and, as stated in [19], a generator is represented by the composite $S^{10} \xrightarrow{\alpha_2} S^3 \to B(3,7)$. On the other hand, consider the homotopy fibration $F \to S^7 \xrightarrow{c} B(3,7)$. In [19], it is shown that F is 10-connected. Therefore c induces an isomorphism on π_{10}. But $\pi_{10}(S^7) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by α_1, so the composite $S^{10} \xrightarrow{\alpha_1} S^7 \xrightarrow{c} B(3,7)$ also represents the generator of $\pi_{10}(B(3,7)) \cong \mathbb{Z}/3\mathbb{Z}$. These two ways of describing the generator of $\pi_{10}(B(3,7))$ gives the asserted homotopy commutative diagram. □

Next, by [15], $\pi_9(B(5,9)) \cong \mathbb{Z}_{(3)}$ and if $d\colon S^9 \to B(5,9)$ represents the generator, then the composite $S^9 \xrightarrow{d} B(5,9) \xrightarrow{s} S^9$ is of degree 3. We give an analogue of Lemma 5.1 for $B(5,9)$, after a preliminary lemma.

In general, let $E^2\colon S^{2n-1} \to \Omega^2 S^{2n+1}$ be the double suspension, which is the double adjoint of the identity map on S^{2n-1}. Oka [15] showed that there is a map $\epsilon\colon B(3,7) \to$

$\Omega^2 B(5, 9)$ and a homotopy fibration diagram

$$\begin{array}{ccccc} S^3 & \longrightarrow & B(3,7) & \xrightarrow{r} & S^7 \\ \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle E^2} \\ \Omega^2 S^5 & \longrightarrow & \Omega^2 B(5,9) & \xrightarrow{\Omega^2 s} & \Omega^2 S^9. \end{array} \tag{3}$$

We will show that ϵ is also compatible with the maps c and $\Omega^2 d$.

Lemma 5.2. *There is a homotopy commutative square*

$$\begin{array}{ccc} S^7 & \xrightarrow{c} & B(3,7) \\ \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \epsilon} \\ \Omega^2 S^9 & \xrightarrow{\Omega^2 d} & \Omega^2 B(5,9). \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} S^7 & \xrightarrow{c} & B(3,7) & \xrightarrow{r} & S^7 \\ \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle E^2} \\ \Omega^2 S^9 & \xrightarrow{\Omega^2 d} & \Omega^2 B(5,9) & \xrightarrow{\Omega^2 s} & \Omega^2 S^9. \end{array} \tag{4}$$

The right square homotopy commutes by (3). An odd dimensional sphere is an H-space when localized at 3, and the degree p map is the p^{th}-power map. So as $r \circ c \simeq 3$ and $\Omega^2 s \circ \Omega^3 d \simeq \Omega^2 3 \simeq 3$, the outer rectangle of the diagram above homotopy commutes. Consider the difference $l = \Omega^2 d \circ E^2 - \epsilon \circ c$. The homotopy commutativity of the right square and outer rectangle implies that $\Omega^2 s \circ l$ is null homotopic. Thus l lifts through the homotopy fibre of $\Omega^2 s$ to a map $\bar{l}\colon S^7 \to \Omega^2 S^5$. By [18], the 3-component of $\pi_9(S^5)$ is 0. Thus $\bar{l}$ is null homotopic, implying that l is null homotopic. Hence $\Omega^2 d \circ E^2 \simeq \epsilon \circ c$. That is, the left square in (4) homotopy commutes, proving the lemma. □

Lemma 5.3. *There is a homotopy commutative square*

$$\begin{array}{ccc} S^{12} & \xrightarrow{\alpha_1} & S^9 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle d} \\ S^5 & \longrightarrow & B(5,9). \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} S^{10} & \xrightarrow{\alpha_1} & S^7 & \xrightarrow{E^2} & \Omega^2 S^9 \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \Omega^2 d} \\ S^3 & \longrightarrow & B(3,7) & \xrightarrow{\epsilon} & \Omega^2 B(5,9) \\ \downarrow{\scriptstyle E^2} & & & & \| \\ \Omega^2 S^5 & & \longrightarrow & & \Omega^2 B(5,9). \end{array}$$

The upper left square homotopy commutes by Lemma 5.1, the upper right square homotopy commutes by Lemma 5.2, and the lower rectangle homotopy commutes by (3). Thus the entire diagram homotopy commutes. In particular, the outer perimeter of the diagram homotopy commutes. But this outer perimeter is the double adjoint of the diagram asserted by the lemma. □

6. The order of ∂_1

In this section we show that $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ has order 120. At issue is the order of ∂_1 when localized at 3, which in Proposition 6.3 we will show is 3.

Recall from Lemma 2.4 that, localized at 3, ∂_1 factors as the composite $SU(5) \xrightarrow{\pi} S^7 \times S^9 \xrightarrow{f} \Omega^3_0 SU(5)$. As well, recall that h is the composite $S^9 \xrightarrow{i_2} S^7 \times S^9 \xrightarrow{f} \Omega^3_0 SU(5)$.

Lemma 6.1. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} S^7 \times S^9 & \xrightarrow{f} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle 3} \\ S^9 & \xrightarrow{3\cdot h} & \Omega^3_0 SU(5). \end{array}$$

Proof. Recall that, in general, $\Sigma(A \times B) \simeq \Sigma A \vee \Sigma B \vee (\Sigma A \wedge B)$. So it is equivalent to adjoint and show that there is a homotopy commutative diagram

$$\begin{array}{ccc} S^8 \vee S^{10} \vee S^{17} & \xrightarrow{f'} & \Omega^2 SU(5) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle 3} \\ S^{10} & \xrightarrow{3\cdot h'} & \Omega^2 SU(5) \end{array} \tag{5}$$

where f', h' are the adjoints of f, h respectively, and q is the pinch map.

To show that (5) homotopy commutes, it suffices to show that it does so when restricted to each of the wedge summands S^8, S^{10} and S^{17}. By (2), the 3-component

of $\pi_8(\Omega^2 SU(5)) \cong \pi_{10}(SU(5))$ is $\mathbb{Z}/3\mathbb{Z}$. Therefore $3 \circ f'$ restricted to S^8 is null homotopic, and clearly the lower direction around (5) is null homotopic when restricted to S^8. Therefore (5) homotopy commutes when restricted to S^8. By (2), the 3-component of $\pi_{10}(\Omega^2 SU(5)) \cong \pi_{12}(SU(5))$ is $\mathbb{Z}/9\mathbb{Z}$. By definition, h' is the restriction of f' to S^{10}. So the restriction of $3 \circ f'$ to S^{10} is $3 \cdot h'$, implying that (5) homotopy commutes when restricted to S^{10}. Finally, by (2) the 3-component of $\pi_{17}(\Omega^2 SU(5)) \cong \pi_{19}(SU(5))$ is 0. Thus both directions around (5) are null homotopic when restricted to S^{19}. Hence (5) homotopy commutes. □

Lemma 6.2. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} S^9 & \xrightarrow{3\cdot h} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle \delta} & & \| \\ C & \xrightarrow{j} & \Omega^3_0 SU(5) \end{array}$$

for some map j.

Proof. By Proposition 4.2, there is a homotopy equivalence $C \simeq S^6 \vee C'$ for some space C', and by Proposition 4.3 the map δ factors as the composite $S^9 \xrightarrow{\alpha_1} S^6 \xrightarrow{i_1} S^6 \vee C' \xrightarrow{\simeq} C$ where i_1 is the inclusion of the first wedge summand. We claim that there is a homotopy commutative square

$$\begin{array}{ccc} S^9 & \xrightarrow{3\cdot h} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle \alpha_1} & & \| \\ S^6 & \xrightarrow{\overline{j}} & \Omega^3_0 SU(5) \end{array}$$

for some map $\overline{j}$. If so then we can define $j\colon C \simeq S^6 \vee C' \to \Omega^3_0 SU(5)$ by taking $j = \overline{j}$ on S^6 and $j = *$ on C', and from the factorization of δ we obtain $j \circ \delta \simeq j \circ (i_1 \circ \alpha_1) \simeq \overline{j} \circ \alpha_1 \simeq 3 \cdot h$. That is, we obtain the homotopy asserted by the lemma.

It remains to prove the claim. By (2) the 3-component of $\pi_{12}(SU(5))$ is $\mathbb{Z}/9\mathbb{Z}$, and this comes from the $B(5,9)$ factor of $SU(5)$. Further, by [15], the generator $a\colon S^{12} \to B(5,9)$ has the property that the composite $S^{12} \xrightarrow{a} B(5,9) \xrightarrow{s} S^9$ is homotopic to α_1. On the other hand, let $h'\colon S^{12} \to SU(5)$ be the adjoint of h. By Lemma 3.4, h has order 9, implying that h' has order 9, and therefore h' represents a generator of $\pi_{12}(SU(5))$, which is equivalent to saying that h' represents a generator of $\pi_{12}(B(5,9))$. Thus, up to multiplication by a unit in $\mathbb{Z}_{(3)}$, h' is homotopic to a and has the property that the composite $S^{12} \xrightarrow{h'} B(5,9) \xrightarrow{s} S^9$ is homotopic to α_1. Observe that $3 \cdot h'$ is nontrivial

since h' has order 9, while $s \circ (3 \cdot h')$ is null homotopic since α_1 has order 3. Thus there is a lift

$$\begin{array}{ccc} & & S^{12} \\ & {\scriptstyle\lambda}\swarrow & \downarrow{\scriptstyle 3\cdot h'} \\ S^5 & \longrightarrow & B(5,9) \end{array}$$

for some map λ. Since $3 \cdot h'$ is nontrivial, λ must be nontrivial. Thus λ must represent a generator of $\pi_{12}(S^5) \cong \mathbb{Z}/3\mathbb{Z}$; that is, up to multiplication by a unit in $\mathbb{Z}_{(3)}$, $\lambda \simeq \alpha_2$. By Lemma 5.3, the composite $S^{12} \xrightarrow{\alpha_2} S^5 \to B(5,9)$ is homotopic to the composite $S^{12} \xrightarrow{\alpha_1} S^9 \xrightarrow{d} B(5,9)$. Hence $3 \cdot h' \simeq d \circ \alpha_1$. Now let $\overline{d}\colon S^6 \to \Omega^3 B(5,9)$ be the triple adjoint of d. Then $3 \cdot h \simeq \overline{d} \circ \alpha_1$. Therefore, if we define $\overline{j}$ as the composite $\overline{j}\colon S^6 \xrightarrow{\overline{d}} \Omega^3 B(5,9) \hookrightarrow \Omega^3_0 SU(5)$, then $\overline{j} \circ \alpha_1 \simeq 3 \cdot h$, giving the homotopy commutative square in the claim. □

Finally, we can improve on the 3-primary upper bound of the order of $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ in Lemma 3.3, and so obtain the precise order of ∂_1.

Proposition 6.3. *Localized at* 3, *the map* $SU(5) \xrightarrow{\partial_1} \Omega^3_0 SU(5)$ *has order* 3.

Proof. Consider the diagram

$$\begin{array}{ccc} SU(5) & \xrightarrow{\partial_1} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle\pi} & & \| \\ S^7 \times S^9 & \xrightarrow{f} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle 3} \\ S^9 & \xrightarrow{3\cdot h} & \Omega^3_0 SU(5) \\ \downarrow{\scriptstyle\delta} & & \| \\ C & \xrightarrow{j} & \Omega^3_0 SU(5). \end{array}$$

The top square homotopy commutes by Lemma 2.4, the middle square homotopy commutes by Lemma 6.1, and the bottom square homotopy commutes by Lemma 6.2. The composite $SU(5) \xrightarrow{\pi} S^7 \times S^9 \xrightarrow{\pi_2} S^9$ is the same as the quotient map $SU(5) \xrightarrow{q} S^9$. As the cofibre of q is given by the map δ, the composite along the left column of the diagram above is null homotopic. Thus the homotopy commutativity of the diagram implies that $3 \circ \partial_1$ is null homotopic. Therefore 3 is an upper bound on the order of ∂_1. On the other hand, by Lemma 2.1 (a), 3 is also a lower bound on the order of ∂_1. Hence the order of ∂_1 is precisely 3. □

Proof of Theorem 2.2. By Lemma 2.1, the order of ∂_1 is a multiple of 120. Combining Lemma 3.3 and Proposition 6.3 shows that the order of ∂_1 divides 120. Hence the order of ∂_1 is exactly 120. □

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