

## ON THE TOTALLY GEODESIC SPACELIKE HYPERSURFACES IN CONFORMALLY STATIONARY SPACETIMES

HENRIQUE F. DE LIMA and MARCO ANTONIO L. VELÁSQUEZ

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### Abstract

We study complete noncompact spacelike hypersurfaces immersed into conformally stationary spacetimes, equipped with either one or two conformal vector fields. In this setting, by using as main analytical tool a suitable maximum principle for complete noncompact Riemannian manifolds, we establish new characterizations of totally geodesic hypersurfaces in terms of their  $r$ -th mean curvatures. For instance, for a timelike geodesically complete conformally stationary spacetime endowed with a closed conformal timelike vector field  $V$ , under appropriate restrictions on the flow and the norm of the tangential component of  $V$ , we are able to prove that totally geodesic spacelike hypersurfaces must be, in fact, leaves of the distribution determined by  $V$ . Applications to the so-called generalized Robertson–Walker spacetimes are also given. Furthermore, we extend our approach in order to obtain a lower estimate of the relative nullity index.

### 1. Introduction

In the last years, the study of spacelike hypersurfaces immersed in Lorentzian spacetimes has been of substantial interest from both physical and mathematical points of view. From the physical one, that interest became clear when Lichnerowicz [29] showed that the Cauchy problem of the Einstein equation with initial conditions on a maximal spacelike hypersurface (that is, with zero mean curvature) has a particularly nice form, reducing to a linear differential system of first order and to a nonlinear second order elliptic differential equation.

From a mathematical point of view, spacelike hypersurfaces are also interesting because of their Calabi–Bernstein type properties. Traditionally, the parametric version of the Calabi–Bernstein theorem asserts that the only complete maximal hypersurfaces in the Lorentz–Minkowski space are the spacelike hyperplanes (cf. [16], [22], [24] and [34]). In this setting, the problem of characterizing maximal, or more generally, constant mean curvature hypersurfaces of different Lorentzian spacetimes has been studied by several authors. For instance, in a series of papers Alías, Romero and Sánchez (cf. [7], [8] and [9]) have studied the uniqueness of spacelike hypersurfaces with constant mean curvature in an important class of Lorentzian manifolds, the so called *conformally stationary spacetimes*. We recall that such a space is a manifold  $\bar{M}^{n+1}$  endowed

with a Lorentzian metric tensor  $\langle \cdot, \cdot \rangle$  equipped with a conformal timelike vector field  $V \in \mathfrak{X}(\bar{M})$ . The fact that  $V$  is conformal means that the Lie derivative of the Lorentzian metric  $\langle \cdot, \cdot \rangle$  with respect to  $V$  satisfies  $\mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle$  for a certain smooth function  $\psi$  on  $\bar{M}^{n+1}$ . In particular, when  $V$  is a Killing vector field (that is,  $\psi$  vanishes identically on  $\bar{M}^{n+1}$ ), then  $\bar{M}^{n+1}$  is classically called a *stationary spacetime*. The reason for the terminology conformally stationary spacetime is due to the fact that  $\bar{M}^{n+1}$  endowed with the conformally related metric  $\langle \cdot, \cdot \rangle^* = |V|^{-2} \langle \cdot, \cdot \rangle$  (cf. Lemma 2.1 of [36]), where  $|V| = \sqrt{-\langle V, V \rangle} > 0$ , is in fact a stationary spacetime, since the timelike field  $V$  is a Killing vector field for  $\langle \cdot, \cdot \rangle^*$ .

In [21], the first author jointly with Caminha derived, for the square operator of Yau [37], an analogue of the Omori–Yau maximum principle for the Laplacian [32, 37] and applied it to obtain nonexistence results concerning complete noncompact spacelike hypersurfaces immersed with some constant  $r$ -th mean curvature in a conformally stationary spacetime. More recently, the authors jointly with Camargo and Caminha have obtained in [17] a new characterization of complete noncompact totally geodesic spacelike hypersurfaces in conformally stationary spacetimes.

We note that the class of conformally stationary spacetimes includes the family of *generalized Robertson–Walker* (GRW) spacetimes. By a GRW spacetime, we mean a Lorentzian warped product  $-I \times_\phi F^n$  with Riemannian fiber  $F^n$  and warping function  $\phi$  (for details, see Section 4). For such a spacetime, the conformal timelike vector field  $V(t, p) = \phi(t)(\partial/\partial t)_{(t,p)}$  is also closed, in the sense that its metrically equivalent 1-form is closed. As it was observed by Montiel in [30], if  $\bar{M}^{n+1}$  is a conformally stationary spacetime equipped with a closed conformal vector field, then it is locally isometric to a GRW spacetime. A global analogue of this assertion was also obtained by Montiel, which showed that a timelike geodesically complete spacetime having a closed conformal timelike vector field must be isometric to an appropriate quotient of a GRW spacetime (see Proposition 2 of [30]).

Recently, many authors have treated the uniqueness problem of spacelike hypersurfaces with constant mean curvature in GRW spacetimes. For instance, we may cite the works [13], [14], [15] and [35], where Romero et al. have obtained rigidity results concerning to spacelike slices of 3-dimensional GRW spacetimes obeying either the null convergence condition or the timelike convergence condition. Also in the 3-dimensional case, Albuje and Alías [3] have established new Calabi–Bernstein results for maximal hypersurfaces immersed into a Lorentzian product space. In [4], the first author jointly with Albuje and Camargo have approached some uniqueness problems concerning complete spacelike hypersurfaces with constant mean curvature immersed in a Robertson Walker (RW) spacetime (that is, a GRW spacetime whose Riemannian fiber has constant sectional curvature), which was supposed to obey the null convergence condition. In [18], by considering the RW model  $-(\pi/2, \pi/2) \times_{\cos t} \mathbb{H}^n$  of an open subset of the anti-de Sitter space  $\mathbb{H}_1^{n+1}$ , where  $\mathbb{H}^n$  stands for the  $n$ -dimensional hyperbolic space, the first author and Camargo obtained new characterizations of totally

geodesic spacelike hypersurfaces in  $\mathbb{H}_1^{n+1}$ .

In this paper, our aim is to establish new characterization theorems of totally geodesic spacelike hypersurfaces immersed in a conformally stationary spacetime. For instance, in Section 3 we prove the following results (cf. Theorems 3.3 and 3.9):

Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a conformal timelike vector field  $V$  and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete, noncompact spacelike hypersurface. Suppose that one of the following conditions is satisfied:

- (a)  $H \geq 0$  and  $\operatorname{div}_{\bar{M}} V \leq 0$  on  $M^n$ ;
- (b)  $H \leq 0$  and  $\operatorname{div}_{\bar{M}} V \geq 0$  on  $M^n$ .

If  $|V^\top|$  is Lebesgue integrable on  $M^n$ , then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is an Einstein spacetime and  $H_2$  is bounded from below on  $M^n$ , then  $M^n$  is totally geodesic.

Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a parallel timelike vector field  $V$  and a homothetic nonparallel timelike vector field  $W$ , and  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete noncompact spacelike hypersurface such that  $H$  does not change sign on  $M^n$ . If  $|V^\top|$  is Lebesgue integrable on  $M^n$ , then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is Einstein and  $H_2$  is bounded from below, then  $M^n$  is totally geodesic.

Here,  $H$  is the mean curvature,  $H_2 = (2/(n(n - 1)))S_2$  denotes the mean value of the second elementary symmetric function  $S_2$  on the eigenvalues of the Weingarten operator (see Section 2 for details about the  $r$ -th mean curvatures) and  $V^\top$  stands for the tangential component of the vector field  $V$  on  $M^n$ . Our approach is based on the use of the Newton transformations (see Section 2) jointly with a suitable extension of a maximum principle at the infinity of [38] due to Caminha in [20] (cf. Lemma 3.2).

In Section 4, we apply those results described above to study the uniqueness of totally geodesic spacelike hypersurfaces in GRW spacetimes. When the spacetime is timelike geodesically complete we can guarantee, with an appropriate set of hypotheses and with the aid of another classic result due to Yau (cf. Lemma 4.2), that complete noncompact spacelike hypersurfaces are leaves of the distribution of vector fields which are orthogonal to a conformal vector field globally defined on the ambient spacetime. Specifically, we obtain the following result that extends Theorem 4.1 of [17], for the case of a single closed conformal field (cf. Theorem 4.3):

Let  $\bar{M}^{n+1}$  be a timelike geodesically complete conformally stationary spacetime, with nonnegative Ricci curvature and endowed with a closed conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a connected complete spacelike hypersurface with  $H$  bounded and  $H_2$  bounded from below. If  $|V^\top|$  is Lebesgue integrable on  $M^n$  and the conformal factor  $\psi$  of  $V$  satisfies

$$\frac{1}{|V|} \frac{\partial \psi}{\partial t} \geq nH^2,$$

where  $t \in \mathbb{R}$  denotes the real parameter of the flow of  $v = V/|V|$ , then  $M^n$  is totally geodesic and the Ricci curvature of  $\bar{M}^{n+1}$  in the direction of  $N$  vanishes identically. Moreover, if  $M^n$  is noncompact,  $\langle V, V \rangle$  is constant on  $M^n$  and the Ricci curvature of

$M^n$  is also nonnegative, then  $x(M^n)$  is contained in a leaf of the distribution of vector fields orthogonal to  $V$ .

In Section 5, concerned with the case of the  $r$ -th mean curvatures, when the ambient spacetime is either a stationary spacetime or a timelike geodesically complete conformally stationary spacetime, we obtain a lower estimate of the index of relative nullity of complete noncompact spacelike hypersurfaces having two consecutive  $r$ -th curvatures which do not change sign. For instance, we get the following extension of Theorem 6.2 of [28] (cf. Theorem 5.3):

Let  $\bar{M}_c^{n+1}$  be a timelike geodesically complete conformally stationary spacetime, with constant sectional curvature  $c$  and endowed with a closed conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}_c^{n+1}$  be a complete, noncompact spacelike hypersurface with bounded second fundamental form  $A$  and whose  $r$ -th mean curvature  $H_r$  does not change sign and such that the  $(r+1)$ -th mean curvature  $H_{r+1}$  is bounded, for some  $r \in \{1, \dots, n-1\}$ . If  $|V^\top|$  is Lebesgue integrable on  $M^n$  and the conformal factor  $\psi$  of  $V$  satisfies

$$\frac{1}{|V|} \frac{\partial \psi}{\partial t} \neq c,$$

where  $t \in \mathbb{R}$  denotes the real parameter of the flow of  $v = V/|V|$ , then the hypersurface  $M^n$  is  $(r-1)$ -maximal. Moreover, if the  $(r+1)$ -th mean curvature  $H_{r+1}$  also does not change sign, then the index of minimum relative nullity  $v_0$  of  $M^n$  is at least  $n - (r-1)$ .

As a consequence of such previous result, we establish a sort of extension of Theorem 1.2 of [18]. More precisely, we obtain the following (cf. Corollary 5.6):

Let  $x: M^n \rightarrow \mathbb{H}_1^{n+1}$  be a complete spacelike hypersurface with bounded second fundamental form, which lies in the chronological future (past) of an equator of  $\mathbb{H}_1^{n+1}$  determined by a unit timelike vector  $a \in \mathbb{R}_2^{n+2}$ . Suppose that, for some  $1 \leq r \leq n-1$ ,  $H_r$  and  $H_{r+1}$  have equal (different) signs and that both of them do not change sign on  $M^n$ . If  $|a^\top|$  is Lebesgue integrable on  $M^n$ , then the index of minimum relative nullity  $v_0$  of  $M^n$  is at least  $n - (r-1)$ .

## 2. Preliminaries

Let  $\bar{M}^{n+1}$  be a  $(n+1)$ -dimensional ( $n \geq 2$ ) manifold with a Lorentzian metric tensor  $\langle \cdot, \cdot \rangle$  and Levi-Civita connection  $\bar{\nabla}$ . We denote by  $\mathfrak{X}(\bar{M})$  the set of vector fields of class  $C^\infty$  on  $\bar{M}^{n+1}$  and by  $C^\infty(\bar{M})$  the ring of real functions of class  $C^\infty$  on  $\bar{M}^{n+1}$ . We recall that a vector field  $V$  on  $\bar{M}^{n+1}$  is said *conformal* if

$$\mathcal{L}_V \langle \cdot, \cdot \rangle = 2\psi \langle \cdot, \cdot \rangle$$

for some function  $\psi \in C^\infty(\bar{M})$ , where  $\mathcal{L}$  stands for the Lie derivative of the Lorentzian metric of  $\bar{M}^{n+1}$ ; the function  $\psi$  is called the *conformal factor* of  $V$ . A Lorentz manifold  $\bar{M}^{n+1}$  endowed with a globally defined conformal timelike vector field is said *conformally stationary spacetime*.

Since  $\mathcal{L}_V(X) = [V, X]$  for all  $X \in \mathfrak{X}(\bar{M})$ , where  $[ \ ]$  denotes the Lie bracket, the tensorial character of  $\mathcal{L}_V$  shows that  $V \in \mathfrak{X}(\bar{M})$  is conformal if, and only if,

$$(2.1) \quad \langle \bar{\nabla}_X V, Y \rangle + \langle X, \bar{\nabla}_Y V \rangle = 2\psi \langle X, Y \rangle,$$

for all  $X, Y \in \mathfrak{X}(\bar{M})$ . In particular,  $V$  is Killing if, and only if,  $\psi \equiv 0$ . Moreover, from equation (2.1) we easily verify that

$$(2.2) \quad \psi = \frac{1}{n + 1} \operatorname{div}_{\bar{M}} V.$$

From now on, let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete noncompact spacelike hypersurface, namely, an isometric immersion from a complete, noncompact, connected,  $n$ -dimensional Riemannian manifold  $M^n$  into  $\bar{M}^{n+1}$ . In this setting, let  $\nabla$  denote the Levi-Civita connection of  $M^n$ . As  $\bar{M}^{n+1}$  is time-orientable by the timelike vector field  $V$  and  $x: M^n \rightarrow \bar{M}^{n+1}$  is a spacelike hypersurface, then  $M^n$  is orientable and one can choose a globally defined unit normal vector field  $N$  on  $M^n$  having the same time-orientation of  $\bar{M}^{n+1}$ , that is,

$$\langle V, N \rangle < 0.$$

Such  $N$  is said the *future-pointing Gauss map* of  $M^n$ . If we let  $A$  denote the Weingarten operator of  $x$  with respect to  $N$ , then  $A$  restricts to a self-adjoint linear map  $A_p: T_p M \rightarrow T_p M$  at each  $p \in M^n$ . Next the *Gauss formula* for  $M^n$  is given by

$$(2.3) \quad \bar{\nabla}_X Y = \nabla_X Y - \langle AX, Y \rangle N$$

and the *Weingarten formula* for  $M^n$  is

$$(2.4) \quad AX = -\bar{\nabla}_X N,$$

for any  $X, Y \in \mathfrak{X}(M)$ .

For  $1 \leq r \leq n$ , let  $S_r(p)$  denote the  $r$ -th elementary symmetric function on the eigenvalues of  $A_p$ , so that one gets  $n$  smooth functions  $S_r: M^n \rightarrow \mathbb{R}$  for which

$$\det(t\operatorname{Id} - A) = \sum_{r=0}^n (-1)^r S_r t^{n-r},$$

where  $S_0 = 1$  by definition. For fixed  $p \in M^n$ , the spectral theorem allows us to choose on  $T_p M$  an orthonormal basis  $\{E_1, \dots, E_n\}$  of eigenvectors of  $A_p$ , with corresponding eigenvalues  $\lambda_1, \dots, \lambda_n$ , respectively. One thus immediately sees that

$$S_r = \sigma_r(\lambda_1, \dots, \lambda_n),$$

where  $\sigma_r \in \mathbb{R}[X_1, \dots, X_n]$  is the  $r$ -th elementary symmetric polynomial on the indeterminates  $X_1, \dots, X_n$ .

For  $1 \leq r \leq n$ , one defines the  $r$ -th mean curvature  $H_r$  of  $x$  by

$$(2.5) \quad \binom{n}{r} H_r = (-1)^r S_r = \sigma_r(-\lambda_1, \dots, -\lambda_n).$$

In particular, when  $r = 1$ ,

$$(2.6) \quad H_1 = -\frac{1}{n} \sum_{i=1}^n \lambda_i = -\frac{1}{n} \operatorname{tr}(A) = H$$

is the mean curvature of  $M^n$ , which is the main extrinsic curvature of the hypersurface.

REMARK 2.1. The choice of the sign  $(-1)^r$  in our definition of  $H_r$  is motivated by the fact that in that case the mean curvature vector is given by  $\mathbf{H} = HN$ . Therefore,  $H(p) > 0$  at a point  $p \in M^n$  if, and only if,  $\mathbf{H}(p)$  is in the time-orientation as  $N(p)$ , and hence as  $V(p)$ .

When  $r = 2$ ,  $H_2$  defines a geometric quantity which is related to the (intrinsic) scalar curvature  $S$  of the hypersurface  $M^n$ . For instance, when the ambient spacetime has constant sectional curvature  $c$ , we obtain that

$$(2.7) \quad S = n(n - 1)(c - H_2).$$

Moreover, a relationship between the squared norm of the second fundamental form  $A$  of the spacelike hypersurface  $M^n$  and their curvatures  $H$  and  $H_2$  is given by

$$(2.8) \quad |A|^2 = n^2 H^2 - n(n - 1)H_2.$$

One also let the  $r$ -th Newton transformation  $P_r$  on  $M^n$  be given by setting  $P_0 = \operatorname{Id}$  and, for  $1 \leq r \leq n$ , via the recurrence relation

$$(2.9) \quad P_r = (-1)^r S_r \operatorname{Id} + A P_{r-1}.$$

A trivial induction shows that

$$P_r = (-1)^r \{S_r \operatorname{Id} - S_{r-1} A + S_{r-2} A^2 - \dots + (-1)^r A^r\},$$

so that Cayley–Hamilton theorem gives  $P_n = 0$ . Moreover, since  $P_r$  is a polynomial in  $A$  for every  $1 \leq r \leq n$ , it is also self-adjoint and commutes with  $A$ . Therefore, all bases of  $T_p M$  diagonalizing  $A$  at  $p \in M^n$  also diagonalize any transformation  $P_r$  at  $p$ .

If  $\{E_1, \dots, E_n\}$  is such a basis and  $A_i$  denotes the restriction of  $A$  to  $\langle E_i \rangle^\perp \subset T_p M$ , it is easy to see that

$$\det(t\text{Id} - A_i) = (-1)^k S_k(A_i) t^{n-1-k},$$

where

$$S_k(A_i) = \sum_{\substack{1 \leq j_1 < \dots < j_k \leq n \\ j_1, \dots, j_k \neq i}} \lambda_{j_1} \cdots \lambda_{j_k}.$$

With the above notations, it is also immediate to check that

$$(2.10) \quad P_r E_i = (-1)^r S_r(A_i) E_i.$$

It follows from (2.10) that for each  $r \in \{0, \dots, n-1\}$ ,

$$(2.11) \quad \text{tr}(P_r) = (-1)^r (n-r) S_r = b_r H_r,$$

$$(2.12) \quad \text{tr}(A P_r) = (-1)^r (r+1) S_{r+1} = -b_r H_{r+1},$$

and

$$(2.13) \quad \begin{aligned} \text{tr}(A^2 P_r) &= (-1)^r (S_1 S_{r+1} - (r+2) S_{r+2}) \\ &= \binom{n}{r+1} (n H H_{r+1} - (n-r-1) H_{r+2}), \end{aligned}$$

where

$$b_r = (n-r) \binom{n}{r} = (r+1) \binom{n}{r+1}.$$

The *divergence* of  $P_r$  is defined by

$$(2.14) \quad \text{div}_M P_r = \text{tr}(\nabla P_r) = \sum_{i=1}^n (\nabla_{E_i} P_r)(E_i),$$

where  $\{E_1, \dots, E_n\}$  is a (local) orthonormal frame on  $M^n$ .

From Lemma 3.1 of [6], we have a suitable formula for the divergence of the Newton transformations. In what follows, as in [31], the curvature tensor  $\bar{R}$  of the ambient spacetime  $\bar{M}^{n+1}$  is given by

$$\bar{R}(X, Y)Z = \bar{\nabla}_{[X, Y]}Z - [\bar{\nabla}_X, \bar{\nabla}_Y]Z,$$

for all  $X, Y, Z \in \mathfrak{X}(\bar{M})$ .

**Proposition 2.2.** *The divergence of the Newton transformations are given by*

$$(2.15) \quad \langle \operatorname{div}_M P_r, X \rangle = \sum_{j=1}^r \sum_{i=1}^n \langle \bar{R}(N, P_{r-j} E_i) E_i, A^{j-1} X \rangle,$$

for all  $X \in \mathfrak{X}(M)$ .

Moreover, from the computations of Section 4 of [6] we get

**Proposition 2.3.** *Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a conformal timelike vector field  $V$  and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface. Then,*

$$\operatorname{div}_M P_r V^\top = \langle \operatorname{div}_M P_r, V^\top \rangle + b_r \{ \psi H_r + \langle V, N \rangle H_{r+1} \},$$

where  $V^\top = V + \langle V, N \rangle N$  is the projection of  $V$  onto  $M^n$ ,  $b_r = (r + 1) \binom{n}{r+1}$  and  $\psi$  is the conformal factor of  $V$ .

### 3. Characterizations of totally geodesic hypersurfaces

Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime with  $n \geq 2$ . If  $\bar{M}^{n+1}$  is also an Einstein manifold, that is, if there exist a constant  $\lambda \in \mathbb{R}$  such that the Ricci tensor  $\operatorname{Ric}_{\bar{M}}$  of  $\bar{M}^{n+1}$  satisfies

$$(3.1) \quad \operatorname{Ric}_{\bar{M}}(X, Y) = \lambda \langle X, Y \rangle,$$

for any  $X, Y \in \mathfrak{X}(\bar{M})$ , then we say that  $\bar{M}^{n+1}$  is a *conformally stationary Einstein spacetime*. In this setting, we have the following

**Lemma 3.1.** *Let  $\bar{M}^{n+1}$  be a conformally stationary Einstein spacetime endowed with a conformal timelike vector field  $V$  and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface. Then,*

$$(3.2) \quad \operatorname{div}_M P_1 V^\top = n(n - 1) \{ \psi H + \langle V, N \rangle H_2 \},$$

where  $\psi$  is the conformal factor of  $V$ . If, in addition,  $\bar{M}_c^{n+1}$  has constant sectional curvature  $c$ , then

$$(3.3) \quad \operatorname{div}_M P_r V^\top = b_r \{ \psi H_r + \langle V, N \rangle H_{r+1} \},$$

where  $b_r = (r + 1) \binom{n}{r+1}$ .

Proof. Since  $\bar{M}^{n+1}$  is an Einstein manifold, from (3.1) and Proposition 2.2 we have

$$\begin{aligned}
 \langle \operatorname{div}_M P_1, V^\top \rangle &= \sum_{i=1}^n \langle \bar{R}(N, E_i)E_i, V^\top \rangle \\
 &= -\operatorname{Ric}_{\bar{M}}(N, V^\top) = -\lambda \langle N, V^\top \rangle = 0,
 \end{aligned}
 \tag{3.4}$$

where  $\{E_1, \dots, E_n\}$  is a (local) orthonormal frame on  $M^n$ . Hence, from equation (3.4) and Proposition 2.3 we obtain equation (3.2).

Now, suppose that  $\bar{M}_c^{n+1}$  has constant sectional curvature  $c$ . Thus, from Proposition 2.2, we have

$$\begin{aligned}
 \langle \operatorname{div}_M P_r, V^\top \rangle &= c \sum_{j=1}^r \sum_{i=1}^n \{ \langle N, E_i \rangle \langle P_{r-j} E_i, A^{j-1} V^\top \rangle \\
 &\quad - \langle P_{r-j} E_i, E_i \rangle \langle N, A^{j-1} V^\top \rangle \} = 0,
 \end{aligned}
 \tag{3.5}$$

where  $\{E_1, \dots, E_n\}$  is a (local) orthonormal frame on  $M^n$ .

Therefore, from (3.5) and Proposition 2.3 we conclude the proof of equation (3.3). □

In the paper [38], Yau established the following version of Stokes’ theorem on an  $n$ -dimensional, complete noncompact Riemannian manifold  $M^n$ : *if  $\omega \in \Omega^{n-1}(M)$  is an integrable  $(n - 1)$ -differential form on  $M^n$ , then there exists a sequence  $B_i$  of domains on  $M^n$  such that  $B_i \subset B_{i+1}$ ,  $M^n = \bigcup_{i \geq 1} B_i$  and*

$$\lim_{i \rightarrow +\infty} \int_{B_i} d\omega = 0.$$

Now, suppose that  $M^n$  is oriented by the volume element  $dM$ . If  $\omega = \iota_X dM$  is the contraction of  $dM$  in the direction of a smooth vector field  $X$  on  $M^n$ , then Caminha [20] obtained a suitable consequence of Yau’s result, which is described below. In what follows,  $\mathcal{L}^1(M)$  stands for the space of Lebesgue integrable functions on  $M^n$ .

**Lemma 3.2.** *Let  $X$  be a smooth vector field on the  $n$ -dimensional complete, non-compact, oriented Riemannian manifold  $M^n$ , such that  $\operatorname{div}_M X$  does not change sign on  $M^n$ . If  $|X| \in \mathcal{L}^1(M)$ , then  $\operatorname{div}_M X = 0$ .*

We recall that a spacelike hypersurface is said *maximal* if its mean curvature vanishes identically. Our first result establishes sufficient conditions to guarantee that a complete maximal spacelike hypersurface is, in fact, totally geodesic.

**Theorem 3.3.** *Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a conformal timelike vector field  $V$  and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete, noncompact spacelike hypersurface. Suppose that one of the following conditions is satisfied:*

- (a)  $H \geq 0$  and  $\operatorname{div}_{\bar{M}} V \leq 0$  on  $M^n$ ;
- (b)  $H \leq 0$  and  $\operatorname{div}_{\bar{M}} V \geq 0$  on  $M^n$ .

*If  $|V^\top| \in \mathcal{L}^1(M)$ , then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is an Einstein spacetime and  $H_2$  is bounded from below on  $M^n$ , then  $M^n$  is totally geodesic.*

Proof. Considering  $r = 0$  in the Proposition 2.3 we have

$$(3.6) \quad \operatorname{div}_M V^\top = n\psi + nH\langle N, V \rangle.$$

From equation (2.2) we see that  $\operatorname{div}_{\bar{M}} V$  and  $\psi$  have the same sign on  $M^n$ . Moreover, since  $N$  and  $V$  has the same time-orientation, then from either item (a) or (b) jointly with equation (3.6) we obtain that  $\operatorname{div}_M V^\top$  does not change sign on  $M^n$ . Since  $|V^\top| \in \mathcal{L}^1(M)$ , Lemma 3.2 gives  $\operatorname{div}_M V^\top = 0$ . Therefore,  $\psi = 0$  and  $H$  vanishes identically on  $M^n$ .

Now, from equation (2.8) we have  $|A|^2 = -n(n - 1)H_2$ , which implies  $H_2 \leq 0$ . Thus, assuming that  $H_2$  is bounded from below on  $M^n$  and the ambient space  $\bar{M}^{n+1}$  is Einstein, we have that  $|A|$  is bounded on  $M^n$  and, from Lemma 3.1, we get

$$(3.7) \quad \operatorname{div}_M P_1 V^\top = n(n - 1)\langle V, N \rangle H_2.$$

Then, from (2.9) we have that  $|P_1|$  is also bounded. Therefore,  $|P_1 V^\top| \in \mathcal{L}^1(M)$ , because  $|V^\top| \in \mathcal{L}^1(M)$ . Next, from (3.7) we obtain that  $\operatorname{div}_M P_1 V^\top$  does not change sign on  $M^n$ . By applying once more Lemma 3.2 we conclude that  $\operatorname{div}_M P_1 V^\top = 0$ . Therefore,  $H_2 = 0$  on  $M^n$  and, hence,  $M^n$  is totally geodesic.  $\square$

When  $V$  is a Killing vector field (namely, when  $\psi \equiv 0$ ), Theorem 3.3 reads as follows

**Corollary 3.4.** *Let  $\bar{M}^{n+1}$  be a stationary spacetime endowed with a Killing timelike vector field  $V$  and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete, noncompact spacelike hypersurface whose mean curvature  $H$  does not change sign. If  $|V^\top| \in \mathcal{L}^1(M)$  then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is Einstein and  $H_2$  is bounded from below on  $M^n$ , then  $M^n$  is totally geodesic.*

Let  $\mathbb{L}^{n+2}$  denote the  $(n + 2)$ -dimensional Lorentz–Minkowski space ( $n \geq 2$ ), that is, the real vector space  $\mathbb{R}^{n+2}$ , endowed with the Lorentz metric

$$\langle v, w \rangle = \sum_{i=1}^{n+1} v_i w_i - v_{n+2} w_{n+2},$$

for all  $v, w \in \mathbb{R}^{n+2}$ . We define the  $(n + 1)$ -dimensional de Sitter space  $\mathbb{S}_1^{n+1}$  as the following hyperquadric of  $\mathbb{L}^{n+2}$

$$\mathbb{S}_1^{n+1} = \{p \in \mathbb{L}^{n+2} : \langle p, p \rangle = 1\}.$$

From the above definition it is easy to show that the metric induced from  $\langle \cdot, \cdot \rangle$  turns  $\mathbb{S}_1^{n+1}$  into a Lorentz space form of constant sectional curvature 1. In  $\mathbb{S}_1^{n+1}$ , from Corollary 3.4, we have the following nonexistence result concerning to complete noncompact spacelike hypersurfaces.

**Corollary 3.5.** *There exists no complete noncompact spacelike hypersurface  $x : M^n \rightarrow \mathbb{S}_1^{n+1}$  in  $(n + 1)$ -dimensional de Sitter space, whose mean curvature  $H$  does not change sign,  $H_2$  is bounded from below and such that, for some pair of orthogonal timelike vectors  $u, v \in \mathbb{L}^{n+2}$ ,  $|u^\top|, |v^\top| \in \mathcal{L}^1(M)$  and  $|\langle u, \cdot \rangle|, |\langle v, \cdot \rangle|$  are bounded on  $M^n$ .*

*Proof.* Suppose, by the sake of contradiction, that there exists such a hypersurface. Following the ideas of [27], for any pair of orthogonal timelike vectors  $u, v \in \mathbb{L}^{n+2}$ , we have that  $W(x) = \langle u, x \rangle v - \langle v, x \rangle u$  is a timelike Killing vector field in  $\mathbb{S}_1^{n+1}$ . Since  $|\langle u, \cdot \rangle|, |\langle v, \cdot \rangle|$  are bounded on  $M^n$  and  $|u^\top|, |v^\top| \in \mathcal{L}^1(M)$ , then  $|W^\top| \in \mathcal{L}^1(M)$ . From Corollary 3.4,  $M^n$  is totally geodesic in  $\mathbb{S}_1^{n+1}$ . Next, from Theorem 5.1 of [1] we obtain that  $M^n$  is isometric to  $n$ -dimensional Euclidean sphere  $\mathbb{S}^n$ , which contradicts the non-compactness of  $M^n$ . □

We recall that the anti-de Sitter spacetime  $\mathbb{H}_1^{n+1}$  is the hyperquadric

$$\mathbb{H}_1^{n+1} = \{p \in \mathbb{R}_2^{n+2} : \langle p, p \rangle = -1\},$$

in the indefinite index two flat space  $\mathbb{R}_2^{n+2}$ . It is also a standard fact that  $\mathbb{H}_1^{n+1}$  is the Lorentz space form of constant sectional curvature  $-1$ . The following result in  $\mathbb{H}_1^{n+1}$  is a sort of extension of Theorems 1.1 and 1.2 of [18].

**Corollary 3.6.** *Let  $x : M^n \rightarrow \mathbb{H}_1^{n+1}$  be a complete, noncompact spacelike hypersurface, whose mean curvature  $H$  does not change sign and such that  $H_2$  is bounded from below. If, for some pair of orthogonal timelike vectors  $u, v \in \mathbb{R}_2^{n+2}$ ,  $|u^\top|, |v^\top| \in \mathcal{L}^1(M)$  and  $|\langle u, \cdot \rangle|, |\langle v, \cdot \rangle|$  are bounded on  $M^n$ , then  $M^n$  is isometric to the  $n$ -dimensional hyperbolic space  $\mathbb{H}^n$ .*

*Proof.* Following once more the ideas of Example 1 of [27], since  $u, v \in \mathbb{R}_2^{n+2}$  are orthogonal timelike vectors, we observe that  $W(x) = \langle u, x \rangle v - \langle v, x \rangle u$  is a timelike Killing vector field in  $\mathbb{H}_1^{n+1}$ . Moreover, since  $|\langle u, \cdot \rangle|, |\langle v, \cdot \rangle|$  are bounded and  $|u^\top|, |v^\top| \in \mathcal{L}^1(M)$ , we have  $|W^\top| \in \mathcal{L}^1(M)$ . Thus, from Corollary 3.4,  $M^n$  is totally

geodesic in  $\mathbb{H}_1^{n+1}$ . Therefore, from Theorem 5.1 of [1] we conclude that  $M^n$  is isometric to  $\mathbb{H}^n$ . □

An important particular case of a conformal vector field  $V$  is that in which

$$(3.8) \quad \bar{\nabla}_Y V = \psi Y$$

for all  $Y \in \mathfrak{X}(\bar{M})$ , where  $\psi$  is the conformal factor of  $V$ . In this case we say that  $V$  is *closed*, in the sense that its dual 1-form is closed. In this setting, a closed conformal vector field  $V$  is said *homothetic* if  $\psi$  is constant, and it is said *parallel* if  $\psi$  vanishes identically.

For any open set  $\mathcal{U} \subset \bar{M}$ , the *distribution on  $\mathcal{U}$  of vector fields orthogonal to  $V$*  is defined by

$$V^\perp(p) = \{w \in T_p \bar{M}; \langle V(p), w \rangle = 0\}, \quad p \in \mathcal{U}.$$

We note that  $V^\perp$  is integrable; in fact, if  $X, Y \in V^\perp$ , then from equation (3.8) we have that

$$\langle [X, Y], V \rangle = \langle \bar{\nabla}_X Y - \bar{\nabla}_Y X, V \rangle = -\langle Y, \bar{\nabla}_X V \rangle + \langle X, \bar{\nabla}_Y V \rangle = 0.$$

Therefore, Frobenius' theorem guarantees that the collection of all connected integral manifolds of  $V^\perp$ , called *leaves*, corresponds to a *spacelike foliation* of  $\bar{M}^{n+1}$ .

Let  $\Sigma^n$  be a *leaf* of  $V^\perp$  furnished with the induced metric. From equation (3.8) we get

$$(3.9) \quad \bar{\nabla} \langle V, V \rangle = 2\psi V.$$

Consequently,  $\langle V, V \rangle$  is constant on connected leaves of  $V^\perp$ . Moreover, computing covariant derivatives in (3.9), we have

$$(\text{Hess}_{\bar{M}} \langle V, V \rangle)(X, Y) = 2X(\psi) \langle V, Y \rangle + 2\psi^2 \langle X, Y \rangle.$$

However, since both  $\text{Hess}_{\bar{M}}$  and the metric are symmetric tensors, we get

$$X(\psi) \langle V, Y \rangle = Y(\psi) \langle V, X \rangle,$$

for all  $X, Y \in \mathfrak{X}(\bar{M})$ . Now, taking  $Y = V$  we arrive at

$$(3.10) \quad \bar{\nabla} \psi = \frac{V(\psi)}{\langle V, V \rangle} V = -\nu(\psi)\nu,$$

where  $\nu = V/|V|$  and  $|V| = \sqrt{-\langle V, V \rangle} > 0$ . Hence,  $\psi$  is also constant on connected leaves of  $V^\perp$ . Furthermore, with a straightforward computation, we verify that the

shape operator  $A_\Sigma$  of a leaf  $\Sigma^n \in V^\perp$  with respect  $\nu$  is given by

$$(3.11) \quad A_\Sigma(X) = -\bar{\nabla}_X \nu = -\frac{\psi}{|V|} X,$$

for any  $X \in \mathfrak{X}(\Sigma)$  and, hence,  $\Sigma^n$  is an umbilical hypersurface with constant mean curvature

$$(3.12) \quad \mathcal{H} = \frac{\psi}{|V|}.$$

REMARK 3.7. In order to conclude that spacelike hypersurfaces are leaves of  $V^\perp$  and taking into account equation (3.12), we consider from now on that the mean curvature of spacelike hypersurfaces studied and the conformal factor  $\psi$  have the same sign.

**Proposition 3.8.** *Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a closed conformal timelike vector field  $V$ , whose conformal factor is  $\psi$ , and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a spacelike hypersurface. If  $W$  is another closed conformal timelike vector field on  $\bar{M}^{n+1}$ , with conformal factor  $\psi_W$ , and  $f: M^n \rightarrow \mathbb{R}$  is given by  $f = \langle V, W \rangle$ , then*

$$(3.13) \quad \nabla f = \psi W^\top + \psi_W V^\top$$

and

$$(3.14) \quad \Delta f = W^\top(\psi) + V^\top(\psi_W) + nH\{\psi\langle W, N \rangle + \psi_W\langle V, N \rangle\} + 2n\psi\psi_W.$$

Proof. If  $Y \in \mathfrak{X}(M)$  then from (3.8) we have

$$\begin{aligned} \langle \nabla f, Y \rangle &= Y(f) = \langle \bar{\nabla}_Y V, W \rangle + \langle V, \bar{\nabla}_Y W \rangle \\ &= \psi \langle Y, W^\top \rangle + \psi_W \langle V^\top, Y \rangle = \langle \psi W^\top + \psi_W V^\top, Y \rangle. \end{aligned}$$

On the other hand, considering  $r = 0$  in the Proposition 2.3,

$$\operatorname{div}_M V^\top = n\psi + nH\langle N, V \rangle.$$

Thus, from (3.13) we obtain

$$\begin{aligned} \Delta f &= \operatorname{div}_M \nabla f = \operatorname{div}_M(\psi W^\top + \psi_W V^\top) \\ &= \psi \operatorname{div}_M W^\top + \langle \nabla \psi, W^\top \rangle + \psi_W \operatorname{div}_M V^\top + \langle \nabla \psi_W, V^\top \rangle \\ &= \psi \{n\psi_W + nH\langle N, W \rangle\} + W^\top(\psi) + \psi_W \{n\psi + nH\langle N, V \rangle\} + V^\top(\psi_W) \\ &= W^\top(\psi) + V^\top(\psi_W) + 2n\psi\psi_W + nH\{\psi\langle N, W \rangle + \psi_W\langle N, V \rangle\}. \quad \square \end{aligned}$$

Now, we are in position to establish the following characterization of totally geodesic spacelike hypersurfaces, which can be regarded as a sort of extension of the Theorem 4.1 of [17].

**Theorem 3.9.** *Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a parallel timelike vector field  $V$  and a homothetic nonparallel timelike vector field  $W$ , and  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete noncompact spacelike hypersurface whose mean curvature  $H$  does not change sign on  $M^n$ . If  $|V^\top| \in \mathcal{L}^1(M)$  then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is Einstein and  $H_2$  is bounded from below, then  $M^n$  is totally geodesic.*

Proof. Since  $V$  is parallel and  $W$  is homothetic and nonparallel, it follows from (3.13) and (3.14) that

$$\nabla f = \psi_W V^\top$$

and

$$\Delta f = nH\psi_W \langle V, N \rangle,$$

with  $\psi_W$  being a nonzero constant. Therefore, the assumption  $|V^\top| \in \mathcal{L}^1(M)$  gives  $|\nabla f| \in \mathcal{L}^1(M)$ , and the assumption on  $H$ , together with the fact that  $\langle V, N \rangle < 0$  on  $M$ , assures that  $\Delta f$  is either nonnegative or nonpositive on  $M^n$ . Therefore, Lemma 3.2 implies  $\Delta f = 0$  on  $M^n$  and, hence,  $H$  vanishes identically on  $M^n$ .

Finally, to prove the second part of the theorem, it is enough to follow the same steps of the end of the proof of Theorem 3.3.  $\square$

#### 4. Applications to GRW spacetimes

According to the terminology introduced in [7], a particular class of conformally stationary spacetimes is that of *generalized Robertson–Walker* (GRW) spacetimes, namely, warped products  $\bar{M}^{n+1} = -I \times_\phi F^n$ , where  $I \subseteq \mathbb{R}$  is an interval with the metric  $-dt^2$ ,  $F^n$  is an  $n$ -dimensional Riemannian manifold and  $\phi: I \rightarrow \mathbb{R}$  is positive and smooth. In particular, when the Riemannian fiber  $F^n$  has constant sectional curvature, then  $-I \times_\phi F^n$  is classically called a *Robertson–Walker* (RW) spacetime.

For such spacetimes, let  $\pi_I: \bar{M}^{n+1} \rightarrow I$  denote the canonical projection onto  $I$ . Then the vector field

$$(4.1) \quad V = (\phi \circ \pi_I) \partial_t$$

is a conformal, timelike and closed, with conformal factor  $\psi = \phi'$ , where the prime denotes differentiation with respect to  $t$ . Moreover, for  $t_0 \in I$ , the slice  $M_{t_0}^n = \{t_0\} \times F^n$  is totally umbilical, with  $r$ -th mean curvature equal to  $(\phi'(t_0)/\phi(t_0))^r$  with respect to  $\partial_t$  (cf. [7]; see also [30] and [6]).

If  $\bar{M}^{n+1} = -I \times_{\phi} F^n$  is a GRW and  $x: M^n \rightarrow \bar{M}^{n+1}$  is a complete spacelike hypersurface of  $\bar{M}^{n+1}$ , such that  $\phi \circ \pi_I$  is bounded on  $M^n$ , then  $\pi_F|_M: M^n \rightarrow F^n$  is necessarily a covering map (cf. [7]). In particular, if  $M^n$  is closed then  $F^n$  is automatically closed. Moreover, from Proposition 7.42 of [31] (see also Corollary 9.107 of [12]), we see that a GRW as above has constant sectional curvature  $c$  if, and only if, its fiber  $F^n$  has constant sectional curvature  $k$  (that is,  $-I \times_{\phi} F^n$  is in fact a RW spacetime) and its warping function  $\phi$  satisfies the following ODE

$$(4.2) \quad \frac{\phi''}{\phi} = c = \frac{(\phi')^2 + k}{\phi^2}.$$

Now, in a GRW space, let  $h$  denote the (vertical) *height function* naturally attached to the spacelike hypersurface  $M^n$ , namely,  $h = (\pi_I)|_M$ . Let  $\bar{\nabla}$  and  $\nabla$  denote gradients with respect to the metrics of  $-I \times_{\phi} F^n$  and  $M^n$ , respectively. A simple computation shows that the gradient of  $\pi_I$  on  $-I \times_{\phi} F^n$  is given by

$$\bar{\nabla}\pi_I = -\langle \bar{\nabla}\pi_I, \partial_t \rangle \partial_t = -\partial_t,$$

so that the gradient of  $h$  on  $M^n$  is

$$(4.3) \quad \nabla h = (\bar{\nabla}\pi_I)^{\top} = -\partial_t^{\top},$$

where  $\partial_t^{\top} = \partial_t + \langle N, \partial_t \rangle N$  is the tangential component of  $\partial_t$  on  $M^n$ .

From Corollary 3.4 we obtain the following

**Corollary 4.1.** *Let  $\bar{M}^{n+1} = -\mathbb{R} \times F^n$  be a stationary spacetime, where  $F^n$  is a complete noncompact Riemannian manifold, and let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a complete noncompact spacelike hypersurface whose mean curvature  $H$  does not change sign. If  $|\nabla h| \in \mathcal{L}^1(M)$  then  $M^n$  is maximal. Moreover, if  $\bar{M}^{n+1}$  is Einstein and  $H_2$  is bounded from below on  $M^n$ , then  $M^n$  is totally geodesic. In particular, when  $F^n$  is the Euclidean space  $\mathbb{R}^n$  then  $M^n$  is a spacelike hyperplane in Lorentz–Minkowski spacetime  $\mathbb{L}^{n+1}$ .*

*Proof.* Since  $V = \partial_t$  is a timelike Killing vector field in  $\bar{M}^{n+1}$ , the first part of the result follows directly from Corollary 3.4. In particular, when  $F^n = \mathbb{R}^n$ , from the classification of the totally geodesic hypersurfaces of  $\mathbb{L}^{n+1}$  (see, for example, [31] or [1]), we conclude that  $M^n$  is a spacelike hyperplane of  $\mathbb{L}^{n+1}$ . □

Let  $\bar{M}^{n+1}$  be a conformally stationary spacetime endowed with a closed conformal timelike vector field  $V$ , whose conformal factor is  $\psi$ . If  $p \in \bar{M}^{n+1}$  and  $\Sigma_p$  is the leaf of  $V^{\perp}$  containing  $p$ , then we can find a neighborhood  $\mathcal{U}_p$  of  $p$  on  $\Sigma_p^n$  and an open interval  $I \subset \mathbb{R}$  containing 0 such that the flow  $\mathcal{F}(t, \cdot)$  of  $\nu = V/|V|$  is defined in  $\mathcal{U}_p$  for any  $t \in I$ . Moreover, when  $\bar{M}^{n+1}$  is *timelike geodesically complete*, that is,

when all timelike geodesic of  $\bar{M}^{n+1}$  is defined for all values of the parameter  $t \in \mathbb{R}$ , S. Montiel [30] proved that the application

$$(4.4) \quad \begin{aligned} \varphi: \mathbb{R} \times \Sigma_p^n &\rightarrow \bar{M}^{n+1} \\ (t, q) &\mapsto \mathcal{F}(t, q) \end{aligned}$$

is a global parametrization of  $\bar{M}^{n+1}$ , such that  $\bar{M}^{n+1}$  is isometric to the GRW

$$-\mathbb{R} \times_\phi \Sigma_p^n,$$

where

$$\phi(t) = \sqrt{-(V(\mathcal{F}(t, q)), V(\mathcal{F}(t, q)))},$$

$t \in \mathbb{R}$ , and  $q \in \Sigma_p^n$  is an arbitrary point.

In order to prove our next result which extends Theorem 4.1 of [17], we will need the following classical result due to S.T. Yau [38].

**Lemma 4.2.** *Every complete noncompact Riemannian manifold with nonnegative Ricci curvature has infinite volume.*

**Theorem 4.3.** *Let  $\bar{M}^{n+1}$  be a timelike geodesically complete conformally stationary spacetime, with nonnegative Ricci curvature and endowed with a closed conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}^{n+1}$  be a connected complete spacelike hypersurface with mean curvature  $H$  bounded and  $H_2$  bounded from below. If  $|V^\top| \in \mathcal{L}^1(M)$  and the conformal factor  $\psi$  of  $V$  satisfies*

$$(4.5) \quad \frac{1}{|V|} \frac{\partial \psi}{\partial t} \geq nH^2,$$

where  $t \in \mathbb{R}$  denotes the real parameter of the flow of  $v = V/|V|$ , then  $M^n$  is totally geodesic and the Ricci curvature of  $\bar{M}^{n+1}$  in the direction of  $N$  vanishes identically. Moreover, if  $M^n$  is noncompact,  $|V|$  is constant on  $M^n$  and the Ricci curvature of  $M^n$  is also nonnegative, then  $x(M^n)$  is contained in a leaf of  $V^\top$ .

*Proof.* Initially we observe that, since  $\bar{M}^{n+1}$  is timelike geodesically complete conformally, we can consider along it the global parametrization (4.4).

If  $f_V: M^n \rightarrow \mathbb{R}$  is given by  $f_V = \langle V, N \rangle$  then  $f_V$  is negative on  $M^n$ . For all  $Y \in \mathfrak{X}(M)$  we have

$$\begin{aligned} \langle \nabla f_V, Y \rangle &= Y(f_V) = Y\langle V, N \rangle \\ &= \langle \bar{\nabla}_Y V, N \rangle + \langle V, \bar{\nabla}_Y N \rangle \\ &= \psi \langle Y, N \rangle - \langle V^\top, A(Y) \rangle = \langle -A(V^\top), Y \rangle. \end{aligned}$$

Thus

$$(4.6) \quad \nabla f_V = -A(V^\top).$$

On the other hand, from Proposition 3.1 of [11],

$$(4.7) \quad \Delta f_V = n\langle \nabla H, V \rangle + \{\text{Ric}_{\bar{M}}(N, N) + |A|^2\}f_V + nH\psi - nN(\psi).$$

From (3.10) we observe that

$$(4.8) \quad N(\psi) = \langle N, \bar{\nabla}\psi \rangle = -\nu(\psi)\langle N, \nu \rangle = -\frac{1}{|V|} \frac{\partial \psi}{\partial t} f_V,$$

where  $t$  is the parameter of the flow of  $\nu = V/|V|$ . Thus, in (4.7) we have

$$(4.9) \quad \Delta f_V = n\langle \nabla H, V \rangle + \{\text{Ric}_{\bar{M}}(N, N) + |A|^2\}f_V + nH\psi + \frac{n}{|V|} \frac{\partial \psi}{\partial t} f_V.$$

From hypothesis (4.5),

$$(4.10) \quad \Delta f_V \leq n\langle \nabla H, V \rangle + \{\text{Ric}_{\bar{M}}(N, N) + |A|^2\}f_V + nH\psi + n^2 H^2 f_V.$$

Now, let us consider on  $M^n$  the tangent vector field

$$X = \nabla f_V - nHV^\top.$$

Since  $H$  is bounded and  $H_2$  is bounded below then, from (2.8), we obtain that the norm of the Weingarten operator  $A$  is bounded. Thus, from (4.6),

$$|X| \leq \{|A| + n|H|\}|V^\top| \in \mathcal{L}^1(M),$$

because  $|V^\top| \in \mathcal{L}^1(M)$ .

Moreover, from the Proposition 2.3, when  $r = 0$ , and (4.10) we have

$$(4.11) \quad \begin{aligned} \text{div}_M X &= \Delta f_V - n\langle \nabla H, V \rangle - nH \text{div}_M V^\top \\ &\leq n\langle \nabla H, V \rangle + \{\text{Ric}_{\bar{M}}(N, N) + |A|^2\}f_V \\ &\quad + nH\psi + n^2 H^2 f_V - n\langle \nabla H, V \rangle - n^2 \psi H - n^2 H^2 f_V \\ &= \{\text{Ric}_{\bar{M}}(N, N) + |A|^2\}f_V - n(n-1)H\psi \leq 0, \end{aligned}$$

where the last inequality we used that  $f_V < 0$ ,  $\text{Ric}_{\bar{M}} \geq 0$  and  $H$  and  $\psi$  has the same sign over  $M^n$  (see Remark 3.7). Thus, Lemma 3.2 gives  $\text{div}_M X = 0$ . Therefore,  $\text{Ric}_{\bar{M}}(N, N) = 0$  and  $M^n$  is totally geodesic.

Now, suppose that  $M^n$  is noncompact,  $|V|$  is constant on  $M^n$  and the Ricci curvature of  $M^n$  is also nonnegative. Since  $A = 0$  and  $\nabla f_V = -A(V^\top)$  then  $f_V = \langle V, N \rangle$  is constant and nonzero on  $M^n$ . Since  $|V|$  also is constant on  $M^n$  and

$$(4.12) \quad |V^\top|^2 = |V + \langle V, N \rangle N|^2 = -|V|^2 + \langle V, N \rangle^2,$$

so that  $|V^\top|$  is constant on  $M^n$ . Therefore,

$$+\infty > \int_M |V^\top| dM = |V^\top| \text{Vol}(M).$$

But since  $M^n$  is noncompact and has nonnegative Ricci curvature, the Lemma 4.2 gives  $\text{Vol}(M) = +\infty$ , and hence the only possibility is  $|V^\top| = 0$ . Then, from (4.12) we have

$$|\langle V, N \rangle| = |V|.$$

Therefore, the inverse Cauchy–Schwarz inequality gives that  $V$  is parallel to  $N$  and, hence,  $x(M^n)$  is contained in a leaf of  $V^\perp$ . □

Let  $\bar{M}^{n+1} = -I \times_\phi F^n$  be a GRW spacetime. According the terminology established in [2], we say that  $M^n$  is *bounded away from the infinity of  $\bar{M}^{n+1}$*  when it lies between two slices of  $\bar{M}^{n+1}$ . From Theorem 4.3, we have the following result.

**Corollary 4.4.** *Let  $\bar{M}^{n+1} = -I \times_\phi F^n$  be a timelike geodesically complete GRW spacetime with nonnegative Ricci curvature and  $x: M^n \rightarrow \bar{M}^{n+1}$  be a connected complete spacelike hypersurface bounded away from the infinity of  $\bar{M}^{n+1}$ , with mean curvature  $H$  bounded and  $H_2$  bounded from below. If the height function  $h$  satisfies  $|\nabla h| \in \mathcal{L}^1(M)$  and*

$$\frac{\phi''}{\phi} \geq nH^2,$$

*then  $M^n$  is totally geodesic and the Ricci curvature of  $\bar{M}^{n+1}$  in the direction of  $N$  vanishes identically. Moreover, if  $M^n$  is noncompact,  $\phi$  is constant on  $M^n$  and the Ricci curvature of  $M^n$  is also nonnegative, then  $x(M^n) \subset \{t_0\} \times F^n$ , for some  $t_0 \in I$ .*

### 5. Estimating the index of relative nullity

Let  $\bar{M}_c^{n+1}$  be a conformally stationary spacetime with constant sectional curvature  $c$  and endowed with a conformal timelike vector field  $V$ , and consider a complete spacelike hypersurface  $x: M^n \rightarrow \bar{M}_c^{n+1}$ , oriented by a vector field  $N$  in the same temporal direction that  $V$ . Let  $A$  be the second fundamental form of  $M^n$  with respect to  $N$ . According to [23], for  $p \in M^n$ , we define the *space of relative nullity*  $\Delta(p)$  of  $M^n$  at  $p$  by

$$\Delta(p) = \{v \in T_p M; v \in \ker(A_p)\},$$

where  $\ker(A_p)$  denotes the kernel of  $A_p$ . The *index of relative nullity*  $\nu(p)$  of  $M^n$  at  $p$  is the dimension of  $\Delta(p)$ , that is,

$$\nu(p) = \dim(\Delta(p)),$$

and the *index of minimum relative nullity*  $\nu_0$  of  $M^n$  is defined by

$$\nu_0 = \min_{p \in M} \nu(p).$$

In order to state our next result, we recall that a spacelike hypersurface  $M^n$  is said *r-maximal* if  $H_{r+1}$  vanishes identically on  $M^n$ .

**Theorem 5.1.** *Let  $\bar{M}_c^{n+1}$  be a stationary spacetime with constant sectional curvature  $c$ , endowed with a Killing timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}_c^{n+1}$  be a complete noncompact spacelike hypersurface with bounded second fundamental form  $A$  and such that the  $(r + 1)$ -th mean curvature  $H_{r+1}$  does not change sign, for some  $r \in \{0, \dots, n - 1\}$ .*

(a) *If  $|V^\top| \in \mathcal{L}^1(M)$  then  $M^n$  is  $r$ -maximal. Moreover, if  $H_{r+2}$  also does not change sign, for some  $r \in \{0, \dots, n - 2\}$ , then the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - r$ .*

(b) *When  $\bar{M}_c^{n+1}$  is the Lorentz–Minkowski space  $\mathbb{L}^{n+1}$ , if  $H_r$  does not vanish on  $M^n$ , for some  $r \in \{0, \dots, n - 2\}$ , then through every point of  $M^n$  there passes an  $(n - r)$ -hyperplane of  $\mathbb{L}^{n+1}$  totally contained in  $M^n$ .*

*Proof.* (a) Since  $V$  is a Killing vector field, then  $\psi = 0$ . Therefore, from Lemma 3.1 we have

$$(5.1) \quad \operatorname{div}_M P_r V^\top = b_r \langle V, N \rangle H_{r+1}.$$

Since  $|A|$  is bounded, from (2.9) we conclude that  $|P_r|$  is also bounded, for any  $1 \leq r \leq n$ . Thus  $|P_r V^\top| \in \mathcal{L}^1(M)$ , because  $|V^\top| \in \mathcal{L}^1(M)$ . Moreover, since  $N$  and  $V$  have the same time-orientation and  $H_{r+1}$  does not change sign on  $M^n$  then from (5.1) we obtain that  $\operatorname{div}_M P_r V^\top$  does not change sign. Thus, Lemma 3.2 gives  $\operatorname{div}_M P_r V^\top = 0$ . Therefore,  $H_{r+1} = 0$  on  $M^n$ .

Replacing  $r$  by  $r + 1$  in equation (5.1) and following the same steps considered above, we obtain that  $H_{r+2} = 0$  on  $M^n$ . Therefore, since  $H_{r+1} = H_{r+2} = 0$ , Proposition 1 of [19] assures that  $H_j = 0$  for all  $j \geq r + 1$  and, hence,  $\nu_0 \geq n - r$ .

(b) Now, suppose that  $\bar{M}_c^{n+1}$  is the Lorentz–Minkowski spacetime  $\mathbb{L}^{n+1}$ . By Theorem 5.3 of [23] (see also [25]), since we are supposing that  $H_r$  does not vanish on  $M^n$ , the distribution  $p \mapsto \Delta(p)$  of minimal relative nullity of  $M^n$  is smooth and integrable with complete leaves, totally geodesic in  $M^n$  and in  $\mathbb{L}^{n+1}$ . Therefore, the result follows from the characterization of complete totally geodesic submanifolds of  $\mathbb{L}^{n+1}$  as spacelike hyperplanes of suitable dimension. □

From the proof of the Theorem 5.1, we observe that we can change the Killing vector field  $V$  by a conformal field, but in this case  $M^n$  has to be  $(r - 1)$ -maximal. Thus, we have the following

**Corollary 5.2.** *Let  $\bar{M}_c^{n+1}$  be a conformally stationary spacetime with constant sectional curvature  $c$  and endowed with a conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}_c^{n+1}$  be a  $(r - 1)$ -maximal complete, noncompact spacelike hypersurface with bounded second fundamental form  $A$  and such that the  $(r + 1)$ -th and  $(r + 2)$ -th mean curvatures  $H_{r+1}$  and  $H_{r+2}$  do not change sign, for some  $r \in \{0, \dots, n - 2\}$ . If  $|V^\top| \in \mathcal{L}^1(M)$  then the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - r$ . In particular, when  $\bar{M}_c^{n+1}$  is the Lorentz–Minkowski spacetime  $\mathbb{L}^{n+1}$ , if  $H_r$  does not vanish on  $M^n$  then through every point of  $M^n$  there passes an  $(n - r)$ -hyperplane of  $\mathbb{L}^{n+1}$  totally contained in  $M^n$ .*

On the other hand, when the ambient space is a timelike geodesically complete conformally stationary spacetime endowed with a closed conformal vector field, we have the following estimate of the index of relative nullity of a complete spacelike hypersurface which extends Theorem 6.2 of [28].

**Theorem 5.3.** *Let  $\bar{M}_c^{n+1}$  be a timelike geodesically complete conformally stationary spacetime, with constant sectional curvature  $c$  and endowed with a closed conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}_c^{n+1}$  be a complete, noncompact spacelike hypersurface with bounded second fundamental form  $A$  and whose  $r$ -th mean curvature  $H_r$  does not change sign and the  $(r + 1)$ -th mean curvature  $H_{r+1}$  is bounded, for some  $r \in \{1, \dots, n - 1\}$ . If  $|V^\top| \in \mathcal{L}^1(M)$  and the conformal factor  $\psi$  of  $V$  satisfies*

$$(5.2) \quad \frac{1}{|V|} \frac{\partial \psi}{\partial t} \neq c,$$

where  $t \in \mathbb{R}$  denotes the real parameter of the flow of  $v = V/|V|$ , then the hypersurface  $M^n$  is  $(r - 1)$ -maximal. Moreover, if the  $(r + 1)$ -th mean curvature  $H_{r+1}$  also does not change sign, then the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - (r - 1)$ .

*Proof.* Since  $\bar{M}_c^{n+1}$  is timelike geodesically complete, we can consider the global parametrization (4.4). So, if we consider the function  $f_V: M^n \rightarrow \mathbb{R}$  given by  $f_V = \langle V, N \rangle$ , then  $f_V$  is negative on  $M^n$  and from (4.6),  $\nabla f_V = -A(V^\top)$ . Moreover, since  $\operatorname{div}_M P_r = 0$  by Proposition 2.2 and the constant sectional curvature condition, from Codazzi equation we obtain

$$\begin{aligned} \operatorname{div}_M P_r \nabla f_V &= \sum_i \langle \nabla_{E_i} \nabla f_V, P_r E_i \rangle \\ &= - \sum_i \langle \nabla_{E_i} (AV^\top), P_r E_i \rangle \end{aligned}$$

$$\begin{aligned}
 &= - \sum_i \langle (\nabla_{E_i} A) V^\top, P_r E_i \rangle - \sum_i \langle A (\nabla_{E_i} V^\top), P_r E_i \rangle \\
 &= - \sum_i \langle (\nabla_{V^\top} A) E_i, P_r E_i \rangle - \sum_i \langle \nabla_{E_i} V^\top, A P_r E_i \rangle \\
 &= - \sum_i \langle (P_r \nabla_{V^\top} A) E_i, E_i \rangle - \sum_i \langle \nabla_{E_i} V^\top, A P_r E_i \rangle \\
 &= - \operatorname{tr}(P_r \nabla_{V^\top} A) - \sum_i \langle \nabla_{E_i} V^\top, A P_r E_i \rangle,
 \end{aligned}$$

where  $\{E_1, \dots, E_n\}$  is a (local) orthonormal frame on  $M^n$ . Thus, by using Lemma A of [33], we get

$$\begin{aligned}
 \operatorname{div}_M P_r \nabla f_V &= (-1)^{r+1} V^\top (S_{r+1}) - \sum_i \langle \bar{\nabla}_{E_i} (f_V N), A P_r E_i \rangle - \sum_i \langle \bar{\nabla}_{E_i} V, A P_r E_i \rangle \\
 &= (-1)^{r+1} \langle V^\top, \nabla S_{r+1} \rangle + \sum_i \langle A E_i, A P_r E_i \rangle f_V - \sum_i \langle \bar{\nabla}_{E_i} V, A P_r E_i \rangle \\
 &= \binom{n}{r+1} \langle V, \nabla H_{r+1} \rangle + \operatorname{tr}(A^2 P_r) f_V - \operatorname{tr}(P_r A (\bar{\nabla} V)^\top).
 \end{aligned}$$

On the other hand, from equation (4) of [10], we have

$$\begin{aligned}
 \operatorname{tr}(P_r A (\bar{\nabla} V)^\top) &= \operatorname{tr}(A P_r (\bar{\nabla} V)^\top) \\
 &= c \operatorname{tr}(P_r) f_V + \operatorname{tr}(A P_r) \psi + \operatorname{tr}(P_r) N(\psi).
 \end{aligned}$$

Then, from equations (2.11), (2.12) and (2.13), we obtain

$$\begin{aligned}
 \operatorname{div}_M P_r \nabla f_V &= \left\{ \binom{n}{r+1} (n H H_{r+1} - (n-r-1) H_{r+2}) - c(r+1) \binom{n}{r+1} H_r \right\} f_V \\
 &\quad - (r+1) \binom{n}{r+1} H_r N(\psi) + (r+1) \binom{n}{r+1} H_{r+1} \psi \\
 &\quad + \binom{n}{r+1} \langle V, \nabla H_{r+1} \rangle.
 \end{aligned}$$

Hence, taking into account equation (4.8), we get

$$\begin{aligned}
 \operatorname{div}_M P_r \nabla f_V &= \left\{ n \binom{n}{r+1} H H_{r+1} - (n-r-1) \binom{n}{r+1} H_{r+2} \right. \\
 (5.3) \quad &\quad \left. - c(r+1) \binom{n}{r+1} H_r \right\} f_V + (r+1) \binom{n}{r+1} \frac{H_r}{|V|} \frac{\partial \psi}{\partial t} f_V \\
 &\quad + (r+1) \binom{n}{r+1} H_{r+1} \psi + \binom{n}{r+1} \langle V, \nabla H_{r+1} \rangle.
 \end{aligned}$$

Considering  $r = 0$  in Proposition 2.3,

$$(5.4) \quad \begin{aligned} \operatorname{div}_M H_{r+1} V^\top &= \langle \nabla H_{r+1}, V \rangle + H_{r+1} \operatorname{div}_M V^\top \\ &= \langle \nabla H_{r+1}, V \rangle + n\psi H_{r+1} + nH H_{r+1} f_V. \end{aligned}$$

Hence, if we consider the field  $Y = P_r \nabla f_V - \binom{n}{r+1} H_{r+1} V^\top \in \mathfrak{X}(M)$ , from (5.3) and (5.4) we get

$$(5.5) \quad \begin{aligned} \operatorname{div}_M Y &= \left\{ -(n-r-1) \binom{n}{r+1} H_{r+2} - c(r+1) \binom{n}{r+1} H_r \right\} f_V \\ &\quad + (r+1) \binom{n}{r+1} \frac{H_r}{|V|} \frac{\partial \psi}{\partial t} f_V - (n-r-1) \binom{n}{r+1} H_{r+1} \psi. \end{aligned}$$

On the other hand, from Lemma 3.1,

$$(5.6) \quad \operatorname{div}_M P_{r+1} V^\top = (n-r-1) \binom{n}{r+1} \psi H_{r+1} + (n-r-1) \binom{n}{r+1} H_{r+2} f_V.$$

Now, let us consider on  $M^n$  the tangent vector field

$$X = Y + P_{r+1} V^\top.$$

Since  $|A|$  is bounded, from (2.9) we have that  $|P_r|$  is also bounded, for any  $1 \leq r \leq n$ . Thus, from (4.6),

$$|X| \leq \{|Y| + |P_{r+1} V^\top|\} \leq \left\{ |P_r| |A| + \binom{n}{r+1} |H_{r+1}| + |P_{r+1}| \right\} |V^\top| \in \mathcal{L}^1(M),$$

because  $|V^\top| \in \mathcal{L}^1(M)$  and  $H_{r+1}$  is bounded. Moreover, from (5.5) and (5.6),

$$\operatorname{div}_M X = (r+1) \binom{n}{r+1} \left\{ \frac{1}{|V|} \frac{\partial \psi}{\partial t} - c \right\} H_r f_V.$$

Since  $H_r$  does not change sign on  $M^n$  and (5.2) is valid, then  $\operatorname{div}_M X$  does not change sign on  $M^n$ . Thus, Lemma 3.2 gives  $\operatorname{div}_M X = 0$ . Therefore,  $H_r = 0$  on  $M^n$ .

Furthermore, if  $H_{r+1}$  does not change sign, from the Lemma 3.1, we obtain that

$$\operatorname{div}_M P_r V^\top = (n-r) \binom{n}{r} H_{r+1} f_V$$

also does not change sign on  $M^n$ . Here, we observe that  $|P_r V^\top| \leq |P_r| |V^\top| \in \mathcal{L}^1(M)$ . Over again, Lemma 3.2 gives  $\operatorname{div}_M P_r V^\top = 0$ . This implies  $H_{r+1} = 0$ .

Finally, since  $H_r = H_{r+1} = 0$ , Proposition 1 of [19] assures us that  $H_j = 0$  for all  $j \geq r$ , so that  $v_0 \geq n - (r - 1)$ . □

Considering  $r = 1$  in Theorem 5.3, we have the following characterization of totally geodesic spacelike hypersurfaces.

**Corollary 5.4.** *Let  $\bar{M}_c^{n+1}$  be a timelike geodesically complete conformally stationary spacetime, with constant sectional curvature  $c$  and endowed with a closed conformal timelike vector field  $V$ . Let  $x: M^n \rightarrow \bar{M}_c^{n+1}$  be a complete, noncompact spacelike hypersurface with bounded second fundamental form  $A$  and whose mean curvature  $H$  does not change sign and such that  $H_2$  is bounded. If  $|V^\top| \in \mathcal{L}^1(M)$  and the conformal factor  $\psi$  of  $V$  satisfies  $(1/|V|)\partial\psi/\partial t \neq c$ , where  $t \in \mathbb{R}$  denotes the real parameter of the flow of  $v = V/|V|$ , then  $M^n$  is maximal. Moreover, if  $H_2$  also does not change sign, then  $M^n$  is totally geodesic.*

We recall that, according to Example 4.2 of [30], fixed an unit timelike vector  $a \in \mathbb{L}^{n+2}$ ,

$$(5.7) \quad V(p) = a - \langle p, a \rangle p, \quad p \in \mathbb{S}_1^{n+1}$$

is a closed conformal timelike vector field in  $\mathbb{S}_1^{n+1}$  which foliates  $\mathbb{S}_1^{n+1}$  by means of totally umbilical round spheres  $M_\tau = \{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle = \tau\}$ ,  $\tau \in \mathbb{R}$ . The level set given by  $\{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle = 0\}$  defines a round sphere of radius one which is a totally geodesic spacelike hypersurface in  $\mathbb{S}_1^{n+1}$ . According to the terminology established in [5], we will refer to that sphere as the *equator* of  $\mathbb{S}_1^{n+1}$  determined by  $a$ . This equator divides  $\mathbb{S}_1^{n+1}$  into two connected components, the *chronological future* which is given by

$$\{p \in \mathbb{S}_1^{n+1} : \langle a, p \rangle < 0\},$$

and the *chronological past*, given by

$$\{p \in \mathbb{S}_1^{n+1} : \langle a, p \rangle > 0\}.$$

Taking into account the previous discussion, from the proof of Theorem 5.3 we get the following

**Corollary 5.5.** *Let  $x: M^n \rightarrow \mathbb{S}_1^{n+1}$  be a complete spacelike hypersurface with bounded second fundamental form, which lies in the chronological future (past) of an equator of  $\mathbb{S}_1^{n+1}$  determined by an unit timelike vector  $a \in \mathbb{L}^{n+2}$ . Suppose that, for some  $1 \leq r \leq n - 1$ ,  $H_r$  and  $H_{r+1}$  have different (equal) signs and that both of them do not change sign on  $M^n$ . If  $|a^\top| \in \mathcal{L}^1(M)$ , then the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - (r - 1)$ .*

**Proof.** From equation (5.6) applied to the vector field  $V$ , taking into account our restrictions on the signs of  $H_r$  and  $H_{r+1}$  as well as the region of  $\mathbb{S}_1^{n+1}$  where  $M^n$  is

supposed to be contained, we get that  $\operatorname{div}_M P_r V^\top$  does not change sign on  $M^n$ . On the other hand, from (5.7) we have that  $|V^\top| = |a^\top| \in \mathcal{L}^1(M)$ . Therefore, we can reason as in the proof of Theorem 5.3 to conclude that  $H_r = H_{r+1} = 0$  on  $M^n$  and, hence, the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - (r - 1)$ .  $\square$

For a fixed timelike unit vector  $a \in \mathbb{R}_2^{n+2}$ , we have that the closed conformal vector field  $V$  given by

$$V(p) = a + \langle a, p \rangle p, \quad p \in \mathbb{H}_1^{n+1},$$

is timelike on the open set consisting of the points  $p \in \mathbb{H}_1^{n+1}$  such that  $\langle a, p \rangle^2 < 1$ . This open set has two connected components and the distribution on  $\mathbb{H}_1^{n+1}$  orthogonal to  $V$  provides a foliation  $\mathcal{F}(V)$  in this spacetime by means of the totally umbilical spacelike hypersurfaces  $M_\tau = \{p \in \mathbb{H}_1^{n+1} : \langle p, a \rangle = \tau\}$ ,  $-1 < \tau < 1$ , which are isometric to two copies of hyperbolic spaces  $\mathbb{H}^n$  with constant curvature  $-1/(1 + \tau^2)$  (see [30], Example 4.3). In this setting, since the level set given by  $\{p \in \mathbb{H}_1^{n+1} : \langle p, a \rangle = 0\}$  defines a totally geodesic spacelike hypersurface in  $\mathbb{H}_1^{n+1}$  which is isometric to  $\mathbb{H}^n$ , we will refer to that level set as the *equator* of  $\mathbb{H}_1^{n+1}$  determined by  $a$ . In a similar way of the de Sitter space, the *chronological future* of  $\mathbb{H}_1^{n+1}$  determined by  $a$  is given by

$$\{p \in \mathbb{H}_1^{n+1} : -1 < \langle a, p \rangle < 0\},$$

and the *chronological past* is given by

$$\{p \in \mathbb{H}_1^{n+1} : 0 < \langle a, p \rangle < 1\}.$$

We can reason as in the proof of Corollary 5.5 to get the following

**Corollary 5.6.** *Let  $x: M^n \rightarrow \mathbb{H}_1^{n+1}$  be a complete spacelike hypersurface with bounded second fundamental form, which lies in the chronological future (past) of an equator of  $\mathbb{H}_1^{n+1}$  determined by an unit timelike vector  $a \in \mathbb{R}_2^{n+2}$ . Suppose that, for some  $1 \leq r \leq n - 1$ ,  $H_r$  and  $H_{r+1}$  have equal (different) signs and that both of them do not change sign on  $M^n$ . If  $|a^\top| \in \mathcal{L}^1(M)$ , then the index of minimum relative nullity  $\nu_0$  of  $M^n$  is at least  $n - (r - 1)$ .*

**REMARK 5.7.** In [26], Ishihara proved that an  $n$ -dimensional complete maximal spacelike hypersurface immersed in  $\mathbb{H}_1^{n+1}$  must have the squared norm of the second fundamental form bounded from above by  $n$ .

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Henrique F. de Lima  
Departamento de Matemática  
Universidade Federal de Campina Grande  
58429-970 Campina Grande, Paraíba  
Brazil  
e-mail: henrique@dme.ufcg.edu.br

Marco Antonio L. Velásquez  
Departamento de Matemática  
Universidade Federal de Campina Grande  
58429-970 Campina Grande, Paraíba  
Brazil  
e-mail: marcolazarovelasquez@gmail.com