Biliotti, L., Ghigi, A. and Heinzner, P.
Osaka J. Math.
51 (2014), 935–968

COADJOINT ORBITOPES

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(Received April 17, 2012, revised February 12, 2013)

Abstract

We study *coadjoint orbitopes*, i.e. convex hulls of coadjoint orbits of compact Lie groups. We show that up to conjugation the faces are completely determined by the geometry of the faces of the convex hull of Weyl group orbits. We also consider the geometry of the faces and show that they are themselves coadjoint orbitopes. From the complex geometric point of view the sets of extreme points of a face are realized as compact orbits of parabolic subgroups of the complexified group.

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Introduction

Let K be a compact Lie group and let $K \to \mathrm{GL}(V)$ be a finite-dimensional representation. An *orbitope* is by definition the convex envelope of an orbit of K in V (see [23]). An interesting class of orbitopes is given by the convex envelope of coadjoint orbits. We call these *coadjoint orbitopes*. The case of an integral orbit has been studied in [6], where it was realised that a remarkable construction introduced by Bourguignon, Li and Yau [8] in the case of complex projective space can be generalized to arbitrary flag manifolds. This allowed to show that the convex envelope of an integral coadjoint orbit is equivariantly homeomorphic to a Satake–Furstenberg compactification. This homeomorphism is constructed by integrating the momentum map, but unfortunately it is not explicit and its nature is not yet well-understood. On the other

2000 Mathematics Subject Classification. 22E46, 53D26.

The first author was partially supported by GNSAGA of INdAM. The second author was partially supported by GNSAGA of INdAM and by PRIN 2009 MIUR “Moduli, strutture geometriche e loro applicazioni”. The third author was partially supported by DFG-priority program SPP 1388 (Darstellungstheorie).

hand, the Satake–Furstenberg compactifications admit a very precise combinatorial description going back to Satake [24].

The aim of this paper is to give a precise description of the boundary structure of coadjoint orbitopes without the integrality assumption and without relying on the connection with Satake–Furstenberg compactifications.

To a coadjoint orbit $\mathcal{O}$ we associate its convex hull $\hat{\mathcal{O}}$. The aim is to describe the *faces* of $\hat{\mathcal{O}}$ and their extremal points in the sense of convex geometry. If we fix a maximal torus T, there is another convex set associated to $\mathcal{O}$, namely the Kostant polytope P, which is the convex hull of a Weyl group orbit in $\mathfrak{t}$. Denote by $\mathscr{F}(\hat{\mathcal{O}})$ the faces of $\mathcal{O}$ and by $\mathscr{F}(P)$ the faces of P. K acts on $\mathscr{F}(\hat{\mathcal{O}})$ and the Weyl group W acts on $\mathscr{F}(P)$. In §4 we show the following.

Theorem 1. *If $\sigma \in \mathscr{F}(P)$ and $\sigma^{\perp}$ is the set of vectors in $\mathfrak{t}$ which are orthogonal to σ, then $Z_K(\sigma^{\perp}) \cdot \sigma$ is a face of $\hat{\mathcal{O}}$. Moreover the map $\sigma \mapsto Z_K(\sigma^{\perp}) \cdot \sigma$ passes to the quotients and the resulting map $\mathscr{F}(P)/W \to \mathscr{F}(\hat{\mathcal{O}})/K$ is a bijection.*

During the proof of Theorem 1 we show that every face is *exposed* (see Definition 3). The extremal points of an exposed face form a symplectic submanifold of $\mathcal{O}$, that has been studied since the important work of Duistermaat, Kolk, Varadarajan and Heckman [10, 14]. In §3 we reformulate their results to describe the structure of exposed faces using the momentum map. It follows that every face is itself a coadjoint orbitope (Theorem 25) and that it is stable under a maximal torus (Theorem 27). For $K = \mathrm{SO}(n)$ a proof of Theorem 1 is given in [23, §3.2]. Their proof relies on the representation of these orbitopes as spectrahedra.

The second main result of the paper deals with the complex geometry of $\mathcal{O}$. Consider the Kähler structure on $\mathcal{O}$ and the holomorphic action of $G = K^{\mathbb{C}}$ (see §2).

Theorem 2. *If F is a face of $\hat{\mathcal{O}}$, then* $\operatorname{ext} F \subset \mathcal{O}$ *is a closed orbit of a parabolic subgroup of G. Conversely, if $P \subset G$ is a parabolic subgroup, then it has a unique closed orbit $\mathcal{O}' \subset \mathcal{O}$ and there is a face F such that* $\operatorname{ext} F = \mathcal{O}'$.

In §5 we show that there is a finite stratification of the boundary of $\hat{\mathcal{O}}$ in terms of face types, where the strata are smooth fibre bundles over flag manifolds. In §6 we give a description of the faces in terms of root data, using the formalism of x-connected subset of simple roots developed by Satake [24]. In the last section we prove that if $\mathcal{O}$ is an integral orbit (i.e. it corresponds to a representation), the same holds for $\operatorname{ext} F$ for any faces $F \subset \hat{\mathcal{O}}$.

We think that many other aspects of these orbitopes are worth studying. It would be interesting to find explicity formulae for the volume, the surface area and the Quermassintegrals. Also, in a future paper we plan to study the following class of orbitopes: G is a real semisimple Lie group with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathcal{O}$ is a K-orbit in $\mathfrak{p}$ and $\hat{\mathcal{O}}$ is the convex hull of $\mathcal{O}$. Coadjoint orbitopes correspond to the special case where $G = K^{\mathbb{C}}$ and $\mathfrak{p} = i\mathfrak{k}$.

1. Preliminaries from convex geometry

It is useful to recall a few definitions and results regarding convex sets (see e.g. [25]). Let V be a real vector space and $E \subset V$ a convex subset. The *relative interior* of E, denoted relint E, is the interior of E in its affine hull. If $x, y \in E$, then $[x, y]$ denotes the closed segment from x to y, i.e. $[x, y] := \{(1-t)x + ty \colon t \in [0, 1]\}$. A face F of E is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and $\operatorname{relint}[x, y] \cap F \neq \emptyset$, then $[x, y] \subset F$. The *extreme points* of E are the points $x \in E$ such that $\{x\}$ is a face. If E is compact the faces are closed [25, p. 62]. If F is a face of E we say that relint F is an *open face* of E. A face distinct from E and $\emptyset$ will be called a *proper face*.

Assume for simplicity that a scalar product $\langle\ ,\ \rangle$ is fixed on V and that $E \subset V$ is a *compact* convex subset with nonempty interior.

DEFINITION 3. The *support function* of E is the function

$$h_E \colon V \to \mathbb{R}, \quad h_E(u) = \max_{x \in E} \langle x, u \rangle. \tag{4}$$

If $u \neq 0$, the hyperplane $H(E, u) := \{x \in V \colon \langle x, u \rangle = h_E(u)\}$ is called the *supporting hyperplane* of E for u. The set

$$F_u(E) := E \cap H(E, u)$$

is a face and it is called the *exposed face* of E defined by u or also the *support set* of E for u.

In using the notation $F_u(E)$ we will tacitly assume that the affine span of E is V. Hence by definition an exposed face is proper. We notice that in general not all faces of a convex subsets are exposed. A simple example is given by the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces.

Lemma 5. *If F is a face of a convex set E, then* $\operatorname{ext} F = F \cap \operatorname{ext} E$.

Proof. It is immediate that $F \cap \operatorname{ext} E \subset \operatorname{ext} F$. The converse follows from the definition of a face. □

Lemma 6. *If G is a compact group, V is a representation space of G and $G \cdot x$ is an orbit of G, then* $\operatorname{conv}(G \cdot x)$ *contains a fixed point of G. Moreover any fixed point contained in* $\operatorname{conv}(G \cdot x)$ *lies in* $\operatorname{relint} \operatorname{conv}(G \cdot x)$.

Proof. Set

$$\bar{x} := \int_G g \cdot x \, dg$$

where dg is the normalized Haar measure. Then $\bar{x}$ is G-invariant and belongs $\mathrm{conv}(G \cdot x)$. Now let y be any fixed point of G that lies in $\mathrm{conv}(G \cdot x)$. By Theorem 8 there is a unique face $F \subset \mathrm{conv}(G \cdot x)$ such that y belongs to $\mathrm{relint}\, F$. Since $\mathrm{conv}(G \cdot x)$ is G-invariant and y is fixed by G, it follows that $a \cdot F = F$ for any $a \in G$. So F is G-invariant, and hence also $\mathrm{ext}\, F$ is G-invariant. Since $\mathrm{ext}\, F \subset \mathrm{ext}(\mathrm{conv}(G \cdot x)) \subset G \cdot x$, it follows that $\mathrm{ext}\, F = G \cdot x$ and hence that $F = \mathrm{conv}(G \cdot x)$. □

Proposition 7. *If $F \subset E$ is an exposed face, the set $C_F := \{u \in V : F = F_u(E)\}$ is a convex cone. If G is a compact subgroup of $O(V)$ that preserves both E and F, then C_F contains a fixed point of G.*

Proof. Let $u_1, u_2 \in C_F$ and $\lambda_1, \lambda_2 \geq 0$ and set $u = \lambda_1 u_1 + \lambda_2 u_2$. We need to prove that if at least one of λ_1, λ_2 is strictly positive, then $F = F_u(E)$. Assume for example that $\lambda_1 > 0$. It is clear that $h_E(u) \leq \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2)$. If $x \in F$, then

$$\langle x, u \rangle = \lambda_1 \langle x, u_1 \rangle + \lambda_2 \langle x, u_2 \rangle = \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2).$$

Hence $h_E(u) = \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2)$ and $F \subset F_u(E)$. Conversely, if $x \in F_u(E)$, then

$$0 = h_E(u) - \langle x, u \rangle = \lambda_1 (h_E(u_1) - \langle x, u_1 \rangle) + \lambda_2 (h_E(u_2) - \langle x, u_2 \rangle).$$

Since $\lambda_1 > 0$ we get $h_E(u_1) - \langle x, u_1 \rangle = 0$, so $x \in F_{u_1}(E) = F$. Thus $F = F_u(E)$. This proves the first fact. To prove the second, pick any vector $u \in C_F$ and apply the previous lemma to the orbit $G \cdot u \subset C_F$: this yields a G-invariant $\bar{u} \in C_F$. □

Theorem 8 ([25, p. 62])**.** *If E is a compact convex set and F_1, F_2 are distinct faces of E then* $\mathrm{relint}\, F_1 \cap \mathrm{relint}\, F_2 = \emptyset$. *If G is a nonempty convex subset of E which is open in its affine hull, then $G \subset \mathrm{relint}\, F$ for some face F of E. Therefore E is the disjoint union of its open faces.*

Lemma 9. *If E is a compact convex set and $F \subsetneq E$ is a face, then* $\dim F < \dim E$.

Proof. If $\dim F = \dim E$, then $\mathrm{relint}\, F$ is open in the affine span of E, so $\mathrm{relint}\, F \subset \mathrm{relint}\, E$. By the previous theorem this implies that $F = E$. □

Lemma 10. *If E is a compact convex set and $F \subset E$ is a face, then there is a chain of faces*

$$F_0 = F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$$

which is maximal, in the sense that for any i there is no face of E strictly contained between F_{i-1} and F_i.

Proof. If $F = E$ there is nothing to prove. Otherwise put $F_0 := F$. If there is no face strictly contained between F_0 and E, just set $F_1 = E$. Otherwise we find a chain $F_0 \subsetneq F_1 \subsetneq F_2 = E$. If this is not maximal, we can refine it. Repeting this step we get a chain with $k+1$ elements. Since $\dim F_{i-1} < \dim F_i$, $k \leq n$. Therefore the chain gotten after at most n steps is maximal. □

Lemma 11. *If E is a convex subset of $\mathbb{R}^n$, $M \subset \mathbb{R}^n$ is an affine subspace and $F \subset E$ is a face, then $F \cap M$ is a face of $E \cap M$.*

Proof. If $x, y \in E \cap M$ and $\operatorname{relint}[x, y] \cap F \cap M \neq \emptyset$ then $[x, y] \subset F$ since F is a face, but $[x, y]$ is also contained in M since M is affine. So $[x, y] \subset F \cap M$ as desired. □

2. Coadjoint orbits

Through the paper we will use the following notation. K denotes a compact connected semisimple Lie group with Lie algebra $\mathfrak{k}$. If $T \subset K$ is a maximal torus and $\Pi \subset \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{C})$ is a set of simple roots, the Weyl chamber of $\mathfrak{t}$ corresponding to Π is defined by

$$C^+ := \{v \in \mathfrak{t} \colon -i\alpha(v) > 0 \text{ for any } \alpha \in \Delta_+\}.$$

B is the Killing form of $\mathfrak{k}^{\mathbb{C}}$ and $\langle\ ,\ \rangle = -B|_{\mathfrak{k}\times\mathfrak{k}}$ is a scalar product on $\mathfrak{k}$. By means of $\langle\ ,\ \rangle$ we identify $\mathfrak{k}$ with $\mathfrak{k}^*$.

Lemma 12. *Let $T \subset K$ be a maximal torus, let Δ be the root system of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{C})$ and let $\Pi \subset \Delta$ be a base. Define $H_\alpha \in \mathfrak{t}^{\mathbb{C}}$ by the formula $B(H_\alpha, \cdot) = \alpha(\cdot)$ and choose a nonzero vector $X_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Delta$. For $\alpha \in \Delta_+$ set*

$$u_\alpha := \frac{1}{\sqrt{2}}(X_\alpha - X_{-\alpha}), \quad v_\alpha := \frac{i}{\sqrt{2}}(X_\alpha + X_{-\alpha}).$$

Then it is possible to choose the vectors X_α in such a way that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and so that the set $\{u_\alpha, v_\alpha \mid \alpha \in \Delta_+\}$ be orthonormal with respect to $\langle\ ,\ \rangle = -B$. Moreover for $y \in \mathfrak{t}$

$$[y, u_\alpha] = -i\alpha(y)v_\alpha, \quad [y, v_\alpha] = i\alpha(y)u_\alpha, \quad [u_\alpha, v_\alpha] = iH_\alpha.$$

For a proof see e.g. [18, pp. 353–354]. Set

$$Z_\alpha = \mathbb{R}u_\alpha \oplus \mathbb{R}v_\alpha. \tag{13}$$

Then

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_+} Z_\alpha.$$

If $\mathcal{O}$ is an adjoint orbit of K and $x \in \mathcal{O}$, then

$$T_x\mathcal{O} = \operatorname{Im} \operatorname{ad} x = \bigoplus_{\alpha \in E} Z_\alpha$$

where $E := \{\alpha \in \Delta_+ : \alpha(x) \neq 0\}$. Denote by $v_{\mathcal{O}}$ the vector field on $\mathcal{O}$ defined by $v \in \mathfrak{k}$. Explicitly $v_{\mathcal{O}}(x) = [v, x]$. Since we identify $\mathfrak{k} \cong \mathfrak{k}^*$ we may regard $\mathcal{O}$ as a coadjoint orbit. As such it is equipped with a K-invariant symplectic form ω, named after Kostant, Kirillov and Souriau, and defined by the following rule. For $u, v \in \mathfrak{k}$

$$\omega_x(u_{\mathcal{O}}(x), v_{\mathcal{O}}(x)) := \langle x, [u, v] \rangle.$$

See e.g. [17, p. 5]. ω is a K-invariant symplectic form on $\mathcal{O}$ and the inclusion $\mathcal{O} \hookrightarrow \mathfrak{k}$ is the momentum map.

If $T \subset K$ is a maximal torus, we denote by $W(K, T)$ or simply by W the Weyl group of (K, T). We let $\pi : \mathfrak{k} \to \mathfrak{t}$ denote the orthogonal projection with respect to the scalar product $\langle \ , \ \rangle = -B$. Its restriction to $\mathcal{O}$ is denoted by $\Phi_T : \mathcal{O} \to \mathfrak{t}$; it is the momentum map for the T-action on $\mathcal{O}$. $P := \Phi_T(\mathcal{O})$ is the momentum polytope. The following convexity theorem of Kostant [20] is the basic ingredient in the whole theory.

Theorem 14 (Kostant)**.** *Let K be a compact connected Lie group, let $T \subset K$ be a maximal torus and let $\mathcal{O}$ be a coadjoint orbit. Then P is a convex polytope,* $\operatorname{ext} P = \mathcal{O} \cap \mathfrak{t}$ *and* $\operatorname{ext} P$ *is a unique W-orbit.*

There is a unique K-invariant complex structure J on $\mathcal{O}$ such that ω be a Kähler form. It can be described as follows (see [16, p. 113] for more information). Fix a maximal torus T and a system of positive roots in such that a way x belongs to the closure of the positive Weyl chamber. Then the complex structure on $T_x\mathcal{O}$ is given by the formula

$$Ju_\alpha = v_\alpha.$$

Set $G = K^{\mathbb{C}}$. The action of K on $\mathcal{O}$ extends to an action $G \times \mathcal{O} \to \mathcal{O}$ which is holomorphic. If $v_{\mathcal{O}}$ denotes the fundamental vector field induced by $v \in \mathfrak{g} = \mathfrak{k}^{\mathbb{C}}$, this implies that

$$(iv)_{\mathcal{O}} = Jv_{\mathcal{O}}.$$

Let

$$\mathfrak{b}_- := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$$

denote the negative Borel subalgebra and let B_- be the corresponding Borel subgroup. The following lemma is well-known.

Lemma 15. *Let $T \subset K$ be a maximal torus and let Δ_+ be a set of positive roots. If $x \in \mathcal{O} \cap \mathfrak{t}$, then $x \in \overline{C^+}$ if and only if B_- is contained in the stabilizer G_x.*

3. Group theoretical description of the faces

In this section we prove that all the faces of a coadjoint orbitope are coadjoint orbitopes and are exposed. These facts will be used throughout the rest of the paper.

Let $\mathcal{O} \subset \mathfrak{k}$ be a coadjoint orbit of K. The *orbitope* $\hat{\mathcal{O}}$ is by definition the convex hull of $\mathcal{O}$.

Lemma 16. $\operatorname{ext} \hat{\mathcal{O}} = \mathcal{O}$. *Moreover for any face $F \subset \hat{\mathcal{O}}$,* $\operatorname{ext} F = F \cap \mathcal{O}$.

Proof. This fact is common to all orbitopes [23, Proposition 2.2]. By construction $\operatorname{ext} \hat{\mathcal{O}} \subset \mathcal{O}$. On the other hand $\mathcal{O}$ lies on a sphere, hence all points of $\mathcal{O}$ are exposed extreme points. This proves the first assertion. The second follows from the first and from Lemma 5. □

A submanifold $M \subset \mathbb{R}^n$ is called *full* if it is not contained in any proper affine subspace of $\mathbb{R}^n$.

Lemma 17. *Let K be a compact connected semisimple Lie group and let $\mathcal{O} \subset \mathfrak{k}$ be a coadjoint orbit. The orbit $\mathcal{O}$ is full if and only if every simple factor of K acts nontrivially on $\mathcal{O}$.*

Proof. Fix $x \in \mathcal{O}$. Let M denote the affine hull of $\mathcal{O}$ in $\mathfrak{k}$ and let V be the associated linear subspace, i.e. $M = x + V$. We claim that M contains the origin. Since $\mathcal{O}$ is K-invariant, so are M and V. Hence V is an ideal and $V^\perp$ is an ideal as well. Write $x = x_0 + x_1$, with $x_0 \in V$ and $x_1 \in V^\perp$. For any $g \in K$, $gx - x \in V$, $gx_0 - x_0 \in V$ and $gx_1 - x_1 \in V^\perp$. So $gx_1 - x_1 \in V \cap V^\perp$, i.e. $gx_1 = x_1$. This means that x_1 is a fixed point of the adjoint action. Since K is semisimple, $x_1 = 0$, $x \in V$ and $M = V$ as desired. Let K_i, $i = 1, \dots, r$ be the simple factors of K. Since V is an ideal, $V = \bigoplus_{i \in I} \mathfrak{k}_i$ for some subset I of $\{1, \dots, r\}$. It is clear that $\mathfrak{k}_j \cap V = \{0\}$ if and only if $[\mathfrak{k}_j, V] = 0$ if and only if K_j acts trivially on $\mathcal{O}$. This proves the first statement. □

Let H be a compact connected Lie group (not necessarily semisimple) and let $\mathcal{O} \subset \mathfrak{h}$ be an orbit. There is a splitting of the algebra

$$\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r \tag{18}$$

where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and $\mathfrak{k}_i$ are simple ideals. Let K_i be the closed connected subgroups of H with Lie $K_i = \mathfrak{k}_i$. So $H = Z \cdot K_1 \cdots K_r$, where Z is the connected component of the identity in the center of H. Any two of these subgroups have finite intersection. We can reorder the factors in such a way that K_i acts nontrivially on $\mathcal{O}$ if and only if $1 \leq i \leq q$ for some q between 1 and r. Set

$$L := K_1 \cdots K_q, \quad L' := K_{q+1} \cdots K_r.$$

By construction there is a decomposition

$$H = Z \cdot L \cdot L'. \tag{19}$$

Any two factors in this decomposition have finite intersection.

Lemma 20. *For any $x \in \mathcal{O}$, there is a unique decomposition $x = x_0 + x_1$ with $x_0 \in \mathfrak{z}$ and $x_1 \in \mathfrak{l}$. Moreover*

$$\mathcal{O} = H \cdot x = x_0 + L \cdot x_1,$$

the affine span of $\mathcal{O}$ is $x_0 + \mathfrak{l}$ and x_0 belongs to relint $\hat{\mathcal{O}}$.

Proof. Write $x = x_0 + x_1 + x_2$ with $x_0 \in \mathfrak{z}$, $x_1 \in \mathfrak{l}$ and $x_2 \in \mathfrak{l}'$. Since $L' \cdot x = x$, the component x_2 is fixed by L'. Since L' is semisimple, this forces $x_2 = 0$. It follows immediately that $H \cdot x = x_0 + L \cdot x_1$. By definition all simple factors of L act nontrivially on $L \cdot x_1$, hence the orbit $L \cdot x_1$ is full in $\mathfrak{l}$ by Lemma 17. This proves that $\operatorname{aff}(\hat{\mathcal{O}}) = x_0 + \mathfrak{l}$. Since $\mathcal{O} \subset x_0 + \mathfrak{l}$ and $\mathfrak{l} \perp x_0$, x_0 is the closest point to the origin. Such a point is unique because $\hat{\mathcal{O}}$ is convex. Since $x_0 \in \mathfrak{z}$, it is fixed by H. The last statement follows from Lemma 6. □

The statement about the affine span is equivalent to $L \cdot x_1$ being full in $\mathfrak{l}$. Therefore after possibily replacing K by L and translating by x_0 we can assume for most part of the paper that $\mathcal{O}$ is full.

We are interested in the facial structure of $\hat{\mathcal{O}}$ and we start by considering the structure of exposed faces.

Lemma 21. *Assume that K is a compact connected Lie group, that $H \subset K$ is a connected Lie subgroup of maximal rank and that $\mathcal{O}$ is a coadjoint orbit of K. Then*
a) *$\mathcal{O} \cap \mathfrak{h}$ is a union of finitely many H-orbits*;
b) *if H is the centralizer of a torus, then $\mathcal{O} \cap \mathfrak{h}$ is a symplectic submanifold of $\mathcal{O}$.*

Proof. Let T be a maximal torus of K contained in H. Since $\mathcal{O} \cap \mathfrak{h}$ is an H-invariant subset of H and T is a maximal torus of H we have $\mathcal{O} \cap \mathfrak{h} = H \cdot (\mathcal{O} \cap \mathfrak{t})$. But

$\mathcal{O} \cap \mathfrak{t}$ coincides with an orbit of the Weyl group and is therefore finite. Hence $\mathcal{O} \cap \mathfrak{h}$ is a finite union of H-orbits. This proves the first statement. For the second assume that $H = Z_K(S)$ where $S \subset K$ is a torus. Then $\mathcal{O} \cap \mathfrak{h} = \mathcal{O}^S$ is the set of fixed points of S, hence it is a symplectic submanifold of $\mathcal{O}$. □

We start the analysis of the face structure of $\hat{\mathcal{O}}$ by looking at the esposed faces. At the end of the section we will prove that all faces are exposed.

Let u be a nonzero vector in $\mathfrak{k}$ and let $\Phi_u \colon \mathcal{O} \to \mathbb{R}$ be the function $\Phi_u(x) := \langle x, u\rangle$. Set

$$\operatorname{Max}(\Phi_u) := \left\{ x \in \mathcal{O} : \Phi_u(x) = \max_{\mathcal{O}} \Phi_u \right\}.$$

Φ_u is just the component of the momentum map along u. Then for $x \in \mathcal{O}$ and $u, v \in \mathfrak{k}$

$$d\Phi_u(x)(v_{\mathcal{O}}) = \omega_x(u_{\mathcal{O}}(x), v_{\mathcal{O}}(x)) = \langle x, [u, v]\rangle = \langle [x, u], v\rangle.$$

This implies that $x \in \mathcal{O}$ is a critical point of Φ_u if and only if $x \in \mathfrak{z}_{\mathfrak{k}}(u)$, i.e.

$$\operatorname{Crit}(\Phi_u) = \mathcal{O} \cap \mathfrak{z}_{\mathfrak{k}}(u).$$

Lemma 22. *Let $H = Z_K(u)$ be the centraliser of u in K and let $F_u(\hat{\mathcal{O}})$ be the exposed face of $\hat{\mathcal{O}}$ defined by u. Then*
a) $\operatorname{Max}(\Phi_u)$ *is an H-orbit*;
b) $\operatorname{ext} F_u(\hat{\mathcal{O}}) = \operatorname{Max}(\Phi_u)$, *so* $\operatorname{ext} F_u(\hat{\mathcal{O}})$ *is an H-orbit*;
c) $F_u(\hat{\mathcal{O}}) \subset \mathfrak{z}_{\mathfrak{k}}(u)$.

Proof. By Atiyah theorem [2] the level sets of Φ_u are connected. In particular $\operatorname{Max}(\Phi_u)$ is a connected component of $\operatorname{Crit}(\Phi_u)$. By the previous lemma it is an H-orbit. This proves (i). Let $h_{\hat{\mathcal{O}}}$ denote the support function of $\hat{\mathcal{O}}$, see (4). Since $\langle \cdot, u\rangle$ is a linear function, its maximum on $\hat{\mathcal{O}}$, that is $h_{\hat{\mathcal{O}}}(u)$, is attained at some extreme point, i.e. on $\mathcal{O}$. Hence

$$\max_{\mathcal{O}} \Phi_u = h_{\hat{\mathcal{O}}}(u).$$

By Lemma 16 $\operatorname{ext} F_u(\hat{\mathcal{O}}) = F_u(\hat{\mathcal{O}}) \cap \mathcal{O} = \{x \in \mathcal{O} \colon \langle x, u\rangle = h_{\hat{\mathcal{O}}}(u)\} = \operatorname{Max}(\Phi_u)$. It follows immediately that $F_u(\hat{\mathcal{O}}) = \operatorname{conv}(\operatorname{Max}(\Phi_u))$. Finally (iii) follows from the fact that $\operatorname{Max}(\Phi_u) \subset \operatorname{Crit}(\Phi_u) = \mathcal{O} \cap \mathfrak{z}_{\mathfrak{k}}(u)$. □

Lemma 23. *Fix a maximal torus $T \subset K$, a nonzero vector $u \in \mathfrak{t}$ and a point $x \in \mathcal{O} \cap \mathfrak{t}$. Then $x \in \operatorname{Crit}(\Phi_u)$ and x is a maximum point of Φ_u if and only if there is a Weyl chamber in $\mathfrak{t}$ whose closure contains both x and u.*

Proof. By assumption $x \in \mathfrak{t} \subset \mathfrak{z}_{\mathfrak{k}}(u)$ and $\mathfrak{z}_{\mathfrak{k}}(u) = \mathrm{Crit}(\Phi_u)$. To check the second assertion recall that Φ_u is a Morse–Bott function with critical points of even index (this is Frankel theorem, see e.g. [3, Theorem 2.3, p. 109] or [21, p. 186]) and any local maximum point is an absolute maximum point (see e.g. [3, p. 112]). Therefore x is a maximum point if and only if the Hessian $D^2\Phi_u(x)$ is negative semidefinite. Recall that $T_x\mathcal{O} = \mathrm{Im}\,\mathrm{ad}\,x$ and that

$$f := \mathrm{ad}\,x|_{T_x\mathcal{O}} \colon T_x\mathcal{O} \to T_x\mathcal{O}$$

is invertible. If $w \in T_x\mathcal{O}$, then $w = z_{\mathcal{O}}(x) = [z, x]$ for some $z \in \mathfrak{k}$. The vector z can be chosen (uniquely) inside $T_x\mathcal{O}$, i.e. $z = -f^{-1}(w)$. Set $\gamma(t) := \mathrm{Ad}(\exp tz) \cdot x$. Then $\gamma(0) = x$, $\dot\gamma(0) = [z, x] = w$, $\ddot\gamma(0) = [z, [z, x]]$, so

$$\begin{aligned} D^2\Phi_u(x)(w, w) &= \frac{\mathrm{d}^2}{\mathrm{dt}^2}\bigg|_{t=0} h(\gamma(t)) = \langle \ddot\gamma(0), u\rangle = \langle [z, x], [u, z]\rangle \\ &= \langle w, [u, z]\rangle = -\langle w, \mathrm{ad}\,u \circ f^{-1}(w)\rangle. \end{aligned}$$

(One can prove the same formula much more generally and by a more geometric argument, see [15, Proposition 2.5].) Thus the quadratic form $D^2\Phi_u(x)$ is negative semidefinite if and only if the operator $\mathrm{ad}\,u \circ f^{-1}$ is positive semidefinite. This operator preserves each Z_α and its restriction to Z_α is just multiplication by $\alpha(u)/\alpha(x)$. Hence it is positive semidefinite iff and only iff $\alpha(u)\alpha(x) \geq 0$ for any $\alpha \in \Delta$. This is equivalent to the condition that x and u lie in the closure of some Weyl chamber (see e.g. [14, p. 11]). □

The computation above goes back to the work [10] of Duistermaat, Kolk and Varadarajan and to Heckman's thesis [14].

Lemma 24. *Let $F = F_u(\hat{\mathcal{O}})$ be an exposed face of $\hat{\mathcal{O}}$. Set $S := \overline{\exp(\mathbb{R}u)}$ and $H = Z_K(S)$. Then*
(a) *S is a nontrivial torus and fixes F pointwise,*
(b) $\mathrm{ext}\,F$ *is an adjoint orbit of $H := Z_K(S)$,*
(c) *$F \subset \mathfrak{h}$.*

Proof. Since $u \neq 0$ by the definition of exposed face, S is a nontrivial torus. (b) follows from Lemma 22. Moreover $\mathrm{ext}\,F = \mathrm{Max}(\Phi_u) \subset \mathrm{Crit}(\Phi_u)$. Since Φ_u is the Hamiltonian function of the fundamental vector field on $\mathcal{O}$ associated to u, $\mathrm{ext}\,F$ is fixed by $\exp(\mathbb{R}u)$ hence by S, thus proving (a). Finally $\mathrm{ext}\,F \subset \mathfrak{z}_{\mathfrak{k}}(u) = \mathfrak{h}$ by Lemma 22. □

Theorem 25. *Let F be a proper face of $\hat{\mathcal{O}}$. Then there is a nontrivial torus $S \subset K$ with the following properties*:
(a) *S fixes F pointwise,*

(b) $\operatorname{ext} F$ *is an adjoint orbit of* $H := Z_K(S)$,
(c) $F \subset \mathfrak{h}$,
(d) $g \cdot F = F$ *for any* $g \in H$.

Proof. (d) is a direct consequence of (c). To prove (a)–(c) fix a chain of faces $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = \hat{\mathcal{O}}$, such that for any i there is no face strictly contained between F_{i-1} and F_i. This is possible by Lemma 10. We will prove (a)–(c) by induction on k. If $k = 1$, then F is a maximal proper face. Since any face is contained in an exposed face, F is necessarily exposed. Thus (a)–(c) follow from the previous lemma. We proceed with the induction. Let $k > 1$ and assume that the theorem is proved for faces contained in a maximal chain of length $k-1$. Fix F with a maximal chain as above of length k. By the inductive hypothesis the theorem holds for F_1, so that there is a nontrivial subtorus $S_1 \subset K$ which pointwise fixes F_1. Moreover if we set $H_1 = Z_K(S_1)$ and $\mathfrak{h}_1 = \operatorname{Lie} H_1 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{s}_1)$, then $F_1 \subset \mathfrak{h}_1$ and $\operatorname{ext} F_1$ is an orbit of H_1. In particular if we choose a point $x \in \operatorname{ext} F \subset \operatorname{ext} F_1$, then $\operatorname{ext} F_1 = H_1 \cdot x$. Split $H_1 = Z \cdot L \cdot L'$ with $Z = Z(H_1)^0$ as in (19) and write $x = x_0 + x_1$ as in Lemma 20, with $x_0 \in \mathfrak{z} = \mathfrak{z}(\mathfrak{h}_1)$ and $x_1 \in \mathfrak{l}$, so that $\operatorname{ext} F_1 = x_0 + L \cdot x_1$. The orbit $\hat{\mathcal{O}}' := L \cdot x_1$ is full in $\mathfrak{l}$ and $F' := F_0 - x_0 = F - x_0$ is a maximal face of $\hat{\mathcal{O}}'$. Therefore F' is an exposed face, i.e. there is some $u \in \mathfrak{l}$ such that $F' = F_u(\hat{\mathcal{O}}')$. Set $S_2 := \overline{\exp(\mathbb{R}u)}$. By the previous lemma $\operatorname{ext} F'$ is an orbit of $Z_L(S_2)$. Moreover $x_1 \in \operatorname{ext} F'$, because $x \in \operatorname{ext} F$. Therefore $\operatorname{ext} F' = Z_L(S_2) \cdot x_1$. Since $u \in \mathfrak{l}$ and $\mathfrak{l} \subset \mathfrak{h}_1$, u commutes with $\mathfrak{s}_1$. So S_1 and S_2 commute and generate a torus S. Set $H := Z_K(S)$. If $g \in H$, then g commutes with S_1, hence $g \in H_1$. It follows that $H \subset Z_{H_1}(S_2)$. Conversely, if $g \in Z_{H_1}(S_2)$, then g also commutes with S, so $g \in H$. Thus we get $H = Z_{H_1}(S_2)$. Since $S_2 \subset L$, $Z \cdot L' \subset Z_{H_1}(S_2) = H$ and $H = Z \cdot Z_L(S_2) \cdot L'$. Since $Z \cdot L'$ fixes x_1 this implies that $H \cdot x_1 = Z_{H_1}(S_2) \cdot x_1 = Z_L(S_2) \cdot x_1 = \operatorname{ext} F'$. Since $x_0 \in \mathfrak{z} = \mathfrak{z}(\mathfrak{h}_1)$, we conclude that $\operatorname{ext} F = \operatorname{ext} F' + x_0 = H \cdot x_1 + x_0 = H \cdot x$. This proves (b). Next observe that the previous lemma also ensures that $F' \subset \mathfrak{z}_{\mathfrak{l}}(u)$ and that $\mathfrak{z}_{\mathfrak{l}}(u) \subset \mathfrak{h}$. Since $x_0 \in \mathfrak{h}$ too, we conclude that $F = F' + x_0 \subset \mathfrak{h}$. This proves (c). By definition $\mathfrak{h} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{s})$, so S fixes any point of $\mathfrak{h}$ and in particular it fixes pointwise F. Thus (a) is proved. □

We remark that the inductive argument used in the previous proof does not imply that all faces are exposed, since being an exposed face is not a transitive relation.

Corollary 26. *If* $F \subset \hat{\mathcal{O}}$ *is a face, then* $\operatorname{ext} F$ *is a symplectic submanifold of* $\mathcal{O}$.

Proof. Let S and H be as in Theorem 25. Then $\operatorname{ext} F \subset \mathcal{O} \cap \mathfrak{h}$ is an H-orbit. The result follows directly from Lemma 21. □

Corollary 27. *If* $F \subset \hat{\mathcal{O}}$ *is a face, there is a maximal torus* $T \subset K$ *that preserves* F.

Proof. A maximal torus of H is also a maximal torus of K. □

REMARK 28. The above results shows that every face of $\hat{\mathcal{O}}$ is a coadjoint orbitope for some subgroup $H \subset K$. One might wonder if a similar property holds for all orbitopes: if a group K acts linearly on V and $\mathcal{O}$ is an orbit, one might ask if every face of $\hat{\mathcal{O}}$ is an orbit of some subgroup of K. The answer is negative in general. Counterexamples are provided e.g. by convex envelopes of orbits of S^1 acting linearly on $\mathbb{R}^n$. These are called *Carathéodory orbitopes*, since their study goes back to [9]. In [26] there is a thorough study of the 4-dimensional case (see also [4]). It turns out (see Theorem 1 in [26]) that there are many 1-dimensional faces whose extreme sets are not orbits of any subgroup of K. Therefore the fact that we just established, namely that the faces of a coadjoint orbitope are all orbitopes of the same kind, seems to be a rather remarkable property.

The subgroups S and H in Theorem 25 are not unique. Later in Theorem 38 (d) we will show that there is a canonical choice. Now we wish to show that one can always assume that $S = Z(H)^0$.

Corollary 29. *In* Theorem 25 *we can assume that* $Z(H)$ *acts trivially on* F *and that* $S = Z(H)^0$.

Proof. Let $p\colon \mathfrak{k} \to \mathfrak{h}$ denote the orthogonal projection. H acts on $\mathcal{O}$ in a Hamiltonian way with momentum map $p|_{\mathcal{O}}$. If $x \in \operatorname{ext} F$, then $H \cdot x = \operatorname{ext} F$ is a symplectic orbit by Corollary 26. Therefore $H_x = H_{p(x)}$, see e.g. [13, Theorem 26.8, p. 196]. Since $p(x) \in \mathfrak{h}$, the stabilizer $H_{p(x)}$ contains the center of H. So $Z(H) \subset H_x$. This proves the first statement. Next set $S' = Z(H)^0$. Then S' is a positive dimensional torus. To prove the second fact it is enough to show that changing S to S' does not change the centralizer, i.e. that $H = Z_K(S')$. Since $S' \subset Z(H)$, H and S' commute, so $H \subset Z_K(S')$. On the other hand H is the centralizer of S, so $S \subset S'$, and $Z_K(S') \subset Z_K(S) = H$. Therefore indeed $H = Z_K(S')$. □

The following is an immediate consequence of Lemma 20.

Lemma 30. *Let* F *be a face of* $\hat{\mathcal{O}}$, $H \subset K$ *a connected subgroup and assume that* $\operatorname{ext} F$ *is an* H*-orbit and that* $F \subset \mathfrak{h}$. *Decompose* H *as in* (19), *i.e.* L *is the product of the simple factors of* (H, H) *that act nontrivially on* F, *while* L' *is the product of those factors that act trivially. If* $x \in \operatorname{ext} F$, *then* $x = x_0 + x_1$ *with* $x_0 \in \mathfrak{z}$ *and* $x_1 \in \mathfrak{l}$. *Moreover*

$$\operatorname{ext} F = H \cdot x = x_0 + L \cdot x_1$$

and $L \cdot x_1 \subset \mathfrak{l}$ *is full.*

Now we fix a maximal torus T and we use the notation of p. 940. We wish to show that the T-stable faces of $\hat{\mathcal{O}}$ and the faces of the momentum polytope are in bijective correspondence. This will be used to prove that all faces of $\hat{\mathcal{O}}$ are exposed. The relation between the T-invariant faces of $\hat{\mathcal{O}}$ and the faces of P will be studied further in the next section.

The following lemma is a consequence of Kostant convexity theorem. See [11, Lemma 7] for a proof in the context of polar representations.

Lemma 31. *Let K be a compact connected Lie group, $T \subset K$ be a maximal torus and let $\pi : \mathfrak{k} \to \mathfrak{t}$ be the orthogonal projection. Then*
(i) *If $E \subset \mathfrak{k}$ is a K-invariant convex subset, then $E \cap \mathfrak{t} = \pi(E)$.*
(ii) *If $A \subset \mathfrak{t}$ is a W-invariant convex subset, then $K \cdot A$ is convex and $\pi(K \cdot A) = A$.*

Lemma 32. *Let $T \subset K$ be a maximal torus and let $F \subset \hat{\mathcal{O}}$ be a nonempty T-invariant face. Set $\sigma := \pi(\operatorname{ext} F)$. Then $\sigma = \pi(F) = F \cap \mathfrak{t}$. Moreover σ is a nonempty face of the momentum polytope P.*

Proof. We prove this lemma in the same way as Kostant theorem is deduced from the Atiyah–Guillemin–Sternberg theorem. By Corollary 26 $\operatorname{ext} F$ is a symplectic submanifold of $\mathcal{O}$. T acts on $\operatorname{ext} F$ with momentum map given by the restriction of π to $\operatorname{ext} F$. By definition $\sigma = \pi(\operatorname{ext} F)$ is the momentum polytope for this action. By the Atiyah–Guillemin–Sternberg theorem

$$\sigma = \operatorname{conv} \pi((\operatorname{ext} F)^T) = \operatorname{conv} \pi(\operatorname{ext} F \cap \mathfrak{t}).$$

This means first of all that σ is convex. Since π is linear it follows that $\pi(F) = \operatorname{conv} \pi(\operatorname{ext} F) = \sigma$. On the other hand, since $\pi(\operatorname{ext} F \cap \mathfrak{t}) = \operatorname{ext} F \cap \mathfrak{t}$, we get

$$\operatorname{ext} \sigma \subset \operatorname{ext} F \cap \mathfrak{t} \quad \sigma \subset F \cap \mathfrak{t}. \tag{33}$$

Conversely $F \cap \mathfrak{t} = \pi(F \cap \mathfrak{t}) \subset \pi(F)$. Since $\pi(F) = \sigma$ we get indeed $F \cap \mathfrak{t} = \sigma$. Thus the first part is proven. In particular we can apply this with $F = \hat{\mathcal{O}}$, and we get that $P = \hat{\mathcal{O}} \cap \mathfrak{t}$. That $F \cap \mathfrak{t}$ is a face of P now follows directly from Lemma 11 without assuming that F be T-invariant. To check that $\sigma \neq \emptyset$, recall that if a torus acts on a compact Kähler manifold in a Hamiltonian way, then it has some fixed points. So $(\operatorname{ext} F)^T = \operatorname{ext} F \cap \mathfrak{t} \neq \emptyset$ and $\sigma \neq \emptyset$. □

Recall the following basic property of Hamiltonian actions (see e.g. [12, Theorem 3.6]).

Lemma 34. *Let M be a symplectic manifold and let T be a torus that acts on M in a Hamiltonian way with momentum map $\Phi \colon M \to \mathfrak{t}$. If $S \subset T$ is a subtorus that acts trivially on M, then $\Phi(M)$ is contained in a translate of $\mathfrak{s}^{\perp}$.*

If $M \subset \mathbb{R}^n$ is an affine subspace, the linear subspace parallel to M is called the *direction* of M [5, p. 42]. Denote by $\sigma^\perp$ the orthogonal space in $\mathfrak{t}$ to the direction of σ.

Lemma 35. *Let F be a proper face of $\hat{\mathcal{O}}$, let H be a subgroup as in* Theorem 25 *and let T be a maximal torus of H. Then $\sigma := F \cap \mathfrak{t}$ is a proper face of P and* $\operatorname{ext} F$ *is a $Z_K(\sigma^\perp)$-orbit.*

Proof. By assumption $\operatorname{ext} F$ is an H-orbit. Hence it is a connected component of $\mathcal{O} \cap \mathfrak{h}$. In particular by Lemma 21 it is a symplectic submanifold of $\mathcal{O}$. By Corollary 29 $S := Z(H)^0$ is a nontrivial subtorus of T, which acts trivially on $\operatorname{ext} F$. The momentum map for the T-action is the restriction of π. So by Lemma 34 $\sigma = \pi(\operatorname{ext} F)$ is contained in a translate of $\mathfrak{s}^\perp$, i.e. $\mathfrak{s} \subset \sigma^\perp$. It follows that $Z_K(\sigma^\perp) \subset Z_K(\mathfrak{s}) = Z_K(S) = H$. Next consider the decomposition (19). We know that $L \cdot x_1 \subset \mathfrak{l}$ is a full orbit. Denoting by $\operatorname{aff}(\,\cdot\,)$ the affine span

$$\operatorname{aff} F = \operatorname{aff}(\operatorname{ext} F) = x_0 + \mathfrak{l}.$$

Since $x_0 \in \mathfrak{t}$, $(x_0 + \mathfrak{l}) \cap \mathfrak{t} = x_0 + (\mathfrak{l} \cap \mathfrak{t})$ and

$$\operatorname{aff} \sigma = \operatorname{aff}(F \cap \mathfrak{t}) \subset (\operatorname{aff} F) \cap \mathfrak{t} = x_0 + (\mathfrak{l} \cap \mathfrak{t}).$$

Since $\mathfrak{l}$ is an ideal of $\mathfrak{k}$, it is the direct orthogonal sum of $\mathfrak{l} \cap \mathfrak{t}$ and some Z_α, see (13). Hence $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{t}) \overset{\perp}{\oplus} (\mathfrak{l} \cap \mathfrak{t}^\perp)$. It follows that $\pi(\mathfrak{l}) = \mathfrak{l} \cap \mathfrak{t}$ and also, since $x_0 \in \mathfrak{t}$, that $\pi(x_0 + \mathfrak{l}) = x_0 + (\mathfrak{l} \cap \mathfrak{t})$. So

$$x_0 + (\mathfrak{l} \cap \mathfrak{t}) = \pi(x_0 + \mathfrak{l}) = \pi(\operatorname{aff} F) \subset \operatorname{aff}(\pi(F)) = \operatorname{aff} \sigma.$$

From these two inclusions we get that $\operatorname{aff} \sigma = x_0 + (\mathfrak{l} \cap \mathfrak{t})$. Therefore $\sigma^\perp$ is the orthogonal complement of $\mathfrak{l} \cap \mathfrak{t}$ in $\mathfrak{t}$. Since $\mathfrak{t} = \mathfrak{z} \overset{\perp}{\oplus} (\mathfrak{l} \cap \mathfrak{t}) \overset{\perp}{\oplus} (\mathfrak{l}' \cap \mathfrak{t})$, we get $\sigma^\perp = \mathfrak{z} \oplus (\mathfrak{l}' \cap \mathfrak{t}) \subset \mathfrak{z} \oplus \mathfrak{l}'$. So $[\mathfrak{l}, \sigma^\perp] \subset [\mathfrak{l}, \mathfrak{z} \oplus \mathfrak{l}'] = 0$ and $L \subset Z_K(\sigma^\perp)$. From the inclusions $L \subset Z_K(\sigma^\perp) \subset H$ and the fact that $L \cdot x = H \cdot x = \operatorname{ext} F$ for any $x \in \operatorname{ext} F$ we immediately get $Z_K(\sigma^\perp) \cdot x = \operatorname{ext} F$. We already know (from Lemma 32) that σ is a nonempty face of P. By Theorem 25 $\mathfrak{z} \neq \{0\}$, so $\sigma^\perp \neq \{0\}$, $\operatorname{aff} \sigma \neq \mathfrak{t}$ and $\sigma \subsetneq P$. This shows that σ is a proper face. □

Corollary 36. *Let F_1, F_2 be a proper faces of $\hat{\mathcal{O}}$, let H_1, H_2 be corresponding subgroups as in* Theorem 25 *and let T be a maximal torus of K which is contained in both H_1 and H_2. If $F_1 \cap \mathfrak{t} = F_2 \cap \mathfrak{t}$, then $F_1 = F_2$.*

Proof. Set $\sigma := F_i \cap \mathfrak{t}$. Recall from (33) that $\operatorname{ext} \sigma \subset \operatorname{ext} F_i$ and pick $x \in \operatorname{ext} \sigma$. Then we can apply the previous lemma to both faces and we get $\operatorname{ext} F_1 = Z_K(\sigma^\perp) \cdot x = \operatorname{ext} F_2$. The result follows. □

If $F \subset \hat{\mathcal{O}}$ is a face set

$$H_F := \{g \in K : gF = F\}, \quad Z_F := Z(H_F)^0, \tag{37}$$
$$C_F := \{u \in \mathfrak{k} \colon F = F_u(\hat{\mathcal{O}})\}.$$

The following is the main result of this section.

Theorem 38. *All proper faces of $\hat{\mathcal{O}}$ are exposed. More precisely, if F is a proper face $F \subset \hat{\mathcal{O}}$, then*

(a) *if $T \subset H_F$ is a maximal torus, $u \in \mathfrak{t}$ and $F \cap \mathfrak{t} = F_u(P)$, then $F = F_u(\hat{\mathcal{O}})$;*

(b) *there is a vector $u \in \mathfrak{z}_F$ such that $F = F_u(\hat{\mathcal{O}})$;*

(c) *if $u \in C_F \cap \mathfrak{z}_F$, then $H_F = Z_K(u)$ (in particular H_F is connected and Z_F has positive dimension);*

(d) *the subgroup H_F satisfies* (a)–(d) *of* Theorem 25.

Proof. We start by proving (a) under the assumption that the maximal torus T is contained in some subgroup H that has the properties listed in Theorem 25. By Lemma 35 $\sigma := F \cap \mathfrak{t} = F \cap P$ is a proper face of P. Since all faces of a polytope are exposed [25, p. 95], there is a vector $u \in \mathfrak{t}$ such that σ equals the exposed face of P defined by u, i.e. $\sigma = F_u(P)$. Since $u \in \mathfrak{t}$ and $P = \pi(\mathcal{O})$, $h_P(u) = \max_{x \in \mathcal{O}} \langle u, x \rangle = h_{\hat{\mathcal{O}}}(u)$. Set $F' := F_u(\hat{\mathcal{O}})$. F' is a T-invariant face since u is fixed by T. We wish to show that $F = F'$. The inclusion $F \subset F'$ is immediate. Indeed if $x \in F$, then $\pi(x) \in \sigma$, so $\langle x, u \rangle = h_P(u) = h_{\hat{\mathcal{O}}}(u)$. It is also immediate that $F' \cap \mathfrak{t} = \sigma$. So we have two faces F and F' with $F \cap \mathfrak{t} = F' \cap \mathfrak{t} = \sigma$. Set $H' := Z_K(u)$. By Lemma 22 $\operatorname{ext} F' = \operatorname{Max}(\Phi_u)$ is an H'-orbit and H' satisfies (a)–(d) of Theorem 25 for F'. Clearly $T \subset H'$ since $u \in \mathfrak{t}$, and by hypothesis also $T \subset H$. We can therefore apply Corollary 36 and we get $F = F'$. In particular $F = F_u(\hat{\mathcal{O}})$ is an exposed face. We have thus proved (a) under the assumption that $T \subset H$ for some H as in Theorem 25. Next we show that the vector u can be chosen inside $\mathfrak{z}_F$. The subgroup $H_F \subset K$ is compact and preserves both $\hat{\mathcal{O}}$ and F. By Proposition 7 there is a vector $u \in C_F$ that is fixed by H_F. Note that H_F is of maximal rank since $H \subset H_F$. If T is a maximal torus contained in H_F, then u is is fixed by T, so $u \in \mathfrak{t} \subset \mathfrak{h}_F$. It follows that $u \in \mathfrak{h}_F$ and since H_F fixes u it follows that $u \in \mathfrak{z}_F$. Thus (b) is proved. To prove (c) assume that $u \in \mathfrak{z}_F$ and that $F = F_u(\hat{\mathcal{O}})$. Then $H_F \subset Z_K(u)$ since $u \in \mathfrak{z}_F$. On the other hand $\operatorname{ext} F = F_u(\hat{\mathcal{O}}) \cap \mathcal{O} = \operatorname{Max}(\Phi_u) = Z_K(u) \cdot x$ by Lemma 22. Therefore $Z_K(u)$ preserves F and therefore $Z_K(u) \subset H_F$ by definition. So $H_F = Z_K(u)$ and (c) is proved. (d) follows from Lemma 24 and the fact that $H_F = Z_K(u)$. Now we know that H_F itself has the properties of Theorem 25. Hence (a) holds for any torus $T \subset H_F$. □

REMARK 39. In general the faces of an orbitope are not necessarily exposed. For example 4-dimensional Carathéodory orbitopes have non-exposed faces, see [26, Theorem 1 (5b)] (note that the author uses the word "facelet" for face and "face" for exposed face). It is important to understand whether an orbitope has only exposed faces. Indeed this is Question 1 in [23]. The previous theorem shows that this is always the case for coadjoint orbitopes.

Corollary 40. *If $\mathcal{O}' \subset \mathcal{O}$ is a smooth submanifold, then* $\operatorname{conv}(\mathcal{O}')$ *is a face of* $\hat{\mathcal{O}}$ *if and only if there is a vector u such that* $\mathcal{O}' = \operatorname{Max}(\Phi_u)$.

Proof. Set $F = \operatorname{conv}(\mathcal{O}')$. From the fact that $\mathcal{O}$ is contained in a sphere, it follows as in Lemma 16 that $\operatorname{ext} F = \mathcal{O}'$. Therefore the statement follows immediately from Lemma 22 and the fact that every face of $\hat{\mathcal{O}}$ is exposed. □

This is a first characterization of the submanifolds that appear as $\operatorname{ext} F$ for some face F. In §7 we will see that this characterization becomes much more transparent using the complex structure of $\mathcal{O}$. An explicit characterization in terms of root data will be given in §6.

Various results about the faces have been established using *some* subgroup H satisfying the properties stated in Theorem 25. Now we know that H_F does satisfy these properties. Hence we can state those results more cleanly. This is done in Theorem 42 below. Next in Lemma 44 we will make precise the possible freedom in the choice of the group H. First of all decompose H_F as in Lemma 30:

$$H_F = Z_F \cdot K_F \cdot K'_F. \tag{41}$$

Z_F is defined in (37), K_F is the product of the simple factors of (H_F, H_F) that act nontrivially on $\operatorname{ext} F$ and K'_F is the product of the remaining factors.

Theorem 42. *Let $T \subset K$ be a maximal torus.*
(a) *If $F \subset \hat{\mathcal{O}}$ is a proper T-invariant face, then $\sigma := F \cap \mathfrak{t} = \pi(F) = \pi(\operatorname{ext} F)$ is a proper face of the momentum polytope P and $\operatorname{ext} F$ is a $Z_K(\sigma^\perp)$-orbit.*
(b) *If F_1 and F_2 are T-invariant proper faces, then $F_1 \subset F_2$ if and only if $F_1 \cap \mathfrak{t} \subset F_2 \cap \mathfrak{t}$.*
(c) *If F_1 and F_2 are T-invariant proper faces, then $F_1 = F_2$ if and only if $F_1 \cap \mathfrak{t} = F_2 \cap \mathfrak{t}$.*
(d) *If $x \in \operatorname{ext} F$, then $x = x_0 + x_1$ with $x_0 \in \mathfrak{z}_F$ and $x_1 \in \mathfrak{k}'_F$. Moreover*

$$\operatorname{ext} F = x_0 + K_F \cdot x_1 \tag{43}$$

and $K_F \cdot x_1 \subset \mathfrak{k}_F$ is full.

Proof. If F is T-stable, then $T \subset H_F$. So (a) follows from Lemma 35. (b) Set $\sigma_i := F_i \cap \mathfrak{t}_i$. If $F_1 \subset F_2$, then clearly $\sigma_1 \subset \sigma_2$. To prove the converse, assume that

$\sigma_1 \subset \sigma_2$ and pick $x \in \sigma_1$. Then $Z_K(\sigma_1^\perp) \subset Z_K(\sigma_2^\perp)$ and $\operatorname{ext} F_i = Z_K(\sigma_i^\perp) \cdot x$. Thus $\operatorname{ext} F_1 \subset \operatorname{ext} F_2$. (c) follows immediately. (d) is just Lemma 30 stated for $H = H_F$. □

Lemma 44. *If $F \subset \hat{\mathcal{O}}$ is a face and $H \subset K$ is a connected subgroup, such that $F \subset \mathfrak{h}$ and $\operatorname{ext} F$ is an H-orbit, then $K_F \subset H \subset H_F$ and $K_F = L$.*

Proof. Necessarily $F \neq \emptyset$. Since $\operatorname{ext} F$ is an H-orbit, H preserves $\operatorname{ext} F$, hence F. So $H \subset H_F$ by definition (37). To prove the opposite inclusion, split as usual $H = Z \cdot L \cdot L'$ and write $x = x_0 + x_1$ as in Lemma 30. The orbit $L \cdot x_1 \subset \mathfrak{l}$ is full, so the affine span of F is $x_0 + \mathfrak{l}$. Since also H_F has the properties stated in Theorem 25 we can repeat the same reasoning for H_F instead of H. Thus we get that the affine span of F is $x_0 + \mathfrak{k}_F$. Therefore $\mathfrak{l} = \mathfrak{k}_F$. So L and K_F are connected subgroups of K with the same Lie algebra and therefore coincide. This implies $K_F = L \subset H$. □

EXAMPLE 45. Set $\mathfrak{k} = \mathfrak{su}(n+1) = \{X \in \mathfrak{gl}(n+1, \mathbb{C})\colon X + X^* = 0,\ \operatorname{Tr}(X) = 0\}$, $\mathcal{H} = \{X \in \mathfrak{gl}(n+1)\colon X = X^*\}$ and $\mathcal{H}_1 = \{X \in \mathcal{H}\colon Tr(X) = 1\}$. We identify $\mathfrak{su}(n+1)$ with $\mathcal{H}_1$ using the map

$$\varphi\colon \mathfrak{su}(n+1) \to \mathcal{H}_1 \quad \varphi(X) = iX + \frac{\mathrm{Id}_{n+1}}{n+1}.$$

The vector space of Hermitian matrices is endowed with an invariant scalar product, given by $\langle A, B\rangle = \operatorname{Tr}(AB)$. Let $\mathcal{O} \subset \mathfrak{su}(n+1)$ be the coadjoint orbit corresponding to $\mathbb{P}^n(\mathbb{C})$ endowed with the Fubini-Study metric. Then $\mathcal{O}' = \varphi(\mathcal{O})$ is the set of orthogonal projectors onto lines, i.e.

$$\mathcal{O}' = \{A \in \mathcal{H}\colon A^2 = A, \operatorname{rank}(A) = 1\}.$$

Using the spectral theorem it is easy to check that

$$\mathcal{O}' = \{A \in \mathcal{H}_1\colon A \geq 0,\ \operatorname{rank}(A) = 1\}$$

and

$$\hat{\mathcal{O}}' = \{A \in \mathcal{H}_1\colon A \geq 0\}.$$

Given a Hermitian matrix $u \neq 0$ we wish to study the face

$$F := F_u(\hat{\mathcal{O}}').$$

We can assume that u be tangent to $\mathcal{H}_1$, i.e. $\operatorname{Tr} u = 0$. Let

$$\mathbb{C}^{n+1} = V_1 \oplus \cdots \oplus V_s$$

be its eigenspace decomposition, i.e. $u_{|V_i} = \mu_i \mathrm{Id}_{V_i}$. Since $u \neq 0$ and $\mathrm{Tr}\, u = 0$ $s > 1$. We assume $\mu_1 < \mu_2 < \cdots < \mu_s$. Let

$$\Phi\colon \mathcal{O}' \to \mathbb{R}, \quad \Phi_u(x) = \langle u, x\rangle$$

be the height function with respect to u. The critical set of Φ_u is $\{A \in \mathcal{O}'\colon [A, u] = 0\}$. Since $[A, u] = 0$ if and only if $A(V_i) \subset V_i$, it follows that this is the set of projectors onto lines that are contained in some of the V_i's, i.e. $\mathrm{Crit}(\Phi_u) = \mathbb{P}(V_1) \sqcup \cdots \sqcup \mathbb{P}(V_s)$. For the same reason

$$Z_{\mathrm{SU}(n+1)}(u) = \mathrm{S}(\mathrm{U}(V_1) \times \cdots \times \mathrm{U}(V_s)).$$

Let v_i be a non zero vector of V_i and let P_{v_i} denote the orthogonal projection onto the complex line $\mathbb{C}v_i$. Then

$$\mathbb{P}(V_i) = Z_{\mathrm{SU}(n+1)}(u) \cdot P_{v_i}.$$

If $A \in \mathrm{Crit}(\Phi_u)$, then

$$\Phi_u(A) = \mu_1 \mathrm{Tr}(A|_{V_1}) + \cdots + \mu_s \mathrm{Tr}(A|_{V_s}).$$

Since $\mathrm{Tr}(A|_{V_i}) \geq 0$ and

$$\sum_{i=1}^{s} \mathrm{Tr}(A|_{V_i}) = \mathrm{Tr}\, A = 1$$

the maximum of Φ_u is equal to μ_s and it is attained exactly on $\mathbb{P}(V_s)$. This means that

$$\begin{aligned} \mathrm{ext}\, F &= \mathrm{Max}(\Phi_u) = \mathbb{P}(V_s) \subset \mathcal{O}', \\ F &= \mathrm{conv}(\mathbb{P}(V_s)) = \{A \in \mathcal{H}_1\colon A \geq 0,\ A|_{V_s^\perp} \equiv 0\}. \end{aligned}$$

So F consists of the operators in $\hat{\mathcal{O}}'$ that are supported on V_s. Notice that $H_F = \mathrm{S}(\mathrm{U}(V_s) \times \mathrm{U}(V_s^\perp))$ and $\mathfrak{z}_F = i\mathbb{R}v$ where v is the Hermitian operator such that

$$v|_{V_s} = \frac{\mathrm{Id}}{\dim V_s}, \quad v|_{V_s^\perp} = -\frac{\mathrm{Id}}{\dim V_s^\perp}.$$

In fact $F = F_v(\hat{\mathcal{O}}')$. In particular in this example C_F is much larger than $\mathfrak{z}_F \cap C_F$. The above computation shows that to each face corresponds a subspace, namely V_s. Vice versa, given a subspace $W \subset \mathbb{C}^{n+1}$, let w be the Hermitian operator such that

$$w|_W = \frac{\mathrm{Id}}{\dim W}, \quad w|_{W^\perp} = -\frac{\mathrm{Id}}{\dim W^\perp}.$$

Then

$$F_w(\hat{\mathcal{O}}) = \{A \in \mathcal{H}_1\colon A \geq 0,\ A|_{W^\perp} = 0\} = \mathrm{conv}(\mathbb{P}(W)).$$

Therefore the faces of $\hat{\mathcal{O}}'$ are in one-to-one correspondence with the subspaces of $\mathbb{C}^{n+1}$.

4. The role of the momentum polytope

In this section we prove Theorem 1. We will start by constructing the inverse of the map considered in Theorem 1 and we will prove in detail that it passes to the quotient. At the end (Theorem 49) we will show that the two maps are inverse to each other.

Consider a full orbit $\mathcal{O} \subset \mathfrak{k}$, a maximal torus $T \subset K$ and the momentum polytope P. In this section we will study in detail the relation between the faces of $\hat{\mathcal{O}}$ and those of P. Denote by $\mathscr{F}(\hat{\mathcal{O}})$ the set of proper faces of $\mathcal{O}$ and by $\mathscr{F}(P)$ the proper faces of the polytope P. If F is a face of $\mathcal{O}$ and $a \in K$, then $a \cdot F$ is still a face, so K acts on $\mathscr{F}(\hat{\mathcal{O}})$. Similarly $W = W(K, T)$ acts on $\mathscr{F}(P)$. We wish to show that $\mathscr{F}(\hat{\mathcal{O}})/K \cong \mathscr{F}(P)/W$.

Lemma 46. *If F is a face of $\mathcal{O}$, there is a T-stable face F' which is conjugate to F, i.e. $F' = a \cdot F$ for some $a \in K$. F' is unique up to conjugation by elements of $N_K(T)$.*

Proof. By Corollary 27 F is preserved by some maximal torus $S \subset K$. There is $a \in K$ such that $S = a^{-1}Ta$. Hence $F' = a \cdot F$ is preserved by T. To prove uniqueness assume that F_1 and F_2 be T-stable faces of $\mathcal{O}$ and that $F_2 = a \cdot F_1$ for some $a \in K$. Then $H_{F_2} = aH_{F_1}a^{-1}$. In particular both T and aTa^{-1} are contained in H_{F_2}, so there is $b \in H_{F_2}$, such that $aTa^{-1} = bTb^{-1}$. Then $w = b^{-1}a \in N_K(T)$ and $w \cdot F_1 = b^{-1}a \cdot F_1 = b^{-1}F_2 = F_2$. □

Define a map

$$\varphi\colon \mathscr{F}(\hat{\mathcal{O}})/K \to \mathscr{F}(P)/W$$

by the following rule: given $[F] \in \mathscr{F}(\hat{\mathcal{O}})$ choose a T-invariant representative F and set $\varphi([F]) := [F \cap \mathfrak{t}]$. By Lemma 32 $F \cap \mathfrak{t}$ is indeed a face of the polytope. By Lemma 46 if F' is T-stable and $[F'] = [F]$ then $F' \cap \mathfrak{t}$ and $F \cap \mathfrak{t}$ are interchanged by some element of W. This shows that the map φ is well-defined.

Now fix a face F of $\hat{\mathcal{O}}$ and a maximal torus $T \subset H_F$. Since $T \cap K_F$ is a maximal torus of K_F and $T \cap K'_F$ is a maximal torus of K'_F, corresponding to the decomposition (41) there is a splitting

$$\mathfrak{t} = \mathfrak{z}_F \oplus (\mathfrak{t} \cap \mathfrak{k}_F) \oplus (\mathfrak{t} \cap \mathfrak{k}'_F).$$

Denote by W_F and W'_F the Weyl groups of $(K_F, K_F \cap T)$ and $(K'_F, K'_F \cap T)$ respectively. W_F and W'_F can be considered as subgroups of W. They commute and have the following sets of invariant vectors:

$$\mathfrak{t}^{W_F} = \mathfrak{z}_F \oplus \mathfrak{k}'_F, \quad \mathfrak{t}^{W'_F} = \mathfrak{z}_F \oplus \mathfrak{k}_F, \quad \mathfrak{t}^{W_F \times W'_F} = \mathfrak{z}_F.$$

Lemma 47. *Let $T \subset K$ be a maximal torus and let F be a nonempty T-invariant face of $\mathcal{O}$. Set $\sigma := F \cap \mathfrak{t}$. Then*
(i) $W_F \times W'_F$ *preserves* σ;
(ii) $F = H_F \cdot \sigma = K_F \cdot \sigma$.

Proof. Recall that $\operatorname{ext} F = x_0 + K_F \cdot x_1$. By Kostant theorem $\sigma = \pi(\operatorname{ext} F) = \pi(x_0 + K_F \cdot x_1) = x_0 + \operatorname{conv}(W_F \cdot x_1) = \operatorname{conv}(W_F \cdot x)$. Hence W_F preserves σ. Moreover $\sigma \subset \mathfrak{z}_F \oplus (\mathfrak{t} \cap \mathfrak{k}_F)$ hence W'_F fixes σ pointwise and (i) follows. Similarly, since $\sigma \subset \mathfrak{z}_F \oplus \mathfrak{k}_F$, $Z_F \cdot K'_F$ fixes σ pointwise. Therefore $H_F \cdot \sigma = K_F \cdot \sigma$. By Lemma 31 $K_F \cdot (\sigma - x_0)$ is convex and the same is true of $x_0 + K_F \cdot (\sigma - x_0) = K_F \cdot \sigma$. So $H_F \cdot \sigma = K_F \cdot \sigma$ is convex. Since $\operatorname{ext} F = H_F \cdot x \subset H_F \cdot \sigma$, it follows that $F \subset H_F \cdot \sigma$. On the other hand $\sigma \subset F$ and F is H_F-invariant, so also $H_F \cdot \sigma \subset F$. This establishes (ii). □

If σ is a face of P set

$$G_\sigma := \{g \in W : g(\sigma) = \sigma\}.$$

Lemma 48. *If $\sigma \in \mathscr{F}(P)$ there is a vector $u \in \mathfrak{t}$ that is fixed by G_σ and such that $\sigma = F_u(P)$. If u is any such vector and $F := F_u(\hat{\mathcal{O}})$, then $F \cap \mathfrak{t} = \sigma$, $G_\sigma = W_F \times W'_F$, $\mathfrak{z}_F = \mathfrak{t}^{G_\sigma}$ and F does not depend on u but only on σ.*

Proof. The existence of u follows directly from Lemma 7. By Lemma 47 (ii) $W_F \times W'_F \subset G_\sigma$, so $u \in \mathfrak{t}^{W_F \times W'_F} = \mathfrak{z}_F$ and using Theorem 38 it follows that $H_F = C_K(u)$. Therefore the subgroup of W that fixes u is the Weyl group of (H_F, T) i.e. $W_F \times W'_F$. It follows that $W_F \times W'_F = G_\sigma$. From this it follows that $\mathfrak{z}_F = \mathfrak{t}^{G_\sigma}$, that $H_F = C_K(\mathfrak{z}_F) = C_K(\mathfrak{t}^{G_\sigma})$ and in particular that H_F and hence $\operatorname{ext} F$ and F only depend on σ. □

Define a map

$$\psi : \mathscr{F}(P)/W \to \mathscr{F}(\hat{\mathcal{O}})/K$$

by the following rule: given σ, fix $u \in \mathfrak{t}^{G_\sigma}$ such that $\sigma = F_u(P)$ and set

$$\psi([\sigma]) := [F_u(\hat{\mathcal{O}})].$$

Thanks to the previous lemma $F_u(\hat{\mathcal{O}})$ depends only on σ, not on u. It is clear that ψ is well-defined on equivalence classes.

Theorem 49. *The maps ψ and φ are inverse to each other and $\psi([\sigma]) = [Z_K(\sigma^\perp) \cdot \sigma]$.*

Proof. Let σ be a face of P. Choose $u \in \mathfrak{t}^{G_\sigma}$ such that $\sigma = F_u(P)$. Then $F_u(\hat{\mathcal{O}})$ is T-stable, so $\varphi \circ \psi([\sigma]) = \varphi([F_u(\hat{\mathcal{O}})]) = [F_u(\hat{\mathcal{O}}) \cap \mathfrak{t}] = [\sigma]$. So $\varphi \circ \psi$ is the identity.

It follows immediately from Theorem 42 (c) that φ is injective. Hence it is a bijection and $\psi = \varphi^{-1}$. By Lemma 35 $\operatorname{ext} F_u(\hat{\mathcal{O}})$ is a $Z_K(\sigma^\perp)$-orbit. Hence $K_F \subset Z_K(\sigma^\perp) \subset H_F$. By Lemma 47 (ii) we get $F_u(\hat{\mathcal{O}}) = Z_K(\sigma^\perp) \cdot \sigma$. □

5. Smooth stratification

As we saw in the previous section the group K acts on $\mathscr{F}(\hat{\mathcal{O}})$, which is the set of faces of $\hat{\mathcal{O}}$ and this action has a finite number of orbits, which are in one-to-one correspondence with the orbits of the Weyl group on the finite set $\mathscr{F}(P)$. Let B denote one of the orbits of K on $\mathscr{F}(\hat{\mathcal{O}})$. We call B a *face type*. The set

$$\mathcal{S}_B := \bigcup_{F \in B} \operatorname{relint} F.$$

is a subset of $\partial\hat{\mathcal{O}}$, because the faces $F \in B$ are proper. Since every boundary point lies in exactly one open face (Theorem 8)

$$\partial\hat{\mathcal{O}} = \bigsqcup_{B \in \mathscr{F}(\hat{\mathcal{O}})/K} \mathcal{S}_B.$$

We call $\mathcal{S}_B$ the *stratum* corresponding to the face type B. The purpose of this section is to show that the strata $\mathcal{S}_B$ yield a stratification of $\hat{\mathcal{O}}$ in the following sense.

Theorem 50. *The strata are smooth embedded submanifolds of $\mathfrak{k}$ and are locally closed in $\partial\hat{\mathcal{O}}$. For any stratum $\mathcal{S}_B$ the boundary $\overline{\mathcal{S}_B} - \mathcal{S}_B$ is the disjoint union of strata of lower dimension.*

There is an obvious map $p\colon \mathcal{S}_B \to B$ which maps a point $x \in \mathcal{S}_B$ to the unique face F such that $x \in \operatorname{relint} F$. To study $\mathcal{S}_B$ it is expedient to fix an element $F \in B$. Thus $B = \{g \cdot F : g \in K\} \cong K/H_F$ and

$$\mathcal{S}_B = K \cdot \operatorname{relint} F = \{g \cdot x : g \in K,\ x \in \operatorname{relint} F\}.$$

$K \to K/H_F$ is a right principal bundle with structure group H_F. Let

$$\mathcal{E}_F = K \times^{H_F} \operatorname{relint} F$$

be the associated bundle gotten from the action of H_F on $\operatorname{relint} F$. Note that $\mathcal{E}_F \to K/H_F$ is a homogeneous bundle in the sense that the left action of K on K/H_F lifts to an action of K on $\mathcal{E}_F$ that is given by the following rule

$$a \cdot [g, x] := [ag, x], \quad a, g \in K,\ x \in \operatorname{relint} F.$$

(Here $[g, x]$ is the point in the associated bundle.)

Proposition 51. *Let B be a face type and let $F \in B$ be a representative. Define a map*

$$f\colon \mathcal{E}_F \to \mathfrak{k}, \quad f([g, x]) = g \cdot x.$$

Then f is a smooth K-equivariant embedding of $\mathcal{E}_F$ into $\mathfrak{k}$ with image $\mathcal{S}_B$. Therefore $\mathcal{S}_B$ is a smooth embedded submanifold of $\mathfrak{k}$. Moreover $p\colon \mathcal{S}_B \to B$ is a smooth fibre bundle.

Proof. It is straightforward to check that f is well-defined, smooth and equivariant. It is also clear that $f(\mathcal{E}_F) = \mathcal{S}_B$. We proceed by showing that f is injective. Recall from Theorem 8 that if F_1 and F_2 are different faces, then $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. If $f([g, x]) = f([g_1, x_1])$ then $g_1^{-1}g \cdot x = x_1$. Since $x_1 \in \operatorname{relint} F$ and $g_1^{-1}g \cdot x \in \operatorname{relint}(g_1^{-1}g \cdot F)$ we get $g_1^{-1}gF = F$, so $[g, x] = [g_1, x_1]$ in $\mathcal{E}_F$. This shows that f is injective. Next we show that f is an immersion. Denote by V the fibre of $\mathcal{E}_F$ over the origin of K/H_F. Since $\mathcal{E}_F$ is a homogeneous bundle and f is equivariant, it is enough to show injectivity of df_p at points $p \in V$, i.e. at points of the form $p = [e, x]$, $x \in \operatorname{relint} F$. At such points

$$T_p\mathcal{E}_F = T_pV \oplus U$$

with

$$U = \left\{ \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} [\exp(tv), x] \colon v \in \mathfrak{h}_F^{\perp} \right\}.$$

Indeed T_pV is the vertical space, while U is the tangent space at p of a local section of $K \to K/H_F$. The injectivity of df_p will follow from the following three facts:
(a) $df_p|_V$ is injective;
(b) $df_p|_U$ is injective;
(c) $df_p(V) \cap df_p(U) = \{0\}$.
(a) follows from the fact that $f|_V$ is a diffeomorphism of V onto $\operatorname{relint} F$. To prove (b) observe first that if $x \in \operatorname{relint} F$, then $\mathfrak{k}_x \subset \mathfrak{h}_F$. Indeed if $g \in K_x$ then $g \cdot x = x \in \operatorname{relint}(g \cdot F) \cap \operatorname{relint} F$, so $g \cdot F = F$ by Theorem 8 and $g \in H_F$. Therefore $K_x \subset H_F$ and $\mathfrak{k}_x \subset \mathfrak{h}_F$, as claimed. Now let u be an element of U. By definition there is $v \in \mathfrak{k}$ such that

$$u := \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} [\exp(tv), x]. \tag{52}$$

Then

$$df_p(u) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} f([\exp(tv), x]) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \exp(tv) \cdot x = [v, x].$$

(The bracket on right is the Lie bracket in $\mathfrak{k}$!) If $df_p(u) = 0$, then $[v, x] = 0$ and $v \in \mathfrak{k}_x \subset \mathfrak{h}_F$. Since $v \in \mathfrak{h}_F^{\perp}$, this means that $v = 0$. Thus (b) is proved. Now observe that $[\mathfrak{h}_F, \mathfrak{h}_F^\perp] \subset \mathfrak{h}_F^\perp$, since the adjoint action of H_F preserves $\mathfrak{h}_F$ and $\mathfrak{h}_F^\perp$. If $v \in \mathfrak{h}_F^\perp$ and $u \in U$ is given by (52), then $df_p(u) = [v, x] \in \mathfrak{h}_F^\perp$ since $x \in F \subset \mathfrak{h}_F$. So $df_p(U) \subset \mathfrak{h}_F^\perp$. On the other hand $df_p(T_pV) = T_{f(p)}(\operatorname{relint} F) \subset \mathfrak{h}_F$. It follows that

$$df_p(T_pV) \cap df_p(U) \subset \mathfrak{h}_F \cap \mathfrak{h}_F^\perp = \{0\}.$$

Thus (c) is proved and f is an immersion. In order to prove that it is an embedding we shall prove that f is proper as a map $f : \mathcal{E}_F \to \mathcal{S}_B = f(\mathcal{E}_F)$. Let $\{y_n\}$ be a sequence in $\mathcal{S}_B$ converging to some point $y \in \mathcal{S}_B$. Set $[g_n, x_n] := f^{-1}(y_n)$. We wish to show that $\{[g_n, x_n]\}$ admits a convergent subsequence. Since K is compact by extracting a subsequence we can assume that $g_n \to g$. Then $y_n = f([g_n, x_n]) = g_n \cdot x_n$. Therefore $x_n = g_n^{-1} \cdot y_n \to x := g^{-1} \cdot y$. Since $y \in \mathcal{S}_B$, $y \in \operatorname{relint}(g^{-1}F)$ and $x \in \operatorname{relint} F$. Therefore $[g_n, x_n] \to [g, x]$ as desired. □

Lemma 53. *If B is the face type of F, then*

$$\dim \mathcal{S}_B = \dim K - \dim K'_F - \dim Z_F.$$

Proof. $\mathcal{S}_B$ is a fibre bundle over K/H_F with fibre $\operatorname{relint} F$. Since $\dim F = \dim \mathfrak{k}_F$ we get the result. □

We introduce a partial order on the face types, as follows: $B_1 \preceq B_2$ if for some (and hence for any) choice of representatives $F_i \in B_i$ there is some $g \in K$ such that $gF_1 \subset F_2$. This is a partial order. We write $B_1 \prec B_2$ if $B_1 \preceq B_2$ and $B_1 \neq B_2$.

Proof of Theorem 50. We already know that the strata are smooth embedded submanifold of $\mathfrak{k}$. In particular they are locally closed subsets both of $\mathfrak{k}$ and of $\hat{\mathcal{O}}$. By Proposition 51 $\mathcal{S}_B = f(\mathcal{E}_F) = f(K \times^{H_F} \operatorname{relint} F)$. So

$$\overline{\mathcal{S}_B} = f(K \times^{H_F} F) = \bigcup_{F \in B} F.$$

Since any face F is the disjoint union of all proper faces contained in F

$$\overline{\mathcal{S}_B} = \bigcup_{F \in B} \operatorname{relint} F \sqcup \bigsqcup_{C \prec B} \bigcup_{G \in C} \operatorname{relint} G = \mathcal{S}_B \sqcup \bigsqcup_{C \prec B} \mathcal{S}_C.$$

To conclude we need to show that $\dim \mathcal{S}_C < \dim \mathcal{S}_B$ if $C \prec B$. Fix representatives $F \in B$ and $G \in C$ such that $G \subsetneq F$. By the previous lemma it is enough to show that $\dim Z_F + \dim K'_F < \dim Z_G + \dim K'_G$. In fact $Z_F \cdot K'_F$ fixes G pointwise since $G \subset F$. Therefore $Z_F \cdot K'_F \subset H_G$. On the other hand if $x \in G$, then $\operatorname{aff}(G) = x + \mathfrak{k}_G \subset \operatorname{aff}(F) = x + \mathfrak{k}_F$. Hence $K_G \subset K_F$. It follows that $[\mathfrak{z}_F \oplus \mathfrak{k}'_F, \mathfrak{k}_G] = 0$. Since $\mathfrak{k}_G$ is semisimple, this shows that $\mathfrak{z}_F \oplus \mathfrak{k}'_F \perp \mathfrak{k}_G$. But $\mathfrak{z}_F \oplus \mathfrak{k}'_F \subset \mathfrak{h}_G$, so in fact $\mathfrak{z}_F \oplus \mathfrak{k}'_F \subset \mathfrak{z}_G \oplus \mathfrak{k}'_G$. This proves the inequality $\dim Z_F + \dim K'_F \leq \dim Z_G + \dim K'_G$. In the case of equality, we would get $Z_F \cdot K'_F = Z_G \cdot K'_G$, so $Z_F = Z_G$, $H_F = H_G$ and hence $\operatorname{ext} F = \operatorname{ext} G$ and $F = G$. □

EXAMPLE 54. We shall describe the strata of the orbitope $\hat{\mathcal{O}}'$ studied in Example 45. We saw there that the faces of $\hat{\mathcal{O}}'$ are in one-to-one correspondence with subspaces of $\mathbb{C}^{n+1}$. Two subspaces are interchanged by an element of $\mathrm{SU}(n+1)$ if and only if they have the same dimension. So the orbit types are indexed by the dimension. Let $W \subset \mathbb{C}^{n+1}$ be a subspace of dimension k, let $F = \operatorname{conv}(\mathbb{P}(W))$ be the corresponding face and let B be the orbit type of F. Then

$$B \cong K/H_F = \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(W) \times \mathrm{U}(W^\perp)).$$

Therefore B is simply the Grassmannian $\mathbb{G}(k, n+1)$. Since $\operatorname{relint} F = \{A \in F \colon \operatorname{rank} A = k\}$, it follows that

$$\mathcal{S}_B = \{A \in \mathcal{H}_1 \colon A \geq 0, \ \operatorname{rank} A = k\}.$$

In fact this is a bundle over the Grassmannian of k-planes. Finally, notice that H_F acts on $\operatorname{relint} F$ simply by the adjoint action of $\mathrm{SU}(W)$.

6. Satake combinatorics of the faces

In this section we describe the faces of $\hat{\mathcal{O}}$ and the faces of the momentum polytope in terms of root data. The description uses the notion of x-connected subset of simple roots, which was introduced in [24]. In that paper Satake introduced certain compactifications of a symmetric space of noncompact type (the Satake–Furstenberg compactifications). The notion of x-connected subset was used in the study of the boundary components of these compactifications. It is no coincidence that faces of $\hat{\mathcal{O}}$ and boundary components admit a description in terms of the same combinatorial data: in fact it was shown in [6] that the Satake compactifications of the symmetric space $K^{\mathbb{C}}/K$ are homeomorphic to convex hulls of integral coadjoint orbit of K. Here we do not use the link with the compactifications. Instead we show directly how to construct all the faces of $\hat{\mathcal{O}}$ (up to conjugation) starting from the root data. This is accomplished for a general coadjoint orbit with no integrality assumption.

Fix a maximal torus T of K and a system of simple roots $\Pi \subset \Delta = \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. As usual we identify $\mathfrak{k}^{\mathbb{C}}$ with its dual using the Killing form B. The roots get identified with elements of $i\mathfrak{t}$.

DEFINITION 55. A subset $E \subset i\mathfrak{t}$ is *connected* if there is no pair of disjoint subsets $D, C \subset E$ such that $D \sqcup C = E$, and $\langle x, y \rangle = 0$ for any $x \in D$ and for any $y \in C$.

(A thorough discussion of connected subsets can be found in [22, §5].) Connected components are defined as usual. For example the connected components of Π are the subsets corresponding to the simple roots of the simple ideals in $\mathfrak{k}$.

DEFINITION 56. If x is a nonzero vector of $\mathfrak{t}$, a subset $I \subset \Pi$ is called *x-connected* if $I \cup \{ix\}$ is connected.

Equivalently $I \subset \Pi$ is x-connected if and only if every connected component of I contains at least one root α such that $\alpha(x) \neq 0$. By definition the empty set is x-connected.

DEFINITION 57. If $I \subset \Pi$ is x-connected, denote by I' the collection of all simple roots orthogonal to $\{ix\} \cup I$. The set $J := I \cup I'$ is called the *x-saturation* of I.

The largest x-connected subset contained in J is I. So J is determined by I and I is determined by J. Given a subset $E \subset \Pi$ we will use the following notation:

$$\begin{aligned}
&\mathfrak{t}_E := \mathfrak{t} \cap \bigcap_{\alpha \in E} \ker \alpha, \\
&\Delta_E = \Delta \cap \operatorname{span}_{\mathbb{R}}(E), \quad \Delta_{E,+} = \Delta_E \cap \Delta_+, \\
&\mathfrak{t}^E = \sum_{\alpha \in E} \mathbb{R} i H_\alpha = \text{orthogonal complement of } \mathfrak{t}_E \text{ in } \mathfrak{t}, \\
&\mathfrak{h}_E := \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta_{E,+}} Z_\alpha, \quad \mathfrak{k}_E := \mathfrak{t}^E \oplus \bigoplus_{\alpha \in \Delta_{E,+}} Z_\alpha.
\end{aligned}$$

We denote by T_E, H_E, K_E the corresponding connected subgroups. Note that H_E is the subgroup associated to the subset $E \subset \Pi$, while H_F is the subset associated to the face $F \subset \hat{\mathcal{O}}$. This should cause no confusion.

Lemma 58. *Let $\mathcal{O}$ be a full coadjoint orbit and let $F \subset \hat{\mathcal{O}}$ be a proper face. Assume that $u \in C_F$ and that $v \in C_F \cap \mathfrak{z}_F$. Let $\alpha \in \Delta$.*
(a) *If $\alpha(u) = 0$, then $\alpha(v) = 0$.*
(b) *If $-i\alpha(u) > 0$, then $-i\alpha(v) \geq 0$.*

Proof. (a) $Z_K(u) \subset H_F$, since $F = F_u(\hat{\mathcal{O}})$, and $H_F = Z_K(v)$ by Theorem 38. If $\alpha(u) = 0$, then $Z_\alpha \subset \mathfrak{z}_{\mathfrak{k}}(u) \subset \mathfrak{h}_F = \mathfrak{z}_{\mathfrak{k}}(v)$, hence $\alpha(v) = 0$. (b) Assume by contradiction that $-i\alpha(v) < 0$. Set $u_t = (1-t)u + tv$. By Proposition 7 C_F is convex, so $u_t \in C_F$ for any $t \in [0, 1]$. Since $-i\alpha(u_0) > 0$ and $-i\alpha(u_1) < 0$, there is some $s \in (0, 1)$ such that $\alpha(u_s) = 0$. Since $u_s \in C_F$ and $\alpha(v) \neq 0$, this would contradict (a). □

Denote by C^+ the positive Weyl chamber associated to Π. The following is immediate and well-known.

Lemma 59. *If $v \in \overline{C^+}$, then $\mathfrak{z}_{\mathfrak{k}}(v) = \mathfrak{h}_E$ with $E = \{\alpha \in \Pi \colon \alpha(v) = 0\}$.*

Theorem 60. *Let $\mathcal{O}$ be a full coadjoint orbit and let x be the unique point in $\mathcal{O} \cap \overline{C^+}$.*
(a) *If $I \subset \Pi$ is x-connected and J is its x-saturation, then*

$$F := \operatorname{conv}(H_J \cdot x)$$

is a face of $\hat{\mathcal{O}}$. If $u \in \mathfrak{t}_J$ and $-i\alpha(u) > 0$ for any $\alpha \in \Pi - J$, then $F = F_u(\hat{\mathcal{O}})$. Moreover

$$H_F = H_J, \quad Z_F = T_J, \quad K_F = K_I, \quad K'_F = K_{I'}. \tag{61}$$

(b) *Given an arbitrary subset $E \subset \Pi$, denote by I the largest x-connected subset contained in E and by J the x-saturation of I. Then $H_E \cdot x = H_I \cdot x = H_J \cdot x$.*
(c) *Any face of $\hat{\mathcal{O}}$ is conjugate to one of the faces constructed in* (a). *More precisely, given a face F and a maximal torus $T \subset H_F$ there are a base $\Pi \subset \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and a subset $I \subset \Pi$ with the following properties*:
(i) *if C^+ is the positive Weyl chamber corresponding to Π, then $\overline{C^+} \cap \operatorname{ext} F \neq \emptyset$;*
(ii) *if x is the unique point in $\overline{C^+} \cap \operatorname{ext} F$, then I is x-connected and $F = \operatorname{conv}(H_J \cdot x)$, where J is the x-saturation of I.*

Proof. (a) Since the set $\{\alpha|_{\mathfrak{t}_J} : \alpha \in \Pi - J\}$ is a basis of $\mathfrak{t}_J^*$, we can pick $u \in \mathfrak{t}_J$ such that $\alpha(u) > 0$ for any $\alpha \in \Pi - J$. Then $Z_K(u) = H_J$. Set $F := F_u(\hat{\mathcal{O}})$. We claim that $x \in F$. Indeed x and u belong to $\overline{C^+}$, so by Lemma 23 x is a maximum point of Φ_u, i.e. $x \in \operatorname{ext} F$. By Lemma 22 $\operatorname{ext} F = Z_K(u) \cdot x$, so $F = \operatorname{conv}(H_J \cdot x)$. This proves that $\operatorname{conv}(H_J \cdot x)$ is indeed a face of $\hat{\mathcal{O}}$. By Lemma 44 $K_F \subset H_J = Z_K(u) \subset H_F$ and $K_F = K_I$. Pick $v \in C_F \cap \mathfrak{z}_F$ (this exists by Theorem 38). By Lemma 58 $-i\alpha(v) \geq 0$ for every $\alpha \in \Delta_+$, i.e. $v \in \overline{C^+}$. By Theorem 38 (c) and Lemma 59 $\mathfrak{h}_F = \mathfrak{z}_{\mathfrak{k}}(v) = \mathfrak{h}_E$, where $E = \{\alpha \in \Pi \colon \alpha(v) = 0\}$. We claim that $E = J$. Indeed $\mathfrak{h}_J \subset \mathfrak{h}_F = \mathfrak{h}_E$, so $J \subset E$. If we write $E = I \sqcup E'$, then $I' \subset E'$. Conversely, if $\alpha \in E'$, then $Z_\alpha \perp \mathfrak{k}_I = \mathfrak{k}_F$ (simply because the root space decomposition is orthogonal), so $Z_\alpha \subset \mathfrak{k}'_F$. This entails on the one hand that $[Z_\alpha, \mathfrak{k}_I] = 0$, i.e. $\alpha \perp I$; on the other hand that Z_α fixes x, i.e. $\alpha(x) = 0$. This means in fact that $\alpha \in I'$. Hence $E = J$ as claimed and (61) follow.

(b) Split E in connected components: $E = E_1 \sqcup \cdots \sqcup E_r$. We can assume that E_j is x-connected iff $j \leq q$ for some q between 1 and r. Then $I = E_1 \sqcup \cdots \sqcup E_q$. Set $E' := E - I = \sqcup_{j>q} E_J$. Then clearly $E' \subset I'$. So $E \subset J$. Let $F = \operatorname{conv}(H_j \cdot x)$ be the face constructed from J as in (a). Then $H_F = H_J$ and $K_F = K_I$. Since $I \subset E \subset J$, $K_I \subset H_E \subset H_J$. But $K_I \cdot x = K_F \cdot x = H_F \cdot x = H_J \cdot x$, so $H_E \cdot x = H_J \cdot x$ as desired.

(c) If $F = \hat{\mathcal{O}}$, then $F = \operatorname{conv}(H_J)$ with $I = J = \Pi$. Otherwise F is a proper face. Fix a point $x \in \operatorname{ext} F \cap \mathfrak{t}$. By Theorem 38 (b) there is a vector $u \in \mathfrak{z}_F$ such that $F = F_u(\hat{\mathcal{O}})$. Then $\operatorname{ext} F = \operatorname{Max}(\Phi_u)$, so there is a Weyl chamber C^+ such that $x, u \in \overline{C^+}$. Let Π be the base corresponding to C^+. By Theorem 38 (c) $H_F = Z_K(u)$. Since $u \in \overline{C^+}$, Lemma 59 says that $H_F = H_E$ with $E = \{\alpha \in \Pi \colon \alpha(u) = 0\}$. Let I and J be as in (b). Then I is x-connected and using (b) we get $\operatorname{ext} F = H_F \cdot x = H_E \cdot x = H_J \cdot x$. Thus $F = \operatorname{conv}(H_J \cdot x)$ as desired. □

REMARK 62. In the proof of (c) we have in fact that $E = J$. Indeed from (a) $H_F = H_J$, so $H_E = H_J$ i.e. $E = J$.

EXAMPLE 63. Let $K = \mathrm{SU}(n+1)$, $n \geq 4$, and let $x \in \mathfrak{su}(n+1)$ be the diagonal matrix $x = \operatorname{diag}(i(n-1), i(n-1), -2i, \ldots, -2i)$. The coadjoint orbit through x is the momentum image of the Grassmannian $\mathbb{G}(2, n+1)$. Let $\mathfrak{t}$ be the set of the diagonal matrices and denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the standard set of simple roots, i.e. $\alpha_i(\operatorname{diag}(x_1, \ldots, x_{n+1})) = x_i - x_{i+1}$. The vector x lies in the closure of the positive Weyl chamber containg x and $\alpha_i(x) \neq 0$ if and only if $i = 2$. Therefore the x-connected subsets of Π are the following:

a) $I_k^1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $2 \leq k \leq n$;
b) $I_k^2 = \{\alpha_2, \ldots, \alpha_k\}$, $2 \leq k \leq n$;
c) the empty set.

For $I = \emptyset$, $\Delta_I = \emptyset$, $H_I = T$ and the x-saturation J of I consists of the simple roots that are orthogonal to ix. Therefore $H_J = Z_K(x)$ and $H_J \cdot x = \{x\}$. The corresponding face is the vertex $F = \{x\}$.

For $i = 1, 2$ let J_k^i be the x-saturation of I_k^i and set $F_k^1 = \operatorname{conv}(H_{J_k^1} \cdot x)$. It is easy to check that $J_k^1 = I_k^1 \cup \{\alpha_{k+2}, \ldots, \alpha_n\}$. $K_{I_k^1}$ is the image of the embedding

$$\mathrm{SU}(k+1) \hookrightarrow \mathrm{SU}(n+1), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

and $H_{J_k^1} = \mathrm{S}(\mathrm{U}(k+1) \times \mathrm{U}(n-k))$. Hence

$$\operatorname{ext} F_k^1 = K_{I_k^1} \cdot x = \mathrm{SU}(k+1)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(k-1)),$$

is the complex Grassmannian $\mathbb{G}(2, k+1)$. The stratum corresponding to F_k^1 is a fibre bundle over $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(k+1) \times \mathrm{U}(n-k)) = \mathbb{G}(k+1, n+1)$.

The x-saturation of I_k^2 is $J_k^2 = I_k^2 \cup \{\alpha_{k+2}, \ldots, \alpha_n\}$. $K_{I_k^2}$ is the image of the embedding

$$\mathrm{SU}(k) \hookrightarrow \mathrm{SU}(n+1), \quad A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

$H_{J_k^2} = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k) \times \mathrm{U}(n-k))$ and

$$\operatorname{ext} F_k^2 = K_{I_k^2} \cdot x = \mathrm{SU}(k)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k-1))$$

is a complex projective space $\mathbb{P}^{k-1}(\mathbb{C})$. The strata corresponding to F_k^2 is a fibre bundle over the flag manifold $\mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(k) \times \mathrm{U}(n-k))$.

7. Complex geometry of the faces

In the previous sections we have described the faces of $\hat{\mathcal{O}}$ in terms of their extreme sets $\operatorname{ext} F$ and have caracterized the submanifolds $\operatorname{ext} F \subset \mathcal{O}$ in various ways. Here we wish to prove Theorem 2, which amounts to the equivalence between (a) and (b) in Theorem 64 below. This will add another characterization in terms of the complex structure of $\mathcal{O}$.

Theorem 64. *Let $\mathcal{O}' \subset \mathcal{O}$ be a submanifold. The following conditions are equivalent.*
(a) *$\mathcal{O}'$ is a compact orbit of a parabolic subgroup of G.*
(b) *There is a face F of $\hat{\mathcal{O}}$ such that $\mathcal{O}' = \operatorname{ext} F$.*
(c) *$\mathcal{O}'$ is compact and the subgroup*

$$P := \{g \in G : g \cdot \mathcal{O}' = \mathcal{O}'\} \tag{65}$$

is a parabolic subgroup of G that acts transitively on $\mathcal{O}'$;
(d) *There are a maximal torus $T \subset K$, a Weyl chamber $C^+ \subset \mathfrak{t}$ and a subset E of the corresponding set of simple roots Π such that $\mathcal{O}' \cap \overline{C^+} \neq \emptyset$ and $\mathcal{O}'$ is an orbit of H_E.*

Proof. That (d) is equivalent to (b) is the content of Theorem 60.

(a) $\Rightarrow$ (c) Since $\mathcal{O}'$ is an orbit of some parabolic subgroup Q, the subgroup P contains Q so it is parabolic.

(c) $\Rightarrow$ (d) Since P is parabolic we can find a maximal torus $T \subset K$ and a system of simple roots in $\mathfrak{t}$ in such a way that $B_- \subset P$. So B_- acts on $\mathcal{O}'$ and by the Borel fixed point theorem B_- has some fixed point $x \in \mathcal{O}'$. Since x is fixed by $T \subset B_-$, $x \in \mathfrak{t}$ and it follows from Lemma 15 that $x \in \overline{C^+}$. If $E \subset \Pi$ set

$$\mathfrak{u}_E := \bigoplus_{\alpha \in \Delta_- - \Delta_E} \mathfrak{g}_\alpha, \quad \mathfrak{p}_E := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_- \cup - \Delta_E} \mathfrak{g}_\alpha.$$

Then $\mathfrak{p}_E = \mathfrak{h}_E^{\mathbb{C}} \oplus \mathfrak{u}_E$ is a parabolic subalgebra. Denote by U_E and P_E the corresponding connected subgroups of G. Then P_E is a parabolic subgroup, U_E is its unipotent radical and $H_E^{\mathbb{C}}$ is a Levi factor. In particular $P_E = H_E^{\mathbb{C}} \cdot U_E$ and $U_E \lhd P_E$. Since $B_- \subset P$ there is some $E \subset \Pi$ such that $P = P_E$. Since $U_E \subset B_- \subset G_x$ we conclude that $\mathcal{O}' = P_E \cdot x = H_E^{\mathbb{C}} \cdot x$. As $\mathcal{O}'$ is compact, the compact form H_E must be transitive on $\mathcal{O}'$. This concludes the proof.

(d) $\Rightarrow$ (a) First observe that $\mathcal{O}' = H_E \cdot x$ is a complex submanifold since it is a connected component of the fixed point set of the torus T_E. Therefore $H_E^{\mathbb{C}}$ preserves $\mathcal{O}'$. By assumption there is $x \in \overline{C^+} \cap \mathcal{O}'$. By Lemma 15 the stabilizer G_x contains the negative Borel subgroup, so U_E fixes x. If $x' \in \mathcal{O}'$, there is $a \in H_E$ such that $x' = a \cdot x$. If $b \in U_E$ then $a^{-1}ba \in U_E$, so $a^{-1}ba \cdot x = x$ and $b \cdot x' = ba \cdot x = a \cdot x = x'$.

Hence U_E fixes pointwise $\mathcal{O}'$. Therefore P_E preserves $\mathcal{O}'$ which is therefore a compact P_E-orbit. □

We notice that in condition (d) the set E can be chosen to be the x-saturation of the maximal x-connected subset $I \subset E$ as shown in Theorem 60 (c).

The above result establishes a one-to-one correspondence between two rather distant classes of objects: on the one side the faces of the orbitope $\hat{\mathcal{O}}$, on the other side the closed orbits of parabolic subgroups of G inside $\mathcal{O}$. To illustrate this correspondence recall the following fact.

Lemma 66. *If $P \subset G$ is a parabolic subgroup, in $\mathcal{O}$ there is only one orbit of P which is closed.*

Proof. Since the action is algebraic and $\mathcal{O}$ is a compact manifold, there is at least one orbit which is closed. Let $\mathcal{O}' \subset \mathcal{O}$ be a closed P-orbit and let $B \subset P$ be a Borel subgroup. Then $\mathcal{O}'$ is B-invariant, so it contains a closed B-orbit. But the B-orbits in $\mathcal{O}$ are just the Schubert cells and the only one which is closed is the fixed point of B. Hence any closed P-orbit contains this fixed point and this implies that the closed P-orbit is unique. □

The above uniqueness statement can also be considered from the point of view of the orbitope, as can be seen from the proof of the implication (c) $\Rightarrow$ (d) in the previous theorem. Indeed, if P is a parabolic subgroup, we write it as $P = P_E$ for some $E \subset \Pi$. Then there is a unique orbit of H_E that is of the form $\operatorname{ext} F$, namely the orbit $H_E \cdot x$ for $x \in \mathcal{O} \cap \overline{C^+}$. Alternatively this orbit can be described as follows: choose $u \in \mathfrak{t}_E = \mathfrak{z}(\mathfrak{h}_E)$ such that $-i\alpha(u) > 0$ for $\alpha \in E$. Then the closed P-orbit is $\operatorname{Max}(\Phi_u)$. In a sense to fix a parabolic subgroup P_E is equivalent to fixing H_E and the vector u. So once P_E is fixed we know both H_E and which component of $\mathcal{O} \cap \mathfrak{h}_E$ corresponds to the maximum of Φ_u.

To conclude we wish to interpret geometrically condition (c) of Theorem 64. Let $\mathcal{O}'$ be a complex submanifold of $\mathcal{O}$. Let $\mathcal{H}$ denote the Hilbert scheme of the projective manifold $\mathcal{O}$. If $Y \subset \mathcal{O}$ is a subscheme, let $[Y]$ be its Hilbert point. (See e.g. [1, Chapter IX].) The group G acts on $\mathcal{H}$ by sending the Hilbert point $[Y]$ of a subscheme $Y \subset \mathcal{O}$ to $[g \cdot Y]$.

Proposition 67. *Let $\mathcal{O}' \subset \mathcal{O}$ be a complex submanifold which is an orbit of some subgroup of K. Let $f : G \to \mathcal{H}$ be the map $f(g) := [g \cdot \mathcal{O}']$. Then the following conditions are all equivalent to condition* (c) *of* Theorem 64:

i) *$f(G)$ is compact*;

ii) *$f(K)$ is a subscheme of $\mathcal{H}$*;

iii) *$f(G) = f(K)$.*

Proof. $f(G)$ is just the orbit of G through the point $p = [\mathcal{O}'] \in \mathcal{H}$, while $f(K)$ is the orbit of K through p. The subgroup P defined in (65) is just the stabilizer G_p. Therefore $f(G) \cong G/P$. It follows immediately that the three conditions are equivalent to P being parabolic, so they are implied by (c). Conversely, if they are satisfied, P is parabolic. By assumption $\mathcal{O}'$ is an orbit of some subgroup $L \subset K$. Then $L \subset P$ and $\mathcal{O}'$ is a P-orbit, thus (c) holds. □

EXAMPLE 68. Consider the orbitope of $\mathbb{P}^2(\mathbb{C})$ as described in Example 45. The complex lines satisfy the conditions in the proposition and in fact they do generate faces of $\hat{\mathcal{O}}$: if $\mathcal{O}' \subset \mathbb{P}^2(\mathbb{C})$ is a line the set $\operatorname{conv}(\mathcal{O}')$ is a face of $\hat{\mathcal{O}}$. Also plane conics are complex submanifolds of $\mathbb{P}^2(\mathbb{C})$ that are homogeneous for a subgroup of $\mathrm{SL}(3,\mathbb{C})$, namely $\mathrm{SO}(3,\mathbb{C})$. Nevertheless the orbit of $\mathrm{SL}(3,\mathbb{C})$ through a conic is not compact since smooth conics degenerate to singular ones. So conics do no satisfy the conditions above and in fact conics do not generate faces of $\hat{\mathcal{O}}$.

EXAMPLE 69. Let $L \subset K$ be the centralizer of a torus and let $\mathcal{O}' \subset \mathcal{O}$ be an orbit of L. As we have shown in general the set $F = \operatorname{conv}(\mathcal{O}')$ is not a face of $\hat{\mathcal{O}}$. One condition is that $\mathcal{O}' \subset \mathfrak{l}$. In fact if $L = Z_K(u)$, and $F = F_u(\hat{\mathcal{O}})$, then $\mathcal{O}' = \operatorname{ext} F = \operatorname{Max}(\Phi_u) \subset \operatorname{Crit}(\Phi_u) = \mathcal{O} \cap \mathfrak{l}$. This condition is not enough either. In fact $\operatorname{Crit}(\Phi_u)$ will contain at least two orbits, one for the maximum and one for the minimum. These are"good" orbits, in the sense that they correspond to faces, namely to $F_u(\hat{\mathcal{O}})$ and $F_{-u}(\hat{\mathcal{O}})$ respectively. The orbits in between in general do not generate faces. Consider the following example. Let $\mathcal{O} \subset \mathfrak{su}(3)$ be the momentum image of the flag manifold of pairs (L_1, L_2) where $L_1 \subset L_2 \subset \mathbb{C}^3$ and $\dim L_i = i$. Let $u = i\,\mathrm{diag}(1,1,-2)$. Set $V = \mathbb{C}^2 \times \{0\}$. Then $\operatorname{Crit}(\Phi_u)$ has the following three connected components:

$$
\begin{aligned}
C_1 &= \{(L_1, L_2) \in \mathcal{O} \colon L_1 \in \mathbb{P}(V), L_2 = L_1 \oplus \mathbb{C}e_3\},\\
C_2 &= \{(L_1, L_2) \in \mathcal{O} \colon L_1 \subset L_2 \subset V\},\\
C_3 &= \{(L_1, L_2) \in \mathcal{O} \colon L_1 = \mathbb{C}e_3\}.
\end{aligned}
$$

Each component is an orbit of $Z_K(u) = S(U(2) \times U(1))$. Let P_i denote the stabilizer of C_i for the action of $G = \mathrm{SL}(3,\mathbb{C})$. Then $P_2 = \{g \in \mathrm{SL}(3,\mathbb{C}) \colon g(V) = V\}$ and $P_3 = \{g \colon ge_3 = e_3\}$. These two subgroups are parabolic. So C_2 and C_3 correspond to faces, by Proposition 64. On the other hand we claim that P_1 is the subgroup of $\mathrm{SL}(3,\mathbb{C})$ of matrices of the form

$$
g = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix}, \quad A \in \mathrm{SL}(2,\mathbb{C}), \lambda \in \mathbb{C}^*.
$$

It is clear that matrices of this form lie in P_1. Conversely assume $g \in P_1$. Then $g(V) = V$. Write $ge_3 = \lambda e_3 + w$ with $w \in V$. For any $v \in V - \{0\}$ the plane $g\,\mathrm{span}(v, e_3) = \mathrm{span}(gv, ge_3)$ contains e_3. Hence $w \in \mathrm{span}(gv, e_3)$. Since $v \in V - \{0\}$ is arbitrary it follows that $w = 0$. The claim is proved, hence P_1 is not parabolic and $\mathrm{conv}(C_1)$ is not a face of $\hat{\mathcal{O}}$.

8. The case of an integral orbit

A coadjoint orbit $\mathcal{O} \subset \mathfrak{k}$ is *integral* if $[\omega]/2\pi$ lies in the image of the natural morphism $H^2(\mathcal{O}, \mathbb{Z}) \to H^2(\mathcal{O}, \mathbb{R})$. (Here ω is the Kostant–Kirillov–Souriau form.) If $\mathcal{O}$ is integral there is a complex line bundle $L \to \mathcal{O}$ such that $[\omega] = 2\pi\, \mathrm{c}_1(L)$. This line bundle can be made K-equivariant and holomorphic with respect to the structure J on $\mathcal{O}$ and it supports a unique K-invariant Hermitian bundle metric h such that $\omega = i R(h)$. With this holomorphic structure the line bundle L turns out to be very ample. Set $V := (H^0(\mathcal{O}, L))^*$. Then V inherits from ω and h an L^2-scalar product. Moreover V is an irreducible representation of K and there is a unique orbit $M \subset \mathbb{P}(V)$ which is a complex submanifold of $\mathbb{P}(V)$. This orbit is simply connected. Fix on M the restriction of the Fubini-Study form gotten from the L^2-scalar product on V. Since K is semisimple there is a unique momentum map $\Phi\colon M \to \mathfrak{k}$ and $\mathcal{O} = \Phi(M)$. Conversely, if there is an irreducible K-representation V such that $\mathcal{O} = \Phi(M)$ for the unique complex orbit $M \subset \mathbb{P}(V)$, then $\mathcal{O}$ is integral. This follows from the fact that the momentum map $\Phi\colon M \to \mathcal{O}$ is a symplectomorphism.

Another way to express integrality of $\mathcal{O}$ is the following. Fix a maximal torus $T \subset K$ and choose a point $x \in \mathcal{O} \cap \mathfrak{t}$. Recall that a linear functional $\lambda \in (i\mathfrak{t})^*$ is an *algebraically integral weight* if

$$\frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} = \frac{\lambda(H_\alpha)}{|H_\alpha|^2} \in \mathbb{Z}$$

for any root $\alpha \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, see e.g. [18, p. 265]. Then $\mathcal{O}$ is integral if and only if $\lambda = \langle ix, \cdot \rangle$ is an algebraically integral weight. (For all this see [17, Chapter 1] or [19].)

Theorem 70. *Let $\mathcal{O} \subset \mathfrak{k}$ be an integral coadjoint orbit and let F be a face of $\hat{\mathcal{O}}$. Write* $\mathrm{ext}\, F = x_0 + K_F \cdot x_1$ *as in* (43). *Denote by $\langle\ ,\ \rangle_F$ the scalar product on $\mathfrak{k}_F$ induced by the Killing form of $\mathfrak{k}_F$. Define $x_1' \in \mathfrak{k}_F$ by the following rule*:

$$\langle x_1', y \rangle_F = \langle x_1, y \rangle \tag{71}$$

for all $y \in \mathfrak{k}_F$. Then $K_F \cdot x_1'$ is an integral coadjoint orbit in $\mathfrak{k}_F$.

Proof. This fact can be proved in a variety of ways using the various characterizations of integrality. One simple way is using the definition, i.e. the condition on the integrality of the Kostant–Kirillov–Souriau form. Let ω_F be the KSS form of $K_F \cdot x_1' \subset \mathfrak{k}_F$. Let $\mu \in \mathfrak{k}_F^*$ be the functional $\mu(y) = \langle x_1, y \rangle = \langle x_1', y \rangle_F$. The stabilizers (for the adjoint action) of x_1 and x_1' are the same, because both coincide with the stabilizer of μ (for the coadjoint action). Moreover the stabilizers in K_F of x and of

x_1 coincide since $x = x_0 + x_1$ and x_0 is fixed by K_F. Summing up we get that the stabilizers in K_F of x_1' and x coincide. Hence the map

$$j\colon K_F \cdot x_1' \hookrightarrow \mathfrak{k}, \quad g \cdot x_1' \mapsto j(g \cdot x_1') := g \cdot x$$

is an embedding of $K_F \cdot x_1'$ onto $\operatorname{ext} F = K_F \cdot x \subset \mathfrak{k}$. We claim that $j^*\omega = \omega_F$. By equivariance it is enough to check that $j^*\omega = \omega_F$ at x_1'. Take $X, Y \in \mathfrak{k}_F$ and set $u = [X, x_1']$, $v = [Y, x_1']$. Then

$$dj_{x_1'}(u) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} j(\operatorname{Ad}(\exp tX)x_1') = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} (\operatorname{Ad}(\exp tX)x) = [X, x]$$

and similarly $dj_{x_1'}(v) = [Y, x]$. Hence $j^*\omega(u, v) = \omega([X, x], [Y, x]) = \langle x, [X, Y]\rangle$. Since $[X, Y] \in \mathfrak{k}_F$ and $x_0 \in \mathfrak{z}_F$, $x_0 \perp [X, Y]$. Therefore $\langle x, [X, Y]\rangle = \langle x_1, [X, Y]\rangle = \langle x_1', [X, Y]\rangle_F = \omega_F(u, v)$. This proves that indeed $\omega_F = j^*\omega$ and thus $[\omega_F]/2\pi$ is integral if $[\omega]/2\pi$ is. □

REMARK 72. Since the various definitions of integrality are equivalent, this theorem ensures that if $\langle ix, \cdot\rangle$ is an integral weight, then $\langle ix_1, \cdot\rangle_F$ is integral as well. Since integral weights give rise to representations, to each face F of an integral coadjoint orbitope is attached an irreducible representation of K_F. If one fixes root data and F is the face corresponding to an x-connected subset $I \subset \Pi$ as in §6, then the representation corresponding to F is the representaion V_I originally described by Satake [24, p. 89] (see also [7, p. 67].

REMARK 73. If $\mathcal{O}$ is an integral orbit, then $\mathcal{O}$ is the momentum image of a flag manifold M provided with an invariant Hodge metric lying in a polarization $L \to M$. The space $H^0(M, L)$ is an irreducible representation τ of K. Out of these data one can construct a Satake–Furstenberg compactification $\overline{X_\tau}^S$ of the symmetric space $K^{\mathbb{C}}/K$ and it is possible to define a homeomorphism (named after Bourguignon–Li–Yau) between this compactification and the orbitope $\hat{\mathcal{O}}$. This was accomplished in [6]. Since this homeomorphism respects the boundary structure, some properties of the faces of $\hat{\mathcal{O}}$ can be deduced in this way. The arguments in the present paper apply also to the non-integral case, give much more information and are more direct and geometric, since no use is made of the Bourguignon–Li–Yau map.

ACKNOWLEDGEMENTS. The first two authors wish to thank Laura Geatti for interesting discussions and the Fakultät für Mathematik of Ruhr-Universität Bochum for hospitality.

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