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Osaka J. Math.
51 (2014), 785–822

INDUCTION AND RESTRICTION FUNCTORS FOR CYCLOTOMIC q-SCHUR ALGEBRAS

Dedicated, with thanks, to Professor Toshiaki Shoji on the occasion of his retirement from Nagoya University

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(Received January 12, 2012, revised January 17, 2013)

Abstract

We define the induction and restriction functors for cyclotomic q-Schur algebras, and study some properties of them. As an application, we categorify a higher level Fock space by using the module categories of cyclotomic q-Schur algebras.

0. Introduction

Let $\mathscr{H}_{n,r}$ be the Ariki–Koike algebra associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ over a commutative ring R. Let $\mathscr{S}_{n,r}$ be the cyclotomic q-Schur algebra associated to $\mathscr{H}_{n,r}$. It is known that $\mathscr{S}_{n,r}$-mod is a highest weight cover of $\mathscr{H}_{n,r}$-mod in the sense of [14] when R is a field. In [14], Rouquier proved that $\mathscr{S}_{n,r}$-mod is equivalent to the category $\mathcal{O}$ of the rational Cherednik algebra associated to $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ as the highest weight covers of $\mathscr{H}_{n,r}$-mod when $R = \mathbb{C}$ with some special parameters.

On the other hand, in [3], Bezrukavnikov and Etingof defined the parabolic induction and restriction functors for rational Cherednik algebras. By using these functors, Shan has categorified a higher level Fock space by using the categories $\mathcal{O}$ of rational Cherednik algebras in [16].

In this paper, we define the induction and restriction functors for cyclotomic q-Schur algebras, and study some properties of them. In §1, we review some known results for cyclotomic q-Schur algebras. In §2, we define the injective homomorphism of algebras $\iota\colon \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$. This injection carries the unit element of $\mathscr{S}_{n,r}$ to a certain idempotent ξ of $\mathscr{S}_{n+1,r}$. Thus, we can regard $\mathscr{S}_{n+1,r}\xi$ (resp. $\xi\mathscr{S}_{n+1,r}$) as an $(\mathscr{S}_{n+1,r}, \mathscr{S}_{n,r})$-bimodule (resp. $(\mathscr{S}_{n,r}, \mathscr{S}_{n+1,r})$-bimodule) by multiplications through the injection ι. By using these bimodules, we define the restriction functor from $\mathscr{S}_{n+1,r}$-mod to $\mathscr{S}_{n,r}$-mod by $\mathrm{Res}_n^{n+1} := \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi, ?)$, and define the induction functors from $\mathscr{S}_{n,r}$-mod to $\mathscr{S}_{n+1,r}$-mod by $\mathrm{Ind}_n^{n+1} := \mathscr{S}_{n+1,r}\xi \otimes_{\mathscr{S}_{n,r}} ?$ and $\mathrm{coInd}_n^{n+1} := \mathrm{Hom}_{\mathscr{S}_{n,r}}(\xi\mathscr{S}_{n+1,r}, ?)$. In

2010 Mathematics Subject Classification. Primary 17B37; Secondary 20C08, 20G42.
This research was supported by JSPS Grant-in-Aid for Research Activity Start-up No. 22840025, and partially supported by GCOE 'Fostering top leaders in mathematics', Kyoto University.

§3, we study the standard (Weyl) and costandard modules of cyclotomic q-Schur algebras applying the functors Res_n^{n+1}, Ind_n^{n+1} and coInd_n^{n+1}. In Theorem 3.4, we prove that the restricted and induced standard (resp. costandard) modules have filtrations whose successive quotients are isomorphic to standard (resp. costandard) modules. In §4, we study some properties of our functors. In particular, we prove the isomorphism of functors $\mathrm{Ind}_n^{n+1} \cong \mathrm{coInd}_n^{n+1}$ (Theorem 4.17). Then, we see that Res_n^{n+1} is left and right adjoint to Ind_n^{n+1}, and both functors are exact. Moreover, these functors commute with the dual functors and Schur functors (Corollary 4.18). In §5, by using the projections to blocks of cyclotomic q-Schur algebras, we refine the induction and restriction functors. As an application, we categorify a level r Fock space by using $\bigoplus_{n\geq 0} \mathscr{S}_{n,r}$-mod with (refined) induction and restriction functors (Corollary 5.7)[1]. In §6, we prove that our induction and restriction functors are isomorphic to the corresponding parabolic induction and restriction functors for rational Cherednik algebras given in [3] when module categories of cyclotomic q-Schur algebras are equivalent to categories $\mathcal{O}$ of rational Cherednik algebras as highest weight covers of module categories of Ariki–Koike algebras (Theorem 6.3).

Notation and conventions: For an algebra $\mathscr{A}$ over a commutative ring R, let $\mathscr{A}$-mod be the category of finitely generated left $\mathscr{A}$-modules, and $K_0(\mathscr{A}\text{-mod})$ be the Grothendieck group of $\mathscr{A}$-mod. For $M \in \mathscr{A}$-mod, we denote by $[M]$ the image of M in $K_0(\mathscr{A}\text{-mod})$.

Let $\theta\colon \mathscr{A} \to \mathscr{A}$ be an algebra anti-automorphism. For a left $\mathscr{A}$-module M, put $M^{\circledast} = \mathrm{Hom}_R(M, R)$, and we define the left action of $\mathscr{A}$ on $M^{\circledast}$ by $(a \cdot \varphi)(m) = \varphi(\theta(a) \cdot m)$ for $a \in \mathscr{A}$, $\varphi \in M^{\circledast}$, $m \in M$. Then we have the contravariant functor $\circledast\colon \mathscr{A}\text{-mod} \to \mathscr{A}\text{-mod}$ such that $M \mapsto M^{\circledast}$. Throughout this paper, we use the same symbol $\circledast$ for contravariant functors defined in the above associated with several algebras since there is no risk to confuse.

1. Review of cyclotomic q-Schur algebras

In this section, we recall the definition and some fundamental properties of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}$ introduced in [4], and we review a presentation of $\mathscr{S}_{n,r}$ by generators and defining relations given in [18].

1.1. Let R be a commutative ring, and we take parameters $q, Q_1, \dots, Q_r \in R$ such that q is invertible in R. The Ariki–Koike algebra ${}_R\mathscr{H}_{n,r}$ associated to the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by

[1]Recently, in [17], Stroppel and Webster gave a categorification of a Fock space by using a quiver Schur algebra, but our methods are totally different from theirs. In the review process, the author received a mail from Professor C. Stroppel. According to her mail, in the revised version of [17], they regard their induction and restriction functors as graded lifts of corresponding our functors, and also obtain graded lifts of some results of this paper. He would like to thank C. Stroppel for her information.

$T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned} &(T_0 - Q_1)(T_0 - Q_2)\dots(T_0 - Q_r) = 0, && \\ &(T_i - q)(T_i + q^{-1}) = 0 && (1 \le i \le n-1), \\ &T_0T_1T_0T_1 = T_1T_0T_1T_0, && \\ &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} && (1 \le i \le n-2), \\ &T_iT_j = T_jT_i && (|i-j| \ge 2). \end{aligned}$$

The subalgebra of ${}_R\mathscr{H}_{n,r}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori–Hecke algebra ${}_R\mathscr{H}_n$ of the symmetric group $\mathfrak{S}_n$ of degree n. For $w \in \mathfrak{S}_n$, we denote by $l(w)$ the length of w, and denote by T_w the standard basis of ${}_R\mathscr{H}_n$ corresponding to w. Let $*: {}_R\mathscr{H}_{n,r} \to {}_R\mathscr{H}_{n,r}$ $(h \mapsto h^*)$ be the anti-isomorphism given by $T_i^* = T_i$ for $i = 0, 1, \dots, n-1$.

1.2. Let $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r_{>0}$ be an r-tuple of positive integers such that $m_k \ge n$ for any $k = 1, \dots, r$. Put

$$\Lambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \dots, \mu^{(r)}) \;\middle|\; \begin{array}{l} \mu^{(k)} = (\mu^{(k)}_1, \dots, \mu^{(k)}_{m_k}) \in \mathbb{Z}^{m_k}_{\ge 0}, \\ \displaystyle\sum_{k=1}^{r}\sum_{i=1}^{m_k} \mu^{(k)}_i = n \end{array} \right\}.$$

We denote by $|\mu^{(k)}| = \sum_{i=1}^{m_k} \mu^{(k)}_i$ (resp. $|\mu| = \sum_{k=1}^{r} |\mu^{(k)}|$) the size of $\mu^{(k)}$ (resp. the size of μ), and call an element of $\Lambda_{n,r}(\mathbf{m})$ an r-composition of size n. Put

$$\Lambda^+_{n,r} = \{\lambda \in \Lambda_{n,r}(\mathbf{m}) \mid \lambda^{(k)}_1 \ge \lambda^{(k)}_2 \ge \dots \ge \lambda^{(k)}_{m_k} \text{ for any } k = 1, \dots, r\}.$$

Then $\Lambda^+_{n,r}$ is the set of r-partitions of size n.

1.3. For $i = 1, \dots, n$, put $L_1 = T_0$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$. For $\mu \in \Lambda_{n,r}(\mathbf{m})$, put

$$m_\mu = \left(\sum_{w \in \mathfrak{S}_\mu} q^{l(w)} T_w\right)\left(\prod_{k=1}^{r}\prod_{i=1}^{a_k}(L_i - Q_k)\right), \quad M^\mu = m_\mu \cdot {}_R\mathscr{H}_{n,r},$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ with respect to μ, and $a_k = \sum_{j=1}^{k-1} |\mu^{(j)}|$ with $a_1 = 0$. The cyclotomic q-Schur algebra ${}_R\mathscr{S}_{n,r}$ associated to ${}_R\mathscr{H}_{n,r}$ is defined by

$${}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m})) = \operatorname{End}_{{}_R\mathscr{H}_{n,r}}\left(\bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} M^\mu\right).$$

REMARK 1.4. Let $\widetilde{\mathbf{m}} = (\widetilde{m}_1, \dots, \widetilde{m}_r) \in \mathbb{Z}^r_{>0}$ be such that $\widetilde{m}_k \ge n$ for any $k = 1, \dots, r$. Then it is known that ${}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}))$ is Morita equivalent to ${}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\widetilde{\mathbf{m}}))$ when R is a field.

1.5. In order to describe a presentation of ${}_R\mathscr{S}_{n,r}$, we prepare some notation.

Put $m = \sum_{k=1}^r m_k$, and let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$. Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, m-1$, then $\Pi = \{\alpha_i \mid 1 \le i \le m-1\}$ is the set of simple roots, and $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Put $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\ge 0}\alpha_i$. We define a partial order "$\ge$" on P, so-called dominance order, by $\lambda \ge \mu$ if $\lambda - \mu \in Q^+$.

Put $\Gamma(\mathbf{m}) = \{(i,k) \mid 1 \le i \le m_k,\ 1 \le k \le r\}$, and $\Gamma'(\mathbf{m}) = \Gamma(\mathbf{m}) \setminus \{(m_r, r)\}$. We identify the set $\Gamma(\mathbf{m})$ with the set $\{1, \dots, m\}$ by the bijection

$$\Gamma(\mathbf{m}) \to \{1, \dots, m\} \quad \text{such that} \quad (i,k) \mapsto \sum_{j=1}^{k-1} m_j + i.$$

Under this identification, we have

$$P = \bigoplus_{i=1}^{m} \mathbb{Z}\varepsilon_i = \bigoplus_{(i,k)\in\Gamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)},$$
$$Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i = \bigoplus_{(i,k)\in\Gamma'(\mathbf{m})} \mathbb{Z}\alpha_{(i,k)}.$$

Then we regard $\Lambda_{n,r}(\mathbf{m})$ as a subset of P by the injective map

$$\Lambda_{n,r}(\mathbf{m}) \to P \quad \text{such that} \quad \lambda \mapsto \sum_{(i,k)\in\Gamma(\mathbf{m})} \lambda_i^{(k)} \varepsilon_{(i,k)}.$$

For convenience, we consider $(m_k + 1, k) = (1, k+1)$ for $(m_k, k) \in \Gamma'(\mathbf{m})$ (resp. $(1 - 1, k) = (m_{k-1}, k-1)$ for $(1,k) \in \Gamma(\mathbf{m}) \setminus \{(1,1)\}$).

For $i = 1, \dots, n-1$, let $s_i = (i, i+1) \in \mathfrak{S}_n$ be the adjacent transposition. For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma'(\mathbf{m})$, put $N^\mu_{(i,k)} = \sum_{l=1}^{k-1} |\mu^{(l)}| + \sum_{j=1}^{i} \mu_j^{(k)}$,

$$X_\mu^{\mu+\alpha_{(i,k)}} = \Big\{1, s_{N^\mu_{(i,k)}}, s_{N^\mu_{(i,k)}} s_{N^\mu_{(i,k)}-1}, \dots, s_{N^\mu_{(i,k)}} s_{N^\mu_{(i,k)}-1} \cdots s_{N^\mu_{(i,k)}-\mu_i^{(k)}+1}\Big\},$$
$$X_\mu^{\mu-\alpha_{(i,k)}} = \Big\{1, s_{N^\mu_{(i,k)}}, s_{N^\mu_{(i,k)}} s_{N^\mu_{(i,k)}+1}, \cdots, s_{N^\mu_{(i,k)}} s_{N^\mu_{(i,k)}+1} \cdots s_{N^\mu_{(i,k)}+\mu_{i+1}^{(k)}-1}\Big\},$$

where we set $\mu^{(k)}_{m_k+1} = \mu_1^{(k+1)}$ if $i = m_k$.

For $(i,k) \in \Gamma'(\mathbf{m})$, we define the elements $E_{(i,k)}, F_{(i,k)} \in {}_R\mathscr{S}_{n,r}$ by

(1.5.1)

$$E_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_{i+1}^{(k)}+1} \left(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)} T_x^* \right) h^\mu_{+(i,k)} m_\mu \cdot h & \text{if} \quad \mu + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{if} \quad \mu + \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}), \end{cases}$$

(1.5.2)

$$F_{(i,k)}(m_\mu \cdot h) = \begin{cases} q^{-\mu_i^{(k)}+1}\left(\sum_{y \in X_\mu^{\mu-\alpha_{(i,k)}}} q^{l(y)} T_y^*\right) m_\mu \cdot h & \text{if} \quad \mu - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \text{if} \quad \mu - \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m}) \end{cases}$$

for $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in {}_R\mathscr{H}_{n,r}$, where $h^\mu_{+(i,k)} = \begin{cases} 1 & (i \neq m_k), \\ L_{N^\mu_{(i,k)}+1} - Q_{k+1} & (i = m_k). \end{cases}$

For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, we define the element $1_\lambda \in {}_R\mathscr{S}_{n,r}$ by

$$1_\lambda(m_\mu \cdot h) = \delta_{\lambda,\mu} m_\lambda \cdot h$$

for $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in {}_R\mathscr{H}_{n,r}$. From this definition, we see that $\{1_\lambda \mid \lambda \in \Lambda_{n,r}(\mathbf{m})\}$ is a set of pairwise orthogonal idempotents, and we have $1 = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda$.

For $\lambda \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma(\mathbf{m})$, we define $\sigma^\lambda_{(i,k)} \in {}_R\mathscr{S}_{n,r}$ by

$$\sigma^\lambda_{(i,k)}(m_\mu \cdot h) = \begin{cases} \delta_{\lambda,\mu}\big(m_\lambda\big(L_{N^\lambda_{(i,k)}} + L_{N^\lambda_{(i,k)}-1} + \cdots + L_{N^\lambda_{(i,k)}-\lambda_i^{(k)}+1}\big)\big) \cdot h & \text{if} \quad \lambda_i^{(k)} \neq 0, \\ 0 & \text{if} \quad \lambda_i^{(k)} = 0 \end{cases}$$

for $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $h \in {}_R\mathscr{H}_{n,r}$. For $(i,k) \in \Gamma(\mathbf{m})$, put

$$\sigma_{(i,k)} = \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} \sigma^\lambda_{(i,k)},$$

then $\sigma_{(i,k)}$ is a Jucys–Murphy element of ${}_R\mathscr{S}_{n,r}$ (See [13] for properties of Jucys–Murphy elements).

1.6. We need some non-commutative polynomials to give a presentation of ${}_R\mathscr{S}_{n,r}$ as follows. Put $\mathcal{A} = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_r]$, where $q, Q_1, \dots, Q_r$ are indeterminate over $\mathbb{Z}$, and let $\mathcal{K} = \mathbb{Q}(q, Q_1, \dots, Q_r)$ be the quotient field of $\mathcal{A}$.

Let $\mathcal{K}\langle\mathbf{x}\rangle$ (resp. $\mathcal{K}\langle\mathbf{y}\rangle$) be the non-commutative polynomial ring over $\mathcal{K}$ with indeterminate variables $\mathbf{x} = \{x_{(i,k)} \mid (i,k) \in \Gamma'(\mathbf{m})\}$ (resp. $\mathbf{y} = \{y_{(i,k)} \mid (i,k) \in \Gamma'(\mathbf{m})\}$). For $g(\mathbf{x}) \in \mathcal{K}\langle\mathbf{x}\rangle$ (resp. $g(\mathbf{y}) \in \mathcal{K}\langle\mathbf{y}\rangle$), let $g(F)$ (resp. $g(E)$) be the element of ${}_\mathcal{K}\mathscr{S}_{n,r}$ obtained by replacing $x_{(i,k)}$ (resp. $y_{(i,k)}$) with $E_{(i,k)}$ (resp. $F_{(i,k)}$). Moreover, for $g(\mathbf{x},\mathbf{y}) = \sum_j r_j g_j^-(\mathbf{x}) \otimes g_j^+(\mathbf{y}) \in \mathcal{K}\langle\mathbf{x}\rangle \otimes_\mathcal{K} \mathcal{K}\langle\mathbf{y}\rangle$ $(r_j \in \mathcal{K})$, put

$$g(F, E) = \sum_j r_j g_j^-(F) g_j^+(E) \in {}_\mathcal{K}\mathscr{S}_{n,r}.$$

Then we have the following lemma.

Lemma 1.7 ([18, Lemma 7.2]). *For $\lambda \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma(\mathbf{m})$, there exists a (non-commutative) polynomial $g^{\lambda}_{(i,k)}(\mathbf{x},\mathbf{y}) \in \mathcal{K}\langle\mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle\mathbf{y}\rangle$ such that*

$$\sigma^{\lambda}_{(i,k)} = g^{\lambda}_{(i,k)}(F,E)1_{\lambda} \tag{1.7.1}$$

in ${}_{\mathcal{K}}\mathscr{S}_{n,r}$.

We remark that a polynomial $g^{\lambda}_{(i,k)}(\mathbf{x},\mathbf{y}) \in \mathcal{K}\langle\mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle\mathbf{y}\rangle$ satisfying (1.7.1) is not unique in general. Thus we fix a polynomial $g^{\lambda}_{(i,k)}(\mathbf{x},\mathbf{y}) \in \mathcal{K}\langle\mathbf{x}\rangle \otimes_{\mathcal{K}} \mathcal{K}\langle\mathbf{y}\rangle$ ($\lambda \in \Lambda_{n,r}(\mathbf{m})$, $(i,k) \in \Gamma(\mathbf{m})$) satisfying (1.7.1).

For an integer $k \in \mathbb{Z}$, put $[k] = (q^k - q^{-k})/(q - q^{-1})$. For a positive integer $t \in \mathbb{Z}_{>0}$, put $[t]! = [t][t-1]\cdots[1]$ and set $[0]! = 1$.

Now we can describe a presentation of cyclotomic q-Schur algebras as follows.

Theorem 1.8 ([18, Theorem 7.16]). *${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is the associative algebra over $\mathcal{K}$ generated by $E_{(i,k)}$, $F_{(i,k)}$ ($(i,k) \in \Gamma'(\mathbf{m})$), 1_{λ} ($\lambda \in \Lambda_{n,r}(\mathbf{m})$) with the following defining relations*:

$$1_{\lambda}1_{\mu} = \delta_{\lambda,\mu}1_{\lambda}, \quad \sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_{\lambda} = 1, \tag{1.8.1}$$

$$E_{(i,k)}1_{\lambda} = \begin{cases} 1_{\lambda+\alpha_{(i,k)}}E_{(i,k)} & \textit{if } \lambda + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \textit{otherwise}, \end{cases} \tag{1.8.2}$$

$$F_{(i,k)}1_{\lambda} = \begin{cases} 1_{\lambda-\alpha_{(i,k)}}F_{(i,k)} & \textit{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \textit{otherwise}, \end{cases} \tag{1.8.3}$$

$$1_{\lambda}E_{(i,k)} = \begin{cases} E_{(i,k)}1_{\lambda-\alpha_{(i,k)}} & \textit{if } \lambda - \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \textit{otherwise}, \end{cases} \tag{1.8.4}$$

$$1_{\lambda}F_{(i,k)} = \begin{cases} F_{(i,k)}1_{\lambda+\alpha_{(i,k)}} & \textit{if } \lambda + \alpha_{(i,k)} \in \Lambda_{n,r}(\mathbf{m}), \\ 0 & \textit{otherwise}, \end{cases} \tag{1.8.5}$$

$$E_{(i,k)}F_{(j,l)} - F_{(j,l)}E_{(i,k)} = \delta_{(i,k),(j,l)} \sum_{\lambda \in \Lambda_{n,r}} \eta^{\lambda}_{(i,k)}, \tag{1.8.6}$$

where $$\eta^{\lambda}_{(i,k)} = \begin{cases} [\lambda^{(k)}_i - \lambda^{(k)}_{i+1}]1_{\lambda} & \textit{if } i \neq m_k, \\ (-Q_{k+1}[\lambda^{(k)}_{m_k} - \lambda^{(k+1)}_1] \\ \quad + q^{\lambda^{(k)}_{m_k} - \lambda^{(k+1)}_1}(q^{-1}g^{\lambda}_{(m_k,k)}(F,E) - qg^{\lambda}_{(1,k+1)}(F,E)))1_{\lambda} & \textit{if } i = m_k, \end{cases}$$

$$\begin{aligned} &E_{(i\pm1,k)}(E_{(i,k)})^2 - (q+q^{-1})E_{(i,k)}E_{(i\pm1,k)}E_{(i,k)} + (E_{(i,k)})^2E_{(i\pm1,k)} = 0, \\ &E_{(i,k)}E_{(j,l)} = E_{(j,l)}E_{(i,k)} \quad (|(i,k)-(j,l)| \geq 2), \end{aligned} \tag{1.8.7}$$

$$\begin{aligned} &F_{(i\pm1,k)}(F_{(i,k)})^2 - (q+q^{-1})F_{(i,k)}F_{(i\pm1,k)}F_{(i,k)} + (F_{(i,k)})^2F_{(i\pm1,k)} = 0, \\ &F_{(i,k)}F_{(j,l)} = F_{(j,l)}F_{(i,k)} \quad (|(i,k)-(j,l)| \geq 2), \end{aligned} \tag{1.8.8}$$

where $(i,k)-(j,l)=\big(\sum_{a=1}^{k-1}m_a+i\big)-\big(\sum_{b=1}^{l-1}m_b+j\big)$ *for* $(i,k),(j,l)\in\Gamma'(\mathbf{m})$.

Moreover, ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ *is isomorphic to the* $\mathcal{A}$*-subalgebra of* ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ *generated by* $E^l_{(i,k)}/[l]!$, $F^l_{(i,k)}/[l]!$ $((i,k)\in\Gamma'(\mathbf{m}),\ l\geq 1)$, 1_λ $(\lambda\in\Lambda_{n,r}(\mathbf{m}))$. *Then we can obtain the cyclotomic q-Schur algebra* ${}_R\mathscr{S}_{n,r}$ *over R as the specialized algebra* $R\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathscr{S}_{n,r}$ *of* ${}_{\mathcal{A}}\mathscr{S}_{n,r}$.

1.9. Weyl modules (see [18] for more details). Let ${}_{\mathcal{A}}\mathscr{S}^+_{n,r}$ (resp. ${}_{\mathcal{A}}\mathscr{S}^-_{n,r}$) be the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by $E^l_{(i,k)}/[l]!$ (resp. $F^l_{(i,k)}/[l]!$) for $(i,k)\in\Gamma'(\mathbf{m})$ and $l\geq 1$. Let ${}_{\mathcal{A}}\mathscr{S}^0_{n,r}$ be the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by 1_λ for $\lambda\in\Lambda_{n,r}(\mathbf{m})$. Then ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ has the triangular decomposition ${}_{\mathcal{A}}\mathscr{S}_{n,r}={}_{\mathcal{A}}\mathscr{S}^-_{n,r}{}_{\mathcal{A}}\mathscr{S}^0_{n,r}{}_{\mathcal{A}}\mathscr{S}^+_{n,r}$ by [18, Proposition 3.2, Theorem 4.12, Theorem 5.6, Proposition 6.4, Proposition 7.7 and Theorem 7.16]. We denote by ${}_{\mathcal{A}}\mathscr{S}^{\geq 0}_{n,r}$ the subalgebra of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ generated by ${}_{\mathcal{A}}\mathscr{S}^+_{n,r}$ and ${}_{\mathcal{A}}\mathscr{S}^0_{n,r}$.

Note that ${}_R\mathscr{S}_{n,r}$ is the specialized algebra $R\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathscr{S}_{n,r}$. We denote by $E^{(l)}_{(i,k)}$ (resp. $F^{(l)}_{(i,k)}$, 1_λ) the elements $1\otimes E^l_{(i,k)}/[l]!$ (resp. $1\otimes F^l_{(i,k)}/[l]!$, $1\otimes 1_\lambda$) of $R\otimes_{\mathcal{A}}{}_{\mathcal{A}}\mathscr{S}_{n,r}$. Then ${}_R\mathscr{S}_{n,r}$ also has the triangular decomposition

$$ {}_R\mathscr{S}_{n,r}={}_R\mathscr{S}^-_{n,r}\,{}_R\mathscr{S}^0_{n,r}\,{}_R\mathscr{S}^+_{n,r} $$

which comes from the triangular decomposition of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$.

For $\lambda\in\Lambda^+_{n,r}$, we define the one-dimensional ${}_R\mathscr{S}^{\geq 0}_{n,r}$-module $\Theta_\lambda=Rv_\lambda$ by $E^{(l)}_{(i,k)}\cdot v_\lambda=0$ $((i,k)\in\Gamma'(\mathbf{m}),\ l\geq 1)$ and $1_\mu\cdot v_\lambda=\delta_{\lambda,\mu}v_\lambda$ $(\mu\in\Lambda_{n,r}(\mathbf{m}))$. Then the Weyl module ${}_R\Delta_n(\lambda)$ of ${}_R\mathscr{S}_{n,r}$ is defined as the induced module of Θ_λ:

$$ {}_R\Delta_n(\lambda)={}_R\mathscr{S}_{n,r}\otimes_{{}_R\mathscr{S}^{\geq 0}_{n,r}}\Theta_\lambda. $$

See also [18, paragraph 3.3 and Theorem 3.4] for definitions of ${}_R\Delta_n(\lambda)$.

It is known that ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is semi-simple, and that $\{{}_{\mathcal{K}}\Delta_n(\lambda)\mid\lambda\in\Lambda^+_{n,r}\}$ gives a complete set of pairwise non-isomorphic (left) simple ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-modules.

1.10. Highest weight modules. Let M be an ${}_R\mathscr{S}_{n,r}$-module. We say that an element $m\in M$ is a primitive vector if $E^{(l)}_{(i,k)}\cdot m=0$ for any $(i,k)\in\Gamma'(\mathbf{m})$ and $l\geq 1$, and say that $m\in M$ is a weight vector of weight μ if $1_\mu\cdot m=m$ for $\mu\in\Lambda_{n,r}(\mathbf{m})$. If $x_\lambda\in M$ is a primitive and a weight vector of weight λ, we say that x_λ is a highest weight vector of weight λ. If M is generated by a highest weight vector $x_\lambda\in M$ of weight λ as an ${}_R\mathscr{S}_{n,r}$-module, we say that M is a highest weight module of highest weight λ. It is clear that the Weyl module ${}_R\Delta_n(\lambda)$ $(\lambda\in\Lambda^+_{n,r})$ is a highest weight module. Moreover, we have the following universality of the Weyl modules.

Lemma 1.11. *Let M be an* ${}_R\mathscr{S}_{n,r}$*-module. If M is a highest weight module of highest weight* λ, *there exists a surjective* ${}_R\mathscr{S}_{n,r}$*-homomorphism* ${}_R\Delta_n(\lambda)\to M$ *such that* $1\otimes v_\lambda\mapsto x_\lambda$, *where* x_λ *is a highest weight vector of M.*

1.12. In [4], it was proven by combinatorial arguments that ${}_R\mathscr{H}_{n,r}$ (resp. ${}_R\mathscr{S}_{n,r}$) is a cellular algebra. We review several properties from [4] for later arguments.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, the diagram $[\mu]$ of μ is the set

$$[\mu] = \{(i, j, k) \in \mathbb{Z}^3 \mid 1 \le i \le m_k,\ 1 \le j \le \mu_i^{(k)},\ 1 \le k \le r\}.$$

For $\lambda \in \Lambda_{n,r}^+$ and $x \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, r\}$, we say that x is a removable node (resp. an addable node) of λ if $[\lambda] \setminus \{x\}$ (resp. $[\lambda] \cup \{x\}$) is the diagram of a certain r-partition $\mu \in \Lambda_{n-1,r}^+$ (resp. $\mu \in \Lambda_{n+1,r}^+$). In such case, we denote the above $\mu \in \Lambda_{n-1,r}^+$ (resp. $\mu \in \Lambda_{n+1,r}^+$) by $\lambda \setminus x$ (resp. $\lambda \cup x$), namely $[\lambda \setminus x] = [\lambda] \setminus \{x\}$ (resp. $[\lambda \cup x] = [\lambda] \cup \{x\}$).

We define a partial order "$\succeq$" on $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \dots, r\}$ by

$$(i, j, k) \succ (i', j', k') \quad \text{if} \quad k < k', \quad \text{or if} \quad k = k' \quad \text{and} \quad i < i'.$$

We also define a partial order "$\succeq$" on $\mathbb{Z}_{>0} \times \{1, \dots, r\}$ by

$$(i, k) \succ (i', k') \quad \text{if} \quad (i, 1, k) \succ (i', 1, k').$$

For $\lambda \in \Lambda_{n,r}^+$, a standard tableau $\mathfrak{t}$ of shape λ is a bijection

$$\mathfrak{t}\colon [\lambda] \to \{1, 2, \dots, n\}$$

satisfying the following two conditions:
(i) $\mathfrak{t}((i, j, k)) < \mathfrak{t}((i, j+1, k))$ if $(i, j+1, k) \in [\lambda]$,
(ii) $\mathfrak{t}((i, j, k)) < \mathfrak{t}((i+1, j, k))$ if $(i+1, j, k) \in [\lambda]$.
We denote by $\mathrm{Std}(\lambda)$ the set of standard tableaux of shape λ.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$, we define the bijection $\mathfrak{t}^\mu\colon [\mu] \to \{1, 2, \dots, n\}$ as

$$\mathfrak{t}^\mu((i, j, k)) = \sum_{c=1}^{k-1} |\mu^{(c)}| + \sum_{a=1}^{i-1} \mu_a^{(k)} + j.$$

It is clear that $\mathfrak{t}^\lambda \in \mathrm{Std}(\lambda)$ for $\lambda \in \Lambda_{n,r}^+$. For $\mathfrak{t} \in \mathrm{Std}(\lambda)$, we define $d(\mathfrak{t}) \in \mathfrak{S}_n$ as

$$\mathfrak{t}((i, j, k)) = d(\mathfrak{t})(\mathfrak{t}^\lambda(i, j, k)) \quad ((i, j, k) \in [\lambda]).$$

For $\lambda \in \Lambda_{n,r}^+$ and $\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$, set $m_{\mathfrak{s}\mathfrak{t}} = T_{d(\mathfrak{s})}^* m_\lambda T_{d(\mathfrak{t})} \in {}_R\mathscr{H}_{n,r}$. Then we have the following theorem.

Theorem 1.13 ([4, Theorem 3.26])**.** *${}_R\mathscr{H}_{n,r}$ is a cellular algebra with a cellular basis $\{m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda)$ for some $\lambda \in \Lambda_{n,r}^+\}$ with respect to the poset $(\Lambda_{n,r}^+, \ge)$. In particular, we have $m_{\mathfrak{s}\mathfrak{t}}^* = m_{\mathfrak{t}\mathfrak{s}}$.*

1.14. For $\lambda \in \Lambda^+_{n,r}$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, a tableau of shape λ with weight μ is a map

$$T : [\lambda] \to \{(a, c) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq 1,\ 1 \leq c \leq r\}$$

such that $\mu_i^{(k)} = \#\{x \in [\lambda] \mid T(x) = (i, k)\}$. We define the reverse lexicographic order on $\mathbb{Z} \times \mathbb{Z}$ by $(a, c) \geq (a', c')$ either if $c > c'$, or if $c = c'$ and $a \geq a'$. For a tableau T of shape λ with weight μ, we say that T is semi-standard if T satisfies the following conditions:
(i) If $T((i, j, k)) = (a, c)$, then $k \leq c$,
(ii) $T((i, j, k)) \leq T((i, j + 1, k))$ if $(i, j + 1, k) \in [\lambda]$,
(iii) $T((i, j, k)) < T((i + 1, j, k))$ if $(i + 1, j, k) \in [\lambda]$.
For $\lambda \in \Lambda^+_{n,r}$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, we denote by $\mathcal{T}_0(\lambda, \mu)$ the set of semi-standard tableaux of shape λ with weight μ. Put $\mathcal{T}_0(\lambda) = \bigcup_{\mu \in \Lambda_{n,r}(\mathbf{m})} \mathcal{T}_0(\lambda, \mu)$.

For $\lambda \in \Lambda^+_{n,r}$, let T^λ be the tableau of shape λ with weight λ such that

$$T^\lambda((i, j, k)) = (i, k).$$

It is clear that T^λ is semi-standard, and it is the unique semi-standard tableau of shape λ with weight λ. Namely, we have $\mathcal{T}_0(\lambda, \lambda) = \{T^\lambda\}$.

For $\mathfrak{t} \in \mathrm{Std}(\lambda)$ $(\lambda \in \Lambda^+_{n,r})$ and $\mu \in \Lambda_{n,r}$, we define the tableau $\mu(\mathfrak{t})$ of shape λ with weight μ by

$$\mu(\mathfrak{t})((i, j, k)) = (a, c), \quad \text{if} \quad \mathfrak{t}^\mu((a, b, c)) = \mathfrak{t}((i, j, k)) \quad \text{for some} \quad b.$$

Namely, $\mu(\mathfrak{t})$ is obtained by replacing each entry $x = \mathfrak{t}((i, j, k))$ in $\mathfrak{t}$ by (a, c) if x appears in the a-th row of the c-th component of $\mathfrak{t}^\mu$ (see [4, Example (4.3)] for examples).

For $S \in \mathcal{T}_0(\lambda, \mu)$, $T \in \mathcal{T}_0(\lambda, \nu)$ $(\lambda \in \Lambda^+_{n,r},\ \mu, \nu \in \Lambda_{n,r}(\mathbf{m}))$, put

$$m_{ST} = \sum_{\substack{\mathfrak{s}, \mathfrak{t} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{s}) = S, \nu(\mathfrak{t}) = T}} q^{l(d(\mathfrak{s})) + l(d(\mathfrak{t}))} m_{\mathfrak{s}\mathfrak{t}},$$

and define the element $\varphi_{ST} \in {}_R\mathscr{S}_{n,r}$ by

$$\varphi_{ST}(m_\tau \cdot h) = \delta_{\nu,\tau} m_{ST} \cdot h \quad (\tau \in \Lambda_{n,r}(\mathbf{m}),\ h \in {}_R\mathscr{H}_{n,r}).$$

Then we have the following theorem.

Theorem 1.15 ([4, Theorem 6.6])**.** *${}_R\mathscr{S}_{n,r}$ is a cellular algebra with a cellular basis $\{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda)$ for some $\lambda \in \Lambda^+_{n,r}\}$ with respect to the poset $(\Lambda^+_{n,r}, \geq)$. In particular, there exists an anti-automorphism $\theta_n : {}_R\mathscr{S}_{n,r} \to {}_R\mathscr{S}_{n,r}$ such that $\theta_n(\varphi_{ST}) = \varphi_{TS}$. Moreover, ${}_R\mathscr{S}_{n,r}$ is a quasi-hereditary algebra when R is a field.*

For the anti-automorphism θ_n of ${}_R\mathscr{S}_{n,r}$ introduced in Theorem 1.15, we have the following lemma.

Lemma 1.16. *For $\mu \in \Lambda_{n,r}(\mathbf{m})$, $(i,k) \in \Gamma'(\mathbf{m})$ and $l \geq 1$, we have that*

(1.16.1) $$\theta_n(1_\mu) = 1_\mu,$$

(1.16.2) $$\theta_n(E^{(l)}_{(i,k)}1_\mu) = q^{l\cdot(\mu^{(k)}_i - \mu^{(k)}_{i+1} + l)} 1_\mu F^{(l)}_{(i,k)},$$

(1.16.3) $$\theta_n(1_\mu F^{(l)}_{(i,k)}) = q^{-l\cdot(\mu^{(k)}_i - \mu^{(k)}_{i+1} + l)} E^{(l)}_{(i,k)} 1_\mu.$$

Proof. Note that θ_n on ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ is the restriction to ${}_{\mathcal{A}}\mathscr{S}_{n,r}$ of θ_n on ${}_{\mathcal{K}}\mathscr{S}_{n,r}$, and it is enough to show the case where $R = \mathcal{K}$ by using the argument of specialization. Moreover, it is enough to show the case where $l = 1$ since we can obtain the statements for $l \geq 2$ by the inductive arguments thanks to the equation

$$\begin{aligned}\theta_n(E^{(l)}_{(i,k)}1_\mu) &= \theta_n(1/[l](E^{(l-1)}_{(i,k)}1_{\mu+\alpha_{(i,k)}})(E_{(i,k)}1_\mu)) \\ &= 1/[l]\theta_n(E_{(i,k)}1_\mu)\theta_n(E^{(l-1)}_{(i,k)}1_{\mu+\alpha_{(i,k)}}).\end{aligned}$$

From the definitions, we see that $1_\lambda = \varphi_{T^\lambda T^\lambda}$, and we obtain $\theta_n(1_\lambda) = 1_\lambda$. By [18, paragraphs 5.5, 6.2, and Lemma 6.10], we see that

$$E_{(i,k)}(m_\mu) = q^{\mu^{(k)}_i - \mu^{(k)}_{i+1} + 1}(F_{(i,k)}(m_{\mu+\alpha_{(i,k)}}))^*.$$

Combining with [4, Proposition 6.9 and Lemma 6.10], we have that

$$\begin{aligned}\theta_n(E_{(i,k)}1_\mu)(m_{\mu+\alpha_{(i,k)}}) &= (E_{(i,k)}(m_\mu))^* \\ &= q^{\mu^{(k)}_i - \mu^{(k)}_{i+1} + 1}((F_{(i,k)}(m_{\mu+\alpha_{(i,k)}}))^*)^* \\ &= q^{\mu^{(k)}_i - \mu^{(k)}_{i+1} + 1}F_{(i,k)}(m_{\mu+\alpha_{(i,k)}}),\end{aligned}$$

and $\theta_n(E_{(i,k)}1_\mu)(m_\tau) = 0$ unless $\tau = \mu + \alpha_{(i,k)}$. Thus, we have that

$$\theta_n(E_{(i,k)}1_\mu) = q^{\mu^{(k)}_i - \mu^{(k)}_{i+1} + 1}F_{(i,k)}1_{\mu+\alpha_{(i,k)}} = q^{\mu^{(k)}_i - \mu^{(k)}_{i+1} + 1}1_\mu F_{(i,k)}.$$

Now we proved (1.16.2), and, by applying θ_n to the equation (1.16.2), we obtain (1.16.3). □

1.17. From the definitions, we have that

(1.17.1) $$1_\tau \varphi_{TS} = \delta_{\mu,\tau}\varphi_{TS} \quad \text{and} \quad \varphi_{TS}1_\tau = \delta_{\nu,\tau}\varphi_{TS}$$

for $T \in \mathcal{T}_0(\lambda,\mu)$, $S \in \mathcal{T}_0(\lambda,\nu)$ (see [4]).

For $\lambda \in \Lambda^+_{n,r}$, let ${}_R\mathscr{S}_{n,r}(> \lambda)$ be the R-submodule of ${}_R\mathscr{S}_{n,r}$ spanned by

$$\{\varphi_{ST} \mid S, T \in \mathcal{T}_0(\lambda') \text{ for some } \lambda' \in \Lambda^+_{n,r} \text{ such that } \lambda' > \lambda\},$$

and ${}_RW(\lambda)$ be the R-submodule of ${}_R\mathscr{S}_{n,r}/{}_R\mathscr{S}_{n,r}(>\lambda)$ spanned by

$$\{\varphi_{TT^\lambda} + {}_R\mathscr{S}_{n,r}(>\lambda) \mid T \in \mathcal{T}_0(\lambda)\}.$$

Then, thanks to the general theory of cellular algebras, ${}_R\mathscr{S}_{n,r}(>\lambda)$ turns out to be a two-sided ideal of ${}_R\mathscr{S}_{n,r}$, and ${}_RW(\lambda)$ turns out to be an ${}_R\mathscr{S}_{n,r}$-submodule of ${}_R\mathscr{S}_{n,r}/{}_R\mathscr{S}_{n,r}(>\lambda)$ whose action comes from the multiplication of ${}_R\mathscr{S}_{n,r}$. Put $\varphi_T = \varphi_{TT^\lambda} + {}_R\mathscr{S}_{n,r}(>\lambda)$ for $T \in \mathcal{T}_0(\lambda)$, then $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$ gives an R-free basis of ${}_RW(\lambda)$, and it is known that ${}_RW(\lambda)$ is generated by φ_{T^λ} as an ${}_R\mathscr{S}_{n,r}$-module.

Lemma 1.18. *For each $\lambda \in \Lambda^+_{n,r}$, there exists the ${}_R\mathscr{S}_{n,r}$-isomorphism*

$$ {}_R\Delta_n(\lambda) \to {}_RW(\lambda) \quad \textit{such that} \quad 1 \otimes v_\lambda \mapsto \varphi_{T^\lambda}.$$

Proof. From the definition of semi-standard tableaux, we see that $\lambda \geq \mu$ if $\mathcal{T}_0(\lambda, \mu) \neq \emptyset$. Thus, we have that $1_\mu \cdot {}_RW(\lambda) = 0$ unless $\lambda \geq \mu$. On the other hand, we have that

$$E^{(l)}_{(i,k)} \cdot \varphi_{T^\lambda} = E^{(l)}_{(i,k)} 1_\lambda \cdot \varphi_{T^\lambda} = 1_{\lambda + l\cdot\alpha_{(i,k)}} E_{(i,k)} \cdot \varphi_{T^\lambda}$$

for $(i,k) \in \Gamma'(\mathbf{m})$ and $l \geq 1$. Noting that $\lambda + l \cdot \alpha_{(i,k)} > \lambda$, these imply that

$$E^{(l)}_{(i,k)} \cdot \varphi_{T^\lambda} = 0 \quad \text{for any} \quad (i,k) \in \Gamma'(\mathbf{m}) \quad \text{and} \quad l \geq 1.$$

Thus, ${}_RW(\lambda)$ is a highest weight module with a highest weight vector φ_{T^λ} of highest weight λ. Then, by Lemma 1.11, we have the surjective homomorphism

$$X_R \colon {}_R\Delta_n(\lambda) \to {}_RW(\lambda) \quad \text{such that} \quad 1 \otimes v_\lambda \mapsto \varphi_{T^\lambda}.$$

We should show that this homomorphism is an isomorphism, and it is enough to show the case where $R = \mathcal{A}$ by the arguments of specializations. First, we consider the case where $R = \mathcal{K}$. In this case, it is known that ${}_\mathcal{K}\mathscr{S}_{n,r}$ is semi-simple, and ${}_\mathcal{K}\Delta_n(\lambda)$ is irreducible ([18, Theorem 2.16 (iv)]). Thus, $X_\mathcal{K}$ is injective. On the other hand, $X_\mathcal{K}$ is obtained from $X_\mathcal{A}$ by applying the right exact functor $\mathcal{K} \otimes_\mathcal{A} ?$, and the restriction of $X_\mathcal{K}$ to ${}_\mathcal{A}\Delta_n(\lambda)$ coincides with $X_\mathcal{A}$. Thus, we have that $X_\mathcal{A}$ is injective, hence, $X_\mathcal{A}$ is an isomorphism. □

REMARK 1.19. Lemma 1.18 was already proved in [18] combined with [7, Theorem 5.16] implicitly. However, this identification is important in the later arguments, we gave the proof by using the universality.

1.20. Recall that ${}_R\mathscr{S}_{n,r}$ has the algebra anti-automorphism θ_n, and we can consider the contravariant functor $\circledast \colon \mathscr{S}_{n,r}\text{-mod} \to \mathscr{S}_{n,r}\text{-mod}$ with respect to θ_n (see conventions in the last of §0). For $\lambda \in \Lambda^+_{n,r}$, put ${}_R\nabla_n(\lambda) = ({}_R\Delta_n(\lambda))^\circledast$. Then $\{{}_R\Delta_n(\lambda) \mid$

$\lambda \in \Lambda^+_{n,r}\}$ (resp. $\{{}_R\nabla_n(\lambda) \mid \lambda \in \Lambda^+_{n,r}\}$ gives a set of standard modules (resp. a set of costandard modules) of ${}_R\mathscr{S}_{n,r}$ in terms of quasi-hereditary algebras when R is a field.

When R is a field, Let ${}_R\mathscr{S}_{n,r}\text{-mod}^{\Delta}$ (resp. ${}_R\mathscr{S}_{n,r}\text{-mod}^{\nabla}$) be the full subcategory of ${}_R\mathscr{S}_{n,r}$-mod consisting of modules which have a filtration such that its successive quotients are isomorphic to standard modules (resp. costandard modules).

2. Induction and restriction functors

In this section, we give an injective homomorphism of algebras from a cyclotomic q-Schur algebra of rank n to one of rank $n+1$. By using this embedding, we define induction and restriction functors between module categories of these two algebras.

2.1. From now on, throughout this paper, we argue under the following setting:

$$\begin{aligned} &\mathbf{m} = (m_1, \dots, m_r) \quad \text{such that} \quad m_k \geq n+1 \quad \text{for all} \quad k = 1, \dots, r, \\ &\mathbf{m}' = (m_1, \dots, m_{r-1}, m_r - 1), \\ &{}_R\mathscr{S}_{n+1,r} = {}_R\mathscr{S}_{n+1,r}(\Lambda_{n+1,r}(\mathbf{m})), \\ &{}_R\mathscr{S}_{n,r} = {}_R\mathscr{S}_{n,r}(\Lambda_{n,r}(\mathbf{m}')). \end{aligned} \tag{2.1.1}$$

We will omit the subscript R when there is no risk to confuse.

REMARK 2.2. We should choose $\mathbf{m}$ and $\mathbf{m}'$ as in (2.1.1) when we consider an embedding from $\mathscr{S}_{n,r}$ to $\mathscr{S}_{n+1,r}$. However, in fact, we can take $\mathbf{m}$ (resp. $\mathbf{m}'$) freely through the Morita equivalence if R is a field (see Remark 1.4).

2.3. We define an injective map

$$\gamma \colon \Lambda_{n,r}(\mathbf{m}') \to \Lambda_{n+1,r}(\mathbf{m}), \quad (\lambda^{(1)}, \dots, \lambda^{(r-1)}, \lambda^{(r)}) \mapsto (\lambda^{(1)}, \dots, \lambda^{(r-1)}, \hat{\lambda}^{(r)}),$$

where $\hat{\lambda}^{(r)} = (\lambda^{(r)}_1, \dots, \lambda^{(r)}_{m_r-1}, 1)$. Put $\Lambda^{\gamma}_{n+1,r}(\mathbf{m}) = \mathrm{Im}\gamma$, and we have

$$\Lambda^{\gamma}_{n+1,r}(\mathbf{m}) = \{\mu = (\mu^{(1)}, \dots, \mu^{(r)}) \in \Lambda_{n+1,r}(\mathbf{m}) \mid \mu^{(r)}_{m_r} = 1\}.$$

For $\lambda \in \Lambda^+_{n+1,r}$ and $\mathfrak{t} \in \mathrm{Std}(\lambda)$, let $\mathfrak{t} \setminus (n+1)$ be the standard tableau obtained by removing the node x such that $\mathfrak{t}(x) = n+1$, and denote the shape of $\mathfrak{t} \setminus (n+1)$ by $|\mathfrak{t} \setminus (n+1)|$. Note that x (in the last sentence) is a removable node of λ, and that $|\mathfrak{t} \setminus (n+1)| = \lambda \setminus x$.

For $\lambda \in \Lambda^+_{n+1,r}$, $\mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})$ and $T \in \mathcal{T}_0(\lambda, \mu)$, let $T \setminus (m_r, r)$ be the tableau obtained by removing the node x such that $T(x) = (m_r, r)$, and denote the shape of $T \setminus (m_r, r)$ by $|T \setminus (m_r, r)|$. Note that x (in the last sentence) is a removable node of λ, and that $|T \setminus (m_r, r)| = \lambda \setminus x$. It is clear that $T \setminus (m_r, r) \in \mathcal{T}_0(\lambda \setminus x, \gamma^{-1}(\mu))$.

For $\lambda \in \Lambda^+_{n+1,r}$ and a removable node x of λ, we define the semi-standard tableau $T^\lambda_x \in \mathcal{T}_0(\lambda)$ by

$$T^\lambda_x(a, b, c) = \begin{cases} (a, c) & \text{if } (a, b, c) \neq x, \\ (m_r, r) & \text{if } (a, b, c) = x. \end{cases} \tag{2.3.1}$$

From the definitions, we see that $T^\lambda_x \in \bigcup_{\mu \in \Lambda^\gamma_{n+1,r}(\mathbf{m})} \mathcal{T}_0(\lambda, \mu)$, and that $T^\lambda_x \setminus (m_r, r) = T^{\lambda \setminus x}$.

2.4. Let $\mathcal{K}\langle \mathbf{x} \rangle$ (resp. $\mathcal{K}\langle \mathbf{x}' \rangle$) be the non-commutative polynomial ring over $\mathcal{K}$ with indeterminate variables $\mathbf{x} = \{x_{(i,k)} \mid (i, k) \in \Gamma'(\mathbf{m})\}$ (resp. $\mathbf{x}' = \{x_{(i,k)} \mid (i, k) \in \Gamma'(\mathbf{m}')\}$). Note that $\Gamma'(\mathbf{m}') = \Gamma'(\mathbf{m}) \setminus \{(m_r - 1, r)\}$, and we have the natural injective map $\mathcal{K}\langle \mathbf{m}' \rangle \to \mathcal{K}\langle \mathbf{m} \rangle$ such that $x_{(i,k)} \mapsto x_{(i,k)}$ for $(i, k) \in \Gamma'(\mathbf{m}')$. Under this injection, we regard a polynomial in $\mathcal{K}\langle \mathbf{m}' \rangle$ as a polynomial in $\mathcal{K}\langle \mathbf{m} \rangle$. It is similar for $\mathcal{K}\langle \mathbf{y} \rangle$ and $\mathcal{K}\langle \mathbf{y}' \rangle$.

For the polynomials $g^\lambda_{(i,k)}(\mathbf{x}', \mathbf{y}') \in \mathcal{K}\langle \mathbf{x}' \rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{y}' \rangle$ ($\lambda \in \Lambda_{n,r}(\mathbf{m}')$, $(i, k) \in \Gamma(\mathbf{m}')$) such as in Lemma 1.7, we have the following lemma.

Lemma 2.5. *Let $\lambda \in \Lambda_{n,r}(\mathbf{m}')$ and $(i, k) \in \Gamma'(\mathbf{m}')$. For $g^\lambda_{(i,k)}(\mathbf{x}', \mathbf{y}') \in \mathcal{K}\langle \mathbf{x}' \rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{y}' \rangle$ such that $g^\lambda_{(i,k)}(F, E)1_\lambda = \sigma^\lambda_{(i,k)}$ in ${}_{\mathcal{K}}\mathscr{S}_{n,r}$, we have*

$$g^\lambda_{(i,k)}(F, E)1_{\gamma(\lambda)} = \sigma^{\gamma(\lambda)}_{(i,k)}$$

in ${}_{\mathcal{K}}\mathscr{S}_{n+1,r}$. In particular, we can take $g^\lambda_{(i,k)}(\mathbf{x}', \mathbf{y}')$ as a polynomial $g^{\gamma(\lambda)}_{(i,k)}(\mathbf{x}, \mathbf{y})$ satisfying (1.7.1) in ${}_{\mathcal{K}}\mathscr{S}_{n+1,r}$.

Proof. Let $\iota : {}_{\mathcal{K}}\mathscr{H}_{n,r} \to {}_{\mathcal{K}}\mathscr{H}_{n+1,r}$ be the natural injective homomorphism defined by $T_i \mapsto T_i$ $(0 \leq i \leq n-1)$. Then we have $\iota(L_j) = L_j$ for $j = 1, \dots, n$.

By (1.5.1) and (1.5.2) (see also [18, Lemma 6.10]) we can check that

$$\iota(E_{(i,k)}(m_\lambda)) = E_{(i,k)}(m_{\gamma(\lambda)}), \quad \iota(F_{(i,k)}(m_\lambda)) = F_{(i,k)}(m_{\gamma(\lambda)}).$$

These imply that, for $g(\mathbf{x}', \mathbf{y}') \in \mathcal{K}\langle \mathbf{x}' \rangle \otimes_{\mathcal{K}} \mathcal{K}\langle \mathbf{y}' \rangle$, we have

$$\iota(g(F, E)(m_\lambda)) = g(F, E)(m_{\gamma(\lambda)})$$

in ${}_{\mathcal{K}}\mathscr{H}_{n+1,r}$. On the other hand, from the definition of $\sigma^\lambda_{(i,k)}$, we have

$$\iota(\sigma^\lambda_{(i,k)}(m_\lambda)) = \sigma^{\gamma(\lambda)}_{(i,k)}(m_{\gamma(\lambda)}).$$

Then we have

$$\begin{aligned} g^\lambda_{(i,k)}(F, E)1_\lambda = \sigma^\lambda_{(i,k)} &\Leftrightarrow g^\lambda_{(i,k)}(F, E)(m_\lambda) = \sigma^\lambda_{(i,k)}(m_\lambda) \\ &\Leftrightarrow \iota(g^\lambda_{(i,k)}(F, E)(m_\lambda)) = \iota(\sigma^\lambda_{(i,k)}(m_\lambda)) \\ &\Leftrightarrow g^\lambda_{(i,k)}(F, E)(m_{\gamma(\lambda)}) = \sigma^{\gamma(\lambda)}_{(i,k)}(m_{\gamma(\lambda)}) \\ &\Leftrightarrow g^\lambda_{(i,k)}(F, E)1_{\gamma(\lambda)} = \sigma^{\gamma(\lambda)}_{(i,k)}. \end{aligned}$$

□

Now, we can define the injective homomorphism from $\mathscr{S}_{n,r}$ to $\mathscr{S}_{n+1,r}$ as the following proposition.

Proposition 2.6. *There exists the algebra homomorphism* $\iota\colon \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$ *such that*

$$E^{(l)}_{(i,k)} \mapsto E^{(l)}_{(i,k)}\xi, \quad F^{(l)}_{(i,k)} \mapsto F^{(l)}_{(i,k)}\xi, \quad 1_\lambda \mapsto 1_{\gamma(\lambda)} \tag{2.6.1}$$

for $(i,k) \in \Gamma'(\mathbf{m}')$, $l \geq 1$, $\lambda \in \Lambda_{n,r}(\mathbf{m}')$, *where* $\xi = \sum_{\lambda \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})} 1_\lambda$ *is an idempotent of* $\mathscr{S}_{n+1,r}$. *In particular, we have that* $\iota(1_{\mathscr{S}_{n,r}}) = \xi$, *and that* $\iota(\mathscr{S}_{n,r}) \subsetneq \xi \mathscr{S}_{n+1,r}\xi$, *where* $1_{\mathscr{S}_{n,r}}$ *is the unit element of* $\mathscr{S}_{n,r}$. *Moreover,* ι *is injective.*

Proof. If ι is well-defined injective homomorphism given by (2.6.1), we easily see that $\iota(1_{\mathscr{S}_{n,r}}) = \xi$, and that $\iota(\mathscr{S}_{n,r}) \subsetneq \xi\mathscr{S}_{n+1,r}\xi$. Hence, it is enough to show the well-definedness and injectivity of ι.

First, we prove the statements for the algebras over $\mathcal{K}$. In order to see the well-definedness of the homomorphism $\iota_{\mathcal{K}}\colon {}_{\mathcal{K}}\mathscr{S}_{n,r} \to {}_{\mathcal{K}}\mathscr{S}_{n+1,r}$ defined by (2.6.1), we should check the relations (1.8.1)–(1.8.8). For the relations except (1.8.6), it is clear, and we can check the relation (1.8.6) by Lemma 2.5.

We show that $\dim_{\mathcal{K}} {}_{\mathcal{K}}\mathscr{S}_{n,r} = \dim_{\mathcal{K}} \iota_{\mathcal{K}}({}_{\mathcal{K}}\mathscr{S}_{n,r})$, then this equality implies that $\iota_{\mathcal{K}}$ is injective.

For $\lambda \in \Lambda^+_{n,r}$, let x be the addable node of λ such that x is minimum for the order $\succeq$ on the set of all addable nodes of λ, and put $\hat{\lambda} = \lambda \cup x$. Thus, we have $\hat{\lambda} \in \Lambda^+_{n+1,r}$. Note that x is a removable node of $\hat{\lambda}$, we can take the semi-standard tableau $T^{\hat{\lambda}}_x$ defined by (2.3.1). From the definitions, we see that $T^{\hat{\lambda}}_x \in \mathcal{T}_0(\hat{\lambda}, \gamma(\lambda))$. When we regard ${}_{\mathcal{K}}\Delta_{n+1}(\hat{\lambda})$ as an ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-module through the homomorphism $\iota_{\mathcal{K}}$, we see that $\varphi_{T^{\hat{\lambda}}_x}$ is a weight vector of weight λ since $\iota_{\mathcal{K}}(1_\lambda) = 1_{\gamma(\lambda)}$ and $T^{\hat{\lambda}}_x \in \mathcal{T}_0(\hat{\lambda}, \gamma(\lambda))$. On the other hand, for $(i,k) \in \Gamma'(\mathbf{m}')$, we have $\gamma(\lambda) + \alpha_{(i,k)} \not\trianglelefteq \hat{\lambda}$ since x is minimum in the set of all addable nodes of λ. Thus, we have $\mathcal{T}_0(\hat{\lambda}, \gamma(\lambda) + \alpha_{(i,k)}) = \emptyset$. This implies that $E_{(i,k)} \cdot \varphi_{T^{\hat{\lambda}}_x} = 0$ since $E_{(i,k)} \cdot \varphi_{T^{\hat{\lambda}}_x} = 1_{\gamma(\lambda)+\alpha_{(i,k)}} E_{(i,k)} \cdot \varphi_{T^{\hat{\lambda}}_x}$ together with (1.17.1), where we consider the actions of ${}_{\mathcal{K}}\mathscr{S}_{n+1,r}$. As a consequence, we see that $E_{(i,k)} \cdot \varphi_{T^{\hat{\lambda}}_x} = 0$ for any $(i,k) \in \Gamma'(\mathbf{m}')$, where we consider the action of ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ through $\iota_{\mathcal{K}}$. This means that $\varphi_{T^{\hat{\lambda}}_x}$ is a highest weight vector of weight λ, and ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-submodule of ${}_{\mathcal{K}}\Delta_{n+1}(\hat{\lambda})$ generated by $\varphi_{T^{\hat{\lambda}}_x}$ is a highest weight module of highest weight λ. Thus, the universality of Weyl modules (Lemma 1.11) implies the isomorphism

$${}_{\mathcal{K}}\Delta_n(\lambda) \cong {}_{\mathcal{K}}\mathscr{S}_{n,r} \cdot \varphi_{T^{\hat{\lambda}}_x} \quad \text{as} \quad {}_{\mathcal{K}}\mathscr{S}_{n,r}\text{-modules}$$

since ${}_{\mathcal{K}}\Delta_n(\lambda)$ is a simple ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-module. Now we proved that, for each $\lambda \in \Lambda^+_{n,r}$, the Weyl module ${}_{\mathcal{K}}\Delta_n(\lambda)$ appears in ${}_{\mathcal{K}}\Delta_{n+1}(\hat{\lambda})$ as an ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-submodule through $\iota_{\mathcal{K}}$. Note that ${}_{\mathcal{K}}\mathscr{S}_{n,r}$ is split semi-simple, the above arguments combined with Wedderburn's theorem implies that

$$\dim_{\mathcal{K}} {}_{\mathcal{K}}\mathscr{S}_{n,r} = \sum_{\lambda \in \Lambda^+_{n,r}} (\dim_{\mathcal{K}} {}_{\mathcal{K}}\Delta_n(\lambda))^2 = \dim_{\mathcal{K}} \iota_{\mathcal{K}}({}_{\mathcal{K}}\mathscr{S}_{n,r}).$$

Hence $\iota_{\mathcal{K}}$ is injective.

By restricting $\iota_{\mathcal{K}}$ to ${}_{\mathcal{A}}\mathscr{S}_{n,r}$, we have the injective homomorphism $\iota_{\mathcal{A}}\colon {}_{\mathcal{A}}\mathscr{S}_{n,r} \to {}_{\mathcal{A}}\mathscr{S}_{n+1,r}$ satisfying (2.6.1). In particular, we have $\iota_{\mathcal{A}}({}_{\mathcal{A}}\mathscr{S}_{n,r}) \subset \xi {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi$. Put

$$\begin{aligned} M(\xi) &= \{\xi x 1_\lambda y \xi \mid x \in {}_{\mathcal{A}}\mathscr{S}^-_{n+1,r},\ y \in {}_{\mathcal{A}}\mathscr{S}^+_{n+1,r},\ \lambda \in \Lambda^\gamma_{n+1,r}(\mathbf{m})\}, \\ M(\not\leq \xi) &= \{\xi x 1_\mu y \xi \mid x \in {}_{\mathcal{A}}\mathscr{S}^-_{n+1,r},\ y \in {}_{\mathcal{A}}\mathscr{S}^+_{n+1,r},\ \mu \in \Lambda_{n+1,r}(\mathbf{m}) \\ &\qquad \text{such that} \quad \mu \not\leq \lambda \text{ for any } \lambda \in \Lambda^\gamma_{n+1,r}(\mathbf{m})\}. \end{aligned}$$

By (2.6.1) and the triangular decomposition of ${}_{\mathcal{A}}\mathscr{S}_{n,r}$, we see that

$$\iota_{\mathcal{A}}({}_{\mathcal{A}}\mathscr{S}_{n,r}) = M(\xi). \tag{2.6.2}$$

Moreover, we claim that

$$\xi {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi = M(\xi) \oplus M(\not\leq \xi). \tag{2.6.3}$$

Thanks to the triangular decomposition of ${}_{\mathcal{A}}\mathscr{S}_{n+1,r}$, we have

$$\xi {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi = \{\xi x 1_\lambda y \xi \mid x \in {}_{\mathcal{A}}\mathscr{S}^-_{n+1,r},\ y \in {}_{\mathcal{A}}\mathscr{S}^+_{n+1,r},\ \lambda \in \Lambda_{n+1,r}(\mathbf{m})\}.$$

Thus, in order to show (2.6.3), it is enough to show that

$$\begin{aligned} &\xi x 1_\mu y \xi = 0 \quad \text{if} \quad \mu < \lambda \\ &\text{for some} \quad \lambda \in \Lambda^\gamma_{n+1,r}(\mathbf{m}) \quad \text{and} \quad \mu \in \Lambda_{n+1,r}(\mathbf{m}) \setminus \Lambda^\gamma_{n+1,r}(\mathbf{m}). \end{aligned} \tag{2.6.4}$$

From the definitions, for $\mu \in \Lambda_{n+1,r}(\mathbf{m}) \setminus \Lambda^\gamma_{n+1,r}(\mathbf{m})$, we see that $\mu^{(r)}_{m_r} \geq 2$ if $\mu < \lambda$ for some $\Lambda^\gamma_{n+1,r}(\mathbf{m})$, and we see that $\xi x 1_\mu y \xi = 0$ ($x \in {}_{\mathcal{A}}\mathscr{S}^-_{n+1,r}$, $y \in {}_{\mathcal{A}}\mathscr{S}^+_{n+1,r}$) if $\mu^{(r)}_{m_r} \geq 2$ from the relations (1.8.1)–(1.8.8). These imply (2.6.4), and we have (2.6.3). Put $\xi' = \sum_{\lambda \in \Lambda_{n+1,r}(\mathbf{m}) \setminus \Lambda^\gamma_{n+1,r}(\mathbf{m})} 1_\lambda$, we have $1_{{}_{\mathcal{A}}\mathscr{S}_{n+1,r}} = \xi + \xi'$. Then, by (2.6.3), we have

$${}_{\mathcal{A}}\mathscr{S}_{n+1,r} = M(\xi) \oplus M(\not\leq \xi) \oplus \xi {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi' \oplus \xi' {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi \oplus \xi' {}_{\mathcal{A}}\mathscr{S}_{n+1,r}\xi'.$$

Combining with (2.6.2), we see that the injective homomorphism $\iota_{\mathcal{A}}\colon {}_{\mathcal{A}}\mathscr{S}_{n,r} \to {}_{\mathcal{A}}\mathscr{S}_{n+1,r}$ is split as a $\mathcal{A}$-homomorphism. Thus, by the specialization of $\iota_{\mathcal{A}}$ to R, we have the injective homomorphism $\iota_R\colon {}_R\mathscr{S}_{n,r} \to {}_R\mathscr{S}_{n+1,r}$ satisfying (2.6.1). □

By (2.6.1) and Lemma 1.16, we have the following corollary.

Corollary 2.7. *When we regard $\mathscr{S}_{n,r}$ as a subalgebra of $\mathscr{S}_{n+1,r}$ through the injective homomorphism $\iota\colon \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$, the anti-involution θ_n on $\mathscr{S}_{n,r}$ coincides with the restriction of the anti-involution θ_{n+1} on $\mathscr{S}_{n+1,r}$.*

2.8. From now on, we regard $\mathscr{S}_{n,r}$ as a subalgebra of $\mathscr{S}_{n+1,r}$ through the injective homomorphism $\iota\colon \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$. As defined in Proposition 2.6, put $\xi = \sum_{\lambda\in\Lambda^{\gamma}_{n+1,r}(\mathbf{m})} 1_\lambda$. Since ξ is an idempotent of $\mathscr{S}_{n+1,r}$, we see that $\xi\mathscr{S}_{n+1,r}\xi$ is a subalgebra of $\mathscr{S}_{n+1,r}$ with the unit element ξ. Thus, $\xi\mathscr{S}_{n+1,r}$ (resp. $\mathscr{S}_{n+1,r}\xi$) is an $(\xi\mathscr{S}_{n+1,r}\xi,\ \mathscr{S}_{n+1,r})$-bimodule (resp. $(\mathscr{S}_{n+1,r}, \xi\mathscr{S}_{n+1,r}\xi)$-bimodule) by the multiplications. Note that $\iota(\mathscr{S}_{n,r}) \subset \xi\mathscr{S}_{n+1,r}\xi$ and $\iota(1_{\mathscr{S}_{n,r}}) = \xi$, we can restrict the action of $\xi\mathscr{S}_{n+1,r}\xi$ to $\mathscr{S}_{n,r}$ through ι. Thus, $\xi\mathscr{S}_{n+1,r}$ (resp. $\mathscr{S}_{n+1,r}\xi$) turns out to be an $(\mathscr{S}_{n,r}, \mathscr{S}_{n+1,r})$-bimodule (resp. $(\mathscr{S}_{n+1,r}, \mathscr{S}_{n,r})$-bimodule) by restriction. We define a restriction functor $\operatorname{Res}^{n+1}_n\colon \mathscr{S}_{n+1,r}$-mod $\to \mathscr{S}_{n,r}$-mod by

$$\operatorname{Res}^{n+1}_n = \operatorname{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi, ?) \cong \xi\mathscr{S}_{n+1,r} \otimes_{\mathscr{S}_{n+1,r}} ?.$$

We also define two induction functors $\operatorname{Ind}^{n+1}_n$, $\operatorname{coInd}^{n+1}_n\colon \mathscr{S}_{n,r}$-mod $\to \mathscr{S}_{n+1,r}$-mod by

$$\operatorname{Ind}^{n+1}_n = \mathscr{S}_{n+1,r}\xi \otimes_{\mathscr{S}_{n,r}} ?,$$
$$\operatorname{coInd}^{n+1}_n = \operatorname{Hom}_{\mathscr{S}_{n,r}}(\xi\mathscr{S}_{n+1,r}, ?).$$

By the definition, we have the following.

- $\operatorname{Res}^{n+1}_n$ is exact, $\operatorname{Ind}^{n+1}_n$ is right exact and $\operatorname{coInd}^{n+1}_n$ is left exact.
- $\operatorname{Ind}^{n+1}_n$ is left adjoint to $\operatorname{Res}^{n+1}_n$.
- $\operatorname{coInd}^{n+1}_n$ is right adjoint to $\operatorname{Res}^{n+1}_n$.

We have the following commutativity with these functors and the contravariant functors $\circledast$ with respect to the anti-involutions θ_n and θ_{n+1}.

Lemma 2.9. *We have the following isomorphisms of functors.*

(i) $\circledast \circ \operatorname{Res}^{n+1}_n \cong \operatorname{Res}^{n+1}_n \circ \circledast$.

(ii) $\circledast \circ \operatorname{Ind}^{n+1}_n \cong \operatorname{coInd}^{n+1}_n \circ \circledast$.

Proof. For a left (resp. right) $\mathscr{S}_{n,r}$-module M, we denote by M^θ the right (resp. left) $\mathscr{S}_{n,r}$-module obtained from M by twisting the action through the anti-involution θ_n. We also use the same notation for left (resp. right) $\mathscr{S}_{n+1,r}$-modules, $(\mathscr{S}_{n+1,r}, \mathscr{S}_{n,r})$-bimodules and $(\mathscr{S}_{n,r}, \mathscr{S}_{n+1,r})$-bimodules when we twist the action through the anti-involutions θ_n and θ_{n+1}. Then we have the isomorphism of $(\mathscr{S}_{n+1,r}, \mathscr{S}_{n,r})$-bimodules

$$g_\theta\colon (\xi\mathscr{S}_{n+1,r})^\theta \to \mathscr{S}_{n+1,r}\xi \tag{2.9.1}$$

by $g_\theta(\xi s) = \theta_{n+1}(s)\xi$ for $s \in \mathscr{S}_{n+1,r}$. From the definition, we also see that

$$\operatorname{Hom}_R((\xi\mathscr{S}_{n+1,r})^\theta, R) = (\xi\mathscr{S}_{n+1,r})^\circledast \quad \text{as} \quad (\mathscr{S}_{n,r}, \mathscr{S}_{n+1,r})\text{-bimodules}. \tag{2.9.2}$$

For $N \in \mathscr{S}_{n,r}$-mod, we have the following natural isomorphisms

$$\begin{aligned}
\circledast \circ \mathrm{Ind}_n^{n+1}(N) &= \mathrm{Hom}_R(\mathscr{S}_{n+1,r}\xi \otimes_{\mathscr{S}_{n,r}} N, R)^\theta \\
&\cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(N, \mathrm{Hom}_R(\mathscr{S}_{n+1,r}\xi, R))^\theta \\
&\cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(N, \mathrm{Hom}_R((\xi\mathscr{S}_{n+1,r})^\theta, R))^\theta \quad \text{(because of (2.9.1))} \\
&\cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(N, (\xi\mathscr{S}_{n+1,r})^\circledast)^\theta \quad \text{(because of (2.9.2))} \\
&\cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(\xi\mathscr{S}_{n+1,r}, N^\circledast) \\
&= \mathrm{coInd}_n^{n+1} \circ \circledast(N),
\end{aligned}$$

and this implies (ii). Note that $\mathrm{Res}_n^{n+1} = \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi, ?) \cong \xi\mathscr{S}_{n+1,r} \otimes_{\mathscr{S}_{n+1,r}} ?$, we can obtain (i) in a similar way. □

The last of this section, we prepare the following lemma for later arguments.

Lemma 2.10. *Assume that R is a field. For $\lambda \in \Lambda_{n,r}^+$, we have*

$$[\mathrm{Ind}_n^{n+1}(\Delta_n(\lambda))] = [\mathrm{Ind}_n^{n+1}(\nabla_n(\lambda))] \quad \textit{in} \quad K_0(\mathscr{S}_{n+1,r}\text{-mod}).$$

Proof. Let $(K, \hat{R}, R)$ be a suitable modular system, namely $\hat{R}$ is a discrete valuation ring such that R is the residue field of $\hat{R}$, and K is the quotient field of $\hat{R}$ such that ${}_K\mathscr{S}_{n,r}$ is semi-simple (e.g. see [12, Section 5.3]). Let X be one of K, $\hat{R}$ or R. For $\lambda \in \Lambda_{n,r}^+$, we see that ${}_X\mathscr{S}_{n+1,r}\xi \otimes_{{}_X\mathscr{S}_{n,r}} {}_X\Delta_n(\lambda)$ is generated by $\{\xi \otimes \varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$ as ${}_X\mathscr{S}_{n+1,r}$-modules, and that $\xi \otimes \varphi_T \neq 0$ for any $T \in \mathcal{T}_0(\lambda)$ since ξ is regarded as the identity element of ${}_X\mathscr{S}_{n,r}$. Thus, ${}_{\hat{R}}\mathscr{S}_{n+1,r}\xi \otimes_{{}_{\hat{R}}\mathscr{S}_{n,r}} {}_{\hat{R}}\Delta_n(\lambda)$ is an ${}_{\hat{R}}\mathscr{S}_{n+1,r}$-submodule of ${}_K\mathscr{S}_{n+1,r}\xi \otimes_{{}_K\mathscr{S}_{n,r}} {}_K\Delta_n(\lambda)$ generated by $\{\xi \otimes \varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$. In particular, ${}_{\hat{R}}\mathscr{S}_{n+1,r}\xi \otimes_{{}_{\hat{R}}\mathscr{S}_{n,r}} {}_{\hat{R}}\Delta_n(\lambda)$ is torsion free, thus it is a full rank $\hat{R}$-lattice of ${}_K\mathscr{S}_{n+1,r}\xi \otimes_{{}_K\mathscr{S}_{n,r}} {}_K\Delta_n(\lambda)$. Similarly, ${}_{\hat{R}}\mathscr{S}_{n+1,r}\xi \otimes_{{}_{\hat{R}}\mathscr{S}_{n,r}} {}_{\hat{R}}\nabla_n(\lambda)$ is a full rank $\hat{R}$-lattice of ${}_K\mathscr{S}_{n+1,r}\xi \otimes_{{}_K\mathscr{S}_{n,r}} {}_K\nabla_n(\lambda)$. Moreover, by the general theory of cellular algebras, we have ${}_K\Delta_n(\lambda) \cong {}_K\nabla_n(\lambda)$ since ${}_K\mathscr{S}_{n,r}$ is semi-simple. Then, the decomposition map implies

$$[{}_R\mathscr{S}_{n+1,r}\xi \otimes_{{}_R\mathscr{S}_{n,r}} {}_R\Delta_n(\lambda)] = [{}_R\mathscr{S}_{n+1,r}\xi \otimes_{{}_R\mathscr{S}_{n,r}} {}_R\nabla_n(\lambda)] \quad \text{in} \quad K_0({}_R\mathscr{S}_{n+1,r}\text{-mod}). \qquad \square$$

3. Restricted and induced Weyl modules

In this section, we describe filtrations of restricted and induced Weyl modules (resp. costandard modules) whose successive quotients are isomorphic to Weyl modules (resp. costandard modules).

3.1. It is known that there exists the injective homomorphism of algebras

(3.1.1) $\qquad \iota^{\mathscr{H}} : \mathscr{H}_{n,r} \to \mathscr{H}_{n+1,r} \quad \text{such that} \quad T_i \mapsto T_i \quad \text{for} \quad i = 0, 1, \dots, n-1.$

We regard $\mathscr{H}_{n,r}$ as a subalgebra of $\mathscr{H}_{n+1,r}$ through $\iota^{\mathscr{H}}$.

3.2. We recall that, for $\lambda \in \Lambda^+_{n+1,r}$, the Weyl module $\Delta_{n+1}(\lambda)$ of $\mathscr{S}_{n+1,r}$ has an R-free basis $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda)\}$. From the definition, we have that

$$\mathrm{Res}^{n+1}_n(\Delta_{n+1,r}(\lambda)) = \xi \cdot \Delta_{n+1,r}(\lambda).$$

Thus, we see that $\mathrm{Res}^{n+1}_n(\Delta_{n+1,r}(\lambda))$ has an R-free basis

$$\{\varphi_T \mid T \in \mathcal{T}_0(\lambda, \mu) \text{ for some } \mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})\}$$

thanks to (1.17.1).

Proposition 3.3. *Let $\lambda \in \Lambda^+_{n+1,r}$, $\mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})$, $T \in \mathcal{T}_0(\lambda,\mu)$. For $(i,k) \in \Gamma'(\mathbf{m}')$, we have the following.*

(i) $E_{(i,k)} \cdot \varphi_T = \sum_{\substack{S \in \mathcal{T}_0(\lambda,\mu+\alpha_{(i,k)}) \\ |S\setminus(m_r,r)| \geq |T\setminus(m_r,r)|}} r_S\varphi_S$ $(r_S \in R)$.

(ii) $F_{(i,k)} \cdot \varphi_T = \sum_{\substack{S \in \mathcal{T}_0(\lambda,\mu-\alpha_{(i,k)}) \\ |S\setminus(m_r,r)| \geq |T\setminus(m_r,r)|}} r_S\varphi_S$ $(r_S \in R)$.

Proof. We prove only (i) since we can prove (ii) in a similar way.

If $\mu + \alpha_{(i,k)} \notin \Lambda_{n+1,r}(\mathbf{m})$, we have $E_{(i,k)} \cdot \varphi_T = E_{(i,k)} 1_\mu \cdot \varphi_T = 0$ by (1.8.2), and there is nothing to prove. Thus, we assume that $\mu + \alpha_{(i,k)} \in \Lambda_{n+1,r}(\mathbf{m})$. Then we have $E_{(i,k)} \cdot \varphi_T = 1_{\mu+\alpha_{(i,k)}} E_{(i,k)} \cdot \varphi_T$, and this implies that $S \in \mathcal{T}_0(\lambda, \mu + \alpha_{(i,k)})$ if $r_S \neq 0$ thanks to (1.17.1). Hence, it is enough to prove that $|S \setminus (m_r, r)| \geq |T \setminus (m_r, r)|$ if $r_S \neq 0$.

By [4, Proposition 6.3], we can write $m_{TT^\lambda} = m_\mu h$ for some $h \in \mathscr{H}_{n+1,r}$. Then, by (1.5.1), we have

$$\begin{aligned}
E_{(i,k)} \cdot \varphi_{TT^\lambda}(m_\lambda) &= E_{(i,k)}(m_{TT^\lambda}) \\
&= E_{(i,k)}(m_\mu h) \\
&= \left(q^{-\mu^{(k)}_{i+1}+1} \left(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)} T^*_x \right) h^\mu_{+(i,k)} m_\mu \right) \cdot h \\
&= q^{-\mu^{(k)}_{i+1}+1} \left(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)} T^*_x \right) h^\mu_{+(i,k)} m_{TT^\lambda} \\
&= \sum_{\substack{\mathfrak{t} \in \mathrm{Std}(\lambda) \\ \mu(\mathfrak{t}) = T}} q^{-\mu^{(k)}_{i+1}+1} \left(\sum_{x \in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)} T^*_x \right) h^\mu_{+(i,k)} m_{\mathfrak{t}\mathfrak{t}^\lambda}.
\end{aligned}$$

(Note that $\mathfrak{t}^\lambda$ is the unique standard tableau $\mathfrak{t} \in \mathrm{Std}(\lambda)$ such that $\lambda(\mathfrak{t}) = T^\lambda$.) Since $\mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})$ and $\mu + \alpha_{(i,k)} \in \Lambda_{n+1,r}(\mathbf{m})$, we have $\mu^{(r)}_{m_r} = 1$ and $\mu^{(k)}_{i+1} \geq 1$. These imply $\sum_{l=1}^{k-1}|\mu^{(l)}| + \sum_{j=1}^{i} \mu_j^{(k)} \le n-1$. Then, from the definitions of $X_\mu^{\mu+\alpha_{(i,k)}}$ and $h^\mu_{+(i,k)}$, we see that $q^{-\mu_{i+1}^{(k)}+1}\big(\sum_{x\in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)}T_x^*\big)h^\mu_{+(i,k)} \in \mathscr{H}_{n,r}$, where we regard $\mathscr{H}_{n,r}$ as a subalgebra of $\mathscr{H}_{n+1,r}$ by (3.1.1). Thus, by [1, Proof of Proposition 1.9], we have

$$
\begin{aligned}
&E_{(i,k)}\cdot \varphi_{TT^\lambda}(m_\lambda)\\
&= \sum_{\substack{\mathfrak{t}\in \mathrm{Std}(\lambda)\\ \mu(\mathfrak{t})=T}} q^{-\mu_{i+1}^{(k)}+1}\left(\sum_{x\in X_\mu^{\mu+\alpha_{(i,k)}}} q^{l(x)}T_x^*\right)h^\mu_{+(i,k)}m_{\mathfrak{t}\mathfrak{t}^\lambda}\\
&\equiv \sum_{\substack{\mathfrak{t}\in \mathrm{Std}(\lambda)\\ \mu(\mathfrak{t})=T}}\left(\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ |\mathfrak{s}\setminus n+1|\ge|\mathfrak{t}\setminus n+1|}} r_{\mathfrak{s}}^{\mathfrak{t}} m_{\mathfrak{s}\mathfrak{t}^\lambda}\right) \bmod \mathscr{H}_{n+1,r}(>\lambda) \quad (r_{\mathfrak{s}}^{\mathfrak{t}}\in R),
\end{aligned}
\tag{3.3.1}
$$

where $\mathscr{H}_{n+1,r}(>\lambda)$ is an R-submodule of $\mathscr{H}_{n+1,r}$ spanned by $\{m_{\mathfrak{u}\mathfrak{v}} \mid \mathfrak{u},\mathfrak{v}\in \mathrm{Std}(\lambda')$ for some $\lambda'\in\Lambda^+_{n+1,r}$ such that $\lambda'>\lambda\}$. Since $\mu^{(r)}_{m_r}=1$, we see that $|\mathfrak{t}\setminus n+1|$ does not depend on a choice of $\mathfrak{t}\in\mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{t})=T$. Then, take and fix a standard tableau $\mathfrak{t}'\in\mathrm{Std}(\lambda)$ such that $\mu(\mathfrak{t}')=T$, and (3.3.1) implies

$$
E_{(i,k)}\cdot\varphi_{TT^\lambda}(m_\lambda) \equiv \sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ |\mathfrak{s}\setminus n+1|\ge|\mathfrak{t}'\setminus n+1|}} r_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}^\lambda} \bmod \mathscr{H}_{n+1,r}(>\lambda) \quad (r_{\mathfrak{s}}\in R).
\tag{3.3.2}
$$

On the other hand, by a general theory of cellular algebras together with (1.17.1), we can write

$$
E_{(i,k)}\cdot\varphi_{TT^\lambda} \equiv \sum_{S\in\mathcal{T}_0(\lambda,\mu+\alpha_{(i,k)})} r_S\varphi_{ST^\lambda} \bmod \mathscr{S}_{n+1,r}(>\lambda) \quad (r_S\in R).
$$

Thus, we have

$$
\begin{aligned}
&E_{(i,k)}\cdot\varphi_{TT^\lambda}(m_\lambda)\\
&\equiv \sum_{S\in\mathcal{T}_0(\lambda,\mu+\alpha_{(i,k)})} r_S m_{ST^\lambda}\\
&= \sum_{S\in\mathcal{T}_0(\lambda,\mu+\alpha_{(i,k)})} r_S\left(\sum_{\substack{\mathfrak{s}\in\mathrm{Std}(\lambda)\\ (\mu+\alpha_{(i,k)})(\mathfrak{s})=S}} q^{l(d(\mathfrak{s}))}m_{\mathfrak{s}\mathfrak{t}^\lambda}\right) \bmod \mathscr{H}_{n+1,r}(>\lambda).
\end{aligned}
\tag{3.3.3}
$$

Note that $\mu+\alpha_{(i,k)}\in\Lambda^\gamma_{n+1,r}(\mathbf{m})$, we can easily check that $|S\setminus(m_r,r)| = |\mathfrak{s}\setminus n+1|$ for $S\in\mathcal{T}_0(\lambda,\mu+\alpha_{(i,k)})$ and $\mathfrak{s}\in\mathrm{Std}(\lambda)$ such that $(\mu+\alpha_{(i,.k)})(\mathfrak{s})=S$. Similarly, we have $|T\setminus(m_r,r)|=|\mathfrak{t}'\setminus n+1|$. Thus, by comparing the coefficients in (3.3.2) and (3.3.3), we have $|S\setminus(m_r,r)|\ge|T\setminus(m_r,r)|$ if $r_S\ne 0$. □

Now we can describe filtrations of restricted and induced Weyl modules (resp. co-standard modules) as follows.

Theorem 3.4. *Assume that R is a field, we have the following.*
(i) *For $\lambda \in \Lambda^+_{n+1,r}$, there exists a filtration of $\mathscr{S}_{n,r}$-modules*

$$\mathrm{Res}_n^{n+1}(\Delta_{n+1}(\lambda)) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0,$$

such that $M_i/M_{i+1} \cong \Delta_n(\lambda \setminus x_i)$, where $x_1, x_2, \dots, x_k$ are all removable nodes of λ such that $x_1 \succ x_2 \succ \cdots \succ x_k$.
(ii) *For $\lambda \in \Lambda^+_{n+1,r}$, there exists a filtration of $\mathscr{S}_{n,r}$-modules*

$$\mathrm{Res}_n^{n+1}(\nabla_{n+1}(\lambda)) = N_k \supset N_{k-1} \supset \cdots \supset N_1 \supset N_0 = 0,$$

such that $N_i/N_{i-1} \cong \nabla_n(\lambda \setminus x_i)$, where $x_1, x_2, \dots, x_k$ are all removable nodes of λ such that $x_1 \succ x_2 \succ \cdots \succ x_k$.
(iii) *For $\mu \in \Lambda^+_{n,r}$, there exists a filtration of $\mathscr{S}_{n+1,r}$-modules*

$$\mathrm{Ind}_n^{n+1}(\Delta_n(\mu)) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

such that $M_i/M_{i+1} \cong \Delta_{n+1}(\mu \cup x_i)$, where $x_1, x_2, \dots, x_k$ are all addable nodes of μ such that $x_k \succ x_{k-1} \succ \cdots \succ x_1$.
(iv) *For $\mu \in \Lambda^+_{n,r}$, there exists a filtration of $\mathscr{S}_{n+1,r}$-modules*

$$\mathrm{coInd}_n^{n+1}(\nabla_n(\mu)) = N_k \supset N_{k-1} \supset \cdots \supset N_1 \supset N_0 = 0$$

such that $N_i/N_{i-1} \cong \nabla_{n+1}(\mu \cup x_i)$, where $x_1, x_2, \dots, x_k$ are all addable nodes of μ such that $x_k \succ x_{k-1} \succ \cdots \succ x_1$.

Proof. (i) Until declining, let R be an arbitrary commutative ring. Put

$$\mathcal{T}_0^{\gamma}(\lambda) = \bigcup_{\mu \in \Lambda^{\gamma}_{n+1,r}(\mathbf{m})} \mathcal{T}_0(\lambda, \mu).$$

Then, $\mathrm{Res}_n^{n+1}(\Delta_{n+1,r}(\lambda))$ has an R-free basis $\{\varphi_T \mid T \in \mathcal{T}_0^{\gamma}(\lambda)\}$. For $T \in \mathcal{T}_0^{\gamma}(\lambda)$, there exists the unique removable node x of λ such that $T(x) = (m_r, r)$ since $\mu^{(r)}_{m_r} = 1$. Let $x_1, x_2, \dots, x_k$ be all removable nodes of λ such that $x_1 \succ x_2 \succ \cdots \succ x_k$ (note that the order $\succeq$ determines a total order on the set of removable nodes of λ). Let M_i be an R-submodule of $\mathrm{Res}_n^{n+1}(\Delta_{n+1}(\lambda))$ spanned by

$$\{\varphi_T \mid T \in \mathcal{T}_0^{\gamma}(\lambda) \text{ such that } T(x_j) = (m_r, r) \text{ for some } j \geq i\}.$$

Then we have a filtration of R-modules

$$\operatorname{Res}_n^{n+1}(\Delta_{n+1}(\lambda)) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0. \tag{3.4.1}$$

For $T \in \mathcal{T}_0^{\gamma}(\lambda)$ such that $T(x_i) = (m_r, r)$, we have $|T \setminus (m_r, r)| = \lambda \setminus x_i$ by the definition. Note that $\lambda \setminus x_j > \lambda \setminus x_i$ if and only if $x_j \prec x_i$ (i.e. $j > i$). Then, for $S, T \in \mathcal{T}_0^{\gamma}(\lambda)$ such that $S(x_j) = T(x_i) = (m_r, r)$, we have that

$$|S \setminus (m_r, r)| > |T \setminus (m_r, r)| \quad \text{if and only if} \quad j > i. \tag{3.4.2}$$

By Proposition 3.3 and (3.4.2), we see that M_i is an $\mathscr{S}_{n,r}$-submodule of $\operatorname{Res}_n^{n+1}(\Delta_{n+1}(\lambda))$ for each $i = 1, \dots, k$, and the filtration (3.4.1) is a filtration of $\mathscr{S}_{n,r}$-modules.

From the definition, M_i/M_{i+1} has an R-free basis

$$\{\varphi_T + M_{i+1} \mid T \in \mathcal{T}_0^{\gamma}(\lambda) \text{ such that } T(x_i) = (m_r, r)\}.$$

Note that $T^{\lambda}_{x_i} \in \mathcal{T}_0^{\gamma}(\lambda)$ and $T^{\lambda}_{x_i}(x_i) = (m_r, r)$. Let $x_i = (a, b, c)$, and put $\tau = \lambda - (\alpha_{(a,c)} + \alpha_{(a+1,c)} + \cdots + \alpha_{(m_r-1,r)})$. Then we have $T^{\lambda}_{x_i} \in \mathcal{T}_0(\lambda, \tau)$. Note that $E_{(j,l)} \cdot \varphi_{T^{\lambda}_{x_i}} = 1_{\tau+\alpha_{(j,l)}} E_{(j,l)} \cdot \varphi_{T^{\lambda}_{x_i}}$ is a linear combination of $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda, \tau+\alpha_{(j,l)})\}$, and that $\mathcal{T}_0(\lambda, \tau + \alpha_{(j,l)}) = \emptyset$ unless $\lambda \geq \tau + \alpha_{(j,l)}$.

If $(j, l) \succ (a, c)$, we have $E_{(j,l)} \cdot \varphi_{T^{\lambda}_{x_i}} = 0$ since $\lambda \ngeq \tau + \alpha_{(j,l)}$.

Assume that $(j, l) \preceq (a, c)$, and we have

$$\sum_{g=1}^{l-1} |\lambda^{(g)}| + \sum_{b=1}^{j} \lambda_b^{(l)} = \sum_{g=1}^{l-1} |(\tau + \alpha_{(j,l)})^{(g)}| + \sum_{b=1}^{j} (\tau + \alpha_{(j,l)})_b^{(l)}. \tag{3.4.3}$$

By (3.4.3) together with the definition of semi-standard tableaux, we can easily check that $S((a', b', c')) \leq (j, l)$ for any $S \in \mathcal{T}_0(\lambda, \tau + \alpha_{(i,k)})$ and any $(a', b', c') \in [\lambda]$ such that $(a', c') \succeq (j, l)$. This implies that

$$|S \setminus (m_r, r)| \neq |T^{\lambda}_{x_i} \setminus (m_r, r)| \quad \text{for any} \quad S \in \mathcal{T}_0(\lambda, \tau + \alpha_{(j,l)}) \tag{3.4.4}$$

since $(a, c) \succeq (j, l)$ and $T^{\lambda}_{x_i}(x_i) = T^{\lambda}_{x_i}((a, b, c)) = (m_r, r) \geq (j, l)$.

Thus, Proposition 3.3 (i) together with (3.4.4) implies

$$E_{(j,l)} \cdot \varphi_{T^{\lambda}_{x_i}} \in M_{i+1} \quad \text{for any} \quad (j, l) \in \Gamma'(\mathbf{m}').$$

Thus, $\varphi_{T^{\lambda}_{x_i}} + M_{i+1} \in M_i/M_{i+1}$ is a highest weight vector of weight $\lambda \setminus x_i$ as an element of the $\mathscr{S}_{n,r}$-module. Thus, by the universality of Weyl modules (Lemma 1.11), we have the surjective ${}_R\mathscr{S}_{n,r}$-homomorphism

$$X_R : {}_R\Delta_n(\lambda \setminus x_i) \to {}_R\mathscr{S}_{n,r} \cdot (\varphi_{T^{\lambda}_{x_i}} + M_{i+1}).$$

Since ${}_{\mathcal{K}}\Delta_n(\lambda \setminus x_i)$ is a simple ${}_{\mathcal{K}}\mathscr{S}_{n,r}$-module, $X_{\mathcal{K}}$ is injective. Note that X_R is determined by $X_R(1 \otimes v_{\lambda\setminus x_i}) = \varphi_{T^{\lambda}_{x_i}} + M_{i+1}$, we see that $X_{\mathcal{A}}$ is the restriction of $X_{\mathcal{K}}$ to ${}_{\mathcal{A}}\Delta_n(\lambda \setminus x_i)$. Thus, $X_{\mathcal{A}}$ is also injective, and $X_{\mathcal{A}}$ is an isomorphism. Then, by the argument of specialization, we have X_R is an isomorphism for an arbitrary commutative ring R.

Assume that R is a field. Since $\Delta_n(\lambda \setminus x_i) \cong \mathscr{S}_{n,r} \cdot (\varphi_{T^{\lambda}_{x_i}} + M_{i+1})$ is an $\mathscr{S}_{n,r}$-submodule of M_i/M_{i+1}, we have that $\Delta_n(\lambda \setminus x_i) \cong M_i/M_{i+1}$ by comparing the dimensions of $\Delta_n(\lambda \setminus x_i)$ and of M_i/M_{i+1}. Now we proved (i).

(ii) is obtained by applying the contravariant functor $\circledast$ to (i) thanks to Lemma 2.9 (i).

Next, we prove (iv). For $\lambda \in \Lambda^+_{n+1,r}$, let $\Delta^{\sharp}_{n+1}(\lambda)$ be an R-submodule of $\mathscr{S}_{n+1,r}/\mathscr{S}_{n+1,r}(> \lambda)$ spanned by $\{\varphi_{T^{\lambda}T} + \mathscr{S}_{n+1,r}(> \lambda) \mid T \in \mathcal{T}_0(\lambda)\}$. Then, by a general theory of cellular algebras, it is known that $\Delta^{\sharp}_{n+1}(\lambda)$ is a right $\mathscr{S}_{n+1,r}$-submodule of $\mathscr{S}_{n+1,r}/\mathscr{S}_{n+1,r}(> \lambda)$, and that

$$\mathrm{Hom}_R(\Delta^{\sharp}_{n+1}(\lambda), R) \cong \Delta_{n+1}(\lambda)^{\circledast} \cong \nabla_{n+1}(\lambda) \tag{3.4.5}$$

as left $\mathscr{S}_{n+1,r}$-modules.

Let $\Lambda^+_{n+1,r} = \{\lambda_1, \lambda_2, \cdots, \lambda_g\}$ be such that $i < j$ if $\lambda_i < \lambda_j$. By a general theory of cellular algebras, we obtain a filtration of $(\mathscr{S}_{n+1,r}, \mathscr{S}_{n+1,r})$-bimodules

$$\mathscr{S}_{n+1,r} = J_1 \supset J_2 \supset \cdots \supset J_g \supset J_{g+1} = 0$$

such that $J_i/J_{i+1} \cong \Delta_{n+1}(\lambda_i) \otimes {}_R\Delta^{\sharp}_{n+1}(\lambda_i)$. By applying the exact functor Res^{n+1}_n to this filtration, we obtain a filtration of $(\mathscr{S}_{n,r}, \mathscr{S}_{n+1,r})$-bimodules

$$\xi\mathscr{S}_{n+1,r} = \xi J_1 \supset \xi J_2 \supset \cdots \supset \xi J_g \supset \xi J_{g+1} = 0$$

such that $\xi J_i/\xi J_{i+1} \cong \mathrm{Res}^{n+1}_n(\Delta_{n+1}(\lambda_i)) \otimes {}_R\Delta^{\sharp}_{n+1}(\lambda_i)$. By (i), we have that $\xi J_i/\xi J_{i+1} \in \mathscr{S}_{n,r}\text{-mod}^{\Delta}$ for each $i = 1, \ldots, g$. Thus, by a general theory of quasi-hereditary algebras, we have a filtration of left $\mathscr{S}_{n+1,r}$-modules

$$\mathrm{Hom}_{\mathscr{S}_{n,r}}(\xi\mathscr{S}_{n+1,r}, \nabla_n(\mu)) = N_g \supset N_{g-1} \supset \cdots \supset N_1 \supset N_0 = 0 \tag{3.4.6}$$

such that $N_i/N_{i-1} \cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(\xi J_i/\xi J_{i+1}, \nabla_n(\mu))$.

On the other hand, we have the following isomorphisms as $\mathscr{S}_{n+1,r}$-modules.

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{S}_{n,r}}(\xi J_i/\xi J_{i+1}, \nabla_n(\mu)) \\
&\cong \mathrm{Hom}_{\mathscr{S}_{n,r}}(\mathrm{Res}^{n+1}_n(\Delta_{n+1}(\lambda_i)) \otimes {}_R\Delta^{\sharp}_{n+1}(\lambda_i), \nabla_n(\mu)) \\
&\cong \mathrm{Hom}_R(\Delta^{\sharp}_{n+1}(\lambda_i), \mathrm{Hom}_{\mathscr{S}_{n,r}}(\mathrm{Res}^{n+1}_n(\Delta_{n+1}(\lambda_i)), \nabla_n(\mu))) \\
&\cong \begin{cases} \mathrm{Hom}_R(\Delta^{\sharp}_{n+1}(\lambda_i), R) & \text{if } \lambda_i = \mu \cup x_i \text{ for some addable node } x_i \text{ of } \mu \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{3.4.7}$$

where the last isomorphism follows from a general theory of quasi-hereditary algebras (see e.g. [6, Proposition A2.2 (ii)]) together with (i).

Suppose $\lambda_i = \mu \cup x_i$ and $\lambda_j = \mu \cup x_j$ for some addable nodes x_i, x_j of μ. Then, we have that $\lambda_i < \lambda_j$ if and only if $x_i \prec x_j$. Thus, by (3.4.6) together with (3.4.7) and (3.4.5), we obtain (iv).

(iii) is obtained by applying the contravariant functor $\circledast$ to (iv) thanks to Lemma 2.9 (ii). □

4. Some properties of induction and restriction functors

In this section, we study some properties for induction and restriction functors. In particular, we will prove that Ind_n^{n+1} and coInd_n^{n+1} are isomorphic.

4.1. First, we prepare some general results. For a finite dimensional associative algebra $\mathscr{A}$ over a field, we denote the full subcategory of $\mathscr{A}$-mod consisting of projective modules by $\mathscr{A}$-proj. Let $I_{\mathscr{A}} : \mathscr{A}\text{-proj} \to \mathscr{A}\text{-mod}$ be the canonical embedding functor.

When $\mathscr{A}$ is a quasi-hereditary algebra, we denote the full subcategory of $\mathscr{A}$-mod consisting of standard (resp. costandard) filtered modules by $\mathscr{A}\text{-mod}^{\Delta}$ (resp. $\mathscr{A}\text{-mod}^{\nabla}$). Let $I^{\Delta}_{\mathscr{A}} : \mathscr{A}\text{-mod}^{\Delta} \to \mathscr{A}\text{-mod}$ (resp. $I^{\nabla}_{\mathscr{A}} : \mathscr{A}\text{-mod}^{\nabla} \to \mathscr{A}\text{-mod}$) be the canonical embedding functor. Let $\mathscr{B}$ be a finite dimensional algebra, and $F : \mathscr{A}\text{-mod} \to \mathscr{B}\text{-mod}$ be a covariant functor. We say that $F \circ I^{\Delta}_{\mathscr{A}}$ (resp. $F \circ I^{\nabla}_{\mathscr{A}}$) is exact if, for any short exact sequence $0 \to N \to M \to L \to 0$ in $\mathscr{A}$-mod such that $N, M, L \in \mathscr{A}\text{-mod}^{\Delta}$ (resp. $L, M, N \in \mathscr{A}\text{-mod}^{\nabla}$), we have the short exact sequence $0 \to F \circ I^{\Delta}_{\mathscr{A}}(N) \to F \circ I^{\Delta}_{\mathscr{A}}(M) \to F \circ I^{\Delta}_{\mathscr{A}}(L) \to 0$.

If $\mathscr{A}$ has an algebra anti-involution $\theta_{\mathscr{A}}$, we consider the contravariant functor $\circledast : \mathscr{A}\text{-mod} \to \mathscr{A}\text{-mod}$ through the anti-involution $\theta_{\mathscr{A}}$ (see conventions in the last of §0).

By the general theory for quasi-hereditary algebras, we have the following lemma.

Lemma 4.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be finite dimensional algebras over a field with algebra anti-involution $\theta_{\mathscr{A}}$ and $\theta_{\mathscr{B}}$ respectively. Let F and G be covariant functors from $\mathscr{A}$-mod to $\mathscr{B}$-mod such that $F \circ \circledast \cong \circledast \circ G$. Assume that $\mathscr{A}$ is a quasi-hereditary algebra. Then we have the following.*

(i) *Assume that $G \cong \mathrm{Hom}_{\mathscr{A}}(M, ?)$ for an $(\mathscr{A}, \mathscr{B})$-bimodule M. If $M \in \mathscr{A}\text{-mod}^{\Delta}$, then $G \circ I^{\nabla}_{\mathscr{A}}$ is exact.*

(ii) *$F \circ I^{\Delta}_{\mathscr{A}}$ is exact if and only if $G \circ I^{\nabla}_{\mathscr{A}}$ is exact.*

Proof. If $M \in \mathscr{A}\text{-mod}^{\Delta}$, we have $\mathrm{Ext}^1_{\mathscr{A}}(M, X) = 0$ for any $X \in A\text{-mod}^{\nabla}$ by [6, Proposition A2.2]. This implies (i).

(ii) follows from the fact that the contravariant functor $\circledast : \mathscr{A}\text{-mod} \to \mathscr{A}\text{-mod}$ associated with $\theta_{\mathscr{A}}$ induce the (exact) contravariant functor form $\mathscr{A}\text{-mod}^{\Delta}$ (resp. $\mathscr{A}\text{-mod}^{\nabla}$) to $\mathscr{A}\text{-mod}^{\nabla}$ (resp. $\mathscr{A}\text{-mod}^{\Delta}$). □

The following lemma plays a fundamental role in the later arguments.

Lemma 4.3. *Let $\mathscr{A}$ and $\mathscr{B}$ be finite dimensional algebras over a field, and F, G be functors from $\mathscr{A}$-*mod *to $\mathscr{B}$-mod. Then we have the following.*
(i) *If F is a right exact functor, the homomorphism of vector spaces*

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(F \circ I_{\mathscr{A}}, G \circ I_{\mathscr{A}}) \quad \textit{given by} \quad \nu \mapsto \nu 1_{I_{\mathscr{A}}}$$

is an isomorphism.
(ii) *Assume that F and G are right exact functors. If $F \circ I_{\mathscr{A}} \cong G \circ I_{\mathscr{A}}$, we have $F \cong G$.*
(iii) *Assume that $\mathscr{A}$ is a quasi-hereditary algebra. If F is a right exact functor, the homomorphism of vector spaces*

$$\mathrm{Hom}(F, G) \to \mathrm{Hom}(F \circ I^{\Delta}_{\mathscr{A}}, G \circ I^{\Delta}_{\mathscr{A}}) \quad \textit{given by} \quad \nu \mapsto \nu 1_{I^{\Delta}_{\mathscr{A}}}$$

is an isomorphism.

Proof. We can prove (i) and (ii) in a similar way as in [16, Lemma 1.2]. We prove (iii).

Since any projective $\mathscr{A}$-module is an object of $\mathscr{A}$-mod$^{\Delta}$, we have $I^{\Delta}_{\mathscr{A}} \circ I_{\mathscr{A}} \cong I_{\mathscr{A}}$, and we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}(F, G) & \xrightarrow{\nu \mapsto \nu 1_{I_{\mathscr{A}}}} & \mathrm{Hom}(F \circ I_{\mathscr{A}}, G \circ I_{\mathscr{A}}) \\ & {\scriptstyle \nu \mapsto \nu 1_{I^{\Delta}_{\mathscr{A}}}} \searrow \quad \nearrow {\scriptstyle \tau \mapsto \tau 1_{I_{\mathscr{A}}}} & \\ & \mathrm{Hom}(F \circ I^{\Delta}_{\mathscr{A}}, G \circ I^{\Delta}_{\mathscr{A}}) & \end{array}$$

By (i), $\mathrm{Hom}(F, G) \to \mathrm{Hom}(F \circ I_{\mathscr{A}}, G \circ I_{\mathscr{A}})$ is an isomorphism. We can also prove that $\mathrm{Hom}(F \circ I^{\Delta}_{\mathscr{A}}, G \circ I^{\Delta}_{\mathscr{A}}) \to \mathrm{Hom}(F \circ I_{\mathscr{A}}, G \circ I_{\mathscr{A}})$ is an isomorphism in a similar way. Thus, the above diagram implies (iii). $\square$

Lemma 4.3 (iii) implies the following lifting arguments on adjointness for functors between module categories of two quasi-hereditary algebras.

Proposition 4.4. *Let $\mathscr{A}$ and $\mathscr{B}$ be quasi-hereditary algebras. Let $E\colon \mathscr{A}$-*mod $\to$ $\mathscr{B}$-mod (*resp.* $F\colon \mathscr{B}$-mod $\to \mathscr{A}$-mod) *be a functor such that E* (*resp.* F) *is right exact. Assume that $E \circ I^{\Delta}_{\mathscr{A}}$* (*resp.* $F \circ I^{\Delta}_{\mathscr{B}}$) *gives a functor* $\mathscr{A}$-mod$^{\Delta} \to \mathscr{B}$-mod$^{\Delta}$ (*resp.* $\mathscr{B}$-mod$^{\Delta} \to \mathscr{A}$-mod$^{\Delta}$).

If $F \circ I^{\Delta}_{\mathscr{B}}$ is right (*resp. left*) *adjoint to $E \circ I^{\Delta}_{\mathscr{A}}$, then F is right* (*resp. left*) *adjoint to E.*

Proof. It is enough to the case $F \circ I^{\Delta}_{\mathscr{B}}$ is right adjoint to $E \circ I^{\Delta}_{\mathscr{A}}$ by replacing a role of F with E in the other case. Assume that $F \circ I^{\Delta}_{\mathscr{B}}$ is right adjoint to $E \circ I^{\Delta}_{\mathscr{A}}$.

Then there exist the morphisms of functors $\tilde{\varepsilon}\colon E \circ I^{\Delta}_{\mathscr{A}} \circ F \circ I^{\Delta}_{\mathscr{B}} \to \mathrm{Id}_{\mathscr{B}} \circ I^{\Delta}_{\mathscr{B}}$ (unit) and $\tilde{\eta}$: $\mathrm{Id}_{\mathscr{A}} \circ I^{\Delta}_{\mathscr{A}} \to F \circ I^{\Delta}_{\mathscr{B}} \circ E \circ I^{\Delta}_{\mathscr{A}}$ (counit) such that $(\tilde{\varepsilon}1_{E\circ I^{\Delta}_{\mathscr{A}}}) \circ (1_{E\circ I^{\Delta}_{\mathscr{A}}}\tilde{\eta}) = 1_{E\circ I^{\Delta}_{\mathscr{A}}}$, and that $(1_{F\circ I^{\Delta}_{\mathscr{B}}}\tilde{\varepsilon}) \circ (\tilde{\eta}1_{F\circ I^{\Delta}_{\mathscr{B}}}) = 1_{F\circ I^{\Delta}_{\mathscr{B}}}$, where we regard the functor $\mathrm{Id}_{\mathscr{B}} \circ I^{\Delta}_{\mathscr{B}}$ (resp. $\mathrm{Id}_{\mathscr{A}} \circ I^{\Delta}_{\mathscr{A}}$) as the identity functor on $\mathscr{B}$-mod$^{\Delta}$ (resp. $\mathscr{A}$-mod$^{\Delta}$). Note that $I^{\Delta}_{\mathscr{A}} \circ F \circ I^{\Delta}_{\mathscr{B}} \cong F \circ I^{\Delta}_{\mathscr{B}}$ etc., we can write simply as $\tilde{\varepsilon}\colon E \circ F \circ I^{\Delta}_{\mathscr{B}} \to \mathrm{Id}_{\mathscr{B}} \circ I^{\Delta}_{\mathscr{B}}$ and $\tilde{\eta}$: $\mathrm{Id}_{\mathscr{A}} \circ I^{\Delta}_{\mathscr{A}} \to F \circ E \circ I^{\Delta}_{\mathscr{A}}$ such that

$$(\tilde{\varepsilon}1_E 1_{I^{\Delta}_{\mathscr{A}}}) \circ (1_E\tilde{\eta}) = 1_E 1_{I^{\Delta}_{\mathscr{A}}}, \tag{4.4.1}$$

$$(1_F\tilde{\varepsilon}) \circ (\tilde{\eta}1_F 1_{I^{\Delta}_{\mathscr{B}}}) = 1_F 1_{I^{\Delta}_{\mathscr{B}}}. \tag{4.4.2}$$

By Lemma 4.3 (iii), there exist the morphisms of functors

$$\varepsilon\colon E \circ F \to \mathrm{Id}_{\mathscr{B}}, \quad \eta\colon \mathrm{Id}_{\mathscr{A}} \to F \circ E$$

such that $\varepsilon 1_{I^{\Delta}_{\mathscr{B}}} = \tilde{\varepsilon}$ and $\eta 1_{I^{\Delta}_{\mathscr{A}}} = \tilde{\eta}$. Moreover, we have

$$\begin{aligned}((\varepsilon 1_E) \circ (1_E\eta))1_{I^{\Delta}_{\mathscr{A}}} &= ((\varepsilon 1_E) \circ (1_E\eta)))(1_{I^{\Delta}_{\mathscr{A}}} \circ 1_{I^{\Delta}_{\mathscr{A}}}) \\ &= (\varepsilon 1_E 1_{I^{\Delta}_{\mathscr{A}}}) \circ (1_E\eta 1_{I^{\Delta}_{\mathscr{A}}}) \\ &= (\tilde{\varepsilon}1_E 1_{I^{\Delta}_{\mathscr{A}}}) \circ (1_E\tilde{\eta}) \\ &= 1_E 1_{I^{\Delta}_{\mathscr{A}}}.\end{aligned} \tag{4.4.3}$$

Similarly, we have

$$\begin{aligned}((1_F\varepsilon) \circ (\eta 1_F))1_{I^{\Delta}_{\mathscr{B}}} &= (1_F\varepsilon 1_{I^{\Delta}_{\mathscr{B}}}) \circ (\eta 1_F 1_{I^{\Delta}_{\mathscr{B}}}) \\ &= (1_F\tilde{\varepsilon}) \circ (\tilde{\eta}1_F 1_{I^{\Delta}_{\mathscr{B}}}) \\ &= 1_F 1_{I^{\Delta}_{\mathscr{B}}}.\end{aligned} \tag{4.4.4}$$

By (4.4.3) and (4.4.4) together with Lemma 4.3 (iii), we have

$$(\varepsilon 1_E) \circ (1_E\eta) = 1_E, \quad (1_F\varepsilon) \circ (\eta 1_F) = 1_F,$$

and F is right adjoint to E. □

REMARK 4.5. By replacing the $\mathscr{A}$-mod$^{\Delta}$ (resp. $\mathscr{B}$-mod$^{\Delta}$) with $\mathscr{A}$-proj (resp. $\mathscr{B}$-proj), we also obtain the lifting argument from the full subcategories consisting of projective objects thanks to Lemma 4.3 (i) in a similar way. (In this case, $\mathscr{A}$ and $\mathscr{B}$ do not need being quasi-hereditary algebras.) In this case, the lifting argument from projectives have already appeared in the proof of [16, Proposition 2.9].

In order to apply Proposition 4.4 to our functors between module categories of cyclotomic q-Schur algebras, we prepare the following technical lemma.

Lemma 4.6. *Let $\mathscr{A}$ and $\mathscr{B}$ be associative algebras over a field. Assume that $\mathscr{B}$ is quasi-hereditary. Let F and G be functors from $\mathscr{B}$-mod to $\mathscr{A}$-mod such that F is right exact. Then we have the following.*
(i) *Assume that the both functors $F \circ I^{\nabla}_{\mathscr{B}}$ and $G \circ I^{\nabla}_{\mathscr{B}}$ are exact. If $F \circ I^{\Delta}_{\mathscr{B}} \cong G \circ I^{\Delta}_{\mathscr{B}}$, then we have $F \circ I^{\nabla}_{\mathscr{B}} \cong G \circ I^{\nabla}_{\mathscr{B}}$.*
(ii) *Assume that $\mathscr{A}$ and $\mathscr{B}$ are equipped with anti-involutions $\theta_{\mathscr{A}}$ and $\theta_{\mathscr{B}}$, and that the both functors $F \circ I^{\nabla}_{\mathscr{B}}$ and $G \circ I^{\nabla}_{\mathscr{B}}$ are exact. Let G' be a functor from $\mathscr{B}$-mod to $\mathscr{A}$-mod. Assume that $F \circ \circledast \cong \circledast \circ F$ and $G' \circ \circledast \cong \circledast \circ G$. If $F \circ I^{\Delta}_{\mathscr{B}} \cong G \circ I^{\Delta}_{\mathscr{B}}$, then we have $F \circ I^{\Delta}_{\mathscr{B}} \cong G' \circ I^{\Delta}_{\mathscr{B}}$.*
(iii) *Assume that G is left exact, and that* $\dim F(M) \geq \dim G(M)$ *for any $M \in \mathscr{B}$*-mod$^{\Delta}$. *If $F \circ I_{\mathscr{B}} \cong G \circ I_{\mathscr{B}}$, then we have $F \circ I^{\Delta}_{\mathscr{B}} \cong G \circ I^{\Delta}_{\mathscr{B}}$.*

Proof. (i). By the assumption, there exists a functorial isomorphism $\tilde{\nu}\colon F \circ I^{\Delta}_{\mathscr{B}} \to G \circ I^{\Delta}_{\mathscr{B}}$. Since F is right exact, by Lemma 4.3 (iii), there exists the unique morphism $\nu\colon F \to G$ such that $\nu 1_{I^{\Delta}_{\mathscr{B}}} = \tilde{\nu}$. We prove $\nu 1_{I^{\nabla}_{\mathscr{B}}}\colon F \circ I^{\nabla}_{\mathscr{B}} \to G \circ I^{\nabla}_{\mathscr{B}}$ is an isomorphism.

By [6, Proposition 4.4], for $N \in \mathscr{B}$-mod$^{\nabla}$, we can take the following exact sequence

$$0 \to T_k \xrightarrow{d'_k} \cdots \xrightarrow{d'_2} T_1 \xrightarrow{d'_1} T_0 \xrightarrow{d'_0} N \to 0,$$

such that T_i is a (characteristic) tilting module, and that $\operatorname{Ker} d'_i \in \mathscr{B}$-mod$^{\nabla}$ for each $i = 0, \dots, k$. By applying the functors F and G to this exact sequence, we have the following commutative diagram

$$\begin{array}{ccccccc}
F(T_1) & \longrightarrow & F(T_0) & \longrightarrow & F(N) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \nu(T_1)} & & \downarrow{\scriptstyle \nu(T_0)} & & \downarrow{\scriptstyle \nu(N)} & & \\
G(T_1) & \longrightarrow & G(T_0) & \longrightarrow & G(N) & \longrightarrow & 0
\end{array}$$

such that each row is exact since $\operatorname{Ker} d'_i \in \mathscr{B}$-mod$^{\nabla}$ for each $i = 0, \dots, k$. By the assumption, we have that both $\nu(T_1)$ and $\nu(T_0)$ are isomorphisms since $T_0, T_1 \in \mathscr{B}$-mod$^{\Delta}$. Thus, the above diagram implies that $\nu(N)$ is an isomorphism. Then $\nu 1_{I^{\nabla}_{\mathscr{B}}}$ is an isomorphism.

We prove (ii). Note that $\circledast \circ I^{\nabla}_{\mathscr{B}} \circ \circledast \circ I^{\Delta}_{\mathscr{B}} \cong I^{\Delta}_{\mathscr{B}}$, we have

$$\begin{aligned}
F \circ I^{\Delta}_{\mathscr{B}} &\cong F \circ \circledast \circ I^{\nabla}_{\mathscr{B}} \circ \circledast \circ I^{\Delta}_{\mathscr{B}} \\
&\cong \circledast \circ F \circ I^{\nabla}_{\mathscr{B}} \circ \circledast \circ I^{\Delta}_{\mathscr{B}} \\
&\cong \circledast \circ G \circ I^{\nabla}_{\mathscr{B}} \circ \circledast \circ I^{\Delta}_{\mathscr{B}} \quad \text{(by (i))} \\
&\cong G' \circ \circledast \circ I^{\nabla}_{\mathscr{B}} \circ \circledast \circ I^{\Delta}_{\mathscr{B}} \\
&\cong G' \circ I^{\Delta}_{\mathscr{B}}.
\end{aligned}$$

Finally, we prove (iii). Let $\tilde{\mu}\colon F \circ I_{\mathscr{B}} \to G \circ I_{\mathscr{B}}$ be the functorial isomorphism which gives the isomorphism $F \circ I_{\mathscr{B}} \cong G \circ I_{\mathscr{B}}$. By Lemma 4.3 (i), there exists the unique morphism $\mu\colon F \to G$ such that $\mu 1_{I_{\mathscr{B}}} = \tilde{\mu}$. We prove that $\mu 1_{I^{\Delta}_{\mathscr{B}}}\colon F \circ I^{\Delta}_{\mathscr{B}} \to G \circ I^{\Delta}_{\mathscr{B}}$ gives an isomorphism of functors.

Note that $\mathscr{B}$ is a quasi-hereditary algebra, in particular, the global dimension of $\mathscr{B}$ is finite. Thus, for any $M \in \mathscr{B}\text{-mod}^{\Delta}$, we can take a projective resolution

$$0 \to P_k \xrightarrow{d_k} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

such that k is equal to the projective dimension of M (denoted by $\mathrm{pdim} M$). By an induction on $\mathrm{pdim} M$, we prove that $\mu 1_{I^{\Delta}_{\mathscr{B}}}(M)$ is an isomorphism.

When $\mathrm{pdim} M = 0$, we have $\mu(M) = \tilde{\mu}(M)$ since M is projective. Thus, $\mu(M)$ is an isomorphism.

Assume that $\mathrm{pdim} M > 0$. For the short exact sequence

$$0 \to \mathrm{Ker} d_0 \to P_0 \xrightarrow{d_0} M \to 0, \tag{4.6.1}$$

we have that $\mathrm{Ker} d_0 \in \mathscr{B}\text{-mod}^{\Delta}$ by [6, Proposition A2.2 (v)]. Moreover, we have $\mathrm{pdimKer} d_0 \leq \mathrm{pdim} M - 1$. By applying the functors F and G to (4.6.1), we have the following commutative diagram

$$\begin{array}{ccccccccc} & & F(\mathrm{Ker} d_0) & \longrightarrow & F(P_0) & \longrightarrow & F(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu(\mathrm{Ker} d_0)} & & \downarrow{\scriptstyle \mu(P_0)} & & \downarrow{\scriptstyle \mu(M)} & & \\ 0 & \longrightarrow & G(\mathrm{Ker} d_0) & \longrightarrow & G(P_0) & \longrightarrow & G(M) & & \end{array} \tag{4.6.2}$$

such that each row is exact. Note that $\mu(\mathrm{Ker} d_0)$ (resp. $\mu(P_0)$) is an isomorphism by the assumption of induction (resp. the fact P_0 is projective). Then, the above diagram implies that $\mu(M)$ is injective. Thus, $\mu(M)$ is an isomorphism since $\dim F(M) \geq \dim G(M)$. Then $\mu 1_{I^{\Delta}_{\mathscr{B}}}$ is an isomorphism. □

REMARK 4.7. In Lemma 4.6 (iii), if G is right exact, then we do not need the assumption $\dim F(M) \geq \dim G(M)$ since the second row of the diagram (4.6.2) becomes right exact in such case.

4.8. We consider the following setting. Let $\mathscr{A}'$ (resp. $\mathscr{B}'$) be a finite dimensional associative algebra over a field, and $\mathscr{A}$ (resp. $\mathscr{B}$) be a quasi-hereditary cover of $\mathscr{A}'$ (resp. $\mathscr{B}'$) in the sense of [14]. Namely, $\mathscr{A}$ is a quasi-hereditary algebra together with a projective $\mathscr{A}$-module $P_{\mathscr{A}}$ satisfying the following two conditions;
(i) $\mathscr{A}' \cong \mathrm{End}_{\mathscr{A}}(P_{\mathscr{A}})^{\mathrm{opp}}$.
(ii) The exact functor $\Omega_{\mathscr{A}} = \mathrm{Hom}_{\mathscr{A}}(P_{\mathscr{A}}, ?)$ from $\mathscr{A}$-mod to $\mathscr{A}'$-mod is fully faithful on projective objects.
Let $\Phi_{\mathscr{A}} = \mathrm{Hom}_{\mathscr{A}'}(\Omega_{\mathscr{A}}(\mathscr{A}), ?)\colon \mathscr{A}'\text{-mod} \to \mathscr{A}\text{-mod}$ be the right adjoint functor of $\Omega_{\mathscr{A}}$. By [14, 4.2.1 and Lemma 4.32], we have

$$\Omega_{\mathscr{A}} \circ \Phi_{\mathscr{A}} \cong \mathrm{Id}_{\mathscr{A}'}, \tag{4.8.1}$$

$$\Phi_{\mathscr{A}} \circ \Omega_{\mathscr{A}} \circ I_{\mathscr{A}} \cong \mathrm{Id}_{\mathscr{A}} \circ I_{\mathscr{A}}. \tag{4.8.2}$$

It is similar for $\mathscr{B}$ and $\mathscr{B}'$.

Let X be an $(\mathscr{A}, \mathscr{B})$-bimodule such that X is projective as left $\mathscr{A}$-module. We define a functor $\mathrm{Res}\colon \mathscr{A}\text{-mod} \to \mathscr{B}\text{-mod}$ by

$$\mathrm{Res} = \mathrm{Hom}_{\mathscr{A}}(X, ?).$$

We also define two functors $\mathrm{Ind}, \mathrm{coInd}\colon \mathscr{B}\text{-mod} \to \mathscr{A}\text{-mod}$ by

$$\mathrm{Ind} = X \otimes_{\mathscr{B}} ? \quad \text{and} \quad \mathrm{coInd} = \mathrm{Hom}_{\mathscr{B}}(\mathrm{Res}(\mathscr{A}), ?).$$

Then we see that Res is exact, and that Ind (resp. coInd) is left (resp. right) adjoint to Res.

Similarly, let X' be an $(\mathscr{A}', \mathscr{B}')$-bimodule such that X' is projective as left $\mathscr{A}'$-module, and define

$$\mathrm{Res}' = \mathrm{Hom}_{\mathscr{A}'}(X', ?)\colon \mathscr{A}'\text{-mod} \to \mathscr{B}'\text{-mod},$$
$$\mathrm{Ind}' = X' \otimes_{\mathscr{B}'} ?\colon \mathscr{B}'\text{-mod} \to \mathscr{A}'\text{-mod},$$
$$\mathrm{coInd}' = \mathrm{Hom}_{\mathscr{B}'}(\mathrm{Res}'(\mathscr{A}'), ?)\colon \mathscr{B}'\text{-mod} \to \mathscr{A}'\text{-mod}.$$

We assume the following conditions;

(A-1) $\mathrm{Ind}' \cong \mathrm{coInd}'$. Hence, Ind' is left and right adjoint to Res'. In particular Ind' is exact.

(A-2) $\mathscr{A}$, $\mathscr{A}'$, $\mathscr{B}$ and $\mathscr{B}'$ are equipped with an anti-involution $\theta_{\mathscr{A}}$, $\theta_{\mathscr{A}'}$, $\theta_{\mathscr{B}}$ and $\theta_{\mathscr{B}'}$ respectively. Moreover, we assume the following isomorphisms;

(A-2a) $\mathrm{Res}(\mathscr{A}) \cong X^{\theta}$ (resp. $\mathrm{Res}'(\mathscr{A}') \cong X'^{\theta}$) as $(\mathscr{B}, \mathscr{A})$-bimodules (resp. $(\mathscr{B}', \mathscr{A}')$-bimodules), where X^{θ} (resp. X'^{θ}) is obtained from X (resp. (X')) by twisting the action through the anti-involutions $\theta_{\mathscr{A}}$ and $\theta_{\mathscr{B}}$ (resp. $\theta_{\mathscr{A}'}$ and $\theta_{\mathscr{B}'}$).

(A-2b) $\Omega_{\mathscr{A}}(\mathscr{A}) \cong P^{\theta}_{\mathscr{A}}$ (resp. $\Omega_{\mathscr{B}}(\mathscr{B}) \cong P^{\theta}_{\mathscr{B}}$) as $(\mathscr{A}', \mathscr{A})$-bimodules (resp. $(\mathscr{B}', \mathscr{B})$-bimodules), where $P^{\theta}_{\mathscr{A}}$ (resp. $P^{\theta}_{\mathscr{B}}$) is obtained from $P_{\mathscr{A}}$ (resp. $P_{\mathscr{B}}$) through the anti-involutions as in (A-2a).

(A-3) $\Omega_{\mathscr{B}} \circ \mathrm{Res} \cong \mathrm{Res}' \circ \Omega_{\mathscr{A}}$.

(A-4) $X^{\theta} \in \mathscr{B}\text{-mod}^{\Delta}$.

(A-5) $\mathrm{Ind} \circ I^{\Delta}_{\mathscr{B}}$ (resp. $\mathrm{Res} \circ I^{\Delta}_{\mathscr{A}}$) gives a functor from $\mathscr{B}\text{-mod}^{\Delta}$ to $\mathscr{A}\text{-mod}^{\Delta}$ (resp. $\mathscr{A}\text{-mod}^{\Delta}$ to $\mathscr{B}\text{-mod}^{\Delta}$).

(A-6) For $M \in \mathscr{B}\text{-mod}^{\Delta}$, we have the following.

(A-6a) $\dim \mathrm{Ind}(M) \geq \dim \mathrm{coInd}(M)$.

(A-6b) $\dim \mathrm{Ind}' \circ \Omega_{\mathscr{B}}(M) \geq \dim \Omega_{\mathscr{A}} \circ \mathrm{coInd}(M)$.

Note that the condition (A-2a) implies that $\mathrm{Res} \cong X^{\theta} \otimes_{\mathscr{A}} ?$ (resp. $\mathrm{Res}' \cong (X')^{\theta} \otimes_{\mathscr{A}} ?$). Similarly, the condition (A-2b) implies that $\Omega_{\mathscr{A}} \cong P^{\theta}_{\mathscr{A}} \otimes_{\mathscr{A}} ?$ (resp. $\Omega_{\mathscr{B}} \cong P^{\theta}_{\mathscr{B}} \otimes_{\mathscr{B}} ?$).

Thanks to the assumptions (A-1) and (A-2), we have the following lemma.

Lemma 4.9. *We have the following isomorphisms of functors.*
(i) $\circledast \circ \mathrm{Res} \cong \mathrm{Res} \circ \circledast$ *and* $\circledast \circ \mathrm{Ind} \cong \mathrm{coInd} \circ \circledast$.
(ii) $\circledast \circ \mathrm{Res}' \cong \mathrm{Res}' \circ \circledast$ *and* $\circledast \circ \mathrm{Ind}' \cong \mathrm{Ind}' \circ \circledast$.
(iii) $\circledast \circ \Omega_{\mathscr{A}} \cong \Omega_{\mathscr{A}} \circ \circledast$ (*resp.* $\circledast \circ \Omega_{\mathscr{B}} \cong \Omega_{\mathscr{B}} \circ \circledast$).

Proof. We can prove the lemma in a similar way as in Lemma 2.9. □

Then we have the following theorem.

Theorem 4.10. *Under the setting in the paragraph* 4.8, *we have the isomorphism of functors* $\mathrm{Ind} \cong \mathrm{coInd}$.

Proof. It is clear that $\mathrm{coInd} \circ \Phi_{\mathscr{B}}$ (resp. $\Phi_{\mathscr{A}} \circ \mathrm{Ind}'$) is right adjoint to $\Omega_{\mathscr{B}} \circ \mathrm{Res}$ (resp. $\mathrm{Res}' \circ \Omega_{\mathscr{A}}$), the uniqueness of the adjoint functor together with the assumption (A-3), we have

$$\mathrm{coInd} \circ \Phi_{\mathscr{B}} \cong \Phi_{\mathscr{A}} \circ \mathrm{Ind}'. \tag{4.10.1}$$

Combining this isomorphism with (4.8.1) and (4.8.2), we have

$$\begin{aligned} \mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}} &\cong \Omega_{\mathscr{A}} \circ \Phi_{\mathscr{A}} \circ \mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}} \quad (\because (4.8.1)) \\ &\cong \Omega_{\mathscr{A}} \circ \mathrm{coInd} \circ \Phi_{\mathscr{B}} \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}} \quad (\because (4.10.1)) \\ &\cong \Omega_{\mathscr{A}} \circ \mathrm{coInd} \circ I_{\mathscr{B}} \quad (\because (4.8.2)). \end{aligned} \tag{4.10.2}$$

By Lemma 4.6 (iii) together with the assumption (A-6b), the isomorphisms (4.10.2) imply the isomorphism

$$\mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}}^{\Delta} \cong \Omega_{\mathscr{A}} \circ \mathrm{coInd} \circ I_{\mathscr{B}}^{\Delta}. \tag{4.10.3}$$

Since $\mathrm{Ind}' \circ \Omega_{\mathscr{B}}$ is exact, $\mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}}^{\nabla}$ is exact. We also see that $\Omega_{\mathscr{A}} \circ \mathrm{coInd} \circ I_{\mathscr{B}}^{\nabla}$ is exact by Lemma 4.2 (i) together with the assumption (A-2a) and (A-4). Then, applying Lemma 4.6 (ii) to the isomorphism (4.10.3) together with Lemma 4.9, we have the isomorphism

$$\mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}}^{\Delta} \cong \Omega_{\mathscr{A}} \circ \mathrm{Ind} \circ I_{\mathscr{B}}^{\Delta}.$$

Combining this isomorphism with (4.10.1), we have

$$\begin{aligned} \mathrm{coInd} \circ \Phi_{\mathscr{B}} \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}}^{\Delta} &\cong \Phi_{\mathscr{A}} \circ \mathrm{Ind}' \circ \Omega_{\mathscr{B}} \circ I_{\mathscr{B}}^{\Delta} \\ &\cong \Phi_{\mathscr{A}} \circ \Omega_{\mathscr{A}} \circ \mathrm{Ind} \circ I_{\mathscr{B}}^{\Delta}. \end{aligned} \tag{4.10.4}$$

Note that $I_{\mathscr{B}}^{\Delta} \circ I_{\mathscr{B}} \cong I_{\mathscr{B}}$, and that the functor Ind preserves projectivity since Ind has the exact right adjoint functor Res. Then, (4.10.4) together with (4.8.2) implies

$$\mathrm{Ind} \circ I_{\mathscr{B}} \cong \mathrm{coInd} \circ I_{\mathscr{B}}. \tag{4.10.5}$$

By Lemma 4.6 (iii) together with the assumption (A-6a), this isomorphism implies

$$\mathrm{Ind}\circ I_{\mathscr{B}}^{\Delta} \cong \mathrm{coInd}\circ I_{\mathscr{B}}^{\Delta}. \tag{4.10.6}$$

By (4.10.6) together with the assumption (A-5), we see that $\mathrm{Ind}\circ I_{\mathscr{B}}^{\Delta}$ is left and right adjoint to $\mathrm{Res}\circ I_{\mathscr{A}}^{\Delta}$. Hence, we have that Ind is left and right adjoint to Res by Proposition 4.4. Then we see that Ind $\cong$ coInd by the uniqueness of the adjoint functor. □

REMARK 4.11. In order to prove Theorem 4.10, we may be able to remove some assumptions from (A-1)–(A-6) in some situations by supposing another condition as follows.
(i) If we can assume the isomorphism $\circledast\circ\mathrm{Ind}\cong\mathrm{Ind}\circ\circledast$, we need only the assumption (A-2a).
(ii) Assume that the functor Ind is exact, and that $\mathscr{A}'$ is symmetric algebra. Then we need only the assumptions (A-1) and (A-3). In this case, we can prove the isomorphism Ind $\cong$ coInd in a similar way as in the proof of [16, Proposition 2.9.] by using the lifting argument from projectives.
(iii) If we can assume the isomorphism

$$\Omega_{\mathscr{A}}\circ\mathrm{Ind}\cong\mathrm{Ind}'\circ\Omega_{\mathscr{B}}, \tag{4.11.1}$$

we need only the condition (A-1), (A-3), (A-5) and (A-6a) since the isomorphism (4.10.5) follows from (4.10.1) and (4.11.1) together with (4.8.2).

4.12. Let's go back to the module categories of cyclotomic q-Schur algebras. From now on, throughout of this paper, we assume that R is a field, and that $Q_k\neq 0$ for any $k=1,\dots,r$.

Put $\omega_n=(\emptyset,\dots,\emptyset,(1,\dots,1,0,\dots,0))\in\Lambda_{n,r}(\mathbf{m}')$ and $\omega_{n+1}=\gamma(\omega_n)\in\Lambda_{n+1,r}(\mathbf{m})$. Then, it is clear that $M^{\omega_n}\cong\mathscr{H}_{n,r}$ as right $\mathscr{H}_{n,r}$-modules (resp. $M^{\omega_{n+1}}\cong\mathscr{H}_{n+1,r}$ as right $\mathscr{H}_{n+1,r}$-modules). Thus, we have the following isomorphisms of algebras;

$$\begin{aligned}&\mathrm{End}_{\mathscr{S}_{n,r}}(\mathscr{S}_{n+1,r}1_{\omega_n})\cong 1_{\omega_n}\mathscr{S}_{n,r}1_{\omega_n}=\mathrm{End}_{\mathscr{H}_{n,r}}(M^{\omega_n})\cong\mathscr{H}_{n,r},\\&\mathrm{End}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}1_{\omega_{n+1}})\cong 1_{\omega_{n+1}}\mathscr{S}_{n+1,r}1_{\omega_{n+1}}=\mathrm{End}_{\mathscr{H}_{n+1,r}}(M^{\omega_{n+1}})\cong\mathscr{H}_{n+1,r}.\end{aligned} \tag{4.12.1}$$

We attach a correspondence between the setting in the paragraph 4.8 and our setting as follows.

- $\mathscr{A}=\mathscr{S}_{n+1,r}$, $\mathscr{A}'=\mathscr{H}_{n+1,r}$, $\mathscr{B}=\mathscr{S}_{n,r}$, $\mathscr{B}'=\mathscr{H}_{n,r}$.
- $P_{\mathscr{A}}=\mathscr{S}_{n+1,r}1_{\omega_{n+1}}$, $P_{\mathscr{B}}=\mathscr{S}_{n,r}1_{\omega_n}$, $\Omega_{\mathscr{A}}=\Omega_{n+1}:=\mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}1_{\omega_{n+1}},?)$, $\Omega_{\mathscr{B}}=\Omega_n:=\mathrm{Hom}_{\mathscr{S}_{n,r}}(\mathscr{S}_{n,r}1_{\omega_n},?)$.
- $X=\mathscr{S}_{n+1,r}\xi$, Res $=\mathrm{Res}_n^{n+1}$, Ind $=\mathrm{Ind}_n^{n+1}$, coInd $=\mathrm{coInd}_n^{n+1}$. (Note that $\mathrm{Res}_n^{n+1}(\mathscr{S}_{n+1,r})\cong\xi\mathscr{S}_{n+1,r}$).
- $X'=\mathscr{H}_{n+1,r}$, where we regard $\mathscr{H}_{n+1,r}$ as a right $\mathscr{H}_{n,r}$-module by the restriction through the injection $\iota^{\mathscr{H}}$ defined by (3.1.1). $\mathrm{Res}'={}^{\mathscr{H}}\mathrm{Res}_n^{n+1}:=\mathrm{Hom}_{\mathscr{H}_{n+1,r}}(\mathscr{H}_{n+1,r},?)$, $\mathrm{Ind}'={}^{\mathscr{H}}\mathrm{Ind}_n^{n+1}:=\mathscr{H}_{n+1,r}\otimes_{\mathscr{H}_{n,r}}?$, $\mathrm{coInd}'={}^{\mathscr{H}}\mathrm{coInd}_n^{n+1}:=\mathrm{Hom}_{\mathscr{H}_{n,r}}(\mathscr{H}_{n+1,r},?)$.

Thanks to the double centralizer property between $\mathscr{S}_{n+1,r}$ and $\mathscr{H}_{n+1,r}$ (see e.g. [12, Theorem 5.3]), we see that Ω_{n+1} (resp. Ω_n) is fully faithful on projective objects.

In order to apply Theorem 4.10 to our setting, we should prove that the assumptions (A-1)–(A-6) hold.

The assumption (A-1) follows from the fact that $\mathscr{H}_{n+1,r}$ (resp. $\mathscr{H}_{n,r}$) is a symmetric algebra as proved in [11] (see e.g. [16, Lemma 2.6]).

Note that the isomorphism $1_{\omega_n}\mathscr{S}_{n,r}1_{\omega_n} \cong \mathscr{H}_{n,r}$ (resp. $1_{\omega_{n+1}}\mathscr{S}_{n+1,r}1_{\omega_{n+1}} \cong \mathscr{H}_{n+1,r}$) is given by $\varphi \mapsto \varphi(m_{\omega_n})$ (resp. $\varphi \mapsto \varphi(m_{\omega_{n+1}})$), we have the following lemma.

Lemma 4.13 ([19, Proposition 6.3])**.** (i) *Under the isomorphism* $\mathscr{H}_{n,r} \cong 1_{\omega_n}\mathscr{S}_{n,r}1_{\omega_n}$, *we have*

$$T_0 = 1_{\omega_n}F_{(m_{r-1},r-1)}E_{(m_{r-1},r-1)}1_{\omega_n} + Q_r 1_{\omega_n},$$
$$T_i = 1_{\omega_n}F_{(i,r)}E_{(i,r)}1_{\omega_n} - q^{-1}1_{\omega_n} \quad (1 \le i \le n-1).$$

(ii) *Under the isomorphism* $\mathscr{H}_{n+1,r} \cong 1_{\omega_{n+1}}\mathscr{S}_{n+1,r}1_{\omega_{n+1}}$, *we have*

$$T_0 = 1_{\omega_{n+1}}F_{(m_{r-1},r-1)}E_{(m_{r-1},r-1)}1_{\omega_{n+1}} + Q_r 1_{\omega_{n+1}},$$
$$T_i = 1_{\omega_{n+1}}F_{(i,r)}E_{(i,r)}1_{\omega_{n+1}} - q^{-1}1_{\omega_{n+1}} \quad (1 \le i \le n-1),$$
$$T_n = 1_{\omega_{n+1}}F_{(m_r-1,r)}\cdots F_{(n+1,r)}F_{(n,r)}E_{(n,r)}E_{(n+1,r)}\cdots E_{(m_r-1,r)}1_{\omega_{n+1}} - q^{-1}1_{\omega_n}.$$

By Proposition 2.6 and Lemma 4.13, we have the following corollary.

Corollary 4.14. *The restriction of* $\iota\colon \mathscr{S}_{n,r} \to \mathscr{S}_{n+1,r}$ *to* $\mathscr{H}_{n,r} \cong 1_{\omega_n}\mathscr{S}_{n,r}1_{\omega_n}$ *coincides with* $\iota^{\mathscr{H}}: \mathscr{H}_{n,r} \to \mathscr{H}_{n+1,r}$.

Recall the anti-involution $*$ on $\mathscr{H}_{n,r}$ (resp. $\mathscr{H}_{n+1,r}$), and we consider the contravariant functor $\circledast\colon \mathscr{H}_{n,r}$-mod $\to \mathscr{H}_{n,r}$-mod (resp. $\circledast\colon \mathscr{H}_{n+1,r}$-mod $\to \mathscr{H}_{n+1,r}$-mod) with respect to $*$. By Proposition 2.6, Lemma 4.13 and Corollary 4.14 together with Lemma 1.16, we see that the assumption (A-2) holds (cf. (2.9.1)).

Lemma 4.15. *The assumption* (A3) *holds, namely we have the following isomorphisms of functors.*

$$\Omega_n \circ \mathrm{Res}_n^{n+1} \cong {}^{\mathscr{H}}\mathrm{Res}_n^{n+1} \circ \Omega_{n+1}.$$

Proof. By Proposition 2.6 and Corollary 4.14, we see that

$$\mathscr{S}_{n+1,r}\xi \otimes_{\mathscr{S}_{n,r}} \mathscr{S}_{n,r}1_{\omega_n} \cong \mathscr{S}_{n+1,r}1_{\omega_{n+1}} \quad \text{as} \quad (\mathscr{S}_{n+1,r}, \mathscr{H}_{n,r})\text{-bimodules.} \tag{4.15.1}$$

For $M \in \mathscr{S}_{n+1,r}$-mod, we have the following natural isomorphisms

$$\Omega_n \circ \mathrm{Res}_n^{n+1}(M) = \mathrm{Hom}_{\mathscr{S}_{n,r}}(\mathscr{S}_{n,r}1_{\omega_n}, \mathrm{Res}_n^{n+1}(M))$$

$$
\begin{aligned}
&\cong \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathrm{Ind}_n^{n+1}(\mathscr{S}_{n,r}1_{\omega_n}), M)\\
&= \mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}\xi \otimes_{\mathscr{S}_{n,r}} \mathscr{S}_{n,r}1_{\omega_n}, M)\\
&\cong {}^{\mathscr{H}}\mathrm{Res}_n^{n+1}(\mathrm{Hom}_{\mathscr{S}_{n+1,r}}(\mathscr{S}_{n+1,r}1_{\omega_{n+1}}, M)) \quad \text{(because of (4.15.1))}\\
&= {}^{\mathscr{H}}\mathrm{Res}_n^{n+1} \circ \Omega_{n+1}(M). \qquad \square
\end{aligned}
$$

Note that $\mathscr{S}_{n+1,r} \in \mathscr{S}_{n+1,r}\text{-mod}^{\Delta}$, Theorem 3.4 (i) implies

$$(\mathscr{S}_{n+1,r}\xi)^{\theta} \cong \mathrm{Res}_n^{n+1}(\mathscr{S}_{n+1,r}) \in \mathscr{S}_{n,r}\text{-mod}^{\Delta}.$$

Thus, the assumption (A-4) holds.

The assumption (A-5) follows from Theorem 3.4 (i) provided that Ind_n^{n+1} is exact on $\mathscr{S}_{n,r}\text{-mod}^{\Delta}$. But Lemma 4.9 (i) holds by (A-2), and Lemma 4.2 asserts that the exactness follows from Lemma 4.9 (i) and (A-4).

Finally, we prove the assumption (A-6).

Lemma 4.16. *For $M \in \mathscr{S}_{n,r}\text{-mod}^{\Delta}$, we have the following.*

(i) $\dim \mathrm{Ind}_n^{n+1}(M) \geq \dim \mathrm{coInd}_n^{n+1}(M)$.

(ii) $\dim {}^{\mathscr{H}}\mathrm{Ind}_n^{n+1} \circ \Omega_n(M) \geq \dim \Omega_{n+1} \circ \mathrm{coInd}_n^{n+1}(M)$.

Proof. By Lemma 2.10, we have that $\dim \mathrm{Ind}_n^{n+1}(\Delta_n(\lambda)) = \dim \mathrm{Ind}_n^{n+1}(\nabla_n(\lambda))$ for $\lambda \in \Lambda_{n,r}^+$. Since $\mathrm{Ind}_n^{n+1}(\nabla_n(\lambda)) \cong \mathrm{Ind}_n^{n+1}(\Delta_n(\lambda)^{\circledast}) \cong \circledast \circ \mathrm{coInd}_n^{n+1}(\Delta_n(\lambda))$ by Lemma 2.9 (ii), we have that $\dim \mathrm{Ind}_n^{n+1}(\Delta_n(\lambda)) = \dim \mathrm{coInd}_n^{n+1}(\Delta_n(\lambda))$. Hence (i) holds for standard modules. Now we argue by induction on the length of Δ-filtration.

Since $M \in \mathscr{S}_{n,r}\text{-mod}^{\Delta}$, we can take a exact sequence

$$0 \to N_1 \to M \to N_2 \to 0 \quad \text{such that} \quad N_1, N_2 \in \mathscr{S}_{n,r}\text{-mod}^{\Delta}. \tag{4.16.1}$$

Recalling that Ind_n^{n+1} is exact on $\mathscr{S}_{n,r}\text{-mod}^{\Delta}$ (see the above of Lemma 4.16), we have the exact sequence

$$0 \to \mathrm{Ind}_n^{n+1}(N_1) \to \mathrm{Ind}_n^{n+1}(M) \to \mathrm{Ind}_n^{n+1}(N_2) \to 0.$$

Then we have

$$\dim \mathrm{Ind}_n^{n+1}(M) = \dim \mathrm{Ind}_n^{n+1}(N_1) + \dim \mathrm{Ind}_n^{n+1}(N_2). \tag{4.16.2}$$

On the other hand, by applying the left exact functor coInd_n^{n+1} to the sequence (4.16.1), we have the exact sequence $0 \to \mathrm{coInd}_n^{n+1}(N_1) \to \mathrm{coInd}_n^{n+1}(M) \to \mathrm{coInd}_n^{n+1}(N_2)$, and we have

$$\dim \mathrm{coInd}_n^{n+1}(M) \leq \dim \mathrm{coInd}_n^{n+1}(N_1) + \dim \mathrm{coInd}_n^{n+1}(N_2). \tag{4.16.3}$$

By the induction on the length of Δ-filtration of M, (4.16.2) and (4.16.3) imply (i).

By a similar way as in (i), we can prove that

$$\dim \Omega_{n+1} \circ \mathrm{Ind}_n^{n+1}(M) \geq \dim \Omega_{n+1} \circ \mathrm{coInd}_n^{n+1}(M). \tag{4.16.4}$$

On the other hand, it is known that $\Omega_n(\Delta_n(\lambda))$ is isomorphic to the Specht module S^λ defined in [4]. Thus, by Theorem 3.4 (iii) and [1, Corollary 1.10], we have

$$\Omega_{n+1} \circ \mathrm{Ind}_n^{n+1}(\Delta_n(\lambda)) \cong {}^{\mathscr{H}}\mathrm{Ind}_n^{n+1}(S^\lambda) \cong {}^{\mathscr{H}}\mathrm{Ind}_n^{n+1} \circ \Omega_n(\Delta_n(\lambda)).$$

By the exactness of Ind_n^{n+1} on $\mathscr{S}_{n,r}$-mod$^\Delta$ (note ${}^{\mathscr{H}}\mathrm{Ind}_n^{n+1}$, Ω_n and Ω_{n+1} are exact), we also have

$$\dim {}^{\mathscr{H}}\mathrm{Ind}_n^{n+1} \circ \Omega_n(M) = \dim \Omega_{n+1} \circ \mathrm{Ind}_n^{n+1}(M).$$

Combining this equation with (4.16.4), we obtain (ii). □

Now we have seen that the assumptions (A-1)–(A-6) hold, we have the following theorem by Theorem 4.10

Theorem 4.17. *We have the following isomorphism of functors*

$$\mathrm{Ind}_n^{n+1} \cong \mathrm{coInd}_n^{n+1}.$$

As corollaries, we have the following properties of induction and restriction functors.

Corollary 4.18. *We have the following.*

(i) Res_n^{n+1} *and* Ind_n^{n+1} *are exact.*

(ii) Ind_n^{n+1} *is left and right adjoint to* Res_n^{n+1}.

(iii) *There exist isomorphisms of functors*

$$\mathrm{Res}_n^{n+1} \circ \circledast \cong \circledast \circ \mathrm{Res}_n^{n+1}, \quad \mathrm{Ind}_n^{n+1} \circ \circledast \cong \circledast \circ \mathrm{Ind}_n^{n+1}.$$

(iv) *There exist isomorphisms of functors*

$$\Omega_n \circ \mathrm{Res}_n^{n+1} \cong {}^{\mathscr{H}}\mathrm{Res}_n^{n+1} \circ \Omega_{n+1}, \quad \Omega_{n+1} \circ \mathrm{Ind}_n^{n+1} \cong {}^{\mathscr{H}}\mathrm{Ind}_\circ^{n+1}\Omega_n.$$

Proof. (i) and (ii) are obtained from the definitions and Theorem 4.17. (iii) is obtained from Lemma 2.9 and Theorem 4.17. The first isomorphism in (iv) is Lemma 4.15 (i). By (4.10.2) and Theorem 4.17, we have

$$\Omega_{n+1} \circ \mathrm{Ind}_n^{n+1} \circ I_n \cong {}^{\mathscr{H}}\mathrm{Ind}_n^{n+1} \circ \Omega_n \circ I_n.$$

Thus, by Lemma 4.3 (ii), we have $\Omega_{n+1} \circ \mathrm{Ind}_n^{n+1} \cong {}^{\mathscr{H}}\mathrm{Ind}_\circ^{n+1}\Omega_n$. □

5. Refinements of induction and restriction functors

In this section, we refine the induction and restriction functors which are defined in the previous sections. As an application, we categorify a Fock space by using categories $\mathscr{S}_{n,r}$-mod ($n \geq 0$).

Throughout this section, we assume that R is a field, and we also assume the following conditions for parameters.

- There exists the minimum positive integer e such that $1 + (q^2) + (q^2)^2 + \cdots + (q^2)^{e-1} = 0$.
- There exists an integer $s_i \in \mathbb{Z}$ such that $Q_i = (q^2)^{s_i}$ for each $i = 1, \ldots, r$.

Thanks to [5, Theorem 1.5], these assumptions make no loss of generality in representation theory of cyclotomic q-Schur algebras.

We also remark that $\mathscr{S}_{n,r}$-mod does not depend on a choice of $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r_{>0}$ such that $m_k \geq n$ for any $k = 1, \ldots, r$ up to Morita equivalence (see Remark 1.4). Then, for each n, we take suitable $\mathbf{m}$ and $\mathbf{m}'$ to consider the induction and restriction functors between $\mathscr{S}_{n,r}$-mod and $\mathscr{S}_{n+1,r}$-mod as in the previous sections.

5.1. For $x = (a, b, c) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \ldots, r\}$, we define the residue of x by

$$\operatorname{res}(x) = (q^2)^{b-a} Q_c = (q^2)^{b-a+s_c}.$$

For $x \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \ldots, r\}$, we say that x is i-node if $\operatorname{res}(x) = (q^2)^i$, where we can regard i as an element of $\mathbb{Z}/e\mathbb{Z}$ since $(q^2)^{i+ke} = (q^2)^i$ for any $k \in \mathbb{Z}$ from the assumption for parameters. We also say that x is removable (resp. addable) i-node of $\lambda \in \Lambda^+_{n,r}$, if x is i-node and removable (resp. addable) node of λ.

For $\lambda \in \Lambda^+_{n,r}$, put $r(\lambda) = (r_0(\lambda), r_1(\lambda), \ldots, r_{e-1}(\lambda)) \in \mathbb{Z}^e_{\geq 0}$, where $r_i(\lambda)$ is the number of i-node in $[\lambda]$. Then it is known that the classification of blocks of $\mathscr{S}_{n,r}$ in [10] as follows.

Theorem 5.2 ([10]). *For $\lambda, \mu \in \Lambda^+_{n,r}$, $\Delta_n(\lambda)$ and $\Delta_n(\mu)$ belong to the same block of $\mathscr{S}_{n,r}$ if and only if $r(\lambda) = r(\mu)$.*

5.3. Put

$$R_{n,e} = \{a = (a_0, a_1, \ldots, a_{e-1}) \in \mathbb{Z}^e \mid a = r(\lambda) \text{ for some } \lambda \in \Lambda^+_{n,r}\}.$$

Then we have a bijection between $R_{n,e}$ and the set of blocks of $\mathscr{S}_{n,r}$ by Theorem 5.2. By using this bijection, for $a = (a_0, a_1, \dots, a_{e-1}) \in \mathbb{Z}^e$ such that $\sum_{j=0}^{e-1} a_j = n$, we define the functor $1_a \colon \mathscr{S}_{n,r}\text{-mod} \to \mathscr{S}_{n,r}\text{-mod}$ as the projection to the corresponding block if $a \in R_{n,e}$, and 0 if $a \notin R_{n,e}$.

We define a refinement of Res_n^{n+1} and Ind_n^{n+1} as follows. For $i \in \mathbb{Z}/e\mathbb{Z}$, put

$$i\text{-}\mathrm{Res}_n^{n+1} = \bigoplus_{a \in R_{n+1,e}} 1_{a-i} \circ \mathrm{Res}_n^{n+1} \circ 1_a,$$

$$i\text{-}\mathrm{Ind}_n^{n+1} = \bigoplus_{a \in R_{n,e}} 1_{a+i} \circ \mathrm{Ind}_n^{n+1} \circ 1_a,$$

where $a \pm i = (a_0, \dots, a_{i-1}, a_i \pm 1, a_{i+1}, \dots, a_{e-1})$ for $a = (a_0, \dots, a_{e-1}) \in \mathbb{Z}^e$. Then, we have $\mathrm{Res}_n^{n+1} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} i\text{-}\mathrm{Res}_n^{n+1}$ and $\mathrm{Ind}_n^{n+1} = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} i\text{-}\mathrm{Ind}_n^{n+1}$. From the definition together with Theorem 3.4 and Theorem 4.17, we have the following corollary.

Corollary 5.4. (i) *For $\lambda \in \Lambda^+_{n+1,r}$, there exists a filtration of $\mathscr{S}_{n,r}$-modules*

$$i\text{-}\mathrm{Res}_n^{n+1}(\Delta_{n+1}(\lambda)) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0,$$

such that $M_i/M_{i+1} \cong \Delta_n(\lambda \setminus x_i)$, where $x_1, x_2, \dots, x_k$ are all removable i-nodes of λ such that $x_1 \succ x_2 \succ \cdots \succ x_k$.
(ii) *For $\mu \in \Lambda^+_{n,r}$, there exists a filtration of $\mathscr{S}_{n+1,r}$-modules*

$$i\text{-}\mathrm{Ind}_n^{n+1}(\Delta_n(\mu)) = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

such that $M_i/M_{i+1} \cong \Delta_{n+1}(\mu \cup x_i)$, where $x_1, x_2, \dots, x_k$ are all addable i-nodes of μ such that $x_k \succ x_{k-1} \succ \cdots \succ x_1$.

5.5. Put $\mathbf{s} = (s_1, s_2, \dots, s_r)$. The Fock space with multi-charge $\mathbf{s}$ is the $\mathbb{C}$-vector space

$$\mathcal{F}[\mathbf{s}] = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigoplus_{\lambda \in \Lambda^+_{n,r}} \mathbb{C}|\lambda, \mathbf{s}\rangle$$

with distinguished basis $\{|\lambda, \mathbf{s}\rangle \mid \lambda \in \Lambda^+_{n,r},\ n \in \mathbb{Z}_{\geq 0}\}$ which admits an integrable $\widehat{\mathfrak{sl}}_e$-module structure with the Chevalley generators acting as follows (cf. [9]): for $i \in \mathbb{Z}/e\mathbb{Z}$,

$$e_i \cdot |\lambda, \mathbf{s}\rangle = \sum_{\substack{\mu = \lambda \setminus x \\ \mathrm{res}(x) = (q^2)^i}} |\mu, \mathbf{s}\rangle, \quad f_i \cdot |\lambda, \mathbf{s}\rangle = \sum_{\substack{\mu = \lambda \cup x \\ \mathrm{res}(x) = (q^2)^i}} |\mu, \mathbf{s}\rangle.$$

5.6. Put

$$i\text{-}\mathrm{Res} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i\text{-}\mathrm{Res}_n^{n+1}, \quad i\text{-}\mathrm{Ind} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i\text{-}\mathrm{Ind}_n^{n+1}.$$

Since i-Res and i-Ind are exact functors from $\bigoplus_{n\geq 0}\mathscr{S}_{n,r}$-mod to itself, these functors imply the well-defined action on $\mathbb{C}\otimes_{\mathbb{Z}} K_0\big(\bigoplus_{n\geq 0}\mathscr{S}_{n,r}\text{-mod}\big)$. Thanks to Corollary 5.4, for $\lambda\in\Lambda^+_{n,r}$, we have that

$$i\text{-Res}\cdot[\Delta_n(\lambda)]=\sum_{\substack{\mu=\lambda\setminus x\\ \mathrm{res}(x)=(q^2)^i}}[\Delta_{n-1}(\mu)],\quad i\text{-Ind}\cdot[\Delta_n(\lambda)]=\sum_{\substack{\mu=\lambda\cup x\\ \mathrm{res}(x)=(q^2)^i}}[\Delta_{n+1}(\mu)].$$

Note that $\{[\Delta_n(\lambda)]\mid\lambda\in\Lambda^+_{n,r},\ n\in\mathbb{Z}_{\geq 0}\}$ gives an $\mathbb{C}$-basis of $\mathbb{C}\otimes_{\mathbb{Z}} K_0\big(\bigoplus_{n\geq 0}\mathscr{S}_{n,r}\text{-mod}\big)$. Then, we have the following corollary.

Corollary 5.7. *The exact functors i-*Res *and i-*Ind *($i\in\mathbb{Z}/e\mathbb{Z}$) give the action of $\widehat{\mathfrak{sl}}_e$ on $\mathbb{C}\otimes_{\mathbb{Z}} K_0\big(\bigoplus_{n\geq 0}\mathscr{S}_{n,r}\text{-mod}\big)$, where i-*Res *(resp. i-*Ind*) is corresponding to the action of the Chevalley generators e_i (resp. f_i) of $\widehat{\mathfrak{sl}}_e$. Moreover, by the correspondence $[\Delta_n(\lambda)]\mapsto|\lambda,\mathbf{s}\rangle$ ($\lambda\in\Lambda^+_{n,r}$, $n\in\mathbb{Z}_{\geq 0}$) of basis, $\mathbb{C}\otimes_{\mathbb{Z}} K_0\big(\bigoplus_{n\geq 0}\mathscr{S}_{n,r}\text{-mod}\big)$ is isomorphic to the Fock space $\mathcal{F}[\mathbf{s}]$ as $\widehat{\mathfrak{sl}}_e$-modules.*

REMARKS 5.8. (i) The results in this section do not depend on the characteristic of the ground field R, namely depend only e and the multi-charge $\mathbf{s}=(s_1,\dots,s_r)$ modulo e.
(ii) By the lifting arguments from the module categories of Ariki–Koike algebras as in [16, Section 5], we obtain the $\widehat{\mathfrak{sl}}_e$-categorification in the sense of [15] in our setting.

6. Relations with category $\mathcal{O}$ of rational Cherednik algebras

In this section, we assume that $R=\mathbb{C}$. We give a relation between our induction and restriction functors for cyclotomic q-Schur algebras and parabolic induction and restriction functors for rational Cherednik algebras given in [3].

6.1. Let $\mathcal{H}_{n,r}$ be the rational Cherednik algebra associated to $\mathfrak{S}_n\ltimes(\mathbb{Z}/r\mathbb{Z})^n$ with the parameters $\mathbf{c}$ (see [14] for definition and parameters $\mathbf{c}$), and $\mathcal{O}_{n,r}$ be the category $\mathcal{O}$ of $\mathcal{H}_{n,r}$ defined in [8]. In [8], they defined the KZ functor $\mathrm{KZ}_n\colon\mathcal{O}_{n,r}\to\mathscr{H}_{n,r}$-mod. Then $\mathcal{O}_{n,r}$ is the highest weight cover of $\mathscr{H}_{n,r}$-mod in the sense of [14] through the KZ functor. In [14], Rouquier proved that $\mathcal{O}_{n,r}$ is equivalent to $\mathscr{S}_{n,r}$-mod as highest weight covers of $\mathscr{H}_{n,r}$-mod under some conditions for parameters.

6.2. Let ${}^{\mathcal{O}}\mathrm{Ind}_n^{n+1}$ (resp. ${}^{\mathcal{O}}\mathrm{Res}_n^{n+1}$) be the parabolic induction (resp. restriction) functors between $\mathcal{O}_{n,r}$ and $\mathcal{O}_{n+1,r}$ defined in [3]. Then, we have the following theorem.

Theorem 6.3. *Assume that $\mathcal{O}_{n,r}$ (resp. $\mathcal{O}_{n+1,r}$) is equivalent to $\mathscr{S}_{n,r}$-*mod *(resp. $\mathscr{S}_{n+1,r}$-*mod*) as highest weight covers of $\mathscr{H}_{n,r}$-*mod *(resp. $\mathscr{H}_{n+1,r}$-*mod*). Then,* *under these equivalences, we have the following isomorphisms of functors*:

$$ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \cong \mathrm{Res}_n^{n+1}, \quad {}^{\mathcal{O}}\mathrm{Ind}_n^{n+1} \cong \mathrm{Ind}_n^{n+1}. $$

Proof. Let $\Theta_n\colon \mathcal{O}_{n,r} \to \mathscr{S}_{n,r}$-mod be the functor giving the equivalence as highest weight covers of $\mathscr{H}_{n,r}$-mod. Then, we have that $\mathrm{KZ}_n \cong \Omega_n \circ \Theta_n$. Let $\Psi_n\colon \mathscr{H}_{n,r}$-mod $\to \mathcal{O}_{n,r}$ be the right adjoint functor of KZ_n. Then, we have that $\Phi_n \cong \Theta_n \circ \Psi_n$ by the uniqueness of the adjoint functors. It is similar for the equivalence $\Theta_{n+1}\colon \mathcal{O}_{n+1,r} \to \mathscr{S}_{n+1,r}$-mod.

By Corollary 4.18 (iv) and [16, Theorem 2.1], we have that

$$ \begin{aligned} \Omega_n \circ \mathrm{Res}_n^{n+1} &\cong {}^{\mathscr{H}}\mathrm{Res}_n^{n+1} \circ \Omega_{n+1} \\ &\cong {}^{\mathscr{H}}\mathrm{Res}_n^{n+1} \circ \mathrm{KZ}_{n+1} \circ \Theta_{n+1}^{-1} \\ &\cong \mathrm{KZ}_n \circ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \circ \Theta_{n+1}^{-1}. \end{aligned} \tag{6.3.1} $$

Recall that $I_{n+1}\colon \mathscr{S}_{n+1,r}$-proj $\to \mathscr{S}_{n+1,r}$-mod is the canonical embedding functor. Then, (6.3.1) together with the isomorphism $\Phi_n \cong \Theta_n \circ \Psi_n$ implies that

$$ \Phi_n \circ \Omega_n \circ \mathrm{Res}_n^{n+1} \circ I_{n+1} \cong \Theta_n \circ \Psi_n \circ \mathrm{KZ}_n \circ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \circ \Theta_{n+1}^{-1} \circ I_{n+1}. \tag{6.3.2} $$

Since Res_n^{n+1} (resp. ${}^{\mathcal{O}}\mathrm{Res}_n^{n+1}$) has the left and right adjoint functor Ind_n^{n+1} (resp. ${}^{\mathcal{O}}\mathrm{Ind}_n^{n+1}$) by Corollary 4.18 (ii) (resp. [16, Proposition 2.9]), Res_n^{n+1} (resp. ${}^{\mathcal{O}}\mathrm{Res}_n^{n+1}$) carries projectives to projectives. Thus, (6.3.2) together with (4.8.2) implies that

$$ \mathrm{Res}_n^{n+1} \circ I_{n+1} \cong \Theta_n \circ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \circ \Theta_{n+1}^{-1} \circ I_{n+1}. \tag{6.3.3} $$

Since both Res_n^{n+1} and $\Theta_n \circ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \circ \Theta_{n+1}^{-1}$ are exact, (6.3.3) together with Lemma 4.3 (ii) implies that

$$ \mathrm{Res}_n^{n+1} \cong \Theta_n \circ {}^{\mathcal{O}}\mathrm{Res}_n^{n+1} \circ \Theta_{n+1}^{-1}. $$

By the uniqueness of adjoint functors (or by a similar arguments), we also have

$$ \mathrm{Ind}_n^{n+1} \cong \Theta_{n+1} \circ {}^{\mathcal{O}}\mathrm{Ind}_n^{n+1} \circ \Theta_n^{-1}. \qquad \square $$

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