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WEGNER ESTIMATE FOR A NONSIGN DEFINITE GENERALIZED ALLOY TYPE POTENTIAL

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Abstract

P.D. Hislop and F. Klopp proved a Wegner estimate for Schrödinger operators with nonsign definite potentials for each fixed position of impurities [12]. In this paper, a similar estimate is proven treating also the position of impurities as random variables.

1. Introduction

In this paper, we will give a Wegner estimate for a Schrödinger operator,

$$H^\omega := H_0 + V^\omega(x) \quad \text{with} \quad V^\omega(x) := \sum_{i\in\mathbf{N}} f_i^{\omega_1} u(x-\xi_i^{\omega_2}) \quad \text{and} \quad H_0 = -\Delta, \tag{1.1}$$

where u is a continuous function with a compact support and does not have a fixed sign, $\{f_i^{\omega_1},\ i \in \mathbf{N},\ \omega_1 \in \Omega_1\}$ on some probability space $(\Omega_1, F_1, \mathbf{P}_1)$ are independently and identically distributed random variables with the probability density function $h_0 \in C_0^1$ such that $\mathbf{P}_1(f_i^{\omega_1} \in d\lambda_i) = h_0(\lambda_i)\,d\lambda_i$ satisfying $\operatorname{supp} h_0 = [m, m']$, and $\{\xi_i^{\omega_2},\ i \in \mathbf{N},\ \omega_2 \in \Omega_2\}$ on some probability space $(\Omega_2, F_2, \mathbf{P}_2)$ is a Poisson point process independent of $\{f_i^{\omega_1}\}$ with the Lebesgue measure as its intensity. We write $\omega = (\omega_1, \omega_2)$. For any $y \in \mathbf{R}^d$ and $L > 0$, we set $\Lambda_L(y) = \{x \in \mathbf{R}^d : |x_i - y_i| < L/2 \text{ for } 1 \le i \le d\}$ and $\Lambda_L := \Lambda_L(0)$. For simplicity we assume $\operatorname{supp} u \subset \Lambda_a$ and $\|u\|_\infty = 1$, where $a > 0$. As in [2, 22], we can prove the essential self-adjointness and the measurability in ω of the Schrödinger operator H^ω, and that the spectrum $\sigma(H^\omega) = \mathbf{R}$ for almost all ω. In the rest of this paper we denote the unique self-adjoint extension by the same symbol H^ω.

We consider the approximation of H^ω defined by the self-adjoint operator

$$H^\omega_{\Lambda_L} := H_0 + V_{\Lambda_L}, \quad V_{\Lambda_L} := \sum_{i:\xi_i^{\omega_2}\in\Lambda_L} f_i^{\omega_1} u(x-\xi_i^{\omega_2})$$

on $L^2(\mathbf{R}^d)$ following Klopp [16] and Hislop and Klopp [12].

The main theorem of this paper is the following:

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Theorem 1.1. *For any $\delta > 0$, $q > 1$ and arbitrarily small $\zeta > 0$, there exists a finite positive constant $C_{q,\delta,\zeta}$ such that*

$$\mathbf{P}\{\mathrm{dist}(\sigma(H_{\Lambda_L}), E_0) \le \eta\} \le C_{q,\delta,\zeta} L^{(1+\zeta)d} \eta^{1/q}, \tag{1.2}$$

for $L > 0$, $E_{0<0}$ and $\eta > 0$ satisfying $\eta < \delta/4$ and $E_0 + \eta < -\delta$.

REMARK 1. We can prove Theorem 1.1 for $H_0 = -\triangle + V$, where V is a periodic non-random potential. However in this paper, we will assume $V = 0$ for simplicity.

REMARK 2. We can extend Theorem 1.1 for more general distributions of impurities' positions $\{\xi_i^{\omega_2}\}$ (see Remark 5, 6 below).

REMARK 3. The initial length scale estimate, which is an important estimate together with a Wegner estimate to prove the Anderson localization, holds in this case as in [22].

As one of the applications, we obtain the strong Hilbert–Schmidt dynamical localization which is deduced in the same way as [22] based on the main theorem.

Corollary 1.1. *There exists $E_0 < 0$ such that*

$$\mathbf{E}\left[\sup_t \| |x|^r e^{-itH^\omega} \mathbf{P}^\omega(I)\chi_0 \|_2^2\right] < \infty$$

for any $r > 0$ and any compact interval I satisfying $\sup I \le E_0$.

The estimate

$$\mathbf{P}\{\mathrm{dist}(\sigma(H_{\Lambda_L}), E_0) \le \eta\} \le C L^{Ad} \eta^B, \tag{1.3}$$

with $A \ge 1$ and $B \le 1$ as (1.2) is called a Wegner estimate. From the fact that $\mathbf{P}\{\mathrm{dist}(\sigma(H_{\Lambda_L}), E_0) \le \eta\}$ is dominated by $\mathbf{E}[\#\{\sigma(H_{\Lambda_L}) \cap [E_0 - \eta, E_0 + \eta]\}]$ and $\mathbf{E}[\#\{\sigma(H_{\Lambda_L}) \cap [E_0 - \eta, E_0 + \eta]\}]/L^d$ converges to the density of states (DS) as $L \to \infty$, we expect that $A = B = 1$ are the best exponents.

Wegner [24] firstly obtained this estimate for the Anderson model. After that the estimate (1.3) with general exponents A and B is applied to the proof of the Anderson localization [6, 8, 21].

There had been many prior results on a Wegner estimate for multidimensional and continuous Schrödinger operators with Anderson-type random potentials whose positions corresponding to $\xi_i^{\omega_2}$ in equation (1.1) were fixed on the lattice [17, 3, 13, 21]. Among them, Combes, Hislop and Nakamura obtained a bound as (1.3) with $A = 1$ and $B < 1$

which is arbitrarily close to 1 for the Schrödinger operators with the Anderson-type positive potentials by the method of the spectral shift function [5]. Kirsch and Veselić used this method to prove a bound as (1.3) with $A = 1$ and B arbitrarily close to 1 for the negative potentials called generalized alloy type potentials where the positions of the impurities were fixed randomly on $\mathbf{R}^d$ [14]. With the method of [14], the author proved a bound as (1.3) with $A = 2$ and B arbitrarily close to 1 at negative energies for Schrödinger operators with negative potentials V^ω under the conditions that $f_i^{\omega_1} \in [0, 1]$ and u is negative and, showed the results of the localization [22]. The exponent A in [22] can be taken arbitrarily nearly to 1 by the method in [22]. These results on a Wegner estimate were obtained by the property that the eigenvalues monotonously depended on the coupling constants. Therefore, this method needs the property that a single-site potential has the definite sign. We would like to drop this condition for this paper.

In 2002, Hislop and Klopp obtained a bound as (1.3) with $A = 1$ and B arbitrarily close to 1 for the nonsign definite Anderson-type potentials using Klopp's vector field method [16] and the spectral shift function method [12]. Applying their method and [14], we obtain Theorem 1.1, which is a bound as (1.3) with A and B arbitrarily close to 1. Theorem 1.1 may not have the optimal exponents of L and η for the Wegner estimate. However, this is the best result that we are able to derive up to now. In 2007, Germinet, Hislop and Klein proved the localization for a sign definite Poisson potential model without $f_i^{\omega_1}$, which is more difficult problem than ours. The inequality corresponding to the Wegner estimate for their proof of the localization is restricted to $\eta \sim L^{-L^\rho}$, for that reason the exponent A depends on L [9, 10].

REMARK 4. The best estimate of Wegner-type for continuous Schrödinger operators with Anderson-type random potentials was proved by Combes, Hislop and Klopp with $A = 1$ and $B \leq 1$ [4]. This estimate was obtained by the method of the spectral averaging mainly for Schrödinger operators with a nonnegative single-site potential. It seems to be very difficult to extend this result to a nonsign definite single-site potential with a randomly distributed position,

Corollary 1.1 is obtained for the first time by treating the positions ξ of the single-site potentials as random variables. This is the different point from [14].

2. The proof of the main theorem

In this section, we will prove a Wegner estimate, Theorem 1.1, using the method in [12].

The resolvent $R_{\Lambda_L}(E_0, \omega) := (H^\omega_{\Lambda_L} - E_0)^{-1}$ is written as

$$R_{\Lambda_L}(E_0, \omega) = (H_0 - E_0)^{-1/2}(1 + \Gamma_{\Lambda_L}(E_0, \omega))^{-1}(H_0 - E_0)^{-1/2},$$

where $\Gamma_{\Lambda_L}(E, \omega)$ is a compact operator defined by $(H_0 - E)^{-1/2} V_{\Lambda_L} (H_0 - E)^{-1/2}$. Then, using the inequality $\|R_{\Lambda_L}(E_0, \omega)\| \le \delta^{-1}\|(1 + \Gamma_{\Lambda_L}(E_0, \omega))^{-1}\|$, we have

$$\mathbf{P}\{\mathrm{dist}(\sigma(H_{\Lambda_L}), E_0) \le \eta\} \le \mathbf{P}\left\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0, \omega)), -1) \le \frac{\eta}{\delta}\right\}.$$

Consequently, we have only to show

$$\mathbf{P}\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0, \omega)), -1) \le \kappa\} \le C_{q,\delta,\zeta} L^{(1+\zeta)d} \eta^{1/q},$$

where $\kappa := \eta/\delta$.

We now apply Chebyshev's inequality,

$$\mathbf{P}\left\{\mathrm{dist}(\sigma(\Gamma_{\Lambda_L}(E_0, \omega)), -1) \le \frac{\eta}{\delta}\right\} = \mathbf{P}[\mathrm{Tr}(P^{\omega}_{\Lambda_L}(I_\kappa)) \ge 1] \le \mathbf{E}[\mathrm{Tr}(P^{\omega}_{\Lambda_L}(I_\kappa))], \tag{2.1}$$

where $P^{\omega}_{\Lambda_L}$ denotes the spectral projection of $\Gamma_{\Lambda_L}(E_0, \omega)$ and $I_\kappa := [-1 - \kappa, -1 + \kappa]$.

We will at first estimate the expectation of the right hand side of (2.1) with respect to the randomness of ω_1. This estimate holds for any point processes. Then we will calculate its expectation with respect to the randomness of ω_2, only which Poisson process affects (see (2.9)).

We define $\mathbf{E}^{\omega_1}$ as the expectation with respect to the randomness of ω_1 and $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$ as $\Gamma_{\Lambda_L}(E_0, \omega)$ in which $f_i^{\omega_1}$ for any $i \in \{i \in \mathbf{N} : \xi_i^{\omega_2} \in \Lambda_L\}$ is replaced by λ_i for any $\lambda = (\lambda_i)_{i\,:\,\xi_i^{\omega_2} \in \Lambda_L} \in [m, m']^{\#\{i : \xi_i^{\omega_2} \in \Lambda_L\}}$. By the method in [12], we have

$$\begin{aligned} &\mathbf{E}^{\omega_1}[\mathrm{Tr}\, P^{\omega}_{\Lambda_L}(I_\kappa)] \\ &\le \mathbf{E}^{\omega_1}\left\{\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt}\, \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \omega) + 1 - t)]\, dt\right\} \\ &= \prod_l \int_m^{m'} h_0(\lambda_l)\, d\lambda_l \left\{\int_{-3\kappa/2}^{3\kappa/2} \frac{d}{dt}\, \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)]\, dt\right\}, \end{aligned} \tag{2.2}$$

where ρ is a nonnegative, smooth, monotone decreasing function such that $\rho(x) = 1$ for $x < -\kappa/2$ and $\rho(x) = 0$ for $x \ge \kappa/2$.

Since $\operatorname{supp}\rho$ is included in $(-\infty, \kappa/2]$, $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$ of the right hand side of (2.2) is restricted to the spectral subspace where the operator is smaller than $(-1 + 2\kappa)$ which is negative. Therefore, noting that ρ' is negative,

$$\begin{aligned} \frac{d}{dt}\, \mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)] &= \sum_{j\,:\,E_j \in [-1-2\kappa, -1+2\kappa]} \frac{d}{dt} \rho(E_j + 1 - t) \\ &\le \sum_{j : E_j \in [-1-2\kappa, -1+2\kappa]} \frac{-E_j}{-1 + 2\kappa} \rho'(E_j + 1 - t) \\ &= \frac{-1}{-1 + 2\kappa}\, \mathrm{Tr}[\rho'(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)], \end{aligned} \tag{2.3}$$

where $\{E_j\}_{j\in\mathbf{N}}$ are the eigenvalues of $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$.

On the other hand, using the Hellmann–Feynman theorem and the equation

$$\sum_{i:\,\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i}\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) = \Gamma_{\Lambda_L}(E_0, \lambda, \omega_2),$$

we can show

$$\begin{aligned}
&\sum_{i:\,\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i} \operatorname{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)] \\
&= \sum_{j:\,E_j\in[-1-2\kappa,-1+2\kappa]} \left\{\rho'(E_j + 1 - t)\cdot \sum_{i:\,\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i} E_j\right\} \qquad (2.4)\\
&= \operatorname{Tr}[\rho'(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)].
\end{aligned}$$

By (2.3) and (2.4), we obtain

$$\begin{aligned}
&\frac{d}{dt} \operatorname{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)] \\
&\le \frac{-1}{-1+2\kappa} \sum_{i:\xi_i^{\omega_2}\in\Lambda_L} \lambda_i \frac{\partial}{\partial\lambda_i} \operatorname{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)].
\end{aligned}$$

By this estimate and the integration by parts with respect to λ_i, the right hand side of (2.2) is less than or equal to

$$\begin{aligned}
&\frac{-1}{-1+2\kappa} \sum_{i:\,\xi_i^{\omega_2}\in\Lambda_L} \int_{-3\kappa/2}^{3\kappa/2} dt \left\{\prod_l \int_m^{m'} h_0(\lambda_l) d\lambda_l\right\} \\
&\times \left\{\lambda_i \frac{\partial}{\partial\lambda_i} \operatorname{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2) + 1 - t)]\right\} \qquad (2.5)\\
&\le 2\frac{((m'-m)\|\tilde{h}_0'\|_\infty) \vee \tilde{h}_0(m')}{(1-2\kappa)} \\
&\quad\times \sum_{i:\,\xi_i^{\omega_2}\in\Lambda} \int_{-3\kappa/2}^{3\kappa/2} dt \prod_{l\neq i} \int_m^{m'} h_0(\lambda_l)\, d\lambda_l |\operatorname{Tr}\{D(i, E_0, m, \lambda_i^+)\}|,
\end{aligned}$$

where $\tilde{h}_0(\lambda)$ is the function $\lambda h_0(\lambda)$ and $D(i, E_0, m, \lambda_0)$ is the operator $\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i} + 1 - t) - \rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_0,i} + 1 - t)$. We denote $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\varpi,i}$ for $\varpi \in [m, m']$ by the operator $\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)$ with the fixed coupling constant $\lambda_i = \varpi$ at the i-th site, and $\lambda_i^+ \in [m, m']$ by the value of the coupling constant λ_i where the maximum of $|\operatorname{Tr}\{D(i, E_0, m, \lambda_0)\}|$ is attained.

Now we use the following proposition on spectral shift functions ([5] Theorem 2.1, [25] Chapter 8 §3 Theorem 3 and Theorem 6):

Proposition 2.1. *Let A_1 and A_0 be self-adjoint operators such that $A_1 - A_0 \in \mathcal{I}_{1/p}$ for $p > 1$, where $\mathcal{I}_{1/p}$ is the family of compact operators of the super trace class, which we define as follows: we say that $A \in \mathcal{I}_{1/p}$ if for some $p > 1$, $|||A|||_{1/p} := \left(\sum_j \mu_j(A)^{1/p}\right)^p < \infty$, where $\mu_j(A)$ denotes the j-th singular value of A.*

Then, there exists some $\pi(\,\cdot\,; A_1, A_0) \in L^p(\mathbf{R})$ such that for $\phi \in C^\infty(\Gamma)$ where $\Gamma \subset \mathbf{R}$: a compact interval which contains $\sigma(A_0)$ and $\sigma(A_1)$,

$$\mathrm{Tr}[\phi(A_1) - \phi(A_0)] = \int_\Gamma \pi(\lambda; A_1, A_0)\phi'(\lambda)\,d\lambda,$$

and

$$\|\pi\|_p \leq |||A_1 - A_0|||_{1/p}^{1/p}.$$

We fix $p > 1$ and set $A_1 = (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_i^+, i})^l$ and $A_0 = (\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i})^l$ for any l greater than $dp/2 + 1$, and $V_i = (\lambda_i^+ - m)R_0(E_0)^{1/2}u(x - \xi_i^{\omega_2})R_0(E_0)^{1/2}$, where $R_0(E_0) := (H_0 - E_0)^{-1}$. Then,

$$\begin{aligned} V_{eff} := A_1 - A_0 &= \sum_{j=0}^{l-1}(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_i^+, i})^{l-j-1} V_i(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i}))^j \\ &= (\lambda_i^+ - m)\sum_{j=0}^{l-1}[J_i^{l-j-1}(R_0(E_0)V_{\Lambda_L}^{\lambda_i^+, i})^{l-j-1}R_0(E_0)^{1/2}]^* u_i \\ &\qquad \times [J_i^j(R_0(E_0)V_{\Lambda_L}^{m,i})^j R_0(E_0)^{1/2}], \end{aligned}$$

where $J \in C_0^\infty$ such that $J(x)u(x) = u(x)$. We denote $u_i := u(x - \xi_i^{\omega_2})$, $J_i := J(x - \xi_i^{\omega_2})$ and for $\varpi \in [m, m']$, $V_{\Lambda_L}^{\varpi, i}$ is the potential V_{Λ_L} with the fixed coupling constant $\lambda_i = \varpi$ at the i-th site.

According to Proposition 12 in [18], for any $\tau \in \operatorname{supp} h_0$ and $r \in \mathbf{N}$,

$$J_i^r(R_0(E_0)\tilde{V}_{\Lambda_L}^{\tau, i})^r R_0(E_0)^{1/2} = \sum_{\alpha=1}^N \left\{\prod_{\beta=1}^r J_i^{\alpha,\beta}R_0(E_0)B_i^{\alpha,\beta}\right\}, \tag{2.6}$$

where the bounded operators $J_i^{\alpha,\beta}$ are combinations of the derivatives of J_i, and the operators $B_i^{\alpha,\beta}$ are the polynomials of the bounded operators containing $V_{\Lambda_L}^{\tau, i}$.

Let $s > d/2$. Using the estimates of the norms in Theorem 4.1 of [20],

$$\|J_i^{\alpha,\beta}R_0(E_0)\|_{\mathcal{I}_s} \leq \|J_i^{\alpha,\beta}\|_{L^s(\mathbf{R}^d)}\left\|\frac{1}{|x|^2 - E_0}\right\|_{L^s(\mathbf{R}^d)},$$

which is bounded by a constant independent of Λ_L. The norms of the operators $B_i^{\alpha,\beta}$ are estimated as follows:

$$\|B_i^{\alpha,\beta}\|_{\mathcal{B}(L^s(\mathbf{R}^d))} \leq C\| \|V_\Lambda^{\tau,i}\|_\infty + 1|^\gamma \leq C\left[\sup_{x\in\Lambda_{L+a}} \#(k \in \mathbf{N}\colon \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1\right]^\gamma,$$

for some γ and C independent of Λ_L. Therefore, we can obtain the estimate of the norm of V_{eff} as follows:

$$\| |V_{eff}| \|_{1/p}^{1/p} \leq C(l-1)\left[\sup_{x\in\Lambda_{L+a}} \#(k \in \mathbf{N}\colon \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1\right]^{\gamma'},$$

for some γ'.

Noting supp $\rho'(\cdot\ + 1 - t)$ is included in $\Sigma := [-1 - 3\kappa, 0)$, we have

$$\begin{aligned}
&|\mathrm{Tr}\, D(i, E_0, m, \lambda_i)| \\
&= |\mathrm{Tr}[\rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{\lambda_i^+, i} - E + t) - \rho(\Gamma_{\Lambda_L}(E_0, \lambda, \omega_2)^{m,i} - E + t)]| \\
&= \left| \int_\Sigma \frac{\partial\rho(\lambda' + 1 - t)}{\partial\lambda'} \pi(\lambda')\, d\lambda' \right| \qquad (2.7) \\
&\leq C\left[\sup_{x\in\Lambda_{L+a}} \#(k \in \mathbf{N}\colon \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1\right]^{\gamma'} \left(\int_\Sigma \left|\frac{\partial\rho}{\partial\lambda}(\lambda + 1 - t)\right|^q d\lambda\right)^{1/q},
\end{aligned}$$

where q is the positive number such that $1/p + 1/q = 1$, and π is the spectral shift function for A_1 and A_0.

By using also

$$\int_{\mathbf{R}} |\rho'(\gamma)|^q\, d\gamma \leq \int_{\mathbf{R}} |\rho'(\gamma)|\, d\gamma\, \sup(\rho')^{(q-1)},$$

we can show that the second factor of the right hand side of (2.7) is dominated by $\kappa^{(1/q-1)}$.

Consequently, according to (2.2), (2.5), (2.7) and above comments, we obtain

$$\begin{aligned}
&\mathbf{E}\{\mathrm{Tr}\, P_{\Lambda_L}^\omega(I_\kappa)\} \\
&\leq C_1 \eta^{1/q} \mathbf{E}^{\omega_2}\left[\#(j \in \mathbf{N}\colon \xi_j^{\omega_2} \in \Lambda_L)\left\{\sup_{x\in\Lambda_{L+a}} \#(k \in \mathbf{N}\colon \xi_k^{\omega_2} \in \Lambda_{3a}(x)) + 1\right\}^{\gamma'}\right]. \qquad (2.8)
\end{aligned}$$

For (2.8), using the Hölder inequality, we obtain

$$\begin{aligned} &\mathbf{E}\{\mathrm{Tr}\, P^{\omega}_{\Lambda_L}(I_\kappa)\} \\ &\le C_1\eta^{1/q}\mathbf{E}^{\omega_2}[\#(j\in\mathbf{N}\colon \xi^{\omega_2}_j \in \Lambda_L)^{1+\theta}]^{1/(1+\theta)} \\ &\qquad \times \mathbf{E}^{\omega_2}\left[\left\{\sup_{x\in\Lambda_{L+a}} \#(k\in\mathbf{N}\colon \xi^{\omega_2}_k\in\Lambda_{3a}(x))+1\right\}^{(\theta+1)\gamma'/\theta}\right]^{\theta/(1+\theta)}, \end{aligned} \tag{2.9}$$

for $\theta > 0$.

Noting that the third factor of the right hand side of (2.9) is bounded by CL^d and the fourth factor is less than or equal to

$$\left[\sum_{e\in\Lambda_{L+a}\cap\mathbf{Z}} \mathbf{E}^{\omega_2}\left[\sup_{x\in\Lambda_1(e)} \#(k\in\mathbf{N}\colon \xi^{\omega_2}_k\in\Lambda_{3a}(x))+1\right]^{(\theta+1)\gamma'/\theta}\right]^{\theta/(1+\theta)} \le CL^{d\theta/(1+\theta)},$$

we obtain the theorem.

REMARK 5. By (2.8), the main theorem holds for a model of which the impurity position $\xi_i(\omega_2) = a_i + y_i(\omega_2)$ with a uniformly bounded $y_i(\omega_2)$ as A=1 similarly to [12], where $\{a_i : i \in \mathbf{N}\} = \mathbf{Z}^d$.

REMARK 6. Our proof of this paper needs only the Z^d stationary property and the finite moments' property for all orders for the number of impurities in the finite cube. Moreover, if we treat point processes with finite moments of some $n > d/2$, then our results hold for $A > 1 + \gamma'/n$. The condition $n > d/2$ is for the essential self-adjointness of our Schrödinger operators on $C_0^\infty(\mathbf{R}^d)$ [14].

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