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Osaka J. Math.
51 (2014), 481–511

LINEAR SYSTEMS ASSOCIATED TO UNICUSPIDAL RATIONAL PLANE CURVES

Dedicated to Antonio Campillo and Ignacio Luengo on the occasion of their 60th birthdays

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(Received February 27, 2012, revised September 19, 2012)

Abstract

Given a unicuspidal rational curve $C \subset \mathbb{P}^2$ with singular point P, we study the unique pencil Λ_C on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\operatorname{Bs}(\Lambda_C) = \{P\}$. We show that the general member of Λ_C is a rational curve if and only if $\tilde{\nu}(C) \geq 0$, where $\tilde{\nu}(C)$ denotes the self-intersection number of C after the minimal resolution of singularities. We also show that if $\tilde{\nu}(C) \geq 0$, then Λ_C has a dicritical of degree 1. Note that all currently known unicuspidal rational curves $C \subset \mathbb{P}^2$ satisfy $\tilde{\nu}(C) \geq 0$.

Introduction

A *unicuspidal rational curve* is a pair (C, P) where C is a curve and $P \in C$ satisfies $C \setminus \{P\} \cong \mathbb{A}^1$. We call P the distinguished point of C.

Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point P. In Section 1 we define an infinite family of linear systems on $\mathbb{P}^2$ determined by (C, P) in a natural way. We are particularly interested in two of these linear systems, denoted Λ_C and N_C, where Λ_C is a pencil and N_C is a net. In fact Λ_C has the following characterization:

(1) Λ_C *is the unique pencil on* $\mathbb{P}^2$ *satisfying* $C \in \Lambda_C$ *and* $\operatorname{Bs}(\Lambda_C) = \{P\}$

where $\operatorname{Bs}(\Lambda_C)$ denotes the base locus of Λ_C on $\mathbb{P}^2$. The existence of this pencil was pointed out to us by A. Campillo and I. Luengo in a friendly conversation. It appeared to us that it would be interesting to understand *how the properties of C are related to those of* Λ_C; this is the underlying theme of the present paper.

Given a curve $C \subset \mathbb{P}^2$, let $\tilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the minimal resolution of singularities of C (this is the "short" resolution, not the "embedded" resolution; see 3.2); let $\tilde{C} \subset \tilde{\mathbb{P}}^2$ be the strict transform of C, and let $\tilde{\nu}(C)$ denote the self-intersection number of $\tilde{C}$ on $\tilde{\mathbb{P}}^2$.

2010 Mathematics Subject Classification. Primary 14C20; Secondary 14J26.
Research of the first author supported by grants RGPIN/104976-2010 from NSERC Canada and SAB2006-0060 from the Spanish Ministry of Education and Science. Research of the second author supported by the grants MTM2010-21740-C02-01 and Grupo Singular CCG07-UCM/ESP-2695-921020.

For a unicuspidal rational curve $C \subset \mathbb{P}^2$, we show (cf. Theorems 4.1, 4.6 and 6.2):

(2) *The general member of* Λ_C *is a rational curve if and only if* $\tilde{\nu}(C) \geq 0$.

(3) *The general member of* N_C *is a rational curve if and only if* $\tilde{\nu}(C) > 0$.

(4) *If* $\tilde{\nu}(C) \geq 0$ *then* Λ_C *has either* 1 *or* 2 *dicriticals, and at least one of them has degree* 1.

In view of these results, it is worth noting that *all currently known unicuspidal rational curves* $C \subset \mathbb{P}^2$ *satisfy* $\tilde{\nu}(C) \geq 0$. See Remark 4.3 for details.

The proofs of the above statements (2) and (3) make use of results from [3], where we solved the following problem: given a curve C on a rational nonsingular projective surface S, find all linear systems $\mathbb{L}$ on S satisfying $C \in \mathbb{L}$, $\dim \mathbb{L} \geq 1$, and the general member of $\mathbb{L}$ is a rational curve.

In statement (4) we claim, in particular, that *if* $\tilde{\nu}(C) \geq 0$ *then* Λ_C *has a dicritical of degree* 1 (see 6.1 for definitions). It seems that the existence of such a dicritical is not an easy fact. Indeed, the proof of this claim takes more than half of the present paper (all of Sections 5 and 6). Note, however, that the graph theoretic tool developed in Section 5 is susceptible of being useful in other settings.

For a survey of open problems related to cuspidal rational plane curves, the reader is referred to [6].

CONVENTIONS. All algebraic varieties are over an algebraically closed field $\mathbf{k}$ of characteristic zero. Varieties (so in particular curves) are irreducible and reduced. A divisor D of a surface is *reduced* if $D = \sum_{i=1}^{n} C_i$ where $C_1, \dots, C_n$ are distinct curves ($n \geq 0$). We write $e_Q(C)$ for the multiplicity of a point Q on a curve C.

1. Definition of Λ_C and N_C

A *unicuspidal rational curve* is a pair (C, P) where C is a curve and P is a point of C such that $C \setminus \{P\} \cong \mathbb{A}^1$. We call P the *distinguished point*, and we consider that the sentence "C is a unicuspical rational curve with distinguished point P" is equivalent to "(C, P) is a unicuspical rational curve". We allow ourselves to speak of a unicuspidal rational curve C without mentioning P, but keep in mind that C always comes equipped with a choice of a point P (that choice being forced when $C \ncong \mathbb{P}^1$).

The aim of this section is to define, given a unicuspidal rational curve $C \subset \mathbb{P}^2$, an infinite family of linear systems $X_{l,j}(C)$ on $\mathbb{P}^2$. This is done in Proposition 1.2. We are particularly interested in two of these linear systems, the pencil Λ_C and the net N_C, defined in Definition 1.3, Corollary 1.4 and Definition 1.5.

NOTATIONS 1.1. Let $C \subset \mathbb{P}^2$ be a unicuspical rational curve with distinguished point P. If D is an effective divisor in $\mathbb{P}^2$, let $i_P(C, D)$ denote the local intersection number of C and D at P (which is defined to be $+\infty$ if C is a component of D). Let $\Gamma = \Gamma_{(C,P)} \subseteq \mathbb{N}$ denote the semigroup of (C, P), i.e., the set of local intersection numbers $i_P(C, D)$ where D is an effective divisor such that $C \not\subseteq \operatorname{supp}(D)$. We also use the standard notation for intervals, $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$.

Proposition 1.2. *Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree d and with distinguished point P. For each pair $(l, j) \in \mathbb{N}^2$ such that $l > 0$ and $j \le ld$, let $X_{l,j}(C)$ be the set of effective divisors D of $\mathbb{P}^2$ such that* $\deg(D) = l$ *and* $i_P(C, D) \ge j$.
(a) *$X_{l,j}(C)$ is a linear system on $\mathbb{P}^2$ for all l, j, and* $\dim X_{l,j}(C) \ge 1$ *whenever $l \ge d$.*
(b) *For each $j \in \mathbb{N}$ such that $j \le d^2$, the dimension of the linear system $X_{d,j}(C)$ is equal to the cardinality of the set $[j, d^2] \cap \Gamma$, where $\Gamma = \Gamma_{(C,P)}$. In particular, for each integer j such that $(d-1)(d-2) \le j \le d^2$,* $\dim X_{d,j}(C) = d^2 - j + 1$. *Consequently, $X_{d,d^2}(C)$ is a pencil and $X_{d,d^2-1}(C)$ is a net.*
For each $l \in \mathbb{N} \setminus \{0\}$, define the abbreviation $X_l(C) = X_{l,ld}(C)$. Note that the above assertions imply that $X_d(C)$ is a pencil and that $\dim X_l(C) \ge 1$ *whenever $l \ge d$. Moreover, if $l \in \mathbb{N}$ is such that $0 < l < d$ then the following hold*:
(c) *$X_l(C)$ contains at most one element and if $X_l(C) \ne \varnothing$ then $ld \in \Gamma$.*
(d) $|\Gamma \cap [0, ld]| \ge (l+1)(l+2)/2$, *and if equality holds and $ld \in \Gamma$ then $X_l(C) \ne \varnothing$.*

REMARK. The proof below is an elaboration of the proof of Proposition 2 of [5]; moreover, the inequality in assertion (d) is part of the cited result.

REMARK. $C \in X_{d,j}(C)$ for all j, because $i_P(C, C) = \infty > j$.

Proof of Proposition 1.2. Choose coordinates (X, Y, Z) for $\mathbb{P}^2$ such that $P = (0 : 0 : 1)$. Let $\mathbf{k}[X, Y, Z]_l$ denote the vector space of homogeneous polynomials of degree l and, given $G \in \mathbf{k}[X, Y, Z]_l \setminus \{0\}$, let $\operatorname{div}_0(G)$ be the effective divisor on $\mathbb{P}^2$, of degree l, with equation "$G = 0$". Let $F \in \mathbf{k}[X, Y, Z]_d$ be an irreducible homogeneous polynomial of degree d whose zero-set is C. Let $x(t), y(t) \in t\mathbf{k}[[t]]$ be a local parametrization of C at P. Then $F(x(t), y(t), 1) = 0$ and, for any $l \in \mathbb{N} \setminus \{0\}$ and $G \in \mathbf{k}[X, Y, Z]_l \setminus \{0\}$, Bezout's theorem gives

$$\operatorname{ord}_t G(x(t), y(t), 1) = i_P(C, \operatorname{div}_0(G)) \begin{cases} \in \Gamma \cap [0, ld], & \text{if } G \in \mathbf{k}[X, Y, Z]_l \setminus (F), \\ = \infty, & \text{if } G \in \mathbf{k}[X, Y, Z]_l \cap (F) \end{cases} \tag{1}$$

where (F) is the principal ideal of $\mathbf{k}[X, Y, Z]$ generated by F. Define a sequence of $\mathbf{k}$-linear maps $L_n \colon \mathbf{k}[X, Y, Z] \to \mathbf{k}$ (for $n \in \mathbb{N}$) by the condition $G(x(t), y(t), 1) = \sum_{n \in \mathbb{N}} L_n(G) t^n$ for any $G \in \mathbf{k}[X, Y, Z]$.

Fix a pair $(l, j) \in \mathbb{N}^2$ such that $l \ge 1$ and $0 \le j \le ld$. Consider the linear map of $\mathbf{k}$-vector spaces

$$T_l \colon \mathbf{k}[X, Y, Z]_l \to \mathbf{k}^{|\Gamma \cap [0, ld]|}, \quad G \mapsto (L_{n_1}(G), \dots, L_{n_p}(G)),$$

where $n_1 < \dots < n_p$ are the elements of $\Gamma \cap [0, ld]$, and define the subspace $E_{l,j}$ of $\mathbf{k}^{|\Gamma \cap [0, ld]|}$ by

$$E_{l,j} = \{(0, \dots, 0, \lambda_1, \dots, \lambda_e) \mid \lambda_1, \dots, \lambda_e \in \mathbf{k}\},$$

where $e = |\Gamma \cap [j, ld]|$. Note that (1) has the following two consequences: firstly, $\ker T_l = \mathbf{k}[X, Y, Z]_l \cap (F)$, so

$$\dim(\ker T_l) = \begin{cases} 0, & \text{if} \quad l < d, \\ 1, & \text{if} \quad l = d; \end{cases} \tag{2}$$

secondly,

$$\begin{aligned} T_l^{-1}(E_{l,j}) \setminus \{0\} &= \{G \in \mathbf{k}[X, Y, Z]_l \setminus \{0\} \mid \operatorname{ord}_t G(x(t), y(t), 1) \geq j\} \\ &= \{G \in \mathbf{k}[X, Y, Z]_l \setminus \{0\} \mid i_P(C, \operatorname{div}_0(G)) \geq j\}, \end{aligned}$$

so

$$X_{l,j}(C) = \{\operatorname{div}_0(G) \mid G \in T_l^{-1}(E_{l,j}) \setminus \{0\}\}. \tag{3}$$

In particular,

$$X_{l,j}(C) \quad \text{is a linear system of dimension} \quad \dim_{\mathbf{k}}(T_l^{-1}(E_{l,j})) - 1. \tag{4}$$

If $l \geq d$ then $\ker(T_l) = \mathbf{k}[X, Y, Z]_l \cap (F)$ has dimension equal to $\dim \mathbf{k}[X, Y, Z]_{l-d} = (l-d)(l-d+3)/2 + 1$, so

$$\dim X_{l,j}(C) = \dim T_l^{-1}(E_{l,j}) - 1 \geq \frac{(l-d)(l-d+3)}{2}.$$

Hence, $\dim X_{l,j}(C) \geq 2$ whenever $l > d$, and $X_{l,j}(C) \neq \varnothing$ when $l = d$. To finish the proof of assertion (a), we still need to show that $\dim X_{l,j}(C) \geq 1$ when $l = d$.

Consider the case $l = d$. It is known (cf. [1] or [8]) that the number $\delta = (d-1)(d-2)/2$ satisfies $2\delta + \mathbb{N} \subseteq \Gamma$ as well as $\delta = |\mathbb{N} \setminus \Gamma|$. As $2\delta < d^2$, it follows that $d^2 + \mathbb{N} \subset \Gamma$ and

$$|\Gamma \cap [0, d^2]| = d^2 + 1 - \delta = (d^2 + 3d)/2 = \dim_{\mathbf{k}} \mathbf{k}[X, Y, Z]_d - 1,$$

so $\dim(V) = \dim(W) + 1$ where we write $V = \mathbf{k}[X, Y, Z]_d$ and $W = \mathbf{k}^{|\Gamma \cap [0, d^2]|}$. As $T_d \colon V \to W$ is a linear map and $\dim(\ker T_d) = 1$ by (2), it follows that T_d is surjective and that (for any $j \leq d^2$) $\dim T_d^{-1}(E_{d,j}) = 1 + \dim E_{d,j} = 1 + |\Gamma \cap [j, d^2]|$, so

$$\dim X_{d,j}(C) = |\Gamma \cap [j, d^2]|. \tag{5}$$

As $d^2 \in \Gamma \cap [j, d^2]$, it follows in particular that $\dim X_{d,j}(C) \geq 1$, which finishes the proof of (a). In the special case where $2\delta \leq j \leq d^2$ we have $[j, d^2] \cap \mathbb{N} \subset \Gamma$, so (5) gives

$$\dim X_{d,j}(C) = d^2 - j + 1.$$

In particular $\dim X_{d,d^2}(C) = 1$ and $\dim X_{d,d^2-1}(C) = 2$, so (b) is proved.

From now-on assume that $0 < l < d$.

Since T_l is injective by (2), and since the definition of $E_{l,j}$ implies

$$\dim E_{l,ld} = |\Gamma \cap \{ld\}| = \begin{cases} 1, & \text{if} \quad ld \in \Gamma, \\ 0, & \text{if} \quad ld \notin \Gamma, \end{cases} \tag{6}$$

we have $\dim T_l^{-1}(E_{l,ld}) \leq 1$, so (3) implies that $X_l(C) = X_{l,ld}(C)$ contains at most one element. Moreover, if $X_l(C) \neq \varnothing$ then $\dim T_l^{-1}(E_{l,ld}) = 1$, so $\dim E_{l,ld} = 1$ and (6) implies that $ld \in \Gamma$. This proves (c).

To prove (d) note that the fact that $T_l\colon \mathbf{k}[X, Y, Z]_l \to \mathbf{k}^{|\Gamma \cap [0,ld]|}$ is injective implies that

$$|\Gamma \cap [0, ld]| \geq (l+1)(l+2)/2. \tag{7}$$

Suppose that equality holds in (7); then T_l is bijective, and if we also assume that $ld \in \Gamma$ then $\dim E_{l,ld} = 1$ by (6), so $T_l^{-1}(E_{l,ld})$ has dimension 1 and (3) implies that $X_l(C) \neq \varnothing$. This completes the proof of (d), and of the proposition. □

DEFINITION 1.3. Let $C \subset \mathbb{P}^2$ be a rational unicuspidal curve, with distinguished point P. We define $\Lambda_C = X_d(C) = X_{d,d^2}(C)$, where $d = \deg(C)$. By Proposition 1.2 (b), Λ_C is a pencil on $\mathbb{P}^2$. The definition of $X_{d,d^2}(C)$ and Bezout's theorem yield the following explicit description of Λ_C:

$$\Lambda_C = \{C\} \cup \{D \in \mathrm{Div}(\mathbb{P}^2) \mid D \geq 0,\ \deg(D) = \deg(C) \text{ and } C \cap \mathrm{supp}(D) = \{P\}\}.$$

The pencil Λ_C can also be characterized as follows:

Corollary 1.4. *Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point P. Then Λ_C is the unique pencil on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{P\}$.*

Proof. From the explicit description of Λ_C given in Definition 1.3, it is clear that $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{P\}$. To prove uniqueness, consider a pencil Λ on $\mathbb{P}^2$ such that $C \in \Lambda$ and $\mathrm{Bs}(\Lambda) = \{P\}$. Let D be any element of Λ other than C. Then (since Λ is a pencil) any point of $\mathrm{supp}(D) \cap C$ is in fact a base point of Λ; so $\mathrm{supp}(D) \cap C = \{P\}$. Using again the explicit description of Λ_C given in Definition 1.3, this gives $D \in \Lambda_C$. This shows that $\Lambda \subseteq \Lambda_C$ and hence that $\Lambda = \Lambda_C$. □

DEFINITION 1.5. Let $C \subset \mathbb{P}^2$ be a rational unicuspidal curve, with distinguished point P. Define $N_C = X_{d,d^2-1}(C)$, where $d = \deg(C)$. By 1.2, N_C is a net. Observe that $\Lambda_C \subset N_C$ and that

$$\mathrm{Bs}(N_C) = \begin{cases} \{P\}, & \text{if} \quad \deg C > 1, \\ \varnothing, & \text{if} \quad \deg C = 1. \end{cases}$$

Also note that the linear systems Λ_C and N_C are primitive (i.e., their general member is irreducible and reduced), because C is irreducible and reduced and is an element of each of them.

REMARK. We shall restrict ourselves to studying the pencil Λ_C and the net N_C associated to a unicuspidal rational curve $C \subset \mathbb{P}^2$, but the other linear systems defined in Proposition 1.2 also deserve some attention. For instance, consider the set $S_C = \{l \in \mathbb{N} \mid 0 \le l < d \text{ and } X_l(C) \neq \varnothing\}$, where $d = \deg(C)$. Parts (c) and (d) of the above proposition indicate that S_C is closely related to the semigroup $\Gamma_{(C,P)}$, and one can see that S_C is also related to the reducible elements of Λ_C. Something interesting can be said about these relations, but this theme is not developed in this paper.

REMARK. The objects $X_{l,j}(C)$, $X_l(C)$, Λ_C and N_C should really be denoted $X_{l,j}(C, P)$, $X_l(C, P)$, $\Lambda_{C,P}$ and $N_{C,P}$, as they depend on the choice of P in the non-singular case.

2. Preliminaries on $\mathbb{P}^1$-rulings on rational surfaces

In this section, S is a rational nonsingular projective surface.

DEFINITION 2.1. A pencil Λ on S is called a $\mathbb{P}^1$*-ruling* if it is base-point-free and if its general member is isomorphic to $\mathbb{P}^1$. If Λ is a $\mathbb{P}^1$-ruling of S then by a *section* of Λ we mean an irreducible curve $\Sigma \subset S$ such that $\Sigma \cdot D = 1$ for any $D \in \Lambda$ (it then follows that $\Sigma \cong \mathbb{P}^1$).

The following is a well-known consequence of the Riemann–Roch theorem for S:

Lemma 2.2. *If $C \subset S$ satisfies $C \cong \mathbb{P}^1$ and $C^2 = 0$ then the complete linear system $|C|$ on S is a $\mathbb{P}^1$-ruling.*

2.3. Recall that, given $k \in \mathbb{N}$, there exists a triple $(\mathbb{F}_k, \mathbb{L}_k, \Delta_k)$ where $\mathbb{F}_k$ is a nonsingular projective rational surface, $\mathbb{L}_k$ is a base-point-free pencil on $\mathbb{F}_k$ each of whose elements is a projective line, and Δ_k is a section of $\mathbb{L}_k$ satisfying $\Delta_k^2 = -k$. Moreover, $(\mathbb{F}_k, \mathbb{L}_k, \Delta_k)$ is uniquely determined by k up to isomorphism. The surface $\mathbb{F}_k$ is called the Nagata–Hirzebruch ruled surface of degree k.

2.4. By an *SNC-divisor* of S we mean a divisor $D = \sum_{i=1}^{n} C_i$ where $C_1, \dots, C_n$ $(n \ge 0)$ are distinct curves on S and:

- each C_i is a nonsingular curve;
- for every choice of $i \neq j$ such that $C_i \cap C_j \neq \varnothing$, $C_i \cap C_j$ is one point and the local intersection number of C_i and C_j at that point is equal to 1;
- if i, j, k are distinct then $C_i \cap C_j \cap C_k = \varnothing$.

The *dual graph* of an SNC-divisor $D = \sum_{i=1}^{n} C_i$ of S is the weighted graph defined by stipulating that the vertex set is $\{C_1, \dots, C_n\}$, that distinct vertices C_i, C_j are joined by an edge if and only if $C_i \cap C_j \neq \varnothing$, and that the weight of the vertex C_i is the self-intersection number C_i^2.

For the following fact, see for instance [9, Chapter 2, 2.2] or [7, Section 2].

Gizatullin's Theorem 2.5. *Let Λ be a $\mathbb{P}^1$-ruling on S. Then Λ has a section and the following hold*:
(a) *Let $D \in \Lambda$. Then each irreducible component of D is a projective line and* $\operatorname{supp}(D)$ *is the support of an SNC-divisor of S whose dual graph is a tree. If* $\operatorname{supp}(D)$ *is irreducible then D is reduced. If* $\operatorname{supp}(D)$ *is reducible then there exists a* (-1)*-component Γ of* $\operatorname{supp}(D)$ *which meets at most two other components of* $\operatorname{supp}(D)$; *moreover, if Γ has multiplicity* 1 *in the divisor D then there exists another* (-1)*-component of* $\operatorname{supp}(D)$ *which meets at most two other components of* $\operatorname{supp}(D)$.
(b) *Let Σ be a section of Λ. Then there exist a nonsingular projective surface $\mathbb{F}$ and a birational morphism $\rho\colon S \to \mathbb{F}$ satisfying*:
- *the exceptional locus of ρ is the union of the irreducible curves $C \subset S$ which are Λ-vertical*[1] *and disjoint from Σ;*
- *the linear system $\mathbb{L} = \rho_*(\Lambda)$ is a base-point-free pencil on $\mathbb{F}$ each of whose elements is a projective line, and the curve $\Delta = \rho(\Sigma)$ is a section of $\mathbb{L}$;*
- *$\mathbb{F} = \mathbb{F}_k$ for some $k \in \mathbb{N}$; moreover, if $\Sigma^2 \leq 0$ then $\Sigma^2 = -k$ and* $(\mathbb{F}, \mathbb{L}, \Delta) = (\mathbb{F}_k, \mathbb{L}_k, \Delta_k)$.

3. Rational linear systems; uniresolvable curves and linear systems

We continue to assume that S is a rational nonsingular projective surface.

DEFINITION 3.1. We say that a linear system $\mathbb{L}$ on S is *rational* if $\dim \mathbb{L} \geq 1$ and the general member of $\mathbb{L}$ is an irreducible rational curve.

DEFINITIONS 3.2. In the following definitions we consider sequences

$$S = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} \cdots \xleftarrow{\pi_n} S_n \tag{8}$$

where, for each $i = 1, \dots, n$, $\pi_i : S_i \to S_{i-1}$ is the blowing-up of the nonsingular projective surface S_{i-1} at a point $P_i \in S_{i-1}$.
(a) Let $C \subset S$ be a curve. The *minimal resolution of singularities* of C is the shortest sequence (8) satisfying:

$$\text{the strict transform of } \quad C \quad \text{on} \quad S_n \quad \text{is a nonsingular curve.}$$

[1] A curve $C \subset S$ is said to be Λ-*vertical* if it is included in the support of an element of Λ.

The *minimal embedded resolution of singularities* of C is the shortest sequence (8) satisfying:

$$\pi^{-1}(C) \quad \text{is the support of an SNC-divisor of} \quad S_n,$$

where $\pi = \pi_1 \circ \cdots \circ \pi_n : S_n \to S$.
(b) Let $C \subset S$ be a curve. Consider the minimal resolution of singularities $X \to S$ of C, let $\tilde{C}$ be the strict transform of C on X, and let $\tilde{\nu}(C)$ denote the self-intersection number of $\tilde{C}$ in X. When $\tilde{\nu}(C) \geq 0$ (resp. $\tilde{\nu}(C) > 0$), we say that C is *of nonnegative type* (resp. *of positive type*). We also consider the minimal embedded resolution of singularities $Y \to S$ of C, and define $\tilde{\nu}_{\text{emb}}(C)$ to be the self-intersection number of the strict transform of C on Y. Clearly, $\tilde{\nu}_{\text{emb}}(C) \leq \tilde{\nu}(C)$.
(c) We say that the sequence (8) is a *chain* if $\pi_{i-1}(P_i) = P_{i-1}$ for all i such that $2 \leq i \leq n$.
(d) A linear system $\mathbb{L}$ on S is *uniresolvable* if $\dim \mathbb{L} \geq 1$, $\mathbb{L}$ is without fixed components and there exists a chain (8) with the property that the strict transform of $\mathbb{L}$ on S_n is base-point-free.
(e) A curve $C \subset S$ is *uniresolvable* if there exists a chain (8) with the property that the strict transform of C on S_n is a nonsingular curve.

Let $C \subset S$ be a curve. It follows from [3, Theorem 2.8] that the existence of a rational pencil Λ on S satisfying $C \in \Lambda$ is equivalent to C being rational and of nonnegative type. Let us now be more precise in the special case where C is uniresolvable.

Note that if $C \subset S$ is uniresolvable then there exists at least one point $P \in C$ such that $\text{Sing}(C) \subseteq \{P\}$.

Theorem 3.3. *Let $C \subset S$ be a uniresolvable curve and let $P \in C$ be such that* $\text{Sing}(C) \subseteq \{P\}$. *Then the following are equivalent*:
(a) *C is rational and of nonnegative type*;
(b) *there exists a rational linear system $\mathbb{L}$ on S satisfying $C \in \mathbb{L}$*;
(c) *there exists a rational and uniresolvable pencil Λ on S such that $C \in \Lambda$ and* $\text{Bs}(\Lambda) \subseteq \{P\}$.

Proof. It follows from [3, Theorem 2.8] that (a) is equivalent to (b), and it is clear that (c) implies (b); so it suffices to prove that (a) implies (c). Assume that (a) is satisfied. Then there exists a chain (8) satisfying:

- the strict transform $C_n \subset S_n$ of C is nonsingular and satisfies $C_n^2 = 0$;
- $P_1 = P$ and, for each $i \geq 2$, P_i lies on the strict transform $C_{i-1} \subset S_{i-1}$ of C.

By Lemma 2.2, $|C_n|$ is a $\mathbb{P}^1$-ruling on S_n. Define $\Lambda = \pi_*|C_n|$, where $\pi = \pi_1 \circ \cdots \circ \pi_n : S_n \to S_0$. Then Λ is a rational pencil on S satisfying $C \in \Lambda$. The strict transform of Λ on S_n is $|C_n|$, which is base-point-free. This has two consequences:
(i) all infinitely near base points of Λ are among $\{P_1, \ldots, P_n\}$, so in particular $\text{Bs}(\Lambda) \subseteq \{P\}$;

(ii) since (8) is a chain, Λ is uniresolvable. $\square$

Let us also mention the following related fact:

Lemma 3.4. *Let Λ be a pencil on S and $C \subset S$ an irreducible component of the support of some member of Λ. If Λ is rational and uniresolvable, then C is rational and uniresolvable.*

Proof. Consider the minimal resolution (8) of the base points of Λ; since Λ is uniresolvable, (8) is a chain. Let Λ_n (resp. C_n) be the strict transform of Λ (resp. of C) on S_n. As Λ is rational, the general member of Λ_n is isomorphic to $\mathbb{P}^1$, so Λ_n is a $\mathbb{P}^1$-ruling. As C_n is included in the support of some element of Λ_n, Gizatullin's Theorem 2.5 implies that C_n is nonsingular and rational. So C is rational and (since (8) is a chain) uniresolvable. $\square$

4. Rationality of Λ_C and N_C

Given a unicuspidal rational curve $C \subset \mathbb{P}^2$ we consider the pencil Λ_C and the net N_C defined in Definition 1.3, and ask when these linear systems are rational (in the sense of Definition 3.1).

Theorem 4.1. *For a unicuspidal rational curve $C \subset \mathbb{P}^2$, the following are equivalent*:
(a) *C is of nonnegative type*
(b) *Λ_C is rational.*
Moreover, if these conditions hold then Λ_C is uniresolvable.

Proof. The fact that (b) implies (a) follows from either one of [3, 2.8] or Theorem 3.3. Conversely, suppose that (a) holds and let P be the distinguished point of C. Then, in particular, C is uniresolvable and $P \in C$ is such that $\mathrm{Sing}(C) \subseteq \{P\}$. By Theorem 3.3, there exists a rational and uniresolvable pencil Λ on $\mathbb{P}^2$ such that $C \in \Lambda$ and $\mathrm{Bs}(\Lambda) \subseteq \{P\}$; then $\mathrm{Bs}(\Lambda) = \{P\}$. By Corollary 1.4, Λ_C is the unique pencil on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{P\}$. Thus $\Lambda = \Lambda_C$. Consequently, Λ_C is rational and uniresolvable. $\square$

REMARK 4.2. Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of nonnegative type, and let $C' \subset \mathbb{P}^2$ be an irreducible component of the support of some member of Λ_C. Then the curve C' is rational and uniresolvable. (This follows from Theorems 4.1 and 3.4.)

REMARK 4.3. In view of 4.1, it is interesting to note:
(a) All unicuspidal rational curves $C \subset \mathbb{P}^2$ satisfying $\bar{\kappa}(\mathbb{P}^2 \setminus C) < 2$ are of nonnegative type.
(b) All currently known unicuspidal rational curves $C \subset \mathbb{P}^2$ are of nonnegative type.

Indeed, let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve and consider $\bar{\kappa} = \bar{\kappa}(\mathbb{P}^2 \setminus C)$, the logarithmic Kodaira dimension of $\mathbb{P}^2 \setminus C$. Then it is a priori clear that $\bar{\kappa} \in \{-\infty, 0, 1, 2\}$.

- If $\bar{\kappa} = -\infty$ then [10] implies that $\tilde{\nu}_{\mathrm{emb}}(C) \geq -1$, and it follows that $\tilde{\nu}(C) > 0$.
- The case $\bar{\kappa} = 0$ cannot occur by a result of Tsunoda [14].
- The case $\bar{\kappa} = 1$ is completely classified in [12], and the multiplicity sequences are given explicitly. A straightforward computation using these sequences shows that $\tilde{\nu}(C) \in \{0, 1\}$, where the two cases occur.
- The case $\bar{\kappa} = 2$ is not classified. The only known examples here are two families of curves (denoted C_{4k} and C_{4k}^*, $k = 1, 2, \ldots$) found by Orevkov in [11]. For these examples the multiplicity sequences are known explicitly, and a straightforward computation shows that $\tilde{\nu}(C) \in \{1, 4\}$ where the two cases occur.

This justifies assertions (a) and (b). Regarding the last case we also mention:

- Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with $\bar{\kappa} = 2$. Then $\tilde{\nu}_{\mathrm{emb}}(C) \leq -2$ by a result of Yoshihara [15]. Moreover, Tono [13] showed that $\tilde{\nu}_{\mathrm{emb}}(C) = -2$ if and only if C is one of Orevkov's curves C_{4k} or C_{4k}^* for some k.

One should also remark that the sets

$$\{\tilde{\nu}(C) \mid C \subset \mathbb{P}^2 \text{cuspidal rational}\},$$
$$\{\tilde{\nu}_{\mathrm{emb}}(C) \mid C \subset \mathbb{P}^2 \text{ unicuspidal rational, } \bar{\kappa}(\mathbb{P}^2 \setminus C) = 1\}$$

are not bounded below, as can be deduced from [4] and [12], respectively.

The next paragraph will be used as a reference, when we want to establish the notation:

NOTATIONS 4.4. Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point P. Then (C, P) determines an infinite sequence

$$\mathbb{P}^2 = S_0 \xleftarrow{\pi_1} S_1 \xleftarrow{\pi_2} S_2 \xleftarrow{\pi_3} \cdots \tag{9}$$

of nonsingular projective surfaces and blowing-up morphisms such that, for each $i \geq 1$, $\pi_i \colon S_i \to S_{i-1}$ is the blowing-up of S_{i-1} at the unique point $P_i \in S_{i-1}$ which lies on the strict transform of C and which is mapped to $P_1 = P$ by $\pi_1 \circ \cdots \circ \pi_{i-1} \colon S_{i-1} \to S_0$. Let $E_i = \pi_i^{-1}(P_i) \subset S_i$ and, if $i < j$, let the strict transform of E_i on S_j be also denoted by $E_i \subset S_j$. Let $C_i \subset S_i$ be the strict transform of $C_0 = C$ on S_i, and let Λ_i be the strict transform of $\Lambda_0 = \Lambda_C$ on S_i. By definition of the sequence (9), it is clear that

$$C_{i-1} \cap E_{i-1} = \{P_i\} \quad \text{in} \quad S_{i-1}, \quad \text{for all} \quad i \geq 2. \tag{10}$$

Let $n \leq N$ be the natural numbers satisfying:

- $S_n \to S_0$ is the minimal resolution of singularities of C;
- $S_N \to S_0$ is the minimal embedded resolution of singularities of C.

Finally, let $r_i = e_{P_i}(C_{i-1})$ (see Conventions) for all $i \geq 1$, and let $d = \deg(C)$. Then the invariants $\tilde{\nu}(C)$ and $\tilde{\nu}_{\mathrm{emb}}(C)$ defined in Definition 3.2 are given by

$$\tilde{\nu}(C) = C_n^2 = d^2 - \sum_{i=1}^{n} r_i^2 \quad \text{and} \quad \tilde{\nu}_{\mathrm{emb}}(C) = C_N^2.$$

It is clear that if C is singular then $N = n + r_n$ and hence $\tilde{\nu}_{\mathrm{emb}}(C) = \tilde{\nu}(C) - r_n$, and that if C is nonsingular (i.e., $d \leq 2$) then $N = n = 0$ and $\tilde{\nu}_{\mathrm{emb}}(C) = \tilde{\nu}(C) = d^2$.

REMARK. If $\tilde{\nu}(C) \geq 0$, the natural number m defined in Proposition 4.5 (below) is to be added to the set of notations introduced in Notations 4.4. Note that the inequality $n \leq \min(N, m)$ always holds, and that the three cases $m < N$, $m = N$ and $m > N$ can occur.

Proposition 4.5. *Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point P, and let the notation be as in* Notations 4.4. *If C is of nonnegative type, then the following hold.*

(a) *There exists a natural number $m \geq n$ such that $S_m \to S_0$ is the minimal resolution of the base points of Λ_C.*

(b) *$C_i \in \Lambda_i$ for all $i \in \{0, \dots, m\}$.*

(c) *Λ_m is a $\mathbb{P}^1$-ruling of S_m (cf.* Definition 2.1).

(d) *$C_m \cong \mathbb{P}^1$ and $C_m^2 = 0$.*

(e) *For all $i \in \{1, \dots, m\}$, the following hold in S_m:*

$$E_i \text{ is horizontal} \iff E_i \cap C_m \neq \varnothing \iff P_{m+1} \in E_i.$$

Here we say that a curve in S_m is vertical if it is included in the support of a member of Λ_m, and horizontal if it is not vertical. The point P_{m+1} is defined by (10).

(f) *E_m is horizontal and at most one $i < m$ is such that $E_i \subset S_m$ is horizontal.*

(g) *E_m is a section of Λ_m if and only if C is of positive type.*

Proof. Let $S = Y_0 \xleftarrow{\rho_1} Y_1 \xleftarrow{\rho_2} \cdots \xleftarrow{\rho_m} Y_m$ be the minimal resolution of the base points of Λ_C, where, for $1 \leq i \leq m$, $\rho_i \colon Y_i \to Y_{i-1}$ is the blowing-up of the nonsingular surface Y_{i-1} at a point $P_i^* \in Y_{i-1}$. As C is of nonnegative type, Theorem 4.1 implies that Λ_C is rational. Let $\tilde{C} \subset Y_m$ (resp. $\tilde{\Lambda}_C$) be the strict transform of C (resp. of Λ_C) on Y_m. By [3, 2.7 (b)], the fact that Λ_C is rational implies that $\tilde{C} \in \tilde{\Lambda}_C$ and that $\tilde{C}$ is nonsingular. From $\tilde{C} \in \tilde{\Lambda}_C$, we deduce that for each i the base point P_i^* lies on the strict transform of C on Y_{i-1}; as P_i^* is infinitely near P (because $\mathrm{Bs}(\Lambda_C) = \{P\}$), it follows that $(P_1^*, \dots, P_m^*) = (P_1, \dots, P_m)$. Thus $S_m \to S_0$ is the minimal resolution of the base points of Λ_C. As we have observed, $\tilde{C} = C_m$ is nonsingular; it follows that $m \geq n$, so (a) is proved.

Then $\mathrm{Bs}(\Lambda_{i-1}) = \{P_i\}$ for all $i \in \{1, \dots, m\}$, and $\mathrm{Bs}(\Lambda_m) = \varnothing$.

We already noted that $\tilde{C} \in \tilde{\Lambda}_C$, which we may rewrite as $C_m \in \Lambda_m$. It follows that assertion (b) holds. As Λ_C is a rational pencil, so is Λ_m; as Λ_m is base-point-free, its general member is a $\mathbb{P}^1$, so (c) holds. By $C_m \in \Lambda_m$ and $\mathrm{Bs}(\Lambda_m) = \varnothing$, we get $C_m^2 = 0$, so assertion (d) holds.

The fact that $C_m \in \Lambda_m$ and that Λ_m is base-point-free implies that if $C' \subset S_m$ is a curve distinct from C_m then C' is horizontal if and only if $C' \cap C_m \neq \varnothing$. In particular, (e) is proved, and (f) immediately follows.

To prove (g), note that E_m is a section of Λ_m if and only if $E_m \cdot C_m = 1$, if and only if C_{m-1} is nonsingular; as $C_{m-1}^2 > C_m^2 = 0$, this is equivalent to C being of positive type. □

Theorem 4.6. *For a unicuspidal rational curve $C \subset \mathbb{P}^2$, the following are equivalent*:
(a) *C is of positive type*;
(b) *N_C is rational*;
(c) *the rational map $\Phi_{N_C}\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, corresponding to the net N_C, is birational.*
Moreover, if the above conditions hold then the Cremona map Φ_{N_C} transforms C into a line, and Λ_C into a pencil of "all lines through some point".

Proof. The fact that (c) implies (b) is trivial. If (b) holds then parts (e) and (f) of [3, 2.8] imply that the linear system $\mathbb{L}_C$ defined in [3, 2.5] satisfies $N_C \subseteq \mathbb{L}_C$ and $\dim \mathbb{L}_C = \tilde{\nu}(C) + 1$; then $\tilde{\nu}(C) > 0$, which shows that (b) implies (a). There remains to show that if (a) holds then Φ_{N_C} is birational and transforms C into a line and Λ_C into a pencil of all lines through some point.

Suppose that C is of positive type and let the notation be as in Notations 4.4 and Proposition 4.5. By Proposition 4.5 (g), E_m is a section of Λ_m. Then Gizatullin's Theorem 2.5 implies that there exists a birational morphism $\rho\colon S_m \to \mathbb{F}_1$ whose exceptional locus $\mathrm{exc}(\rho) \subset S_m$ is a union of Λ_m-vertical curves in S_m and $\mathrm{exc}(\rho) \cap E_m = \varnothing$. Moreover, in the notation of 2.3, $\rho_*(\Lambda_m)$ is the standard ruling $\mathbb{L}_1$ of $\mathbb{F}_1$ and $\rho(E_m)$ is the (-1)-section of that ruling. As the exceptional loci of the two morphisms $S_{m-1} \xleftarrow{\pi_m} S_m \xrightarrow{\rho} \mathbb{F}_1$ are disjoint, we have the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{P}^2 = S_0 & \xleftarrow{\pi_1} & \cdots & \xleftarrow{\pi_{m-1}} & S_{m-1} & \xleftarrow{\pi_m} & S_m \\
 & & & & \downarrow{\scriptstyle \bar{\rho}} & & \downarrow{\scriptstyle \rho} \\
 & & & & \mathbb{P}^2 & \xleftarrow{\bar{\pi}_m} & \mathbb{F}_1
\end{array}$$

where $\bar{\pi}_m\colon \mathbb{F}_1 \to \mathbb{P}^2$ is the contraction of $\rho(E_m)$. Define the birational map $\Phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as the composition

$$S_0 \overset{(\pi')^{-1}}{\dashrightarrow} S_{m-1} \xrightarrow{\bar{\rho}} \mathbb{P}^2,$$

where $\pi' = \pi_1 \circ \cdots \circ \pi_{m-1} \colon S_{m-1} \to S_0$. It is clear that Φ transforms C into a line in $\mathbb{P}^2$. Also, Φ determines a net N on $\mathbb{P}^2$ (without fixed components); let us show that $N = N_C$.

Consider the group homomorphisms

$$\operatorname{Div}(S_0) \xleftarrow{\pi'_*} \operatorname{Div}(S_{m-1}) \xleftarrow{\bar\rho^*} \operatorname{Div}(\mathbb{P}^2)$$

where $\bar\rho^*$ is the operation of taking the total transform with respect to $\bar\rho$ and π'_* takes direct image with respect to π'. Let $Q = \bar\rho(P_m) \in \mathbb{P}^2$ and let $\mathbb{L}$ be the linear system on $\mathbb{P}^2$ consisting of all lines through Q. Then the strict transform of $\mathbb{L}$ on S_{m-1} (via $\bar\rho$) is Λ_{m-1}. As $\bar\rho$ restricts to an isomorphism from a neighborhood of P_m to a neighborhood of Q (because $\operatorname{exc}(\rho) \cap E_m = \varnothing$), the strict transform of $\mathbb{L}$ coincides with the total transform of $\mathbb{L}$, so $\bar\rho^*$ transforms $\mathbb{L}$ into Λ_{m-1} and consequently $\pi'_* \circ \bar\rho^*$ transforms $\mathbb{L}$ into Λ_C. Now we note that $\pi'_* \circ \bar\rho^*$ transforms $\mathbb{M}$ into N, where $\mathbb{M}$ is the linear system of all lines in $\mathbb{P}^2$. As $\mathbb{L} \subset \mathbb{M}$, it follows that $\Lambda_C \subset N$ (in particular the elements of N have degree $d = \deg C$).

Let $\mathbb{M}^\circ$ be the set of $M \in \mathbb{M}$ such that $Q \notin M$ and $\bar\rho^{-1}(M)$ is an irreducible curve in S_{m-1}. Then the image of $\mathbb{M}^\circ$ via $\pi'_* \circ \bar\rho^*$ is a dense subset of N. Since N and N_C have the same dimension, in order to show that $N = N_C$ it suffices to show that $\pi'_* \circ \bar\rho^*$ maps $\mathbb{M}^\circ$ into N_C. Let $M \in \mathbb{M}^\circ$ and consider the curve $D = (\pi'_* \circ \bar\rho^*)(M) = \pi'(\bar\rho^{-1}(M)) \subset S_0$.

Let $L = \bar\rho(C_{m-1}) \in \mathbb{L}$ and note that $\bar\rho$ restricts to an isomorphism from a neighbourhood of C_{m-1} to a neighbourhood of L. As $(M \cdot L)_{\mathbb{P}^2} = 1$ and the point $M \cap L$ is not Q, it follows that

$$(\bar\rho^{-1}(M) \cdot C_{m-1})_{S_{m-1}} = 1$$

and that the point $\bar\rho^{-1}(M) \cap C_{m-1} = \{R\}$ belongs to $C_{m-1} \setminus P_m$, so $R \notin \operatorname{exc}(\pi')$. Consequently, $D \cap C \subseteq \{\pi'(R), P\}$ and $i_{\pi'(R)}(D, C) = 1$, where the point $\pi'(R)$ is distinct from P. By Bezout, $i_P(D, C) = d^2 - 1$, so $D \in N_C$. This shows that $\pi'_* \circ \bar\rho^*$ maps $\mathbb{M}^\circ$ into N_C; it follows that $N = N_C$, as desired.

So $\Phi_{N_C} = \Phi$ and consequently Φ_{N_C} is birational. We already noted that Φ transforms C into a line and that $\pi'_* \circ \bar\rho^*$ transforms $\mathbb{L}$ into Λ_C, so the last assertions follow. $\square$

5. Intermezzo: erasable weighted pairs

The aim of this section is to prove Proposition 5.15, which is needed in the proof of Theorem 6.2. Our proof of Proposition 5.15 makes use of a theory of "erasable weighted pairs" which we develop in this section; in fact Proposition 5.14 is the only fact from this graph theory which is needed, but its proof requires several preliminary lemmas.

We stress that the present section is completely self-contained. Except for the fact that Proposition 5.15 is used in the proof of Theorem 6.2, this section is completely independent from the rest of the paper.

Our graphs have finitely many vertices and edges, edges are not directed, no edge connects a vertex to itself, and at most one edge exists between a given pair of vertices. A *weighted graph* is a graph in which each vertex is assigned an integer (called the weight of the vertex). Note that the empty graph is a weighted graph. We assume that the reader is familiar with the classical notion of blowing-up of a weighted graph, and refer to 1.1 and 1.2 of [2] for details. In particular, recall that there are three ways to blow-up a weighted graph $\mathcal{G}$: one can blow-up $\mathcal{G}$ *at a vertex*, or *at an edge*, or one can perform the *free blowing-up* of $\mathcal{G}$ (in the last case, one takes the disjoint union of $\mathcal{G}$ and of a vertex of weight -1). In all cases, blowing-up $\mathcal{G}$ produces a new weighted graph $\mathcal{G}'$ whose vertex-set is obtained from that of $\mathcal{G}$ by adding one new vertex e of weight -1 (one says that e is the vertex "created" by the blowing-up). If $\mathcal{G}'$ is a blowing-up of $\mathcal{G}$ and e is the vertex of $\mathcal{G}'$ created by the blowing-up, then one says that $\mathcal{G}$ is the blowing-down of $\mathcal{G}'$ at e. Two weighted graphs $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* (denoted $\mathcal{A} \sim \mathcal{B}$) if one can be obtained from the other by a finite sequence of blowings-up and blowings-down. Note that if $\mathcal{G}$ is a weighted graph without edges, and in which each vertex has weight -1, then $\mathcal{G}$ is equivalent to the empty weighted graph $\varnothing$.

DEFINITIONS 5.1. (a) By a *weighted pair*, we mean an ordered pair $(\mathcal{G}, v)$ where $\mathcal{G}$ is a nonempty weighted graph and v is a vertex of $\mathcal{G}$ (called the distinguished vertex).
(b) A *blowing-up* of a weighted pair $(\mathcal{G}, v)$ is a weighted pair $(\mathcal{G}', v')$ satisfying:

- the weighted graph $\mathcal{G}'$ is obtained by blowing-up the weighted graph $\mathcal{G}$ either at the vertex v or at an edge incident[2] to v;
- v' is the unique vertex of $\mathcal{G}'$ which is not a vertex of $\mathcal{G}$ (i.e., v' is the vertex of weight -1 which is created by the blowing-up).

We write $(\mathcal{G}, v) \leftarrow (\mathcal{G}', v')$ to indicate that $(\mathcal{G}', v')$ is a blowing-up of $(\mathcal{G}, v)$.
(c) A weighted pair $(\mathcal{G}, v)$ is said to be *erasable* if there exists a finite sequence

$$(\mathcal{G}, v) = (\mathcal{G}_0, e_0) \leftarrow (\mathcal{G}_1, e_1) \leftarrow \cdots \leftarrow (\mathcal{G}_n, e_n) \quad (\text{with } n \geq 0) \tag{11}$$

of blowings-up of weighted pairs such that the weighted graph $\mathcal{G}_n \setminus \{e_n\}$ is equivalent to the empty weighted graph.

REMARK 5.2. In contrast with the theory of weighted graphs, we do not define a "blowing-down" of weighted pairs. The contraction of weighted pairs defined in Definition 5.7 is not the inverse operation of the blowing-up of weighted pairs.

REMARK 5.3. Let $(\mathcal{G}, v)$ be a weighted pair. The following claims are obvious:
(a) *If $\mathcal{G}$ has a vertex w of nonnegative weight such that $w \neq v$ and w is not a neighbor of v, then $(\mathcal{G}, v)$ is not erasable.*

[2] An edge α is *incident* to a vertex v if v is one of the endpoints of α.

(b) *If $\mathcal{G}$ has at least two vertices, v has negative weight and all weights in $\mathcal{G}\setminus\{v\}$ are strictly less than -1, then $(\mathcal{G}, v)$ is not erasable.*

DEFINITION 5.4. For any weighted pair $(\mathcal{G}, v)$ we define $l(\mathcal{G}, v) \in \mathbb{N} \cup \{\infty\}$ as follows. If $(\mathcal{G}, v)$ is not erasable, we set $l(\mathcal{G}, v) = \infty$. If $(\mathcal{G}, v)$ is erasable, then we define $l(\mathcal{G}, v)$ to be the least $n \in \mathbb{N}$ for which there exists a sequence (11) satisfying $\mathcal{G}_n \setminus \{e_n\} \sim \varnothing$. Thus a weighted pair $(\mathcal{G}, v)$ is erasable if and only if $l(\mathcal{G}, v) < \infty$. Also note that the condition $l(\mathcal{G}, v) = 0$ is equivalent to $\mathcal{G} \setminus \{v\} \sim \varnothing$.

DEFINITION 5.5. Let $(\mathcal{G}, v)$ be an erasable weighted pair such that $l(\mathcal{G}, v) > 0$. A blowing-up $(\mathcal{G}', v')$ of $(\mathcal{G}, v)$ is said to be *good* if it satisfies $l(\mathcal{G}', v') < l(\mathcal{G}, v)$.

Lemma 5.6. *Let $(\mathcal{G}, v)$ be an erasable weighted pair such that $l(\mathcal{G}, v) > 0$. Then there exists a good blowing-up of $(\mathcal{G}, v)$. Moreover, if $(\mathcal{G}', v')$ is a good blowing-up of $(\mathcal{G}, v)$ then $(\mathcal{G}', v')$ is erasable and $l(\mathcal{G}', v') = l(\mathcal{G}, v) - 1$.*

Proof. Obvious. □

DEFINITION 5.7. Let $(\mathcal{G}, v)$ be a weighted pair. A *contractible vertex* of $(\mathcal{G}, v)$ is a vertex w of $\mathcal{G}$ satisfying:

- the weight of w is (-1) and w has at most two neighbours in $\mathcal{G}$
- if w has two neighbours v_1 and v_2, then v_1, v_2 are not joined by an edge
- $w \neq v$ and w is not a neighbour of v.

If w is a contractible vertex of $(\mathcal{G}, v)$ then the *contraction of* $(\mathcal{G}, v)$ *at* w is the weighted pair $(\bar{\mathcal{G}}, \bar{v})$ defined by taking $\bar{\mathcal{G}}$ to be the blowing-down of the weighted graph $\mathcal{G}$ at w and by setting $\bar{v} = v$.

Lemma 5.8. *Suppose that $(\bar{\mathcal{G}}, \bar{v})$ is the contraction of a weighted pair $(\mathcal{G}, v)$ at some contractible vertex. Then $l(\mathcal{G}, v) = l(\bar{\mathcal{G}}, \bar{v})$.*

Proof. We proceed by induction on $n = \min(l(\mathcal{G}, v), l(\bar{\mathcal{G}}, \bar{v}))$, noting that the lemma is true whenever $n = \infty$. Let w be the contractible vertex of $(\mathcal{G}, v)$ at which the contraction is performed. Then $\bar{\mathcal{G}} \setminus \{\bar{v}\}$ is the blowing-down of $\mathcal{G} \setminus \{v\}$ at w, so there is an equivalence of weighted graphs $\mathcal{G} \setminus \{v\} \sim \bar{\mathcal{G}} \setminus \{\bar{v}\}$. In particular, the lemma is true whenever $n = 0$.

Consider $n \in \mathbb{N} \setminus \{0\}$ such that the lemma is true for all $(\mathcal{G}, v)$ and $(\bar{\mathcal{G}}, \bar{v})$ satisfying $\min(l(\mathcal{G}, v), l(\bar{\mathcal{G}}, \bar{v})) < n$. Consider $(\mathcal{G}, v)$ and $(\bar{\mathcal{G}}, \bar{v})$ such that $\min(l(\mathcal{G}, v), l(\bar{\mathcal{G}}, \bar{v})) = n$.

Choose an element $(\mathcal{G}_0, v_0)$ of the set $\{(\mathcal{G}, v), (\bar{\mathcal{G}}, \bar{v})\}$ such that $l(\mathcal{G}_0, v_0) = n$, and let $(\mathcal{G}_0', v_0')$ denote the other element of the set. By Lemma 5.6, there exists a blowing-up $(\mathcal{G}_0, v_0) \leftarrow (\mathcal{G}_1, v_1)$ such that $l(\mathcal{G}_1, v_1) = n - 1$. Then $\mathcal{G}_1$ is the blowing-up of $\mathcal{G}_0$ at x, where x is either the distinguished vertex v_0 or an edge $\{v_0, u\}$ with u a neighbour of v_0 in $\mathcal{G}_0$. As the distinguished vertices of $(\mathcal{G}_0, v_0)$ and $(\mathcal{G}_0', v_0')$ are the same ($v_0 = v_0'$

because $v = \bar{v}$), and the neighbours of that vertex are the same in $\mathcal{G}_0$ and in $\mathcal{G}'_0$, it makes sense to blow-up $\mathcal{G}'_0$ at x, and this gives rise to a blowing-up $(\mathcal{G}'_0, v'_0) \leftarrow (\mathcal{G}'_1, v'_1)$ of weighted pairs. Let us change the notation again and represent the two blowings-up $(\mathcal{G}_0, v_0) \leftarrow (\mathcal{G}_1, v_1)$ and $(\mathcal{G}'_0, v'_0) \leftarrow (\mathcal{G}'_1, v'_1)$ as

$$(\mathcal{G}, v) \leftarrow (\mathcal{H}, e) \quad \text{and} \quad (\bar{\mathcal{G}}, \bar{v}) \leftarrow (\bar{\mathcal{H}}, \bar{e}) \quad \text{(in some order).}$$

Note that w is a contractible vertex of $(\mathcal{H}, e)$, and that $(\bar{\mathcal{H}}, \bar{e})$ is the contraction of $(\mathcal{H}, e)$ at w. We have

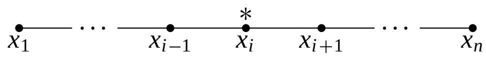
$$\min(l(\mathcal{H}, e), l(\bar{\mathcal{H}}, \bar{e})) = \min(l(\mathcal{G}_1, v_1), l(\mathcal{G}'_1, v'_1)) \leq l(\mathcal{G}_1, v_1) = n - 1,$$

so the inductive hypothesis implies that $l(\mathcal{H}, e) = l(\bar{\mathcal{H}}, \bar{e})$, which is equal to $n-1$. Thus $l(\mathcal{G}, v) \leq 1 + l(\mathcal{H}, e) = n$ and $l(\bar{\mathcal{G}}, \bar{v}) \leq 1 + l(\bar{\mathcal{H}}, \bar{e}) = n$, so

$$\max(l(\mathcal{G}, v), l(\bar{\mathcal{G}}, \bar{v})) \leq n = \min(l(\mathcal{G}, v), l(\bar{\mathcal{G}}, \bar{v}))$$

and consequently $l(\mathcal{G}, v) = l(\bar{\mathcal{G}}, \bar{v})$). $\square$

NOTATION 5.9. Given integers $x_1, \dots, x_n$ and $i \in \{1, \dots, n\}$, the weighted pair



(where the asterisk $*$ indicates the distinguished vertex) is denoted by

$$[x_1, \dots, x_{i-1}, x_i^*, x_{i+1}, \dots, x_n].$$

Observe that there is an equality of weighted pairs

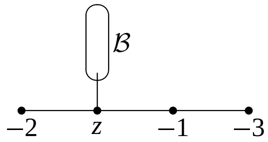
$$[x_1, \dots, x_{i-1}, x_i^*, x_{i+1}, \dots, x_n] = [x_n, \dots, x_{i+1}, x_i^*, x_{i-1}, \dots, x_1].$$

Lemma 5.10. *If* $l[-2, -1^*, -1, -3] < \infty$, *then*

$$l[-3, -1^*, -1, -2] < l[-2, -1^*, -1, -3].$$

Proof. Suppose that $(\mathcal{G}, v) = [-2, -1^*, -1, -3]$ is erasable and observe that $l(\mathcal{G}, v) > 0$. Pick a sequence (11) such that $\mathcal{G}_n \setminus \{e_n\} \sim \varnothing$ and such that $n = l(\mathcal{G}, v)$. Then $(\mathcal{G}_1, e_1)$ is a good blowing-up of $(\mathcal{G}, v)$ and one of the following holds:
(a) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at v
(b) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at the edge $[-1^*, -1]$
(c) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at the edge $[-2, -1^*]$.

In case (a), one of the connected components of $\mathcal{G}_n \setminus \{e_n\}$ has the form

(12) 

(for some $z \in \mathbb{Z}$)

where every vertex in the branch $\mathcal{B}$ has weight strictly less than -1 (and $\mathcal{B}$ might be empty). This is absurd, because the weighted graph (12) is not equivalent to $\varnothing$. Thus case (a) does not occur.

In case (b), Remark 5.3 (b) implies that $(\mathcal{G}_1, e_1)$ is not erasable, which is absurd; so case (b) does not occur either.

In case (c) we have $(\mathcal{G}_1, e_1) = [-3, -1^*, -2, -1, -3]$, and the contraction of $(\mathcal{G}_1, e_1)$ at its contractible vertex is $(\bar{\mathcal{G}}_1, \bar{e}_1) = [-3, -1^*, -1, -2]$. Consequently

$$l[-3, -1^*, -1, -2] = l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}_1, e_1) < l(\mathcal{G}, v) = l[-2, -1^*, -1, -3]$$

(where we used Lemma 5.8), and this proves the lemma. □

Lemma 5.11. *If $x \le -2$ and $l[-1^*, -1, x, -4] < \infty$, then*

$$l[-1^*, -1, x, -4] > l[-3, -1^*, -1, -2].$$

Proof. Let $x \le -2$, let $(\mathcal{G}, v) = [-1^*, -1, x, -4]$ and suppose that $l(\mathcal{G}, v) < \infty$. As $l(\mathcal{G}, v) > 0$, there exists a good blowing-up $(\mathcal{G}', v')$ of $(\mathcal{G}, v)$. By Remark 5.3 (b), $(\mathcal{G}', v')$ cannot be the blowing-up of $(\mathcal{G}, v)$ at the edge $[-1^*, -1]$; so $(\mathcal{G}', v')$ is the blowing-up of $(\mathcal{G}, v)$ at v, i.e., $(\mathcal{G}', v') = [-1^*, -2, -1, x, -4]$. The contraction of $(\mathcal{G}', v')$ at its contractible vertex is $(\bar{\mathcal{G}}', \bar{v}') = [-1^*, -1, x + 1, -4]$, so

$$l[-1^*, -1, x + 1, -4] = l(\bar{\mathcal{G}}', \bar{v}') = l(\mathcal{G}', v') < l(\mathcal{G}, v) = l[-1^*, -1, x, -4].$$

More precisely, we have shown that if $x \le -2$ and $l[-1^*, -1, x, -4] < \infty$ then

$$l[-1^*, -1, x, -4] > l[-1^*, -1, x + 1, -4].$$

By induction it follows that if $x \le -2$ and $l[-1^*, -1, x, -4] < \infty$, then

$$l[-1^*, -1, x, -4] > l[-1^*, -1, -1, -4] = l[-1^*, 0, -3]$$

(where the equality follows from Lemma 5.8); so there only remains to show that

$$l[-1^*, 0, -3] \ge l[-3, -1^*, -1, -2]. \tag{13}$$

This is obvious if $l[-1^*, 0, -3] = \infty$, so let us assume that $l[-1^*, 0, -3] < \infty$. Let $(\mathcal{G}, v) = [-1^*, 0, -3]$. As $l(\mathcal{G}, v) > 0$, there exists a good blowing-up $(\mathcal{G}', v')$ of $(\mathcal{G}, v)$.

By Remark 5.3 (a), $(\mathcal{G}', v')$ cannot be the blowing-up of $(\mathcal{G}, v)$ at v, so it must be the blowing-up of $(\mathcal{G}, v)$ at the edge $[-1^*, 0]$; so $(\mathcal{G}', v') = [-2, -1^*, -1, -3]$ and consequently

$$l[-2, -1^*, -1, -3] = l(\mathcal{G}', v') < l(\mathcal{G}, v) = l[-1^*, 0, -3] < \infty. \tag{14}$$

As $l[-2, -1^*, -1, -3] < \infty$, Lemma 5.10 implies that $l[-3, -1^*, -1, -2] < l[-2, -1^*, -1, -3]$, so (14) gives

$$l[-3, -1^*, -1, -2] < l[-2, -1^*, -1, -3] < l[-1^*, 0, -3].$$

So (13) is proved and we are done. $\square$

Lemma 5.12. *If $x \le -2$ and $l[-1, -1^*, x, -4] < \infty$, then*

$$l[-1, -1^*, x, -4] > l[-3, -1^*, -1, -2].$$

Proof. Let E be the set of $x \in \mathbb{Z}$ satisfying $x \le -2$ and

$$l[-1, -1^*, x, -4] < \infty \quad \textit{and} \quad l[-1, -1^*, x, -4] \le l[-3, -1^*, -1, -2]. \tag{15}$$

It suffices to show that $E = \varnothing$. By contradiction, suppose that $E \neq \varnothing$ and pick $x \in E$. Let $(\mathcal{G}, v) = [-1, -1^*, x, -4]$. Then $l(\mathcal{G}, v) < \infty$ and $l(\mathcal{G}, v) > 0$, so there exists a good blowing-up $(\mathcal{G}', v')$ of $(\mathcal{G}, v)$. By Remark 5.3 (b), $(\mathcal{G}', v')$ cannot be the blowing-up of $(\mathcal{G}, v)$ at the edge $[-1, -1^*]$; so one of the following conditions must hold:
(a) $(\mathcal{G}', v')$ is the blowing-up of $(\mathcal{G}, v)$ at v
(b) $(\mathcal{G}', v')$ is the blowing-up of $(\mathcal{G}, v)$ at the edge $[-1^*, x]$.
In case (a), the contraction of $(\mathcal{G}', v')$ at its contractible vertex is

$$(\bar{\mathcal{G}}', \bar{v}') = [-1^*, -1, x, -4].$$

Thus $l[-1^*, -1, x, -4] = l(\bar{\mathcal{G}}', \bar{v}') = l(\mathcal{G}', v') < l(\mathcal{G}, v) < \infty$, so Lemma 5.11 implies that $l[-1^*, -1, x, -4] > l[-3, -1^*, -1, -2]$. This gives

$$l[-3, -1^*, -1, -2] < l[-1^*, -1, x, -4] < l(\mathcal{G}, v) = l[-1, -1^*, x, -4],$$

which contradicts (15) (and (15) holds since $x \in E$). Thus case (a) does not occur.

In case (b), $(\mathcal{G}', v') = [-1, -2, -1^*, x-1, -4]$. The contraction of $(\mathcal{G}', v')$ at its contractible vertex is $(\bar{\mathcal{G}}', \bar{v}') = [-1, -1^*, x-1, -4]$, so $l[-1, -1^*, x-1, -4] = l(\bar{\mathcal{G}}', \bar{v}') = l(\mathcal{G}', v') < l(\mathcal{G}, v) = l[-1, -1^*, x, -4]$. In fact we have shown:

$$\text{if} \quad x \in E \quad \text{then} \quad l[-1, -1^*, x-1, -4] < l[-1, -1^*, x, -4] \quad \text{and} \quad x - 1 \in E.$$

This implication together with $E \neq \varnothing$ imply the existence of an infinite descending sequence

$$l[-1, -1^*, x, -4] > l[-1, -1^*, x-1, -4] > l[-1, -1^*, x-2, -4] > \cdots$$

of natural numbers, which is absurd. So $E = \varnothing$ and we are done. □

Lemma 5.13. $[-3, -1^*, -1, -2]$ *is not erasable.*

Proof. We prove this by contradiction. Let $(\mathcal{G}_0, e_0) = [-3, -1^*, -1, -2]$ and assume that $(\mathcal{G}_0, e_0)$ is erasable. As $l(\mathcal{G}_0, e_0) > 0$, there exists a good blowing-up $(\mathcal{G}_1, e_1)$ of $(\mathcal{G}_0, e_0)$. There are three possibilities:
(a) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}_0, e_0)$ at e_0
(b) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}_0, e_0)$ at the edge $[-1^*, -1]$
(c) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}_0, e_0)$ at the edge $[-3, -1^*]$.

Consider case (a). Let $(\bar{\mathcal{G}}_1, \bar{e}_1)$ be obtained from $(\mathcal{G}_1, e_1)$ by performing two contractions at contractible vertices. Then $(\bar{\mathcal{G}}_1, \bar{e}_1) = [-1^*, 0, -3]$, so $l(\bar{\mathcal{G}}_1, \bar{e}_1) > 0$, so $(\bar{\mathcal{G}}_1, \bar{e}_1)$ has a good blowing-up $(\bar{\mathcal{G}}_2, \bar{e}_2)$. By Remark 5.3 (a), the blowing-up of $(\bar{\mathcal{G}}_1, \bar{e}_1)$ at $\bar{e}_1$ is not good; so $(\bar{\mathcal{G}}_2, \bar{e}_2)$ must be the blowing-up of $(\bar{\mathcal{G}}_1, \bar{e}_1)$ at the edge $[-1^*, 0]$, i.e., $(\bar{\mathcal{G}}_2, \bar{e}_2) = [-2, -1^*, -1, -3]$. Then

$$\begin{aligned} l[-2, -1^*, -1, -3] &= l(\bar{\mathcal{G}}_2, \bar{e}_2) < l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}_1, e_1) < l(\mathcal{G}_0, e_0) \\ &= l[-3, -1^*, -1, -2], \end{aligned}$$

so $l[-2, -1^*, -1, -3] < l[-3, -1^*, -1, -2] < \infty$, which contradicts Lemma 5.10. So case (a) cannot occur.

In case (b) we have $(\mathcal{G}_1, e_1) = [-3, -2, -1^*, -2, -2]$, which is not erasable by Remark 5.3 (b). So case (b) does not occur either.

In case (c) we have $(\mathcal{G}_1, e_1) = [-4, -1^*, -2, -1, -2]$. Let $(\bar{\mathcal{G}}_1, \bar{e}_1)$ be obtained from $(\mathcal{G}_1, e_1)$ by performing two contractions at contractible vertices. Then $(\bar{\mathcal{G}}_1, \bar{e}_1) = [-4, -1^*, 0]$, so $l(\bar{\mathcal{G}}_1, \bar{e}_1) > 0$, so $(\bar{\mathcal{G}}_1, \bar{e}_1)$ has a good blowing-up $(\bar{\mathcal{G}}_2, \bar{e}_2)$. In fact $(\bar{\mathcal{G}}_2, \bar{e}_2)$ must be the blowing-up of $(\bar{\mathcal{G}}_1, \bar{e}_1)$ at the edge $[-1^*, 0]$, otherwise Remark 5.3 (a) gives a contradiction. So $(\bar{\mathcal{G}}_2, \bar{e}_2) = [-4, -2, -1^*, -1] = [-1, -1^*, -2, -4]$ and consequently

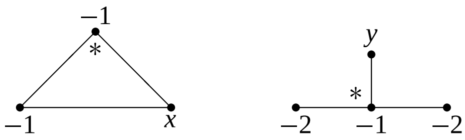
$$\begin{aligned} l[-1, -1^*, -2, -4] &= l(\bar{\mathcal{G}}_2, \bar{e}_2) < l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}_1, e_1) \\ &< l(\mathcal{G}_0, e_0) = l[-3, -1^*, -1, -2]. \end{aligned}$$

We conclude that

$$l[-1, -1^*, -2, -4] < l[-3, -1^*, -1, -2] < \infty,$$

which contradicts Lemma 5.12. So we are done. □

Proposition 5.14. *Let $x \in \mathbb{Z} \setminus \{-2\}$ and $y \in \mathbb{Z}$. Then the two weighted pairs*

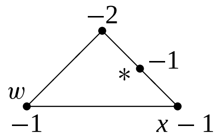


are not erasable.

Proof. Let $(\mathcal{G}, v)$ be the weighted pair which looks like a triangle, in the statement of the proposition, and (proceeding by contradiction) assume that $(\mathcal{G}, v)$ is erasable. Since $x \neq -2$, we have $\mathcal{G} \setminus \{v\} \not\sim \varnothing$, so $l(\mathcal{G}, v) > 0$. Pick a sequence (11) such that $\mathcal{G}_n \setminus \{e_n\} \sim \varnothing$ and such that $n = l(\mathcal{G}, v)$; note that $(\mathcal{G}_1, e_1)$ is a good blowing-up of $(\mathcal{G}, v)$. If $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at v then $\mathcal{G}_n \setminus \{e_n\}$ contains a simple circuit, which contradicts $\mathcal{G}_n \setminus \{e_n\} \sim \varnothing$; so $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at one of the two edges incident to v. Consequently, $(\mathcal{G}_1, e_1)$ is either as in (16) or as in (17), below.

Consider the case where $(\mathcal{G}_1, e_1)$ is as follows:

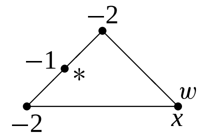
(16)



Then w is a contractible vertex and if $(\bar{\mathcal{G}}_1, \bar{e}_1)$ denotes the contraction of $(\mathcal{G}_1, e_1)$ at w then $(\bar{\mathcal{G}}_1, \bar{e}_1)$ is isomorphic[3] to $(\mathcal{G}, v)$. This isomorphism implies that $l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}, v)$ but on the other hand Lemma 5.8 implies that $l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}_1, e_1) < l(\mathcal{G}, v)$. This contradiction shows that $(\mathcal{G}_1, e_1)$ cannot be as in (16).

The only other possibility is that $(\mathcal{G}_1, e_1)$ be as follows:

(17)



Now we must have $x = -1$, otherwise $\mathcal{G}_n \setminus \{e_n\}$ would not contain any vertex of weight (-1) and hence would not be equivalent to the empty weighted graph. So w is a contractible vertex and the contraction $(\bar{\mathcal{G}}_1, \bar{e}_1)$ of $(\mathcal{G}_1, e_1)$ at w is isomorphic to $(\mathcal{G}, v)$. This leads to the same contradiction as in the first case, so we have shown that $(\mathcal{G}, v)$ is not erasable.

From now-on let $(\mathcal{G}, v)$ be the weighted pair on the right-hand-side, in the statement of the proposition; proceeding again by contradiction, assume that $(\mathcal{G}, v)$ is erasable. It

[3]The definition of *isomorphism of weighted pairs* is the obvious one.

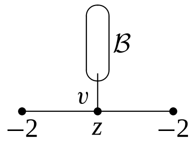
is clear that $\mathcal{G} \setminus \{v\} \not\sim \varnothing$, so $l(\mathcal{G}, v) > 0$. Pick a sequence (11) such that $\mathcal{G}_n \setminus \{e_n\} \sim \varnothing$ and such that $n = l(\mathcal{G}, v)$, and note that $(\mathcal{G}_1, e_1)$ is a good blowing-up of $(\mathcal{G}, v)$. One of the following holds:

(a) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at the edge which contains the vertex of weight y

(b) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at the distinguished vertex v

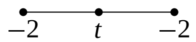
(c) $(\mathcal{G}_1, e_1)$ is the blowing-up of $(\mathcal{G}, v)$ at an edge which does not contain the vertex of weight y.

In case (a), one of the connected components of $\mathcal{G}_n \setminus \{e_n\}$ has the following shape:

(18) 

(for some $z \in \mathbb{Z}$)

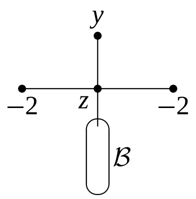
where $\mathcal{B}$ represents a (possibly empty) branch of $\mathcal{G}_n \setminus \{e_n\}$ at v; so the weighted graph (18) is equivalent to $\varnothing$. However, (18) is not equivalent to $\varnothing$. Indeed, if it were, then we would have $\mathcal{B} \sim \varnothing$ and in fact (18) would contract to

(19) 

(for some $t \in \mathbb{Z}$)

but clearly the graph (19) is not equivalent to $\varnothing$. So (18) is not equivalent to $\varnothing$ either, which rules out case (a).

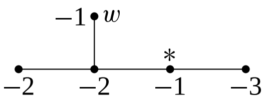
In case (b), $\mathcal{G}_n \setminus \{e_n\}$ has a connected component as follows:

(20) 

(for some $z \in \mathbb{Z}$)

where $\mathcal{B}$ might be empty and all vertices of $\mathcal{B}$ have weight strictly less than -1. This implies that the weighted graph (20) is equivalent to the empty graph. However, (20) is not equivalent to $\varnothing$. Indeed, if it were then we would have $\mathcal{B} \sim \varnothing$, so in fact $\mathcal{B} = \varnothing$, then (20) would be of the form (18) and hence would not be equivalent to $\varnothing$. So (20) is not equivalent to $\varnothing$ and case (b) is ruled out.

Consequently case (c) must occur, i.e., $(\mathcal{G}_1, e_1)$ must be the blowing-up of $(\mathcal{G}, v)$ at an edge which does not contain the vertex of weight y. Note that, although there are two such edges, only one case needs to be considered because an automorphism of $(\mathcal{G}, v)$ interchanges the two edges. Also observe that, if the vertex of weight y is called w, then w has the same weight in $\mathcal{G}$ and in $\mathcal{G}_n$; consequently $y = -1$, because $\mathcal{G}_n \setminus \{e_n\}$ must have a vertex of weight -1 and all vertices of $\mathcal{G}_n \setminus \{e_n, w\}$ have weight strictly less than -1. So $(\mathcal{G}_1, e_1)$ is the following weighted pair:



Then w is a contractible vertex and the contraction of $(\mathcal{G}_1, e_1)$ at w is $(\bar{\mathcal{G}}_1, \bar{e}_1) = [-3, -1^*, -1, -2]$. Then $l[-3, -1^*, -1, -2] = l(\bar{\mathcal{G}}_1, \bar{e}_1) = l(\mathcal{G}_1, e_1) < l(\mathcal{G}, v) < \infty$, which implies that $[-3, -1^*, -1, -2]$ is erasable. This contradicts Lemma 5.13, so the proof is complete. $\square$

The next proof requires familiarity with the classical notion of dual graph (see for instance 2.4). If D is an SNC-divisor of a nonsingular projective surface S, we write $\mathcal{G}(D, S)$ for the dual graph of D in S. Recall in particular that $\mathcal{G}(D, S)$ is a weighted graph. See Definition 3.2 for the definition of "chain".

Proposition 5.15. *No triple (Y_0, D, L) satisfies the following conditions* (i)–(iii):

(i) *Y_0 is a nonsingular projective surface and $D, L \subset Y_0$ are irreducible curves.*

(ii) *L is nonsingular, $L^2 = 0$ and $D \cdot L = 2$.*

(iii) *There exists a chain $Y_0 \xleftarrow{\sigma_1} Y_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_N} Y_N$ such that $N \geq 1$ and, if $D_N \subset Y_N$, $L_N \subset Y_N$, and $G_i \subset Y_N$ denote respectively the strict transforms of D, of L, and of the exceptional curve of σ_i, then*:

- *the subset $D_N \cup L_N \cup G_1 \cup \cdots \cup G_{N-1}$ of Y_N is the exceptional locus of a birational morphism $Y_N \to S$ where S is a nonsingular projective surface*;
- *$L_N^2 \neq -1$ in Y_N.*

Proof. By contradiction, assume that (Y_0, D, L) exists and consider $Y_0 \xleftarrow{\sigma_1} Y_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_N} Y_N$ as in the statement, where $\sigma_i \colon Y_i \to Y_{i-1}$ is the blowing-up at the point $Q_i \in Y_{i-1}$. Let $D_i, L_i \subset Y_i$ be the strict transforms of $D_0 = D$ and $L_0 = L$ respectively; we write $G_i \subset Y_i$ for the exceptional curve of σ_i and, if $i < j \leq N$, the strict transform of G_i in Y_j is also denoted by $G_i \subset Y_j$. For each $i \in \{1, \ldots, N\}$, let Δ_i denote the reduced divisor $D_i + L_i + G_1 + \cdots + G_i$ of Y_i. Let Ω denote the reduced divisor $D_N + L_N + G_1 + \cdots + G_{N-1}$ of Y_N, i.e., $\Omega = \Delta_N - G_N$.

As $\mathrm{supp}(\Omega)$ is the exceptional locus of a birational morphism, Ω is an SNC-divisor of Y_N which has at least one (-1)-component. Because $L_N^2 \neq -1$, it follows that $D_N^2 = -1$ and that D_N is the only (-1)-component of Ω. Moreover, there must hold $L_N^2 < -1$ (so $N \geq 2$, $Q_1 \in L_0$ and $Q_2 \in L_1$). Also note that $D_N \cdot L_N \leq 1 < 2 = D_0 \cdot L_0$, so $Q_1 \in D_0 \cap L_0$. We record:

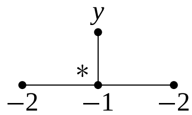
$$Q_1 \in L_0 \cap D_0 \quad \text{and} \quad Q_2 \in L_1 \cap G_1. \tag{21}$$

Suppose that Q_1 is a singular point of D_0. Then $D_0 \cdot L_0 = 2$ implies that $D_1 \cap L_1 = \varnothing$ and that $D_1 \cdot G_1 = 2$; then (21) implies that $Q_2 \notin D_1$ and hence that $(D_N \cdot G_1)_{Y_N} =$

$(D_1 \cdot G_1)_{Y_1} > 1$, which contradicts the fact that Ω is an SNC-divisor. This shows that Q_1 is a regular point of D_0. As D_N is nonsingular and $\sigma_1 \circ \cdots \circ \sigma_N$ is a chain, it follows that D_0 is nonsingular.

Consider the case where $D_0 \cap L_0$ is one point (so it is Q_1). Then it follows from (21) that $\Delta_2 = D_2 + L_2 + G_1 + G_2$ is an SNC-divisor of Y_2 whose dual graph is

(22) $\mathcal{G}(\Delta_2, Y_2)$:



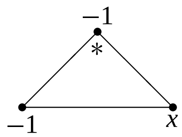
where $y = D_2^2 \in \mathbb{Z}$ and where G_2 is the vertex indicated by an asterisk $*$. Then $\Delta_i = D_i + L_i + G_1 + \cdots + G_i$ is an SNC-divisor of Y_i for each $i \in \{2, \ldots, N\}$, and

$$(\mathcal{G}(\Delta_2, Y_2), G_2) \leftarrow \cdots \leftarrow (\mathcal{G}(\Delta_N, Y_N), G_N) = (\mathcal{G}, v)$$

is a sequence of blowings-up of weighted pairs (cf. Definition 5.1). The weighted graph $\mathcal{G} \setminus \{v\}$ is equal to $\mathcal{G}(\Omega, Y_N)$, which is equivalent to the empty weighted graph since $\operatorname{supp}(\Omega)$ is the exceptional locus of a birational morphism. So the weighted pair $\big(\mathcal{G}(\Delta_2, Y_2), G_2\big)$ is erasable, i.e., the weighted pair pictured in (22) is erasable, and this contradicts Proposition 5.14.

This shows that $D_0 \cap L_0$ contains more than one point. Then it follows from (21) that $\Delta_1 = D_1 + L_1 + G_1$ is an SNC-divisor of Y_1 whose dual graph is

(23) $\mathcal{G}(\Delta_1, Y_1)$:



where $x = D_1^2 \geq D_N^2 = -1$ and where G_1 is the vertex indicated by the asterisk. Then

$$(\mathcal{G}(\Delta_1, Y_1), G_1) \leftarrow \cdots \leftarrow (\mathcal{G}(\Delta_N, Y_N), G_N) = (\mathcal{G}, v)$$

is a sequence of blowings-up of weighted pairs such that $\mathcal{G} \setminus \{v\} = \mathcal{G}(\Omega, Y_N) \sim \varnothing$. So the weighted pair $\big(\mathcal{G}(\Delta_1, Y_1), G_1\big)$ is erasable, i.e., the weighted pair pictured in (23) is erasable. This contradicts Proposition 5.14, so the proof is complete. $\square$

6. Existence of a dicritical of degree 1

6.1. Dicriticals. Let Λ be a pencil without fixed components on a nonsingular projective surface S and $\Phi_\Lambda \colon S \dashrightarrow \mathbb{P}^1$ the rational map given by Λ. Choose a commutative diagram

$$
\begin{tikzcd}
S \arrow[d, dashed, "\Phi_\Lambda"'] & \tilde{S} \arrow[l, "\pi"'] \arrow[dl, "\Psi_\Lambda"] \\
\mathbb{P}^1
\end{tikzcd}
\tag{24}
$$

where $\tilde{S}$ is a nonsingular projective surface, π is a birational morphism and Ψ_Λ is a morphism, and consider the exceptional locus $\mathcal{E} = \mathrm{exc}(\pi) \subset \tilde{S}$ of π. The horizontal[4] curves included in $\mathcal{E}$ are called the *dicriticals* of diagram (24). If $E \subseteq \mathcal{E}$ is a dicritical of (24) then the composition $E \hookrightarrow \tilde{S} \xrightarrow{\Psi_\Lambda} \mathbb{P}^1$ is a surjective morphism $f_E : \mathbb{P}^1 \to \mathbb{P}^1$; the positive integer $\deg(f_E)$ is called the *degree of the dicritical* E.

Suppose that diagram (24) has $s \geq 0$ dicriticals, of degrees $d_1, \dots, d_s$ respectively. Then the number s and the unordered s-tuple $[d_1, \dots, d_s]$ are uniquely determined by Λ, i.e., are independent of the choice of a diagram (24) which resolves the points of indeterminacy of Φ_Λ. So it makes sense to speak of the number of dicriticals "of Λ", and of the degrees of these dicriticals.

The main objective of this section is to prove:

Theorem 6.2. *Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve with distinguished point P and let Λ_C be the unique pencil on $\mathbb{P}^2$ such that $C \in \Lambda_C$ and* $\mathrm{Bs}(\Lambda_C) = \{P\}$. *If C is of nonnegative type then Λ_C has either* 1 *or* 2 *dicriticals, and at least one of them has degree* 1.

The fact that Λ_C has either one or two dicriticals easily follows from Proposition 4.5 (f); the real contents of the theorem is the claim that there exists a dicritical of degree 1.

The proof of the Theorem makes use of Proposition 5.15 (see the last sentence of the proof). The following notation is also needed:

6.3. Let $(a,b) \in \mathbb{Z}^2$ be such that $\min(a,b) \geq 1$. Consider the Euclidean algorithm of (a, b):

$$
\begin{aligned}
x_0 &= q_1 x_1 + x_2, \\
&\dots \\
x_{p-2} &= q_{p-1} x_{p-1} + x_p, \\
x_{p-1} &= q_p x_p
\end{aligned}
$$

[4]A curve $E \subset \tilde{S}$ is *vertical* if $\Psi_\Lambda(E)$ is a point, *horizontal* otherwise.

where $x_0 = b$, $x_1 = a$, all x_i and q_i are positive integers and $x_1 > \cdots > x_p \geq 1$ (so that $\gcd(a, b) = x_p$). We define the finite sequence $S(a, b)$ by

$$S(a,b) = (\underbrace{x_1, \dots, x_1}_{q_1 \text{ times}}, \dots, \underbrace{x_{p-1}, \dots, x_{p-1}}_{q_{p-1} \text{ times}}, \underbrace{x_p, \dots, x_p}_{q_p \text{ times}}).$$

Note that $S(a, b) = S(b, a)$. It is well known and easy to verify that if we change the notation to $S(a, b) = (r_1, r_2, \dots, r_n)$ then

$$\sum_{i=1}^{n} r_i = a + b - \gcd(a, b) \quad \text{and} \quad \sum_{i=1}^{n} r_i^2 = ab. \tag{25}$$

The proof of Theorem 6.2 also requires the following fact.

6.4. Consider $S_m \xrightarrow{\pi_m} S_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_1} S_0$ where, for each $i = 1,\dots,m$, $\pi_i \colon S_i \to S_{i-1}$ is the blowing-up of the nonsingular projective surface S_{i-1} at a point $P_i \in S_{i-1}$. Let $E_i = \pi_i^{-1}(P_i) \subset S_i$ for each $i = 1, \dots, m$. Given a curve $H_0 \subset S_0$, and given i, j such that $1 \leq i \leq j \leq m$, let $(E_i \cdot H_j)_{S_j}$ denote the intersection number in S_j of the curves $E_i \subset S_j$ and $H_j \subset S_j$, where E_i and H_j denote the strict transforms of $E_i \subset S_i$ and $H_0 \subset S_0$, respectively.

Lemma 6.5. *Let the setup and notation be as in* 6.4*. Then, for each $j \in \{1,\dots,m\}$, there exists a $\mathbb{Z}$-linear map $T_j \colon \mathbb{Z}^m \to \mathbb{Z}^j$ with the following property*:

$$\text{for every curve} \quad H_0 \subset S_0, \quad T_j \begin{pmatrix} (E_1 \cdot H_m)_{S_m} \\ \vdots \\ (E_m \cdot H_m)_{S_m} \end{pmatrix} = \begin{pmatrix} (E_1 \cdot H_j)_{S_j} \\ \vdots \\ (E_j \cdot H_j)_{S_j} \end{pmatrix}.$$

Proof. If $j = m$ then the claim is trivial. Assume that $j < m$ (in particular $m \geq 2$). For each $k = 2, \dots, m$, define the $\mathbb{Z}$-linear map $L_k \colon \mathbb{Z}^k \to \mathbb{Z}^{k-1}$ by

$$L_k \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_1 + e_{P_k}(E_1) x_k \\ \vdots \\ x_{k-1} + e_{P_k}(E_{k-1}) x_k \end{pmatrix},$$

where $e_{P_k}(E_i)$ is the multiplicity of the point $P_k \in S_{k-1}$ on the curve $E_i \subset S_{k-1}$. Note that $L_2, \dots, L_m$ are completely determined by the sequence $S_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_1} S_0$. We leave it to the reader to verify that $T_j = L_{j+1} \circ \cdots \circ L_m$ has the desired property. □

Proof of Theorem 6.2. Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of nonnegative type, with distinguished point P. Let the notation be as in Notations 4.4 and Proposition 4.5, and note that Λ_m is a $\mathbb{P}^1$-ruling by Proposition 4.5 (c). The dicriticals of

Λ_C are the $E_i \subset S_m$ which are horizontal, i.e., which are not included in the support of an element of Λ_m. So, by Proposition 4.5 (f), Λ_C has either one or two dicriticals. To prove that at least one dicritical has degree 1, we have to show that there exists $i \in \{1, \dots, m\}$ such that E_i is a section of Λ_m, i.e., $(E_i \cdot D)_{S_m} = 1$ for all $D \in \Lambda_m$. Note that Λ_m does have a section by Gizatullin's Theorem 2.5; however, we don't know a priori whether a section can be found among the E_i. Proceeding by contradiction, we assume that no E_i is a section of Λ_m. As $C_m \in \Lambda_m$ by Proposition 4.5 (b), it follows that

$$\text{for all} \quad i \in \{1, \dots, m\}, \quad E_i \cdot C_m \neq 1 \quad (\text{in } S_m). \tag{26}$$

Then in S_m we have

$$E_m \cdot C_m > 1 \quad \text{and for all} \quad i < m \quad \text{we have} \quad E_i \cap C_m = \varnothing. \tag{27}$$

Indeed, $E_m \cdot C_m = e_{P_m}(C_{m-1}) \geq 1$ and (26) implies that the inequality is strict. If for some $i < m$ we have $E_i \cap C_m \neq \varnothing$ then the fact that $E_i \cap C_m = \{P_{m+1}\} = E_m \cap C_m$ implies that $\min(E_i \cdot C_m, E_m \cdot C_m) = e_{P_{m+1}}(C_m) = 1$, which contradicts (26). So (27) is true.

Consider the multiplicity sequence $(r_1, \dots, r_m)$ where $r_i = e_{P_i}(C_{i-1}) = (E_i \cdot C_i)_{S_i}$, and note that

$$r_m > 1$$

by the first part of (27). Let $d = \deg(C)$. As $C_m^2 = 0$ and $C_m \cong \mathbb{P}^1$, we have $0 = C_m^2 = C_0^2 - \sum_{i=1}^m r_i^2 = d^2 - \sum_{i=1}^m r_i^2$ and (by the genus formula) $(d-1)(d-2) = \sum_{i=1}^m r_i(r_i - 1)$. It follows that

$$d^2 = \sum_{i=1}^m r_i^2 \quad \text{and} \quad 3d - 2 = \sum_{i=1}^m r_i. \tag{28}$$

Note that $(r_1,\dots,r_m)$ cannot be a constant sequence $(a,\dots,a)$ because equations (28) would then read $d^2 = ma^2$ and $3d - 2 = ma$, and these have no solution in integers with $a > 1$. We point out that $m \geq 2$, for otherwise $(r_1, \dots, r_m)$ would be constant. From the second part of (27) and the fact that $(r_1, \dots, r_m)$ is not constant, we deduce that $(r_1, \dots, r_m)$ has the following description: there exist $(a_1, b_1), \dots, (a_h, b_h) \in \mathbb{Z}^2$ (for some $h \geq 1$) such that

- $\min(a_i, b_i) \geq 1$ for all $i \in \{1, \dots, h\}$
- $a_{i+1} = \gcd(a_i, b_i)$ for all $i \in \{1, \dots, h - 1\}$
- $a_1 > \cdots > a_h > a_{h+1}$, where we define $a_{h+1} = \gcd(a_h, b_h)$
- $(r_1,\dots,r_m) = \big(S(a_1,b_1),\dots, S(a_h,b_h), (a_{h+1})_e\big)$ for some $e \geq 0$, where each sequence $S(a_i, b_i)$ is defined as in 6.3 and where $(a_{h+1})_e$ is the sequence $(a_{h+1}, \dots, a_{h+1})$ where a_{h+1} occurs e times.

By 6.3, the last term of the sequence $S(a_h, b_h)$ is $\gcd(a_h, b_h) = a_{h+1}$; so $r_m = a_{h+1}$ holds when $e = 0$, and obviously it also holds when $e \neq 0$. So

$$a_{h+1} = r_m > 1$$

in all cases. By (28) and (25),

$$d^2 = \sum_{i=1}^{h} a_i b_i + e a_{h+1}^2$$

and since a_{h+1} divides each a_i and each b_i it follows that $a_{h+1}^2 \mid d^2$ and hence that $a_{h+1} \mid d$. The other part of (28) gives

$$3d - 2 = \sum_{i=1}^{h}(a_i + b_i - a_{i+1}) + e a_{h+1} = a_1 + (e-1)a_{h+1} + \sum_{i=1}^{h} b_i,$$

so $a_{h+1} \mid 2$ and consequently

(29) $$r_m = a_{h+1} = 2.$$

Define the integers $\delta = d/2$, $\alpha_i = a_i/2$ $(1 \le i \le h+1)$ and $\beta_i = b_i/2$ $(1 \le i \le h)$. Then $\alpha_{i+1} = \gcd(\alpha_i, \beta_i)$ for all $i \in \{1, \dots, h\}$ and $\alpha_1 > \cdots > \alpha_h > \alpha_{h+1} = 1$. The above equations yield:

$$\delta^2 = \sum_{i=1}^{h} \alpha_i \beta_i + e, \quad 3\delta = \alpha_1 + e + \sum_{i=1}^{h} \beta_i.$$

Suppose that p is a prime number which divides both e and α_h. Then $\delta^2 \equiv 0 \pmod{p}$ and $3\delta \equiv \beta_h \pmod{p}$, so $p \mid \beta_h$ and consequently $p \mid \gcd(\alpha_h, \beta_h) = \alpha_{h+1} = 1$, which is absurd. This contradiction shows that $\gcd(e, \alpha_h) = 1$, and since $\alpha_h > 1$ we have shown that $e > 0$. This has the following consequence:

(30) the only $i < m$ which satisfies $E_i \cap E_m \neq \varnothing$ (in S_m) is $i = m-1$.

As $P_i \in E_{i-1}$ for all $i > 1$ (cf. (10)), we see in particular that $\bigcup_{i=1}^{m} E_i$ is connected; by (30), it follows that the subset $\mathcal{E} = \bigcup_{i=1}^{m-1} E_i$ of S_m is connected. As each irreducible component of $\mathcal{E}$ is vertical by (27) and Proposition 4.5 (e), it follows that

(31) $$\mathcal{E} \subseteq \operatorname{supp}(F) \quad \text{for some} \quad F \in \Lambda_m$$

because distinct elements of Λ_m have disjoint supports. We claim:

(32) if $G \in \Lambda_m$ and $G \neq F$ then G is irreducible and reduced.

By contradiction, suppose that $G \in \Lambda_m \setminus \{F\}$ is not irreducible and reduced. Then the support of G is a union of at least two curves (otherwise we would have $G = nG_0$ for some $n \geq 2$ and some divisor G_0 of S_m, and this would contradict the fact Theorem 2.5 that Λ_m has a section). Let $L \subset S_m$ be an irreducible component of G. As E_m is horizontal and $\mathcal{E} \subseteq \text{supp}(F)$, G does not contain any E_i, so the image of L in S_0 (via $\pi_1 \circ \cdots \circ \pi_m$) is a curve $L_0 \subset S_0$. As $\varnothing \neq L_0 \cap C \subseteq \text{Bs}(\Lambda_0) = \{P_1\}$, we have $P_1 \in L_0$, so $L \cap (\mathcal{E} \cup E_m) \neq \varnothing$; as $L \cap \mathcal{E} \subseteq \text{supp}(G) \cap \text{supp}(F) = \varnothing$, we have $L \cdot E_m > 0$ (for any irreducible component L of G). As $G \cdot E_m = C_m \cdot E_m = r_m = 2$, and since G has at least two irreducible components, it follows that $G = L + M$ where L, M are distinct prime divisors, $L \cdot E_m = 1 = M \cdot E_m$ and $L \cap \mathcal{E} = \varnothing = M \cap \mathcal{E}$. Moreover, Gizatullin's Theorem 2.5 implies that $L \cong \mathbb{P}^1 \cong M$ and that $L^2 = -1 = M^2$.

Let $L_i \subset S_i$ be the strict transform of L_0 on S_i and note that $L_m = L$. By the above observations we have $P_i \in L_{i-1}$ for all $i \in \{1, \dots, m\}$ and L_m satisfies $L_m \cong \mathbb{P}^1$ and $L_m^2 = -1$. Define $\mathbf{m}(L_0) = (r'_1, \dots, r'_m)$ by $r'_i = e_{P_i}(L_{i-1}) = (E_i \cdot L_i)_{S_i}$ and let us compare $\mathbf{m}(L_0)$ with the sequence $\mathbf{m}(C) = (r_1, \dots, r_m)$ which we have already considered. We claim:

$$(r_1, \dots, r_m) = 2(r'_1, \dots, r'_m). \tag{33}$$

To see this, note that $\begin{pmatrix} (E_1 \cdot C_m)_{S_m} \\ \vdots \\ (E_m \cdot C_m)_{S_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} (E_1 \cdot L_m)_{S_m} \\ \vdots \\ (E_m \cdot L_m)_{S_m} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$, so

$$\begin{pmatrix} (E_1 \cdot C_m)_{S_m} \\ \vdots \\ (E_m \cdot C_m)_{S_m} \end{pmatrix} = 2 \begin{pmatrix} (E_1 \cdot L_m)_{S_m} \\ \vdots \\ (E_m \cdot L_m)_{S_m} \end{pmatrix}. \tag{34}$$

By Lemma 6.5, for each $j \in \{1, \dots, m\}$ there exists a $\mathbb{Z}$-linear map $T_j : \mathbb{Z}^m \to \mathbb{Z}^j$ which is completely determined by the sequence $S_m \xrightarrow{\pi_m} \cdots \xrightarrow{\pi_1} S_0$ and which has the following property: given a curve $H_0 \subset S_0$ and its strict transform H_j on S_j,

$$T_j \begin{pmatrix} (E_1 \cdot H_m)_{S_m} \\ \vdots \\ (E_m \cdot H_m)_{S_m} \end{pmatrix} = \begin{pmatrix} (E_1 \cdot H_j)_{S_j} \\ \vdots \\ (E_j \cdot H_j)_{S_j} \end{pmatrix}.$$

By (34) and linearity of T_j it follows that $(E_i \cdot C_j)_{S_j} = 2(E_i \cdot L_j)_{S_j}$ for all i, j such that $1 \leq i \leq j \leq m$, so in particular $r_j = (E_j \cdot C_j)_{S_j} = 2(E_j \cdot L_j)_{S_j} = 2r'_j$ for all $j \in \{1, \dots, m\}$. This proves (33).

Let $d' = \deg(L_0)$. As $L_m \cong \mathbb{P}^1$ and $L_m^2 = -1$, $(d'-1)(d'-2) = \sum_{i=1}^m r'_i(r'_i - 1)$ and $(d')^2 = \sum_{i=1}^m (r'_i)^2 - 1$, so $3d' = 1 + \sum_{i=1}^m r'_i$. Doubling the last equation and using the second part of (28) gives

$$6d' = 2 + \sum_{i=1}^{m}(2r_i') = 2 + \sum_{i=1}^{m} r_i = 3d,$$

so $d = 2d'$. Then

$$d^2 = 4(d')^2 = 4\sum_{i=1}^{m}(r_i')^2 - 4 = \sum_{i=1}^{m} r_i^2 - 4$$

contradicts (28), and hence (32) is proved.

By Gizatullin's result 2.5 we may choose a section $\Sigma \subset S_m$ of Λ_m and consider the birational morphism $\rho\colon S_m \to \mathbb{F}$ whose exceptional locus is the union of the curves in S_m which are Λ_m-vertical and disjoint from Σ. Recall from the same result Theorem 2.5 that $\mathbb{F}$ is one of the Nagata–Hirzebruch ruled surfaces and that $\mathbb{L} = \rho_*(\Lambda_m)$ is a base-point-free pencil on $\mathbb{F}$ each of whose elements is a projective line. We have $\operatorname{exc}(\rho) \subseteq \operatorname{supp} F$ by (32), so the number of irreducible components of $\operatorname{exc}(\rho)$ is 1 less than the number of irreducible components of $\operatorname{supp} F$ (as exactly one component of F meets Σ). Recall that the canonical divisors $K_{\mathbb{P}^2}$ and $K_{\mathbb{F}}$ satisfy $K_{\mathbb{F}}^2 = K_{\mathbb{P}^2}^2 - 1$; so, consideration of

$$\mathbb{P}^2 = S_0 \xleftarrow{\pi} S_m \xrightarrow{\rho} \mathbb{F}$$

(where $\pi = \pi_1 \circ \cdots \circ \pi_m$) shows that ρ contracts exactly $m-1$ curves, and hence that F has exactly m irreducible components. As $\mathcal{E} \subseteq \operatorname{supp}(F)$, it follows that $\operatorname{supp}(F) = \Gamma \cup \mathcal{E}$ for some curve $\Gamma \subset S_m$ such that $\Gamma \not\subseteq \mathcal{E}$, and where we must have $\Gamma^2 = -1$ since no component of $\mathcal{E}$ has that property. We have $\Gamma \cap \Sigma = \varnothing$, for otherwise Theorem 2.5 would imply that F has a (-1)-component other than Γ, which is not the case. Note that $\Gamma \neq E_m$ since E_m is horizontal, so Γ is not an E_i. It also follows that exactly one element $j \in \{1, \dots, m-1\}$ is such that $\rho(E_j)$ is a curve; in fact E_j is the unique component of F which meets Σ and consequently $\rho(E_j)$ is an element of $\mathbb{L}$. Let us also observe that $\operatorname{exc}(\rho) = \Gamma \cup \bigcup_{i\in I} E_i$, where $I = \{1, \dots, m-1\} \setminus \{j\}$, so $\rho(E_m)$ is a curve.

Let us state some properties of the triple (Y_0, D, L), where we define $Y_0 = \mathbb{F}$, $D = \rho(E_m)$ and $L = \rho(E_j)$ (the symbol "L" was used in an earlier part of the proof, but we give it a new meaning here). Obviously,

(i) *Y_0 is a nonsingular projective surface and $D, L \subset Y_0$ are irreducible curves.*

We also observe:

(ii) *L is nonsingular, $L^2 = 0$ and $D \cdot L = 2$.*

Indeed, we have already noted that $L \in \mathbb{L}$, so L is nonsingular and $L^2 = 0$. As $E_m \cdot C_m = 2$ and (since $\operatorname{exc}(\rho) \subseteq \operatorname{supp} F$) ρ is an isomorphism in a neighborhood of C_m, it follows that $D \cdot \rho(C_m) = 2$; noting that $\rho(C_m) \in \mathbb{L}$, it follows that $D \cdot L' = 2$ for any $L' \in \mathbb{L}$ and in particular (ii) is true. Next we note:[5]

[5] See Definition 3.2 for the definition of "chain".

(ii) *There exists a chain* $Y_0 \xleftarrow{\sigma_1} Y_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_N} Y_N$ *such that* $N \geq 1$ *and, if* $D_N \subset Y_N$, $L_N \subset Y_N$, *and* $G_i \subset Y_N$ *denote respectively the strict transforms of* D, *of* L, *and of the exceptional curve of* σ_i, *then*:

- *the subset* $D_N \cup L_N \cup G_1 \cup \cdots \cup G_{N-1}$ *of* Y_N *is the exceptional locus of a birational morphism* $Y_N \to S$ *where* S *is a nonsingular projective surface*;
- $L_N^2 \neq -1$ *in* Y_N.

This is obtained from $\rho \colon S_m \to \mathbb{F}$ by changing the notation: let $N = m - 1$ and factor ρ as $S_m = Y_N \xrightarrow{\sigma_N} \cdots \xrightarrow{\sigma_1} Y_0 = \mathbb{F}$, where each σ_i is a blowing-up at a point. Just after (28) we noted that $m \geq 2$, so $N \geq 1$. The fact that the blowing-up sequence $(\sigma_1, \ldots, \sigma_N)$ is a chain follows from the fact that $\operatorname{exc}(\rho) = \Gamma \cup \bigcup_{i \in I} E_i$ (where $I = \{1, \ldots, m-1\} \setminus \{j\}$) has exactly one (-1)-component. We have $G_N = \Gamma$, $D_N = E_m$, and $L_N = E_j$, so in particular $L_N^2 \neq -1$. The subset $D_N \cup L_N \cup G_1 \cup \cdots \cup G_{N-1}$ of $Y_N = S_m$ is equal to $\bigcup_{i=1}^m E_i$, which is the exceptional locus of the birational morphism $\pi_1 \circ \cdots \circ \pi_m$. So (iii) is true.

By Proposition 5.15, no triple (Y_0, D, L) satisfies (i)–(iii). This contradiction completes the proof of the theorem. □

ACKNOWLEDGEMENTS. The first author would like to thank the Department of Algebra of the Universidad Complutense de Madrid for its hospitality. The research which led to this article was initiated during a 7 months stay of the first author at that institution.

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