Jentsch, T.
Osaka J. Math.
51 (2014), 285–336

PARALLEL SUBMANIFOLDS OF THE REAL 2-GRASSMANNIAN

TILLMANN JENTSCH

(Received August 29, 2011, revised August 24, 2012)

Abstract

A submanifold of a Riemannian symmetric space is called parallel if its second fundamental form is parallel. We classify parallel submanifolds of the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ which parameterizes the oriented 2-planes of the Euclidean space $\mathbb{R}^{n+2}$. Our main result states that every complete parallel submanifold of $G_2^+(\mathbb{R}^{n+2})$, which is not a curve, is contained in some totally geodesic submanifold as a symmetric submanifold. The analogous result holds if the ambient space is the Riemannian product of two Euclidean spheres of equal curvature or the non-compact dual of one of the previously considered spaces. We also give a characterization of parallel submanifolds with curvature isotropic tangent spaces of maximal possible dimension in any symmetric space of compact or non-compact type.

1. Introduction

Let N be a Riemannian symmetric space. A submanifold[1] of N is called *parallel* if its second fundamental form is parallel. D. Ferus [6] has shown that every compact parallel submanifold of a Euclidean space is a symmetric orbit of some s-representation, called a *symmetric R-space*. In particular, such a submanifold is *extrinsically symmetric* which means that it is invariant under the reflections in its affine normal spaces. More generally, every complete parallel submanifold of a space form has this property (see [2, 7, 25, 26]). This should be seen as an extrinsic analog of the following well known fact: every complete and simply connected Riemannian manifold with parallel curvature tensor is already a symmetric space.

Further, increasing the complexity of the ambient space step by step, consider parallel submanifolds of rank-one symmetric spaces. Their classification was achieved by various authors, cf. the overview given in [1, Chapter 9.4]. Under slight restrictions, parallelity of the second fundamental form implies extrinsic symmetry also here. More precisely, recall that a submanifold is called *full* if it is not contained in any proper totally geodesic submanifold of the ambient space. On the one hand, one can show that in all simply connected rank-one spaces of non-constant sectional curvature (i.e. the

2010 Mathematics Subject Classification. 53C35, 53C40, 53C42.

[1]We are implicitly dealing with immersed submanifolds, i.e. we consider isometric immersions defined from a connected Riemannian manifold M into N. In particular, a "submanifold" may have self-intersections.

projective spaces over the complex numbers or the quaternions, the Cayley plane, and their non-compact duals) there exist one-dimensional complete parallel submanifolds (so called *extrinsic circles*) which are full but not extrinsically symmetric. On the other hand, it turns out that every complete parallel submanifold of dimension at least two is contained in some totally geodesic submanifold as a symmetric submanifold. Hence, at least every higher-dimensional full complete parallel submanifold of a rank-one space is extrinsically symmetric.

The classification of symmetric submanifolds in ambient symmetric spaces of higher rank was finally achieved by H. Naitoh in a series of papers. His result is surprisingly simple in its statement, but the proof seems rather lengthy. Very roughly said, he considers subspaces $W \subset T_pN$ such that both W and $W^\perp$ are curvature invariant and decides whether there exists some non totally geodesic symmetric submanifold $M \subset N$ with $T_pM = W$. For this, he uses a case by case strategy which is mainly based on [22, Lemma 1.1]. In fact, he obtains assertions on a larger class of submanifolds (see [1, Chapter 9.3]).

In contrast, there is "not much known" about parallel submanifolds of symmetric spaces of higher rank. As a particular case, the classification of totally geodesic submanifolds is still an open problem. But for ambient rank-two spaces, the classification was obtained by B. Chen and T. Nagano [3, 4][2] and later by S. Klein [14, 15, 16, 17, 18] using different methods. Thus, one may ask also for the classification of parallel submanifolds in rank-two spaces.

In this article, we classify the parallel submanifolds of the Grassmannian $G_2^+(\mathbb{R}^{n+2})$—which parameterizes the oriented 2-planes of the Euclidean space $\mathbb{R}^{n+2}$—and its non-compact dual, the symmetric space $G_2^+(\mathbb{R}^{n+2})^*$, i.e. the Grassmannian of time-like 2-planes in the pseudo Euclidean space $\mathbb{R}^{n,2}$ equipped with the indefinite inner product $dx_1^2 + \cdots + dx_n^2 - dx_{n+1}^2 - dx_{n+2}^2$. Note, these are simply connected symmetric spaces of rank two if $n \geq 2$.

Theorem 1 (Main theorem)**.** *If M is a complete parallel submanifold of the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ with $\dim(M) \geq 2$, then there exists a totally geodesic submanifold $\bar{M} \subset G_2^+(\mathbb{R}^{n+2})$ such that M is a symmetric submanifold of $\bar{M}$. In particular, every full complete parallel submanifold of $G_2^+(\mathbb{R}^{n+2})$, which is not a curve, is a symmetric submanifold. The analogous result holds for ambient space $G_2^+(\mathbb{R}^{n+2})^*$.*

We also obtain the classification of higher-dimensional parallel submanifolds in a product of two Euclidean spheres or two real hyperbolic spaces of equal curvature (see Corollary 1). Further, we conclude that every higher-dimensional complete parallel submanifold of $G_2^+(\mathbb{R}^{n+2})$ is extrinsically homogeneous (see Corollary 2).

[2]However, the claimed classification of totally geodesic submanifolds of $G_2^+(\mathbb{R}^{n+2})$ from [3] is incomplete.

Here, we focus our attention mainly on the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ (and its non-compact dual). Nevertheless, we also establish a splitting theorem for parallel submanifolds with curvature isotropic tangent spaces of maximal possible dimension in any symmetric space (of compact or non-compact type), see Corollary 5.

The proof of Theorem 1 is based on the results given by S. Klein and B.-Y. Chen–T. Nagano for $N := G_2^+(\mathbb{R}^{n+2})$, see above. Namely, recall that these authors actually classified *curvature invariant subspaces* of the tangent spaces of N (since complete totally geodesic submanifolds through a point $p \in N$ correspond to curvature invariant subspaces of T_pN via the exponential map $\exp^N : T_pN \to N$). Thereupon, we classify *orthogonal curvature invariant pairs*. Then we decide case by case on their *integrability*. For more details see Section 1.1.

Essentially, this method should work for any ambient symmetric space whose curvature invariant subspaces are known. Hence, one may hope that it is also possible to classify parallel submanifolds of the other rank-two symmetric spaces (e.g. the Grassmannians of complex or quaternionic 2-planes). It would be an interesting question whether some analogue of Theorem 1 remains true for such ambient spaces.

1.1. Overview and outline of the proof of the main theorem. This section gives a detailed overview on the results presented in this article, an outline of the proof of Theorem 1 included. For a Riemannian symmetric space N with metric tensor $\langle \cdot, \cdot \rangle$ and a submanifold M, let TM, $\bot M$, $h\colon TM \times TM \to \bot M$ and $S\colon TM \times \bot M \to TM$ denote the tangent bundle, the normal bundle, the second fundamental form and the shape operator, respectively. Let ∇^M and ∇^N denote the Levi-Civita connections of M and N, respectively, and $\nabla^\bot$ be the usual connection on $\bot M$ (obtained by orthogonal projection of $\nabla^N \xi$ along TM for every section ξ of $\bot M$). Let $\mathrm{Sym}^2(TM, \bot M)$ denote the vector bundle whose sections are $\bot M$-valued symmetric bilinear maps on TM. Then there is a linear connection on $\mathrm{Sym}^2(TM, \bot M)$ induced by ∇^M and $\nabla^\bot$ in a natural way, often called *Van der Waerden–Bortolotti connection*.

DEFINITION 1. The submanifold M is called *parallel* if h is a parallel section of $\mathrm{Sym}^2(TM, \bot M)$.

EXAMPLE 1. A unit speed curve $c\colon J \to N$ is parallel if and only if it satisfies the equation

$$\nabla^N_\partial \nabla^N_\partial \dot{c} = -\kappa^2 \dot{c} \tag{1}$$

for some constant $\kappa \in \mathbb{R}$. For $\kappa = 0$ these curves are geodesics; otherwise, due to K. Nomizu and K. Yano [24], c is called an *extrinsic circle*.

Recall that for every unit vector $x \in T_pN$ and every $\eta \in T_pN$ with $\eta \perp x$ there exists a unique unit speed curve c satisfying (1) with $c(0) = p$, $\dot{c}(0) = x$ and $\nabla_\partial \dot{c}(0) = \eta$.

EXAMPLE 2. Let $\bar{M} \subset N$ be totally geodesic (i.e. $h^{\bar{M}} = 0$). A submanifold of $\bar{M}$ is parallel if and only if it is parallel in N.

DEFINITION 2. A submanifold $M \subset N$ is called (*extrinsically*) *symmetric* if M is a symmetric space (whose geodesic symmetries are denoted by σ_p^M, where p ranges over M) and for every $p \in M$ there exists an involutive isometry $\sigma_p^\perp$ of N such that

- $\sigma_p^\perp(M) = M$,
- $\sigma_p^\perp | M = \sigma_p^M$,
- the differential $T_p\sigma_p^\perp$ is the reflection in the normal space $\perp_p M$.

As mentioned already before, every symmetric submanifold is parallel. However, in the situation of Example 2, we do not necessarily obtain a symmetric submanifold of N even if M is symmetric in $\bar{M}$.

Let M be a parallel submanifold of the symmetric space N. Then the linear space $\perp_p^1 M := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$ is called the *first normal space* at p.

QUESTION. Given a pair of linear spaces (W, U) both contained in T_pN and such that $W \perp U$, does there exist some parallel submanifold M through p with $W = T_pM$ and $U = \perp_p^1 M$? In particular, are there natural obstructions against the existence of such a submanifold?

Let R^N denote the curvature tensor of N and recall that a linear subspace $V \subset T_pN$ is called *curvature invariant* if $R^N(V \times V \times V) \subset V$ holds. It is well known that T_pM is a curvature invariant subspace of T_pN for every parallel submanifold M. In Section 2.2, we will show that also $\perp_p^1 M$ is curvature invariant. Moreover, the curvature endomorphisms of T_pN generated by T_pM leave $\perp_p^1 M$ invariant and vice versa. This means that $(T_pM, \perp_p^1 M)$ is an *orthogonal curvature invariant pair*, see Definition 4 and Proposition 1. As a first illustration of this concept, we classify the orthogonal curvature invariant pairs (W, U) of the complex projective space $\mathbb{C}\mathrm{P}^n$, see Example 3. We observe that here the linear space $W \oplus U$ is complex or totally real (in particular, curvature invariant) unless $\dim(W) = 1$. Hence, following the proof of Theorem 1 given below, we obtain the well known result that the analogue of Theorem 1 is true for ambient space $\mathbb{C}\mathrm{P}^n$.

In Section 3.1, we will determine the orthogonal curvature invariant pairs of $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$. Our result is summarized in Table 1. Note, even if we assume additionally that $\dim(W) \geq 2$, there do exist certain orthogonal curvature invariant pairs (W, U) for which the linear space $W \oplus U$ is not curvature invariant (in contrast to the situation where the ambient space is $\mathbb{C}\mathrm{P}^n$, see above). Hence, at least at the level of curvature invariant pairs, we can not yet give the proof of Theorem 1.

Therefore, it still remains to decide whether there actually exists some parallel submanifold M such that $(W, U) = (T_pM, \perp_p^1 M)$ in which case the orthogonal curvature invariant pair (W, U) will be called *integrable*. In Section 3.2, by means of a case by case analysis, we will show that if (W, U) is integrable and $\dim(W) \geq 2$, then the linear space $W \oplus U$ is curvature invariant. For this, we will need some more intrinsic properties of the second fundamental form of a parallel submanifold of a symmetric space which are derived in Section 2.

Further, note that (orthogonal) curvature invariant pairs of N and N^*, respectively, are the same.[3] Moreover, it turns out that all arguments from Section 3.2 remain valid for ambient space N^*.

Proof of Theorem 1. Since $G_2^+(\mathbb{R}^3)$ is isometric to a 2-dimensional Euclidean sphere, we can assume that $n \geq 2$. Given some $p \in M$, the curvature invariant pair $(T_pM, \perp^1_p M)$ is integrable (by definition). Thus, using the above mentioned results, we conclude that the *second osculating space* $\mathcal{O}_p M := T_pM \oplus \perp^1_p M$ is a curvature invariant subspace of T_pN. By means of *reduction of the codimension* (see [5]), we obtain that M is already contained in the totally geodesic submanifold $\bar{M} := \exp^N(\mathcal{O}_p M)$. Let M^{uc} and $\bar{M}^{\mathrm{uc}}$ denote the universal covering spaces of M and $\bar{M}$, respectively. Then the immersion of M into $\bar{M}$ admits a lift $M^{\mathrm{uc}} \to \bar{M}^{\mathrm{uc}}$, i.e. M^{uc} becomes a complete parallel submanifold of $\bar{M}^{\mathrm{uc}}$. By construction, $\perp^1_q M^{\mathrm{uc}} = \perp_q M^{\mathrm{uc}}$ for all $q \in M^{\mathrm{uc}}$ which means that M^{uc} is 1-*full* in $\bar{M}^{\mathrm{uc}}$. Thus, $M^{\mathrm{uc}} \subset \bar{M}^{\mathrm{uc}}$ is even a symmetric submanifold according to Corollary 3. Further, since the case $M = \bar{M}$ is obvious, we can henceforth assume that $\dim(\bar{M}) \geq 3$. Then, checking the list of isometry classes of symmetric spaces occurring as complete totally geodesic submanifolds of N from [14, Section 5], we immediately see that any isometry of $\bar{M}^{\mathrm{uc}}$ goes down to $\bar{M}$ via the covering map. Hence, according to Definition 2, also $M \subset \bar{M}$ is symmetric. The same arguments apply to ambient space N^*. □

Next, we consider the Riemannian product $S^k \times S^l$ of two Euclidean unit-spheres with $k + l = n \geq 2$ and $k \leq l$. Set $0_k := (0, \ldots, 0) \in \mathbb{R}^k$. If we choose the metric on $G_2^+(\mathbb{R}^{n+2})$ in accordance with [17, Section 2] (cf. also Section 3 of this article), then

$$\tau : S^k \times S^l \to G_2^+(\mathbb{R}^{n+2}), \quad (p, q) \mapsto \{(p, 0_{l+1}), (0_{k+1}, q)\}_{\mathbb{R}}$$

defines an isometric 2-fold covering onto a totally geodesic submanifold of $G_2^+(\mathbb{R}^{n+2})$ (see [17, Section 2]). Hence every parallel submanifold of $S^k \times S^l$ is also parallel in $G_2^+(\mathbb{R}^{n+2})$. Further, the embedding ι: $S^k_{\sqrt{2}} \to S^k \times S^l$, $p \mapsto (p/\sqrt{2}, p/\sqrt{2})$ is an isometry onto its totally geodesic image.

[3]However, there is no duality between parallel submanifolds of N and N^*, respectively. This is due to the semi-parallelity condition on the second fundamental form (see (4) with $R = R^N$) which is not preserved if one multiplies R^N by the factor minus one.

Corollary 1 (Parallel submanifolds of $\mathrm{S}^k \times \mathrm{S}^l$). *Let M be a complete parallel submanifold of $\mathrm{S}^k \times \mathrm{S}^l$ with $\dim(M) \geq 2$. Then M is a product $M_1 \times M_2$ of two symmetric submanifolds $M_1 \subset \mathrm{S}^k$ and $M_2 \subset \mathrm{S}^l$, or conjugate to a symmetric submanifold of $\iota\big(\mathrm{S}^k_{\sqrt{2}}\big)$ via some isometry of $\mathrm{S}^k \times \mathrm{S}^l$. In the first case, M is extrinsically symmetric in $\mathrm{S}^k \times \mathrm{S}^l$. In the second case, M is not symmetric in $\mathrm{S}^k \times \mathrm{S}^l$ unless $k = l$ and $M \cong \iota\big(\mathrm{S}^k_{\sqrt{2}}\big)$. The analogous result holds for ambient spaces $\mathrm{H}^k \times \mathrm{H}^l$, the Riemannian product of two hyperbolic spaces of constant sectional curvature -1 (for $2 \leq k \leq l$), and $\mathbb{R} \times \mathrm{H}^l$, the Riemannian product of the real line and the hyperbolic space.*

Proof. Let M be a complete parallel submanifold of $\tilde{N} := \mathrm{S}^k \times \mathrm{S}^l$ through (p, q) with $\dim(M) \geq 2$. Then M is parallel also in $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ (via τ). Hence, according to Theorem 1 and its proof, the second osculating space $V := T_{(p,q)}M \oplus \perp^1_{(p,q)}M$ is a curvature invariant subspace of both $T_{\tau(p,q)}N$ and $T_{(p,q)}\tilde{N}$ such that M is contained in the totally geodesic submanifold $\bar{M} := \exp^N(V)$ as a full symmetric submanifold. Further, the curvature invariant space $T_{(p,q)}\tilde{N}$ is of Type $(\mathrm{tr}_{k,l})$. Thus, using the classification of curvature invariant subspaces in $T_{\tau(p,q)}N$ (see Theorem 5 below), we obtain that there are only two possibilities:

- We have $V = W_1 \oplus W_2$ where W_1 and W_2 are i- and j-dimensional subspaces of $T_p\mathrm{S}^k$ and $T_q\mathrm{S}^l$, respectively (Type $(\mathrm{tr}_{i,j})$). Hence, the totally geodesic submanifold $\bar{M}$ is the Riemannian product of the Euclidean unit-spheres S^i and S^j. If $i = j = 1$, then $\dim(\bar{M}) = 2$ and $M = \bar{M}$. Otherwise, at least one of the factors of $\bar{M}$ is a higher-dimensional Euclidean sphere. It follows from a result of H. Naitoh (see Theorem 4 below) that $M = M_1 \times M_2$ where $M_1 \subset \mathrm{S}^i$ and $M_2 \subset \mathrm{S}^j$ are symmetric submanifolds. Therefore, the product $M_1 \times M_2$ is symmetric in $\tilde{N}$.
- There exists an i-dimensional linear space $W_0' \subset T_p\mathrm{S}^k$ and some linear isometry I' defined from W_0' onto its image $I'(W_0') \subset T_q\mathrm{S}^l$ such that $V = \{(x, -I'x) \mid x \in W_0'\}$ (Type (tr_i')). Then, up to an isometry of $\tilde{N}$, we can assume that M is a complete parallel submanifold of $\iota\big(\mathrm{S}^k_{\sqrt{2}}\big)$, i.e. a symmetric submanifold. Further, it follows from Theorem 4 that M is not symmetric in $\tilde{N}$ unless $i = k = l$ and $M \cong \iota\big(\mathrm{S}^k_{\sqrt{2}}\big)$.

The hyperbolic case is handled in a similar way. Our result follows. □

Submanifolds in a product of two space forms where recently studied by B. Mendoca and R. Tojeiro [19]. By means of different methods they where able to prove a more general version of Corollary 1.

Recall that a submanifold $M \subset N$ is called *extrinsically homogeneous* if a suitable subgroup of the isometry group $\mathrm{I}(N)$ acts transitively on M. In [11, 12] we have dealt with the question whether a complete parallel submanifold of a symmetric space of compact or non-compact type is automatically extrinsically homogeneous. One can show that a generic extrinsic circle of any symmetric space of rank at least two is not extrinsically homogeneous. Further, if the rank of the ambient space N is exactly two (e.g. $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})$ or $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})^*$), then it follows a priori from [12, Corollary 1.4] that every complete parallel submanifold M is extrinsically homogeneous provided that the Riemannian space M does not split off (not even locally) a factor of dimension one or two (e.g. M is locally irreducible and $\dim(M) \geq 3$). Moreover, then M has even *extrinsically homogeneous holonomy bundle*. The latter means the following: there exists a subgroup $G \subset \mathrm{I}(N)$ such that $g(M) = M$ for every $g \in G$ and $G|_M$ is the group which is generated by the *transvections* of M. For ambient spaces $N = \mathrm{G}_2^+(\mathbb{R}^{n+2})$ and its non-compact dual, we obtain by means of our previous result:

Corollary 2 (Homogeneity of parallel submanifolds). *Every complete parallel submanifold of* $\mathrm{G}_2^+(\mathbb{R}^{n+2})$*, which is not a curve, has extrinsically homogeneous holonomy bundle. In particular, every such submanifold is extrinsically homogeneous in* $\mathrm{G}_2^+(\mathbb{R}^{n+2})$*. This result holds also for ambient space* $\mathrm{G}_2^+(\mathbb{R}^{n+2})^*$.

Proof. We can assume that $n \geq 2$. Let M be a complete parallel submanifold of $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ with $\dim(M) \geq 2$. Then there exists a totally geodesic submanifold $\bar{M} \subset N$ such that M is a symmetric submanifold of $\bar{M}$. In particular, $\bar{M}$ is intrinsically a symmetric space. Furthermore, since the rank of N is two, the rank of $\bar{M}$ is less than or equal to two. It follows immediately that there are no more than the following possibilities:

- The totally geodesic submanifold $\bar{M}$ is the 2-dimensional flat torus. Then we automatically have $M = \bar{M}$ (since $\dim(M) \geq 2$). Hence, we have to show that the totally geodesic flat $\bar{M}$ has extrinsically homogeneous holonomy bundle: let $\bar{\mathfrak{i}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$ and $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decompositions of the Lie algebras of $\mathrm{I}(\bar{M})$ and $\mathrm{I}(N)$, respectively. Then $[\bar{\mathfrak{p}}, \bar{\mathfrak{p}}] = \{0\}$, since $\bar{M}$ is flat. Let $\bar{G} \subset \mathrm{I}(\bar{M})$ denote the connected subgroup whose Lie algebra is $\bar{\mathfrak{p}}$. Then $\bar{G}$ is the transvection group of $\bar{M}$. Moreover, $\bar{\mathfrak{p}} \subset \mathfrak{p}$, because $\bar{M}$ is totally geodesic. Hence, we may take G as the connected subgroup of $\mathrm{I}(N)$ whose Lie algebra is $\bar{\mathfrak{p}}$.
- The totally geodesic submanifold $\bar{M}$ is locally the Riemannian product $\mathbb{R} \times \tilde{M}$ where $\tilde{M}$ is a locally irreducible symmetric space with $\dim(\tilde{M}) \geq 2$. Since $M \subset \bar{M}$ is symmetric, there exists a distinguished reflection $\sigma_p^\perp$ of $\bar{M}$ whose restriction to M is the geodesic reflection in p for every $p \in M$, see Definition 2. Therefore, these reflections generate a subgroup of $\mathrm{I}(\bar{M})$ whose connected component acts transitively on M and gives the full transvection group of M. Thus, it suffices to show that there exists a suitable subgroup of $\mathrm{I}(N)$ whose restriction to $\bar{M}$ is the connected component of $\mathrm{I}(\bar{M})$: let $\bar{\mathfrak{i}} = \bar{\mathfrak{k}} \oplus \bar{\mathfrak{p}}$, $\tilde{\mathfrak{i}} = \tilde{\mathfrak{k}} \oplus \tilde{\mathfrak{p}}$ and $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ denote the Cartan decompositions of the Lie algebras of $\mathrm{I}(\bar{M})$, $\mathrm{I}(\tilde{M})$ and $\mathrm{I}(N)$, respectively. Then $\bar{\mathfrak{k}} = \tilde{\mathfrak{k}} = [\tilde{\mathfrak{p}}, \tilde{\mathfrak{p}}] = [\bar{\mathfrak{p}}, \bar{\mathfrak{p}}]$, where the first and the last equality are related to the special product structure of $\bar{M}$ and the second one uses the fact that the Killing form of $\tilde{\mathfrak{i}}$ is non-degenerate. It follows that $\bar{\mathfrak{i}} = [\bar{\mathfrak{p}}, \bar{\mathfrak{p}}] \oplus \bar{\mathfrak{p}}$. Moreover, we have $\bar{\mathfrak{p}} \subset \mathfrak{p}$, see above. Hence, every Killing vector field of $\bar{M}$ is the restriction of some Killing vector field of N.
- The totally geodesic submanifold $\bar{M}$ is locally irreducible or locally the Riemannian product of two higher dimensional locally irreducible symmetric spaces: then we

have $\bar{\mathfrak{i}} = [\bar{\mathfrak{p}}, \bar{\mathfrak{p}}] \oplus \bar{\mathfrak{p}}$ because the Killing form of $\bar{\mathfrak{i}}$ is non-degenerate. Hence we can use arguments given in the previous case.
The hyperbolic case is handled in exactly the same way. Our result follows. □

2. Parallel submanifolds of symmetric spaces

We establish (or only rephrase) some general facts on parallel submanifolds of symmetric spaces. If possible, we also give alternative proofs of some results from [10, 11] which might fit better into the framework developed here.

First, we solve the existence problem for parallel submanifolds of symmetric spaces by means of giving necessary and sufficient tensorial "integrability conditions" on the 2-jet (see Theorem 2 and Remark 1).[4] It remains to find a way to make efficiently use of those conditions.

Thus, we will derive from the previous that $(T_pM, \perp^1_pM)$ is a curvature invariant pair for every parallel submanifold M. Further, we establish a property of the 2-jet of a parallel submanifold which is related to the linearized isotropy representation of the ambient space (see Theorem 3 and Corollary 4).

Moreover, we give two results on reduction of the codimension: for parallel submanifolds with curvature isotropic tangent spaces of maximal possible dimension in any symmetric space of compact or non-compact type (see Proposition 3 and Corollary 5) and for certain parallel submanifolds with 1-dimensional first normal spaces (see Proposition 4). Note, the first result is apparently new (whereas the second is somehow well known).

Finally, we recall a result of H. Naitoh on the classification of symmetric submanifolds in products of symmetric spaces (see Theorem 4).

2.1. Existence of parallel submanifolds in symmetric spaces. It was first shown by W. Strübing [25] that a (simply connected and complete) parallel submanifold M of an arbitrary Riemannian manifold N is uniquely determined by its 2-*jet* (T_pM, h_p) at some point $p \in M$. Conversely, let a prescribed 2-jet (W, h) at some point $p \in N$ be given (i.e. $W \subset T_pN$ is a subspace and $h\colon W \times W \to W^\perp$ is a symmetric bilinear map). If there exists some parallel submanifold through p whose 2-jet is given by (W, h), then the latter (or simply h) will be called *integrable*. Note, according to [13, Theorem 7], for every integrable 2-jet at p, there exists a unique simply connected and complete parallel submanifold through p having this 2-jet.

Let U be the subspace of $W^\perp$ which is spanned by the image of h and set $V := W \oplus U$, i.e. U and V play the roles of the "first normal space" and the "second osculating space", respectively. Then the orthogonal splitting $V := W \oplus U$ turns $\mathfrak{so}(V)$ into a naturally $\mathbb{Z}_2$-graded algebra $\mathfrak{so}(V) = \mathfrak{so}(V)_+ \oplus \mathfrak{so}(V)_-$ where $A \in \mathfrak{so}(V)_+$ or

[4] Note, such conditions were already claimed in [13]. However, the tensorial conditions stated in [13, Theorem 2] are not very handy.

$A \in \mathfrak{so}(V)_-$ according to whether A respects the splitting $V = W \oplus U$ or $A(W) \subset U$ and $A(U) \subset W$. Further, consider the linear map $\boldsymbol{h}\colon W \to \mathfrak{so}(T_pN)$ given by

$$\forall x, y \in W, \xi \in W^\perp\colon \boldsymbol{h}_x(y + \xi) = -S_\xi x + h(x, y) \tag{2}$$

(where S_ξ denotes the shape operator associated with h for every $\xi \in U$ in the usual way). Since $S_\xi = 0$ holds for every $\xi \in W^\perp$ which is orthogonal to U, we actually have

$$\forall x \in W\colon \boldsymbol{h}_x \in \mathfrak{so}(V)_-. \tag{3}$$

DEFINITION 3. Let a curvature like tensor R on T_pN and an R-invariant subspace W of T_pN (i.e. $R(W \times W \times W) \subset W$) be given. A symmetric bilinear map $h\colon W \times W \to W^\perp$ will be called *R-semi-parallel* if

$$\boldsymbol{h}_{R_{x,y}z - [\boldsymbol{h}_x, \boldsymbol{h}_y]z} v = [R_{x,y} - [\boldsymbol{h}_x, \boldsymbol{h}_y], \boldsymbol{h}_z]v \tag{4}$$

holds for all $x, y, z \in W$ and $v \in T_pN$. Here $R_{u,v}\colon T_pN \to T_pN$ denotes the *curvature endomorphism* $R(u, v, \cdot)$ for all $u, v \in T_pN$. If W is a curvature invariant subspace of T_pN and (4) holds for $R = R^N_p$, then h is simply called *semi-parallel*.

In the situation of Definition 3, it is easy to see that h is R-semi-parallel if and only if (4) holds for all $x, y, z \in W$ and $v \in V$.

Clearly, each linear map A on V induces an endomorphism $A\cdot$ on $\Lambda^2 V$ by means of the usual rule of derivation, i.e. $A\cdot u \wedge v = Au \wedge v + u \wedge Av$. Let $(A\cdot)^k$ denotes the k-th power of $A\cdot$ on $\Lambda^2 V$. Similarly, $[A, \cdot]$ defines an endomorphism on $\mathfrak{so}(V)$ whose k-th power will be denoted by $[A, \cdot]^k$. Furthermore, every curvature like tensor $R\colon T_pN \times T_pN \times T_pN \to T_pN$ can be seen as a linear map $R\colon \Lambda^2 T_pN \to \mathfrak{so}(V)$ characterized by $R(u \wedge v) = R_{u,v}$. The following theorem states the necessary and sufficient “integrability conditions”:[5]

Theorem 2. *Let N be a symmetric space. The 2-jet (W, h) is integrable if and only if the following conditions together hold*:

- *W is a curvature invariant subspace of T_pN,*
- *h is semi-parallel,*
- *we have*

$$[\boldsymbol{h}_x, \cdot\,]^k R^N_{y,z} v = R^N((\boldsymbol{h}_x \cdot)^k y \wedge z)v \tag{5}$$

for all $x, y, z \in W$, $k = 1, 2, 3, 4$ and each $v \in V$.

[5] This result and the following remark were also obtained in an unpublished paper by E. Heintze.

Proof. In order to apply the main result of [13], consider the space $\mathfrak{C}$ of all curvature like tensors on T_pN and the affine subspace $\tilde{\mathfrak{C}} \subset \mathfrak{C}$ which consists, by definition, of all curvature like tensors R on T_pN such that W is R-invariant and h is R-semi-parallel. Then we define the one-parameter subgroup $R_x(t)$ of curvature like tensor on T_pN characterized by

$$\exp(t\boldsymbol{h}_x)R_x(t)(u, v, w) = R^N(\exp(t\boldsymbol{h}_x)u, \exp(t\boldsymbol{h}_x)v, \exp(t\boldsymbol{h}_x)w) \tag{6}$$

for all $u, v, w \in T_pN$ and $x \in W$. According to [13, Theorem 1] and [13, Remark 2], the 2-jet (W, h) is integrable if and only if $R_x(t) \in \tilde{\mathfrak{C}}$ for all $x \in W$ and $t \in \mathbb{R}$ (since R^N is a parallel tensor). Moreover, if (W, h) is integrable, then, by considering explicitly the corresponding parallel submanifold of N, one can show that $R_x(t)(y, z, v)$ is constant in t for all $x, y, z \in W$ and $v \in V$ (combine [10, Example 3.7 (a)] with [10, Lemma 3.8 (b)]). Conversely, if $R^N_p \in \tilde{\mathfrak{C}}$ and $R_x(t)(y, z, v)$ is constant in t for all $x, y, z \in W$ and $v \in V$, then $R_x(t)$ in $\tilde{\mathfrak{C}}$ for all t for simple reasons.

Let us first assume that (W, h) is integrable. Then the previous implies that

$$\exp(t\boldsymbol{h}_x)R^N_{y,z} \exp(-t\boldsymbol{h}_x)v = R^N_{\exp(t\boldsymbol{h}_x)y,\exp(t\boldsymbol{h}_x)z}\, v \tag{7}$$

Taking the derivatives up to k-th order of (7) with respect to t, we now see that (5) holds for all $k \geq 1$.

Conversely, suppose that $R^N_p \in \tilde{\mathfrak{C}}$ holds. It suffices to show that (5) implies that the function $t \mapsto R_x(t)(y, z, v)$ is constant for all $x, y, z \in W$ and $v \in V$:

Put $A := \boldsymbol{h}_x$, set $\Sigma := \sum_{i=0}^{3}(A\,\cdot)^i(\Lambda^2 W)$ and note that

$$A \cdot y \wedge z = Ay \wedge z + y \wedge Az, \tag{8}$$

$$(A\,\cdot)^2 y \wedge z = A^2 y \wedge z + 2Ay \wedge Az + y \wedge A^2 z, \tag{9}$$

$$(A\,\cdot)^3 y \wedge z = A^3 y \wedge z + 3A^2 y \wedge Az + 3Ay \wedge A^2 z + y \wedge A^3 z, \tag{10}$$

$$(A\,\cdot)^4 y \wedge z = A^4 y \wedge z + 4A^3 y \wedge Az + 6A^2 y \wedge A^2 z + 4Ay \wedge A^3 z + y \wedge A^4 z \tag{11}$$

for all $y, z \in W$. Since $A^2(W) \subset W$, we hence see that $(A\,\cdot)^4(\Lambda^2 W) \subset \Lambda^2 W + (A\,\cdot)^2(\Lambda^2 W)$. Therefore, $A \cdot \Sigma \subset \Sigma$ and, furthermore, since (5) holds for $k = 1, 2, 3, 4$, the natural map $\Lambda^2 T_pN \to \mathfrak{so}(T_pN)$, $u \wedge v \mapsto R^N_{u,v}$ restricts to an equivariant linear map $\Sigma \to \mathfrak{so}(V)$, $\lambda \mapsto R^N(\lambda)|_V$ with respect to the linear actions of the 1-dimensional Lie algebra $\mathbb{R}$ induced by $A\,\cdot$ and $[A, \cdot\,]$ on Σ and $\mathfrak{so}(V)$, respectively. Switching to the level of one-parameter subgroups, we obtain that $R_x(t)(\lambda)v$ is constant in t for all $\lambda \in \Sigma$ and $v \in V$, in particular $R_x(t)(y, z, v)$ is constant in t for all $x, y, z \in W$, $v \in V$. □

In fact, there exist "seemingly more" necessary integrability conditions:

REMARK 1. In the situation of Theorem 2, suppose that (W, h) is integrable. Then we even have

$$[\boldsymbol{h}_{x_1}, \ldots [\boldsymbol{h}_{x_k}, R^N_{y,z}] \cdots]|_V = R^N_{\boldsymbol{h}_{x_1} \cdot \ldots \cdot \boldsymbol{h}_{x_k} \cdot y \wedge z}|_V \tag{12}$$

for all $x_1, \ldots, x_k, y, z \in W$ (possibly with $x_i \neq x_j$) for $k = 1, 2, \ldots$.

Proof. For $k = 1$, this is simply (7). However, for $k \geq 2$, there does not seem to exist a way to deduce (12) very easily from Theorem 2. Rather, we consider again the corresponding parallel submanifold. By means of considering suitable k-times broken geodesics on the submanifold emanating from p, then adapting the ideas from [10, Example 3.7] and finally using [10, Lemma 3.9 (b)], we can show that

$$g \circ R^N_{y,z} \circ g^{-1} = R^N_{gy,gz} \tag{13}$$

holds on V where $g := \exp(t_1 \boldsymbol{h}_{x_1}) \circ \cdots \circ \exp(t_k \boldsymbol{h}_{x_k})$ for arbitrary $(t_1, \ldots, t_k) \in \mathbb{R}^k$ (one should note that (7) does not imply (13) since, say for $k = 2$, $\exp(t_2 \boldsymbol{h}_{x_2})y$ or $\exp(t_2 \boldsymbol{h}_{x_2})z$ might not be elements of W). Considering g as a function of $(t_1, \ldots, t_k)$ and taking the partial derivatives at zero, $\partial/\partial t_1 \cdots \partial/\partial t_k|_{t_1 = \cdots = t_k = 0}$ of (13), we immediately see that (12) holds. □

2.2. Curvature invariant pairs. The first crucial concept of this article is the following:

DEFINITION 4. Let linear spaces W and U both contained in $T_p N$ be given. We call (W, U) a *curvature invariant pair* if

$$R^N(W \times W \times W) \subset W \quad \text{and} \quad R^N(W \times W \times U) \subset U, \tag{14}$$

$$R^N(U \times U \times U) \subset U \quad \text{and} \quad R^N(U \times U \times W) \subset W. \tag{15}$$

In particular, then both W and U are curvature invariant subspaces of $T_p N$. If additionally $W \perp U$ holds, then (W, U) is called an *orthogonal* curvature invariant pair.

If $U = W^\perp$, then (W, U) is an orthogonal curvature invariant pair if and only if both W and U are curvature invariant subspaces. But if U is strictly contained in $W^\perp$, then the previous definition requires more.

We obtain the first obstruction against the existence of a parallel submanifold with prescribed tangent- and first normal spaces (cf. [10, Corollary 13]):

Proposition 1. *Let an integrable* 2*-jet* (W, h) *of the symmetric space* N *be given. Set* $U := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$. *Then* (W, U) *is an orthogonal curvature invariant pair.*

Proof. For (14): recall that W is a curvature invariant subspace of T_pN and that $h\colon W \times W \to W^\perp$ is semi-parallel according to Theorem 2. Then (14) immediately follows.

For (15): using (12) with $k = 2$, we obtain that

$$R^N_{h(x,x),h(y,y)}|_V = [\boldsymbol{h}_x, [\boldsymbol{h}_y, R^N_{x,y}|_V]] + R^N_{S_{h(x,y)}x,y}|_V + R^N_{x,S_{h(y,y)}x}|_V \tag{16}$$

for all $x, y \in W$ (since h is symmetric). By means of (14), we further have that $R^N_{x,y}(V) \subset V$ and $R^N_{x,y}|_V \in \mathfrak{so}(V)_+$. Using (3) and the rules for $\mathbb{Z}_2$-graded Lie algebras, we thus see that r.h.s. of (16) defines an element of $\mathfrak{so}(V)_+$, too, and so does l.h.s. Finally, because h is symmetric, $\Lambda^2(U) = \{h(x,x) \wedge h(y,y) \mid x, y \in W\}_{\mathbb{R}}$ holds. We conclude that $R^N_{\xi,\eta}(V) \subset V$ and $R^N_{\xi,\eta}|_V \in \mathfrak{so}(V)_+$ actually for all $\xi, \eta \in U$, i.e. (15) holds. This finishes our proof. □

An (orthogonal) curvature invariant pair (W, U) which is induced by some integrable 2-jet as in Proposition 1 is called *integrable*.

Furthermore, it is known that every complete parallel submanifold of a simply connected symmetric space whose normal spaces are curvature invariant is even a symmetric submanifold (cf. [1, Proposition 9.3]). Hence we see (cf. [10]):

Corollary 3. *Every* 1*-full complete parallel submanifold of a simply connected symmetric space is a symmetric submanifold.*

In order to determine the curvature invariant pairs involving a given curvature invariant subspace $W \subset T_pN$, note that

$$\mathfrak{h}_W := \{R^N_{x,y} \mid x, y \in W\}_{\mathbb{R}} \tag{17}$$

is a subalgebra of $\mathfrak{so}(T_pN)$ equipped with a natural representation on $W^\perp$ (by restriction). In view of (14), we aim to determine the $\mathfrak{h}_W$-invariant subspaces of $W^\perp$. For this, let us recall some basics from linear algebra: given an irreducible representation ρ of some real Lie algebra $\mathfrak{h}$ on the Euclidean space V via skew-symmetric endomorphisms, the representation is called *real* if V is irreducible even over $\mathbb{C}$, *complex* if the complexified space $V \otimes \mathbb{C}$ decomposes into two non-isomorphic $\mathfrak{h}$-modules and *quaternionic* otherwise. Note that in the complex or quaternionic case, there exists uniquely the underlying structure of a unitary or quaternionic Hermitian space on V such that $\rho(\mathfrak{h}) \subset \mathfrak{u}(V)$ or $\rho(\mathfrak{h}) \subset \mathfrak{sp}(V)$, respectively. Conversely, if $\rho(\mathfrak{h}) \subset \mathfrak{sp}(V)$, then ρ is quaternionic; furthermore, $\rho(\mathfrak{h}) \subset \mathfrak{u}(V)$ implies that ρ is not real.

Next we consider the $(k+1)$-fold orthogonal direct sum of V, i.e. the Euclidean space $\tilde{V} := \bigoplus_{i=0}^k V$. Then $\mathfrak{h}$ acts on $\tilde{V}$ via skew-symmetric endomorphisms, too. Further, the irreducible $\mathfrak{h}$-invariant subspaces of $\tilde{V}$ are parameterized by the real projective space $\mathbb{R}\mathrm{P}^k$ (if V is real), the complex projective space $\mathbb{C}\mathrm{P}^k$ (if V is complex) or the quaternionic projective space $\mathbb{H}P^k$ (otherwise). More precisely, let $\lambda_i := V \hookrightarrow \tilde{V}, v \mapsto (0, \dots, \underbrace{v}_{i}, \dots, 0)$ be the canonical embedding onto the i-th factor, choose $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ according to the type of V and set $\lambda_c := \sum_{j=0}^{k} c_j \lambda_j$ for every $c = (c_0, \dots, c_k) \in \mathbb{K}^{k+1}$. Then $\lambda_c(V)$ is an irreducible $\mathfrak{h}$-invariant subspace of $\tilde{V}$. Using Schur's Lemma, this gives the claimed parameterization.

Finally, for any representation ρ of some real Lie algebra $\mathfrak{h}$ on the Euclidean space V via skew-symmetric endomorphisms, there is still some orthogonal decomposition into $\mathfrak{h}$-irreducible subspaces $V = \bigoplus_{i=1}^{k} V_i$. Moreover, after a permutation of the index set, there exists some $r \geq 1$ and a sequence $1 = k_1 < k_2 < \cdots < k_{r+1} = k + 1$ such that $V_{k_i} \cong V_{k_i+1} \cong \cdots \cong V_{k_{i+1}-1}$ for $i = 1, \dots, r$ but V_{k_i} is not isomorphic to V_{k_j} for $i \neq j$. Hence, there is also the decomposition $V = \bigoplus_{i=1}^{r} \mathbf{V}_i$ with $\mathbf{V}_i := V_{k_i} + V_{k_i+1} + \cdots + V_{k_{i+1}-1}$. Using again Schur's Lemma, we see that every irreducible $\mathfrak{h}$-invariant subspace $U \subset V$ is contained in a unique $\mathbf{V}_i$. Further, we can apply the previous in order to describe the $\mathfrak{h}$-irreducible subspaces of $\mathbf{V}_i$.

EXAMPLE 3 (Curvature invariant pairs of $\mathbb{C}P^n$). Consider the complex projective space $N := \mathbb{C}P^n$ of constant holomorphic sectional curvature four. Its curvature tensor is given by $R^N_{u,v} = -u \wedge v - Ju \wedge Jv - 2\omega(u, v)J$ for all $u, v \in T_p\mathbb{C}P^n$ (where J denotes the complex structure of T_pN and $\omega(u, v) := \langle Ju, v\rangle$ is the Kähler form). The curvature invariant subspaces of T_pN are known to be precisely the totally real and the complex subspaces. Let us determine the orthogonal curvature invariant pairs (W, U): if W is totally real, then $R^N_{x,y} = -x \wedge y - Jx \wedge Jy$ for all $x, y \in W$. Hence the Lie algebra $\mathfrak{h}_W$ (see (17)) is given by the linear space $\{x \wedge y + Jx \wedge Jy \mid x, y \in W\}_{\mathbb{R}}$. In the following, we assume that $\dim(W) \geq 2$. By definition of a totally real space W, there is the decomposition $W^\perp = JW \oplus (\mathbb{C}W)^\perp$ (here $(\mathbb{C}W)^\perp$ means the orthogonal complement of $\mathbb{C}W$ in T_pN). Then $\mathfrak{h}_W$ acts irreducibly on $J(W)$ and trivially on $(\mathbb{C}W)^\perp$. Further, Equation 14 shows that U is $\mathfrak{h}_W$-invariant. Considering also the decomposition of U into $\mathfrak{h}_W$-invariant subspaces, it follows that either $J(W) \subset U$ or $U \perp \mathbb{C}W$. In the first case, we claim that actually $U = J(W)$ (and hence $V := W \oplus U$ is a complex subspace of T_pN): let $\tilde{U} \subset (\mathbb{C}W)^\perp$ be chosen such that $U = JW \oplus \tilde{U}$. Clearly, U is not complex, hence U is necessarily totally real, because U is curvature invariant. Moreover, we have $\dim(U) \geq 2$, thus $\mathfrak{h}_U$ (defined as above) acts irreducibly on $J(U) = W \oplus J(\tilde{U})$. Since W is $\mathfrak{h}_U$-invariant (see (15)), we see that this is not possible unless $J(\tilde{U}) = \{0\}$. The claim follows.

In the second case, we claim that U is totally real (and thus V is totally real, too): in fact, otherwise U would be a complex subspace of $(\mathbb{C}W)^\perp$. Then the Lie algebra $\mathfrak{h}_U$ is given by $\mathbb{R}J \oplus \{\xi \wedge \eta + J\xi \wedge J\eta \mid \xi, \eta \in U\}_{\mathbb{R}}$. Thus $\mathfrak{h}_U$ acts on $U^\perp$ via $\mathbb{R}J$. Further, W is invariant under the action of $\mathfrak{h}_U$ according to (15) implying that W is complex, a contradiction. The claim follows.

Anyway, the linear space V is curvature invariant unless $\dim(W) = 1$. Therefore, by means of arguments given in the proof of Theorem 1, we see that every higher dimensional totally real parallel submanifold of $\mathbb{C}\mathrm{P}^n$ is a Lagrangian symmetric submanifold of some totally geodesically embedded $\mathbb{C}\mathrm{P}^k$ or a symmetric submanifold of some totally geodesically embedded $\mathbb{R}\mathrm{P}^k$.

If W is a complex subspace of $T_p\mathbb{C}\mathrm{P}^n$, then $\mathfrak{h}_W|_{W^\perp} = \mathbb{R}J|_{W^\perp}$. Hence, if (W,U) is an orthogonal curvature invariant pair, then both U and $V := W \oplus U$ are complex subspaces, too. This shows that every complex parallel submanifold of $\mathbb{C}\mathrm{P}^n$ is a complex symmetric submanifold of some totally geodesically embedded $\mathbb{C}\mathrm{P}^k$.

Note, several conclusions from the previous example can also be made by explicit calculations but without using the notion of curvature invariant pairs, cf. [23, Proposition 2.3], [23, Lemma 3.2] and [23, Lemma 4.1].

2.3. Further necessary integrability conditions. There remains the problem to decide on the integrability of a given orthogonal curvature invariant pair (W,U). Recall that integrability of (W,U) means by definition that there exists at least one integrable symmetric bilinear map $h\colon W \times W \to W^\perp$ such that $U = \{h(x,y) \mid x, y \in W\}_{\mathbb{R}}$. We will see below that there exist certain restrictions on any such h.

The first observation is the following: given a 2-jet (W,h) at p, set

$$\mathrm{Kern}(h) := \{x \in W \mid \forall y \in W\colon h(x,y) = 0\}. \tag{18}$$

Using (2), (3) and (27), we immediately see that

$$\mathrm{Kern}(h) = \{x \in W \mid \boldsymbol{h}(x) = 0\}. \tag{19}$$

Proposition 2. *Let N be a symmetric space and an integrable 2-jet (W,h) be given. Then* Kern(h) *is invariant under the action of $\mathfrak{h}_W$ on W.*

Proof. This follows from the curvature invariance of W, the symmetry of h and (4) (with $R = R^N$), cf. [21, Proof of Lemma 5.1]. □

Further, let K denote the isotropy subgroup of $\mathrm{I}(N)$ at some fixed point p, $\mathfrak{k}$ denote its Lie algebra and $\rho\colon \mathfrak{k} \to \mathfrak{so}(T_pN)$ be the linearized isotropy representation. Recall that

$$R^N_{u,v} \in \rho(\mathfrak{k}) \tag{20}$$

for all $u, v \in T_pN$ (since N is a symmetric space). For a symmetric submanifold with second fundamental form h at p, as mentioned in [22, p. 657], the image of $\boldsymbol{h}$ is contained in $\rho(\mathfrak{k})$. For a parallel submanifold, this is no longer true in general. Nevertheless, there is still some relation between $\boldsymbol{h}$ and $\rho(\mathfrak{k})$, as follows.

Given a 2-jet (W, h) at p, we set $U := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$, $V := W \oplus U$ and

$$\mathfrak{k}_V := \{X \in \mathfrak{k} \mid \rho(X)(V) \subset V\}. \tag{21}$$

Then ρ induces a representation of $\mathfrak{k}_V$ on V. Further, recall that the centralizer of a subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ is given by

$$Z(\mathfrak{g}) := \{A \in \mathfrak{so}(V) \mid \forall B \in \mathfrak{g}\colon [A, B] = 0\}. \tag{22}$$

Theorem 3. *Let N be a symmetric space and an integrable 2-jet (W, h) at p be given. Set $U := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$ and $V := W \oplus U$. Further, let $\rho\colon \mathfrak{k} \to \mathfrak{so}(T_pN)$ be the linearized isotropy representation. In the following, we view h as a linear map $\boldsymbol{h}\colon T_pM \to \mathfrak{so}(V)_-$ via* (2), (3).
(a) *For every $k \geq 0$ and $x_1, \ldots, x_k, y, z \in W$, the skew-symmetric endomorphism of T_pN given by*

$$[\boldsymbol{h}_{x_1}, [\ldots, [\boldsymbol{h}_{x_k}, R^N_{y,z}], \ldots]] \tag{23}$$

leaves V invariant. The so generated subalgebra of $\mathfrak{so}(V)$, denoted by $\mathfrak{g}$, is contained in $\rho(\mathfrak{k}_V)|_V$. Further, it bears the structure of a $\mathbb{Z}_2$-graded subalgebra of $\mathfrak{so}(V)$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ with $\mathfrak{g}_+ := \mathfrak{g} \cap \mathfrak{so}(V)_+$ and $\mathfrak{g}_- := \mathfrak{g} \cap \mathfrak{so}(V)_-$.
(b) *Set*

$$\mathfrak{h} := \mathfrak{h}_W|_V + \mathfrak{h}_U|_V. \tag{24}$$

Then $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}_+$.[6]
(c) *For every $x \in W$ there exist $A_x \in \mathfrak{g}_-$, $B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_-$ such that $\boldsymbol{h}_x = A_x + B_x$.*

Proof. For (a): since (W, U) is a curvature invariant pair, we have $R^N_{x,y}(V) \subset V$ for all $x, y \in W$ according to (14). Thus (23) leaves V invariant for $k = 0$ and then also for $k > 0$ because of (2). Further, we claim that $\mathfrak{g} \subset \rho(\mathfrak{k}_V)|_V$: because of (20), r.h.s. of (12) belongs to $\rho(\mathfrak{k}_V)|_V$ and so does l.h.s. Thus, the restriction to V of (23) belongs to $\rho(\mathfrak{k}_V)|_V$ for every k, which gives our claim.

Furthermore, applying $[\boldsymbol{h}_x, \cdot\,]$ to (23) leaves the form of (23) invariant with the natural number k increased by one for every $x \in W$. Hence $[\boldsymbol{h}_x, \mathfrak{g}] \subset \mathfrak{g}$. Thus the restriction to V of (23) belongs to $\mathfrak{so}(V)_+$ or $\mathfrak{so}(V)_-$ according to whether k is even or odd, see (3) and (14). Therefore, we have $\mathfrak{g} = \mathfrak{g} \cap \mathfrak{so}(V)_+ \oplus \mathfrak{g} \cap \mathfrak{so}(V)_-$.

For (b): because (W, U) is a curvature invariant pair, it is easy to see that the linear space $\mathfrak{h}$ is actually a subalgebra of $\mathfrak{so}(V)_+$. Further, recall that the restriction to V of (23) belongs to $\mathfrak{g}_+$ if k is even. In particular, for $k = 0$, we see that $A(V) \subset V$

[6]It is actually true that $\mathfrak{h} = \mathfrak{g}_+$ holds, cf. [10, Proof of Theorem 5.2 (b)].

and $A|_V \in \mathfrak{g}_+$ for all $A \in \mathfrak{h}_W$. It remains to show that the same is true for each $A \in \mathfrak{h}_U$, which follows by means of (16) as in the proof of Proposition 1.

For (c): let A_x denote the orthogonal projection of $\boldsymbol{h}_x$ onto $\mathfrak{g}$ with respect to the positive definite symmetric bilinear form on $\mathfrak{so}(V)$ which is given by $-\mathrm{trace}(A \circ B)$ for all $A, B \in \mathfrak{so}(V)$. Since the splitting $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ is orthogonal and $\boldsymbol{h}_x \in \mathfrak{so}(V)_-$ holds, we immediately see that $A_x \in \mathfrak{so}(V)_-$ (cf. [11, Lemma 4.19]). Furthermore, using the invariance property of the trace form (i.e. $\mathrm{trace}([A, B] \circ C) = \mathrm{trace}(A \circ [B, C])$), we conclude from $[\boldsymbol{h}_x, \mathfrak{g}] \subset \mathfrak{g}$ that $B_x := \boldsymbol{h}_x - A_x$ centralizes $\mathfrak{g}$. It also follows that $B_x \in \mathfrak{so}(V)_-$. This proves the theorem. □

Thus, the Lie algebra $\mathfrak{g}$ from Theorem 3 (a) gives the link between the linear map $\boldsymbol{h}$ and the Lie algebra $\rho(\mathfrak{k})$. Further, note the Lie algebra $\mathfrak{h}$ defined in Part (b) of this theorem depends only on the orthogonal curvature invariant pair in question. Since $\mathfrak{h} \subset \mathfrak{so}(V)_+$, restricting the elements of $\mathfrak{h}$ to W or U defines representations of $\mathfrak{h}$ on W and U, respectively. Hence, we introduce the linear spaces of homomorphisms

(25) $\mathrm{Hom}(W, U) := \{\lambda : W \to U \mid \lambda \text{ is } \mathbb{R}\text{-linear}\},$

(26) $\mathrm{Hom}_{\mathfrak{h}}(W, U) := \{\lambda \in \mathrm{Hom}(W, U) \mid \forall A \in \mathfrak{h} : \lambda \circ A|_W = A|_U \circ \lambda\}.$

Recall that

$$\mathfrak{so}(V)_- \to \mathrm{Hom}(W, U),\ A \mapsto A|_W \tag{27}$$

is actually a linear isomorphism inducing an equivalence

$$Z(\mathfrak{h}) \cap \mathfrak{so}(V)_- \cong \mathrm{Hom}_{\mathfrak{h}}(W, U), \tag{28}$$

where $Z(\mathfrak{h})$ denotes the centralizer of $\mathfrak{h}$ in $\mathfrak{so}(V)$. Further, mapping λ to its adjoint λ^* defines an isomorphism

$$\mathrm{Hom}_{\mathfrak{h}}(W, U) \cong \mathrm{Hom}_{\mathfrak{h}}(U, W). \tag{29}$$

Corollary 4. *In the situation of* Theorem 3, *suppose additionally that* $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$. *Let* $\mathfrak{h}$ *be the Lie algebra defined in Part* (*b*) *of the theorem. Then*

$$\forall x \in W : h(x, \,\cdot\,) \in \mathrm{Hom}_{\mathfrak{h}}(W, U). \tag{30}$$

Proof. Consider the decomposition $\boldsymbol{h}_x = A_x + B_x$ described in Theorem 3 (c). Then, by means of Part (a) of the same theorem, $A_x \in \mathfrak{g}_- \subset \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$. Hence $\boldsymbol{h}_x = B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_- \subset Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$ according to Theorem 3 (b). We conclude that $h(x, \cdot) \in \mathrm{Hom}_{\mathfrak{h}}(W, U)$ for each $x \in W$ because of (2), (3), (27) and (28). This finishes our proof. □

"In practice", having an orthogonal curvature curvature invariant pair (W, U) described in a concrete way, it is not very difficult to determine the linear spaces $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ and $\mathrm{Hom}_{\mathfrak{h}}(W, U)$ explicitly. However, if W is curvature isotropic, then the last theorem and its corollary do not provide any further information. Thus, in the next Section, we will examine this particular situation in case $\dim(W) = \mathrm{rank}(N)$ (which is clearly sufficient for ambient rank-2 spaces).

2.4. Parallel submanifolds with curvature isotropic tangent spaces. Let N be a symmetric space of compact or non-compact type.

DEFINITION 5. (a) A linear subspace $W \subset T_pN$ is called *curvature isotropic* if the curvature endomorphism $R^N_{x,y}$ vanishes identically for all $x, y \in W$.
(b) The *rank* of N is the dimension of any maximal curvature isotropic subspace of T_pN.

Lemma 1. *Suppose that N is of compact or non-compact type. Let a linear subspace $W \subset T_pN$ be given. The following is equivalent*:
(a) *The linear space W is curvature isotropic.*
(b) *The sectional curvature of N vanishes on every 2-plane of W, i.e. $\langle R^N(x,y,y),x\rangle = 0$ for all $x, y \in W$.*

Proof. Let $\mathfrak{i} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of $\mathrm{I}(N)$ with respect to the base point p. Recall that N is of compact or non-compact type if and only if the Killing form of $\mathfrak{i}$ restricted to $\mathfrak{p}$ is negative or positive definite, respectively. Hence (b) $\Rightarrow$ (a) follows from [9, Chapter V, §3, Equation 1]. The direction (a) $\Rightarrow$ (b) is obvious. □

Given a submanifold M, we thus see that the sectional curvature of N vanishes identically on any 2-plane of T_pM for every $p \in M$ if and only if T_pM is curvature isotropic in T_pN for each p. In this situation, if we also assume that the dimension of M is equal to the rank of N, then D. Ferus and F. Pedit [8] have shown that M is intrinsically flat and hence called it a "curved flat".

Proposition 3. *Let N be a symmetric space of compact or non-compact type, (W, h) be an integrable 2-jet at p and set $U := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$. Suppose that $d := \dim(W)$ is equal to the rank of N and that W is a curvature isotropic subspace of T_pN. Then there exists an orthonormal basis $\{x_1, \ldots, x_d\}$ of W such that*

$$h(x_i, x_j) = 0 \quad \text{whenever} \quad i \neq j, \tag{31}$$

$$\eta_i := h(x_i, x_i) \quad \text{satisfies} \quad \langle \eta_i, \eta_j \rangle = 0 \text{ whenever } i \neq j, \tag{32}$$

$$R^N_{x_i,x_j} = R^N_{x_j,\eta_i} = R^N_{\eta_i,\eta_j} = R^N_{\eta_j,x_i} = 0 \quad \text{for all} \quad i \neq j. \tag{33}$$

In particular, both W and U are curvature isotropic.

Proof. Let a parallel submanifold M be given such that $T_pM = W$ and $h_p = h$. Since W is curvature isotropic, the sectional curvature of N vanishes on every 2-plane of T_pM and then even identically on any 2-plane of the parallel submanifold M (see [10, Proposition 3.14]), i.e. M is a curved flat. Thus, $R^M_{x,y} = 0$ for all $x, y \in T_pM$ according to [8], implying in the parallel case that $R^\perp_{x,y}\xi = 0$ for all $\xi \in U$ because of (4). Using the Equations of Gauß, Codazzi and Ricci for a parallel submanifold, i.e.

$$\forall x, y \in T_pM\colon\ R^N_{x,y} = R^M_{x,y} \oplus R^\perp_{x,y} + [\boldsymbol{h}_x, \boldsymbol{h}_y], \tag{34}$$

we obtain that $[\boldsymbol{h}_x, \boldsymbol{h}_y] = 0$ for all $x, y \in W$. Further, we claim that there exists an orthonormal basis $\{x_1, \dots, x_d\}$ of W such that (31), (32) hold: since $\{h_x \mid x \in W\}$ is a set of pairwise commuting, skew-symmetric operators which map W to U and vice versa, there exist an orthonormal basis $\{x_1, \dots, x_d\}$ of W and some $d_0 \le d$ such that $\mathrm{Kern}(h) = \{x_1, \dots, x_{d_0}\}_{\mathbb{R}}$ (see (18), (19)), an orthonormal basis $\{\xi_{d_0+1}, \dots, \xi_d\}$ of U and linear maps $\lambda_i\colon W \to \mathbb{R}$ such that $\boldsymbol{h}_x = \sum_{i=d_0+1}^d \lambda_i(x) x_i \wedge \xi_i$. Using the symmetry of h,

$$\lambda_j(x_i)\xi_j = \sum_{l=d_0+1}^{d} \lambda_l(x_i) x_l \wedge \xi_l(x_j) = h(x_i, x_j) = h(x_j, x_i) = \lambda_i(x_j)\xi_i.$$

It follows that $\lambda_i(x_j) = 0$ for $i \ne j$. This gives our claim.

Moreover, by means of (31), we have

$$\forall i \ne j\colon\ R^N_{x_i,\eta_j}|_V = R^N_{x_i,h(x_j,x_j)}|_V = -R^N_{h(x_j,x_i),x_j}|_V = 0$$

where the second equality uses (5) (with $k = 1$), i.e. the curvature endomorphism $R^N_{x_i,\eta_j}$ vanishes on V whenever $i \ne j$. Furthermore, (16) implies that then also $R^N_{\eta_i,\eta_j}$ vanishes on V. Using Lemma 1 once more, $R^N_{x_i,\eta_j}$ and $R^N_{\eta_i,\eta_j}$ both vanish on T_pN unless $i = j$. The result now follows. □

In the notation of Proposition 3, set $V_i := \{x_i, \eta_i\}_{\mathbb{R}}$ for $i = 1, \dots, d$. Note, (32) can be rephrased by saying that the linear spaces V_i are pairwise orthogonal and (33) means that $R^N(u, v) = 0$ holds whenever $(u, v) \in V_i \times V_j$ with $i \ne j$.

Lemma 2. *Let $\{V_i\}_{i=1,\dots,d}$ be a collection of pairwise orthogonal subspaces of T_pN such that $R^N_{u,v} = 0$ whenever $(u,v) \in V_i \times V_j$ with $i \ne j$. Then there exist pairwise orthogonal curvature invariant subspaces of T_pN, denoted by $\bar{V}_i$, such that*

$$V_i \subset \bar{V}_i \quad \textit{for} \quad i = 1, \dots, d, \tag{35}$$

$$R^N_{u,v} = 0 \quad \textit{whenever} \quad (u, v) \in \bar{V}_i \times \bar{V}_j \quad \textit{with} \quad i \ne j. \tag{36}$$

Moreover, then also the linear space

$$\bar{V} := \bigoplus_{i=1}^{d} \bar{V}_i \tag{37}$$

is a curvature invariant subspace of T_pN.

Proof. Consider collections $\{\bar{V}_i\}_{i=1,\dots,d}$ of pairwise orthogonal subspaces of T_pN with properties (35), (36). Such collections exist, since at least one is given by $\bar{V}_i := V_i$. Hence, for obvious reasons, there exists $\{\bar{V}_i\}_{i=1,\dots,d}$ which is maximal in the following sense: if $\{\tilde{V}_i\}_{i=1,\dots,d}$ is another collection of pairwise orthogonal subspaces of T_pN with properties (35), (36) and such that $\bar{V}_i \subset \tilde{V}_i$ for $i = 1, \dots, d$, then $\bar{V}_i = \tilde{V}_i$ holds for all i.

Suppose that $\{\bar{V}_i\}_{i=1,\dots,d}$ is maximal. We claim that the linear space $\bar{V}_i$ is curvature invariant in T_pN for $i = 1, \dots, d$: let i be arbitrary but fixed and $u_i, v_i, w_i \in \bar{V}_i$. Further, let j with $i \neq j$ and $w_j \in \bar{V}_j$. Then, using a symmetry of R^N,

$$\langle R^N(u_i, v_i, w_i), w_j\rangle = \langle R^N(w_i, w_j, u_i), v_i\rangle \overset{(36)}{=} 0.$$

Therefore, the linear space $\tilde{V}_i := \bar{V}_i + \mathbb{R}R^N(u_i, v_i, w_i)$ is contained in the orthogonal complement of $\bar{V}_j$, too. Further, note that

$$R^N(u_i, v_i, w_j) = -R^N(w_j, u_i, v_i) - R^N(v_i, w_j, u_i) \overset{(36)}{=} 0 + 0 = 0 \tag{38}$$

by the first Bianchi-identity. Thus, the Jacobi-identity for the Lie bracket on $\mathfrak{i}(N)$ shows that

$$R^N_{w_j, R^N(u_i,v_i,w_i)} = -R^N_{R^N(u_i,v_i,w_j),w_i} + [R^N_{u_i,v_i}, R^N_{w_j,w_i}] = 0 + 0 = 0.$$

Therefore, the curvature endomorphism $R^N_{u,v}$ vanishes whenever $(u, v) \in \tilde{V}_i \times \bar{V}_j$ for each j different from i. Consider the collection of linear spaces $\{\tilde{V}_j\}_{j=1,\dots,d}$ defined by $\tilde{V}_j := \bar{V}_j$ (for $j \neq i$) and $\tilde{V}_i := \bar{V}_i + \mathbb{R}R^N(u_i, v_i, w_i)$. By maximality of $\{\bar{V}_j\}_{j=1,\dots,d}$, we have $\tilde{V}_j = \bar{V}_j$ for all j. In particular, $\bar{V}_i = \tilde{V}_i$, i.e. $R^N(u_i, v_i, w_i) \in \bar{V}_i$. Since $u_i, v_i, w_i \in \bar{V}_i$ were chosen arbitrary, we see that $\bar{V}_i$ is curvature invariant. Letting i vary, we conclude that $\bar{V}_i$ is curvature invariant for $i = 1, \dots, d$.

Further, we claim that then also $\bar{V}$ is curvature invariant: let $u, v, w \in \bar{V}$ be given. We have to show that $R^N(u, v, w) \in \bar{V}$. For this, we can assume, by multilinearity of R^N, that each of these three vectors belongs to some $\bar{V}_i$. If $(u_i, v_j, w_k) \in \bar{V}_i \times \bar{V}_j \times \bar{V}_k$ with $i \notin \{j, k\}$, then $R^N(u_i, v_j, w_k) = R^N(w_k, u_i, v_j) = 0$ by means of (36) and hence also $R^N(v_j, w_k, u_i) = 0$ because of the first Bianchi-identity. Therefore, $R^N(u, v, w) = 0$ unless all three vectors u, v, w belong to the same $\bar{V}_i$ in which case $R^N(u, v, w) \in \bar{V}_i \subset \bar{V}$ by the curvature invariance of $\bar{V}_i$. □

Corollary 5. *In the situation of* Proposition 3, *let M be the simply connected complete parallel submanifold of N through p whose* 2*-jet is given by* (W, h). *Then M is contained in some totally geodesic submanifold $\bar{M} \subset N$ whose universal covering space $\bar{M}^{\mathrm{uc}}$ splits as a Riemannian product $\mathbb{R}^{d_0} \times M_{d_0+1} \times \cdots \times M_d$ where $d_0 := \mathrm{Kern}(h)$. Moreover, we have $M \cong \mathbb{R}^d$ and there exist extrinsic circles c_i into M_i such that the immersion of M into $\bar{M}$ is given by the product map $\mathrm{Id}_{\mathbb{R}^{d_0}} \times c_{d_0+1} \times \cdots \times c_d$ followed by the covering map $\bar{M}^{\mathrm{uc}} \to \bar{M}$. In particular, the parallel curved flat M^d with $d =$ $\mathrm{rank}(N) \geq 2$ is never full if N is simply connected and irreducible.*

Proof. Following the notation of Proposition 3, set $V_i := \{x_i, \eta_i\}_{\mathbb{R}}$ for $i = 1,\ldots,d$ such that $\{x_1, \ldots, x_{d_0}\}_{\mathbb{R}}$ is an orthonormal basis of $\mathrm{Kern}(h)$. By means of Lemma 2, there exist curvature invariant spaces $\bar{V}_i \subset T_pN$ which satisfy (35), (36) for $i = 1,\ldots,d$. Since $V_i = \mathbb{R}x_i$ is already curvature invariant for $i = 1,\ldots,d_0$, we can assume that $\bar{V}_i = V_i$ for $i \leq d_0$. Further, consider the totally geodesic submanifolds $\bar{M}_i := \exp^N(\bar{V}_i)$ and their universal covering spaces M_i. Thus, $M_i \cong \mathbb{R}$ for $i = 1, \ldots, d_0$ and $\mathbb{R}^{d_0} \times M_{d_0+1} \times \cdots \times M_d$ is the universal covering space of the totally geodesic submanifold $\bar{M} := \exp^N(\bar{V})$ according to (36), (37). Therefore, by means of (31), (32) we have $\mathcal{O}_pM = \bigoplus_{i=1}^d V_i \subset \bar{V}$ which implies that M is contained in $\bar{M}$ (reduction of the codimension). Further, let $c_i\colon \mathbb{R} \to M_i$ be the extrinsic circle with $\dot{c}_i(0) = x_i$ and $\nabla^{M_i}_\partial \dot{c}_i(0) = \eta_i$ for $i = d_0+1,\ldots,d$. Thus, the product map $\mathrm{Id}_{\mathbb{R}^{d_0}} \times c_{d_0+1} \times \cdots \times c_d$ followed by the covering map $\bar{M}^{\mathrm{uc}} \to \bar{M}$ defines an isometric immersion of $\mathbb{R}^d$ as a parallel submanifold of $\bar{M}$ whose 2-jet at 0 is identical with the 2-jet of M at p according to (31), (32). Our first assertion follows, since a simply connected complete parallel submanifold of $\bar{M}$ is uniquely determined by its 2-jet at one point. In particular, if already N is simply connected and irreducible and, moreover, M is full in N, then $M \cong \mathbb{R}$. □

2.5. Parallel submanifolds with 1-dimensional first normal spaces. For certain integrable 2-jets, one implicitly knows that the second osculating space is curvature invariant.

Proposition 4. *Let N be a symmetric space,* (W, h) *be an integrable* 2*-jet and $U := \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$. Assume that* $\dim(U) = 1$ *and* $\dim(W) \geq 2$. *Suppose additionally that $\mathfrak{h}_W$ acts irreducibly on W. Then $V := W \oplus U$ is a curvature invariant subspace of T_pN.*

Proof. Using Proposition 2, we obtain that $\mathrm{Kern}(h) = \{0\}$. Thus $\tilde{h}(x, y) := \langle h(x, y), \eta\rangle$ defines a non-degenerate bilinear form on W for any unit vector $\eta \in U$. Further, in view of Proposition 1, it remains to show that $R^N_{x,\eta}(V) \subset V$ holds. For this, we may proceed as in the proof of [1, Theorem 9.2.2]: we can assume that $x \neq 0$ in which case there exist $y, z \in W$ with $h(x, z) = \eta$ and $h(y, z) = 0$ (since $\tilde{h}$ is non-degenerate and $\dim(W) \geq 2$). Hence, using (5) with $k = 1$, we see that $R^N_{x,\eta} = [\boldsymbol{h}_z, R^N_{x,y}]$ holds on V. The result follows by means of (2) and the curvature invariance of W. □

2.6. Symmetric submanifolds of product spaces. In order to show that certain orthogonal curvature invariant pairs of $G_2^+(\mathbb{R}^{n+2})$ are not integrable (cf. the proof of Corollary 20), we will apply the following theorem to ambient space $S^k \times S^k$:

Theorem 4 (H. Naitoh). *Suppose that N is a simply connected symmetric space and that the de Rham decomposition of N has precisely two factors, $N = N_1 \times N_2$. If $M \subset N$ is a symmetric submanifold, then either $N_1 = N_2$ and $M = \{(p, g(p)) \mid p \in N_1\}$ where g is an isometry of N_1 (in particular, then M is totally geodesic) or M is a product $M_1 \times M_2$ of symmetric submanifolds $M_i \subset N_i$ for $i = 1, 2$.*

Proof. In case both factors of N are of compact type, we can immediately apply [23, Theorem 2.2]. In case both factors of N are of non-compact type, we use the duality between compact and non-compact spaces to pass to the previous case (note that the results of [23] are mainly based on [22, Lemma 1.1] which is preserved under duality). In the general case, we decompose $N \cong N_c \times N_{nc} \times N_e$ into its compact, non-compact and Euclidean factor (where one or more factors may be trivial) and show as in [23, p. 562/563] that M splits as the Riemannian product $M = M_c \times M_{nc} \times M_e$ of symmetric submanifolds $M_c \subset N_c$, $M_{nc} \subset N_{nc}$ and $M_e \subset N_e$, which finally establishes Theorem 4. □

3. Parallel submanifolds of $G_2^+(\mathbb{R}^{n+2})$

Let $n \geq 2$ and consider the simply connected compact $2n$-dimensional symmetric space $N := G_2^+(\mathbb{R}^{n+2})$ of rank two which is given by the oriented 2-planes of $\mathbb{R}^{n+2}$. In accordance with [17, Section 2], we choose the metric on N such that the shortest restricted root has length equal to one for $n \geq 3$ and $G_2^+(\mathbb{R}^4) \cong S^2_{\sqrt{2}} \times S^2_{\sqrt{2}}$. Further, set $T := T_pN$ for some fixed $p \in N$ and let $\rho\colon \mathfrak{k} \to \mathfrak{so}(T)$ be the linearized isotropy representation. Then one knows (cf. [14]):

- N is a Hermitian symmetric space. Hence, there exists a complex structure J^N compatible with the inner product on T. This turns T into a Hermitian vector space of complex dimension n.
- Recall that a real form $\mathfrak{R} \subset T$ is an n-dimensional real subspace of T with $\mathfrak{R} \perp \mathrm{i}\mathfrak{R}$. By a *circle of real forms* we mean the set $\{e^{\mathrm{i}\varphi}\mathfrak{R} \mid \varphi \in [0, 2\pi]\}$ defined by some real form $\mathfrak{R} \subset T$. As a special feature of $G_2^+(\mathbb{R}^{n+2})$, there exists a distinguished circle of real forms of T, denoted by $\mathcal{U}$, see [14, Section 3].
- Consider the orthogonal splitting $v = \Re(v) + \mathrm{i}\Im(v)$ into real and imaginary parts for every $v \in T$ depending on $\mathfrak{R} \in \mathcal{U}$. Then, for any $\mathfrak{R} \in \mathcal{U}$, the curvature tensor of N can be described via

$$\forall u, v \in T:\quad R^N_{u,v} = (\langle \Re(v), \Im(u)\rangle - \langle \Re(u), \Im(v)\rangle) J^N - \Re(u) \wedge \Re(v) - \Im(u) \wedge \Im(v). \tag{39}$$

This is consistent with [14, p.84, Equation (16)] (but there the inner product gets a factor 1/2).

- Further, let the Lie algebra $\mathfrak{so}(\mathfrak{R})$ act on T via $Av = A\mathfrak{R}(v) + \mathrm{i}A\mathfrak{I}(v)$. Then $\rho(\mathfrak{k}) = \{R^N_{u,v} \mid u, v \in T\}_{\mathbb{R}} = \mathbb{R}J^N \oplus \mathfrak{so}(\mathfrak{R})$ for any $\mathfrak{R} \in \mathcal{U}$.

Recall that a subspace $W \subset T$ is called curvature invariant if $R^N(x, y, z) \in W$ for all $x, y, z \in W$.

Theorem 5 (S. Klein). *For $N := \mathrm{G}_2^+(\mathbb{R}^{n+2})$ with $n \geq 2$, there are precisely the following curvature invariant subspaces of T:*

Type (c_k) *Let $\mathfrak{R} \in \mathcal{U}$ and a k-dimensional subspace $W_0 \subset \mathfrak{R}$ be given. Then $W := \mathbb{C}W_0$ is curvature invariant. Here we assume that $k \geq 1$.*

Type ($\mathrm{tr}_{k,l}$) *Let $\mathfrak{R} \in \mathcal{U}$ and an orthogonal pair of subspaces W_1, W_2 of $\mathfrak{R}$ be given. Then $W := W_1 \oplus \mathrm{i}W_2$ is curvature invariant. Here the dimensions k and l of W_1 and W_2, respectively, are supposed to satisfy $k + l \geq 2$.*

Type (c'_k) *Let $\mathfrak{R} \in \mathcal{U}$ and a subspace $W' \subset \mathfrak{R}$ equipped with a Hermitian structure I' be given. Then $W := \{x - \mathrm{i}I'x \mid x \in W'\}$ is curvature invariant. Here $k \geq 1$ denotes the complex dimension of (W', I').*

Type (tr'_k) *Let $\mathfrak{R} \in \mathcal{U}$, a subspace $W' \subset \mathfrak{R}$ equipped with a Hermitian structure I' and a real form W'_0 of the Hermitian vector space (W', I') be given. Then $W := \{x - \mathrm{i}I'x \mid x \in W'_0\}$ is curvature invariant. Here $k \geq 2$ denotes the dimension of W'_0.*

Type (ex_3) *Let $\mathfrak{R} \in \mathcal{U}$ and an orthonormal system $\{e_1, e_2\} \subset \mathfrak{R}$ be given. The 3-dimensional linear space $W := \{e_1 - \mathrm{i}e_2, e_2 + \mathrm{i}e_1, e_1 + \mathrm{i}e_2\}_{\mathbb{R}}$ is curvature invariant.*

Type (ex_2) (only for $n \geq 3$) *Let $\mathfrak{R} \in \mathcal{U}$ and an orthonormal system $\{e_1, e_2, e_3\} \subset \mathfrak{R}$ be given. The 2-dimensional linear space $W := \{2e_1 + \mathrm{i}e_2, e_2 + \mathrm{i}(e_1 + \sqrt{3}e_3)\}_{\mathbb{R}}$ is curvature invariant.*

Type (tr_1) *Let u be a unit vector of T. The 1-dimensional space $\mathbb{R}u$ is curvature invariant.*

For a proof see [14, Theorem 4.1]. The corresponding totally geodesic submanifolds are described in [14, Section 5] or [17, Section 2.1].

Our notation emphasizes that spaces of Types (c_k) and (c'_k) both are complex of dimension k over $\mathbb{C}$ and those of Types ($\mathrm{tr}_{k,l}$) and (tr'_k) are totally real of dimensions $k + l$ and k, respectively. The spaces of Types (ex_3) and (ex_2) are "exceptional" (in the sense that they do not occur in a series).

3.1. Curvature invariant pairs of $\mathrm{G}_2^+(\mathbb{R}^{n+2})$. In this section, we determine the orthogonal curvature invariant pairs of T. Note that (W, U) is a curvature invariant pair if and only if (U, W) has this property. Since Theorem 5 provides seven types of curvature invariant subspaces of T, there are $(7\cdot 8)/2 = 28$ possibilities to consider. Our approach is briefly explained as follows: given a curvature invariant subspace W of T, we will first determine the Lie algebra $\mathfrak{h}_W$ (see (17)) and the $\mathfrak{h}_W$-invariant subspaces of $W^{\perp}$. Second, we will determine those skew-symmetric endomorphisms of T which belong to $\rho(\mathfrak{k})$ and leave W invariant, see (20). Once this information is available for

Table 1. Orthogonal curvature invariant pairs of $G_2^+(\mathbb{R}^{n+2})$.

Type	Data	Conditions
(c_k, c_l)	$(\mathfrak{R}, W_0; \mathfrak{R}^*, U_0)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_0 \perp U_0$
$(tr_{i,j}, tr_{k,l})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = e^{i\varphi}\mathfrak{R}^*,\ W_1 \oplus W_2 \perp e^{i\varphi}(U_1 \oplus U_2)$
$(tr_{j,k}, tr_{l,j})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_1 = U_2,\ W_2 \perp U_1$
$(tr_{k,l}, tr_{l,k})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_1 = U_2,\ W_2 = U_1$
$(tr_{k,1}, tr_{1,l})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_1 \perp U_2,\ W_2 \perp U_2,\ W_1 \perp U_1$
$(tr_{k,1}, tr_{1,k})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_1 = U_2$
$(tr_{1,1}, tr_{1,1})$	$(\mathfrak{R}, W_1, W_2; \mathfrak{R}^*, U_2, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W_1 \perp U_1,\ W_2 \perp U_2$
$(tr_{k,l}, tr_1)$	$(\mathfrak{R}, W_1, W_2; u)$	$u \perp \mathbb{C}W_1 \oplus \mathbb{C}W_2$
$(tr_{k,1}, tr_1)$	$(\mathfrak{R}, W_1, W_2; u)$	$u \perp \mathbb{C}W_1,\ \Im(u) \perp W_2$
$(tr_{1,1}, tr_1)$	$(\mathfrak{R}, W_1, W_2; u)$	$\Re(u) \perp W_1,\ \Im(u) \perp W_2$
(c_k, c'_l)	$(\mathfrak{R}, W_0; \mathfrak{R}^*, U', I')$	$\mathfrak{R} = \mathfrak{R}^*,\ W_0 \perp U'$
(c'_k, c'_l)	$(\mathfrak{R}, W', I'; \mathfrak{R}^*, U', J')$	$\mathfrak{R} = \mathfrak{R}^*,\ W' \perp U'$
(c'_k, c'_k)	$(\mathfrak{R}, W', I'; \mathfrak{R}^*, U', J')$	$\mathfrak{R} = \mathfrak{R}^*,\ W' = U',\ I' = -J'$
(c'_1, tr_1)	$(\mathfrak{R}, W', I'; u)$	$u \in \overline{W}$
$(tr'_j, tr_{k,l})$	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U_1, U_2)$	$\mathfrak{R} = \mathfrak{R}^*,\ W' \perp U_1 \oplus U_2$
(tr'_k, tr'_l)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*,\ W' \perp U'$
(tr'_k, tr'_k)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*,\ W' = U',\ U'_0 = I'(W'_0),\ J' = I'$
(tr'_k, tr'_k)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*,\ W' = U',$ $U'_0 = \exp(\theta I')(W'_0),\ J' = -I'$
(tr'_2, tr'_2)	$(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$	$\mathfrak{R} = \mathfrak{R}^*,\ W' = U'$ and there exists $\tilde{J} \in SU(W', \tilde{I}) \cap \mathfrak{so}(W')$ such that $U'_0 = \tilde{J}(W'_0)$ and $J' = \tilde{J} \circ I' \circ \tilde{J}^{-1}$ $(*)$
(tr'_k, tr_1)	$(\mathfrak{R}, W', I', W'_0; u)$	$u \perp \mathbb{C}W'$
(ex_3, tr_1)	$(\mathfrak{R}, \{e_1, e_2\}; u)$	$u = \pm(1/\sqrt{2})(e_2 - ie_1)$
(tr_1, tr_1)	$(u; \xi)$	$u \perp \xi$

$(*)$ If W is of Type tr'_2 defined by $(\mathfrak{R}, W', I', W'_0)$, then a second Hermitian structure on W' is given by $\tilde{I} := e_1 \wedge e_2 + I'e_1 \wedge I'e_2$ for some orthonormal basis $\{e_1, e_2\}$ of W'_0.

curvature invariant spaces of Types x and y, we will determine all curvature invariant pairs of Type (x, y).

Lemma 3. *Let W be curvature invariant of Type* (c_k) *defined by the data* $(\mathfrak{R}, W_0)$.
(a) *We have*

$$\mathfrak{h}_W = \mathbb{R} J^N \oplus \mathfrak{so}(W_0). \tag{40}$$

(b) *A subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is a complex subspace.*
(c) *Let $A \in \mathfrak{so}(\mathfrak{R})$ and $a \in \mathbb{R}$ be given. The linear map $aJ^N + A$ leaves W invariant if and only if $A(W_0) \subset W_0$.*

Proof. By means of (39), the curvature endomorphism $R^N_{\mathrm{i}x,x}$ is given by J^N for every unit vector $x \in W_0$. Further, $R^N_{x,y} = R^N_{\mathrm{i}x,\mathrm{i}y} = -x \wedge y$ for all $x, y \in W_0$ and $R^N_{x,\mathrm{i}y} = 0$ if $x, y \in W_0$ with $\langle x, y\rangle = 0$. Part (a) follows. For (b), note that $\mathfrak{h}_W|_{W^\perp} = \mathbb{R} J^N|_{W^\perp}$. Part (c) is obvious. □

Corollary 6. *Let W and U be curvature invariant of Types* (c_k) *and* (c_l) *defined by the data* $(\mathfrak{R}, W_0)$ *and* $(\mathfrak{R}^*, U_0)$, *respectively. If $\mathfrak{R} = \mathfrak{R}^*$ and $W_0 \perp U_0$, then (W, U) is an orthogonal curvature invariant pair. Conversely, every orthogonal curvature invariant pair of Type* $(\mathrm{c}_k, \mathrm{c}_l)$ *can be obtained in this way.*

Proof. Using Lemma 3, the first part of the corollary is obvious. For the last assertion, since the linear space W is determined also by the tuple $(e^{\mathrm{i}\varphi}\mathfrak{R}, e^{\mathrm{i}\varphi} W_0)$ for all $\varphi \in \mathbb{R}$, we can assume that $\mathfrak{R} = \mathfrak{R}^*$. Thus the condition $W \perp U$ implies that $W_0 \perp U_0$. □

Corollary 7. *There are no orthogonal curvature invariant pairs of Types* $(\mathrm{c}_j, \mathrm{tr}_{k,l})$, $(\mathrm{c}_j, \mathrm{tr}'_k)$, $(\mathrm{c}_j, \mathrm{ex}_3)$, $(\mathrm{c}_j, \mathrm{ex}_2)$ *and* $(\mathrm{c}_j, \mathrm{tr}_1)$.

Proof. If W is of Type (c_j), then any $\mathfrak{h}_W$-invariant subspace of $W^\perp$ is complex according to Lemma 3 (b). Since spaces of Types $(\mathrm{tr}_{k,l})$, (tr'_k), (ex_3), (ex_2) and (tr_1) are not complex, this proves the result. □

Lemma 4. *Let W be of Type* $(\mathrm{tr}_{k,l})$ *defined by the data* $(\mathfrak{R}, W_1, W_2)$.
(a) *We have*

$$\mathfrak{h}_W = \mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2). \tag{41}$$

In particular, if $k = l = 1$, then W is curvature isotropic.
(b) *If $k, l \neq 1$, then a subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to* $\mathrm{i}W_1$, W_2, *a subspace of the orthogonal complement of* $\mathrm{i}W_1 \oplus W_2$, *or a sum of such spaces.*

If $k = 1$ and $l \geq 2$, then a subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to W_2, a subspace of $W_2^\perp$ or a sum of such spaces. If $k = l = 1$, then any subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant.
(c) *Let $A \in \mathfrak{so}(\Re)$ and $a \in \mathbb{R}$ be given. The linear map $a J^N + A$ leaves W invariant if and only if $a = 0$ and $A(W_i) \subset W_i$ for $i = 1, 2$.*

Proof. For (a), see the proof of Lemma 3. For (b), consider the decomposition $W^\perp = \mathrm{i}W_1 \oplus W_2 \oplus (\mathbb{C}W_1 \oplus \mathbb{C}W_2)^\perp$ into $\mathfrak{h}_W$-invariant subspaces. Then $\mathfrak{h}_W$ acts trivially on $(\mathbb{C}W_1 \oplus \mathbb{C}W_2)^\perp$ and irreducibly on both $\mathrm{i}W_1$ and W_2. In particular, the linear spaces $\mathrm{i}W_1$ and W_2 are trivial $\mathfrak{h}_W$-modules only if $k = 1$ or $l = 1$, respectively. Moreover, they are non-isomorphic $\mathfrak{h}_W$-modules unless $k = l = 1$. The result follows. Part (c) is straightforward. □

Corollary 8. *Let W and U be curvature invariant of Types $(\mathrm{tr}_{i,j})$ and $(\mathrm{tr}_{k,l})$ defined by the data $(\Re, W_1, W_2)$ and $(\Re^*, U_1, U_2)$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:*
- *the real number φ is chosen such that $\Re = e^{\mathrm{i}\varphi}\Re^*$ and $e^{\mathrm{i}\varphi}(U_1 \oplus U_2)$ belongs to the orthogonal complement of $W_1 \oplus W_2$;*
- $\Re = \Re^*$, $W_2 = U_1$ *and* $W_1 = U_2$;
- $\Re = \Re^*$, $W_2 \perp U_1$ *and* $W_1 = U_2$;
- $j = k = 1$, $\Re = \Re^*$, $W_1 \perp U_2$, $W_1 \perp U_1$ *and* $W_2 \perp U_2$;
- $j = k = 1$, $\Re = \Re^*$, $W_1 = U_2$;
- $(i, j) = (k, l) = (1, 1)$, $\Re = \Re^*$, $W_1 \perp U_1$ *and* $W_2 \perp U_2$.

Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{tr}_{i,j}, \mathrm{tr}_{k,l})$ can be obtained in this way.

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that they are curvature invariant pairs is verified by means of Lemma 4. Conversely, let us see that these conditions are also necessary: we have

$$u_1 = e^{-\mathrm{i}\varphi}e^{\mathrm{i}\varphi}u_1 = \cos(\varphi)e^{\mathrm{i}\varphi}u_1 - \mathrm{i}\sin(\varphi)e^{\mathrm{i}\varphi}u_1,$$
$$\mathrm{i}u_2 = e^{-\mathrm{i}\varphi}e^{\mathrm{i}\varphi}\mathrm{i}u_2 = \sin(\varphi)e^{\mathrm{i}\varphi}u_2 + \mathrm{i}\cos(\varphi)e^{\mathrm{i}\varphi}u_2,$$

with $e^{\mathrm{i}\varphi}u_1 \in \Re$ and $\mathrm{i}e^{\mathrm{i}\varphi}u_2 \in \mathrm{i}\Re$ for all $(u_1, u_2) \in U_1 \times U_2$. Thus, the condition $U \perp W$ implies that

$$0 = \langle x_1, u_1\rangle = \cos(\varphi)\langle x_1, e^{\mathrm{i}\varphi}u_1\rangle,$$
$$0 = \langle x_1, \mathrm{i}u_2\rangle = \sin(\varphi)\langle x_1, e^{\mathrm{i}\varphi}u_2\rangle,$$
$$0 = \langle \mathrm{i}x_2, u_1\rangle = -\sin(\varphi)\langle x_2, e^{\mathrm{i}\varphi}u_1\rangle,$$
$$0 = \langle \mathrm{i}x_2, \mathrm{i}u_2\rangle = \cos(\varphi)\langle x_2, e^{\mathrm{i}\varphi}u_2\rangle$$

for all $(x_1, x_2) \in W_1 \times W_2$ and $(u_1, u_2) \in U_1 \times U_2$. Hence, in case $\varphi \notin (\pi/2)\mathbb{Z}$, the condition $W \perp U$ necessarily implies that $e^{i\varphi}(U_1 \oplus U_2) \perp W_1 \oplus W_2$.

In case $\varphi \in (\pi/2)\mathbb{Z}$, interchanging, if necessary, U_1 with U_2, we can assume that $\mathfrak{R} = \mathfrak{R}^*$. Clearly, then $W_1 \perp U_1$ and $W_2 \perp U_2$ by the condition $W \perp U$. Further, suppose that $j \geq 2$. On the one hand, since (W, U) is a curvature invariant pair and $\mathfrak{h}_W \subset \mathfrak{so}(\mathfrak{R})$ by means of Lemma 4 (a), the linear space U_1 is an $\mathfrak{h}_W$-invariant subspace of $W_1^\perp \cap \mathfrak{R}$. Using Lemma 4 (b), we conclude that $U_1 \perp W_1 \oplus W_2$ or $U_1 = W_2 \oplus \tilde{U}$ for some $\tilde{U} \subset \mathfrak{R}$ which belongs to the orthogonal complement of $W_1 \oplus W_2$. We claim that the second possibility can not occur unless $\tilde{U} = \{0\}$: since (W, U) is a curvature invariant pair and $\mathfrak{h}_U \subset \mathfrak{so}(\mathfrak{R})$, the linear space iW_2 is an $\mathfrak{h}_U$-invariant subspace of $U^\perp \cap i\mathfrak{R}$. Moreover, the condition $U_1 = W_2 \oplus \tilde{U}$ implies that $k \geq j \geq 2$. Therefore, by means of Lemma 4 (b), we have $W_2 \perp U_1 \oplus U_2$ (which is clearly not given) or $W_2 = U_1 \oplus \tilde{W}$ for some $\tilde{W} \perp U_1 \oplus U_2$. Hence $U_1 = U_1 \oplus \tilde{W} \oplus \tilde{U}$, thus $\tilde{W} = \tilde{U} = \{0\}$.

We conclude that $U_1 \perp W_1 \oplus W_2$ or $U_1 = W_2$ unless $j = 1$. Similarly, we can show that $U_2 \perp W_1 \oplus W_2$ or $U_2 = W_1$ unless $i = 1$. Clearly, the same conclusions hold with the roles of W and U interchanged. This finishes the proof. □

Corollary 9. *Let W and U be curvature invariant of Types $(\mathrm{tr}_{k,l})$ and (tr_1) defined by the data $(\mathfrak{R}, W_1, W_2)$ and a unit vector $u \in T$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair:*

- *u belongs to the orthogonal complement of $\mathbb{C}W_1 \oplus \mathbb{C}W_2$;*
- *$l = 1$, $u \perp \mathbb{C}W_1$ and $\Im(u) \perp W_2$;*
- *$k = l = 1$, $\Re(u) \perp W_1$ and $\Im(u) \perp W_2$.*

Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{tr}_{k,l}, \mathrm{tr}_1)$ can be obtained in this way.

Proof. Note, the pair (W, U) is an orthogonal curvature invariant pair if and only if $u \in W^\perp$ and $\mathfrak{h}_W$ annihilates the vector u. If $k, l \neq 1$, this is equivalent to $u \perp \mathbb{C}W_1 \oplus \mathbb{C}W_2$ according to Lemma 4 (b). Further, we can assume that $k \geq l$. If $k \geq 2$ and $l = 1$, we use the same argument as before; however, now it is allowed that $\Re(u)$ has a component in W_2. In case $k = l = 1$, the Lie algebra $\mathfrak{h}_W$ is trivial and the only condition is $u \in W^\perp$. □

Lemma 5. *Let W be of Type (c'_k) determined by the data $(\mathfrak{R}, W', I')$. Further, let $\overline{W}$ denote its complex conjugate in T with respect to the real form $\mathfrak{R}$.*

(a) *We have*

$$\mathfrak{h}_W = \mathfrak{su}(W') \oplus \mathbb{R}(I' + kJ^N). \tag{42}$$

(b) *In case $k \geq 2$, a subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is equal to $\overline{W}$, a complex subspace of $(\mathbb{C}W')^\perp$ or a sum of such spaces. In case $k = 1$, the previous statement remains true if we replace the phrase "equal to $\overline{W}$" by "contained in*

$\overline{W}$". Anyway, the linear space $\overline{W}$ as well as any $\mathfrak{h}_W$-invariant subspace of $(\mathbb{C}W')^\perp$ is complex.
(c) *Let $a \in \mathbb{R}$ and $A \in \mathfrak{so}(\mathfrak{R})$. The linear map $aJ^N + A$ leaves W invariant if and only if $A(W') \subset W'$ and $A|_{W'} \in \mathfrak{u}(W', I')$.*

Proof. For (a), note that

$$R^N_{x-\mathrm{i}I'x,y-\mathrm{i}I'y} = -2\langle I'x, y\rangle J^N - x \wedge y - I'x \wedge I'y \tag{43}$$

for all $x, y \in W'$ because of (39). In particular,

$$R^N_{x-\mathrm{i}I'x,y-\mathrm{i}I'y} = -2J^N - 2x \wedge I'x \tag{44}$$

for every unit vector $x \in W'$ and $y = I'x$. Similarly, if $x, y \in W'$ are unit vectors with $\langle x, I'y\rangle = 0$, then

$$R^N_{x-\mathrm{i}I'x,y-\mathrm{i}I'y} = -x \wedge y - I'x \wedge I'y. \tag{45}$$

It follows from (44), (45) that $A \in \mathfrak{h}_W$ if and only if there exists some $B \in \mathfrak{u}(W', I')$ such that $A = -\mathrm{i}\,\mathrm{trace}_{\mathbb{C}}(B)J^N + B$ (where $\mathrm{trace}_{\mathbb{C}}(B)$ means the complex trace of B in (U', I')). Now (42) is straightforward.

For (b), note that

$$\forall x \in W'\colon \mathrm{i}(x \mp \mathrm{i}I'x) = \pm I'x + ix = \pm(I'x \mp \mathrm{i}I'I'x) = \pm I'(x \mp \mathrm{i}I'x), \tag{46}$$

hence $J^N|_W = I'|_W$ and $J^N|_{\overline{W}} = -I'|_{\overline{W}}$. In particular, the linear space $\overline{W}$ is a complex subspace of T. The fact that $I' = -J^N$ on $\overline{W}$ and part (a) together imply that $\mathfrak{h}_W|_{\overline{W}} = \{0\}$ for $k = 1$ and $\mathfrak{h}_W|_{\overline{W}} = \mathfrak{u}(W', I')$ for $k \geq 2$. Further, we have $\mathfrak{u}(W', I') = \mathfrak{u}(W', -I') \cong \mathfrak{u}(\overline{W})$ where the second equality uses (46). Therefore, the linear space $\overline{W}$ is an irreducible $\mathfrak{h}_W$-module of real dimension $2k$ for $k \geq 2$. Furthermore, the Lie algebra $\mathfrak{h}_W$ acts on $(\mathbb{C}W')^\perp$ via $\mathbb{R}J^N$. Part (b) easily follows.

For (c), recall that $J^N|_W = I'|_W$ according to (46). Thus W is actually complex and we can assume in the following that $a = 0$. Since

$$\forall x \in W'\colon A(x + \mathrm{i}I'x) = Ax + \mathrm{i}AI'x, \tag{47}$$

for all $A \in \mathfrak{so}(\mathfrak{R})$, we see that $A(W) \subset W$ if and only if $A(W') \subset W'$ and $A|_{W'} \in \mathfrak{u}(W', I')$. Part (c) follows. □

Corollary 10. *Let W and U be of Types (c_k) and (c'_l) determined by the data $(\mathfrak{R}, W_0)$ and $(\mathfrak{R}^*, U', I')$, respectively. If $\mathfrak{R} = \mathfrak{R}^*$ and $W_0 \perp U'$, then (W, U) is an orthogonal curvature invariant pair. Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{c}_k, \mathrm{c}'_l)$ can be obtained in this way.*

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemmas 3 and 5, parts (a) and (c). Conversely, let us see that the conditions are also necessary: here we can assume that $\Re = \Re^*$ (cf. the proof of Corollary 6). Since $U \perp W_0$,

$$0 = \langle u - \mathrm{i}I'u, x\rangle = \langle u, x\rangle \tag{48}$$

for all $u \in U'$ and $x \in W_0$, i.e. $W_0 \perp U'$. □

Corollary 11. *Let W and U be of Types (c'_k) and (c'_l) determined by the data $(\Re, W', I')$ and $(\Re^*, U', J')$, respectively. If one of the following conditions holds, then (W, U) is an orthogonal curvature invariant pair*:

- $\Re = \Re^*$, $U' = W'$ *and* $I' = -J'$;
- $\Re = \Re^*$ *and* $U' \perp W'$.

Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{c}'_k, \mathrm{c}'_l)$ can be obtained in this way.

Proof. Note, if $\Re = \Re^*$, $U' = W'$ and $I' = -J'$, then $U = \overline{W}$. Further, if $\Re = \Re^*$ and $U' \perp W'$, then $\mathbb{C}W' \perp \mathbb{C}U'$. Thus the fact that these are orthogonal curvature invariant pairs follows by means of Lemma 5 (b).

Conversely, the Hermitian structure I' extends to $W' \oplus \mathrm{i}W'$ (via complexification) and the linear space W is determined also by the data $(e^{\mathrm{i}\varphi}\Re, e^{\mathrm{i}\varphi}W', I'|_{e^{\mathrm{i}\varphi}W'})$. Hence, we can assume that $\Re = \Re^*$. In the following, we further suppose that $k \geq l$. If also $k \geq 2$, then by means of Lemma 5, either $U \perp \mathbb{C}W'$ or $U = \overline{W} \oplus \tilde{U}$ with $\tilde{U} \perp \mathbb{C}W'$. In the first case, obviously $W' \perp U'$. In the second case, we have $\tilde{U} = \{0\}$ (since $l \leq k$), i.e. $U = \overline{W}$.

In case $k = l = 1$, by means of Lemma 5 we have $U = \tilde{U} \oplus U^\#$ for some $\tilde{U} \subset \overline{W}$ and a complex subspace $U^\#$ of the orthogonal complement of $\mathbb{C}W'$. Thus, since $\dim(U) = \dim(\bar{W}) = 2$, either $U^\# = \{0\}$ (and hence $U = \tilde{U} = \bar{W}$) or $U = U^\#$. This finishes the proof. □

Corollary 12. (a) *There are no orthogonal curvature invariant pairs of Type $(\mathrm{c}'_j, \mathrm{tr}_{k,l})$.*

(b) *There are no curvature invariant pairs of Type $(\mathrm{c}'_k, \mathrm{tr}_1)$ for $k \geq 2$.*

(c) *Let W and U be of Types (c'_1) and (tr_1) determined by the data $(\mathcal{R}, W', I', W'_0)$ and a unit vector $u \in T$, respectively. Then (W, U) is a curvature-invariant pair if and only if $u \in \overline{W}$.*

Proof. For (a), let W and U be of Types (c'_j) and $(\mathrm{tr}_{k,l})$ defined by the data $(\Re, W', I')$ and $(\Re^*, U_1, U_2)$. Then we can assume that $\Re = \Re^*$, cf. the proof of

Corollary 6. Therefore, the condition $W \perp U$ implies that

$$0 = \langle u_1, x - \mathrm{i}I'x\rangle = \langle u_1, x\rangle, \tag{49}$$

$$0 = \langle \mathrm{i}u_2, I'x + \mathrm{i}x\rangle = \langle u_2, x\rangle \tag{50}$$

for all $(u_1, u_2) \in U_1 \times U_2$ and $x \in W'$. Thus U_1, U_2 and W' are mutually orthogonal subspaces of $\mathfrak{R}$. In particular, the linear space U is contained in the orthogonal complement of $\mathbb{C}W'$. We hence see by the $\mathfrak{h}_W$-invariance of U that the latter would be complex according to Lemma 5 (b), a contradiction.

For (b) and (c), according to Lemma 5 (b), the 1-dimensional subspace $\mathbb{R}u$ of $W^\perp$ is $\mathfrak{h}_W$ invariant if and only if $k = 1$ and $u \in \overline{W}$. □

Lemma 6. *Let W be of Type* (tr'_k) *determined by the data* $(\mathfrak{R}, W', I', W'_0)$.
(a) *The Lie algebra $\mathfrak{h}_W$ is given by*

$$\{A \in \mathfrak{u}(W', I') \mid A(W'_0) \subset W'_0\}. \tag{51}$$

(b) *An $\mathfrak{h}_W$-invariant subspace of $W^\perp$ is contained in the orthogonal complement of the complex space $\mathbb{C}W'$, belongs to a distinguished family of k-dimensional totally real subspaces of $\mathbb{C}W' \cap W^\perp$—which can be parameterized by the real projective space* $\mathbb{R}\mathrm{P}^2$ *(for $k \geq 3$) or the complex projective space* $\mathbb{C}\mathrm{P}^2$ *(for $k = 2$)—or is a direct sum of such spaces.*
(c) *Let $A \in \mathfrak{so}(\mathfrak{R})$, $a \in \mathbb{R}$ be given and set $B := aI' + A$. Then $B \in \mathfrak{so}(\mathfrak{R})$ holds and the linear map $aJ^N + A$ leaves W invariant if and only if $B(W'_0) \subset W'_0$ and $BI'x = I'Bx$ for all $x \in W'_0$.*

Proof. For (a), we use that the curvature endomorphism $R^N_{x-\mathrm{i}I'x,y-\mathrm{i}I'y}$ is given by $-x \wedge y - I'x \wedge I'y$ for all $x, y \in W'_0$ according to (45). For (b), in order to avoid any confusion in case $k = 2$ (see below), we temporarily drop the notation $\mathrm{i}x$ for $J^N x$ with $x \in T$. Thus, set $\lambda_0 x := x + J^N I'x$, $\lambda_1 x := J^N x$ and $\lambda_2 x := I'x$ for all $x \in W'_0$. Then λ_i is an isomorphism of $\mathfrak{h}_W$-modules defined from W'_0 onto $\overline{W}$, $J^N(W'_0)$ and $I'(W'_0)$, respectively. Therefore,

$$(W' \oplus J^N(W')) \cap W^\perp = \overline{W} \oplus J^N(W'_0) \oplus I'(W'_0) \tag{52}$$

is an orthogonal decomposition into three irreducible, pairwise equivalent $\mathfrak{h}_W$-modules each being isomorphic to W'_0. Moreover, we note that $\mathfrak{h}_W|_{W'_0} = \mathfrak{so}(W'_0)$. Hence the linear space W'_0 is an irreducible $\mathfrak{so}(W'_0)$-module even over $\mathbb{C}$ for $k \geq 3$. For $k = 2$, let $\{e_1, e_2\}$ be an orthonormal basis of W'_0 and consider the Hermitian structure on W' given by

$$\tilde{I} := e_1 \wedge e_2 + I'e_1 \wedge I'e_2. \tag{53}$$

Then $\tilde{I}$ extends to $W' \oplus J^N(W')$ (via complexification by J^N) such that $\mathfrak{h}_W \subset \mathfrak{u}(W' \oplus J^N(W'), \tilde{I})$. Further, then $\tilde{I}(W'_0) \subset W'_0$ and λ_i commutes with $\tilde{I}$ for $i = 0, 1, 2$. Therefore, as was mentioned in Section 2.2, there exists $(c_0 : c_1 : c_2) \in \mathbb{K}\mathrm{P}^2$ with $\mathbb{K} = \mathbb{R}$ (for $k \geq 3$) or $\mathbb{K} = \mathbb{C}$ (for $k = 2$) such that

$$U = \{c_0 x + c_2 I' x + J^N(c_0 I' x + c_1 x) \mid x \in W'_0\} \tag{54}$$

(where in case $k = 2$ multiplication with the complex numbers c_i is now defined via $\tilde{I}$). Part (b) follows.

For (c): since W is totally real and the complexification $W \oplus \mathrm{i}W$ is of Type (c'_k) defined by the data $(\mathfrak{R}, W', I')$, we have $J^N|_W = I'|_W$ in accordance with (46). In particular, the linear map $J^N - I'$ leaves W invariant, which reduces the question to the case $a = 0$. It remains to determine those $A \in \mathfrak{so}(\mathfrak{R})$ which leave the linear space W'_0 invariant and satisfy $AI'x = I'Ax$ for all $x \in W'_0$, i.e. those for which $A(W') \subset W'$, $A|_{W'} \in \mathfrak{u}(W')$ and $A(W'_0) \subset W'_0$ holds. This proves our result. □

Corollary 13. *Let W and U be of Types (tr'_j) and $(\mathrm{tr}_{k,l})$ defined by the data $(\mathfrak{R}, W', I', W'_0)$ and $(\mathfrak{R}^*, U_1, U_2)$, respectively. If $\mathfrak{R} = \mathfrak{R}^*$ and the linear space $U_1 \oplus U_2$ is contained in the orthogonal complement of W', then (W, U) is an orthogonal curvature invariant pair. Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{tr}'_j, \mathrm{tr}_{k,l})$ can be obtained in this way.*

Proof. Obviously, the pairs (W, U) mentioned above satisfy $W \perp U$. Further, the fact that these are curvature invariant pairs is verified by means of Lemmas 4 and 6, parts (a) and (c). Conversely, let us see that our conditions are also necessary: suppose that (W, U) is an orthogonal curvature invariant pair. Note that W is defined also by the data $(e^{\mathrm{i}\varphi}\mathfrak{R}, e^{\mathrm{i}\varphi}W', I', e^{\mathrm{i}\varphi}W'_0(-\varphi))$ with $W'_0(-\varphi) := \{\cos(\varphi)x - \sin(\varphi)I'x \mid x \in W'_0\}$ for every $\varphi \in \mathbb{R}$, hence we can assume that $\mathfrak{R} = \mathfrak{R}^*$. Since U is $\mathfrak{h}_W$-invariant, there exists a decomposition $U = U^\# \oplus \tilde{U}$ into $\mathfrak{h}_W$-invariant subspaces $U^\# \subset \mathbb{C}W' \cap W^\perp$ and $\tilde{U} \subset (\mathbb{C}W')^\perp$ according to Lemma 6 (b). We claim that the only possibilities are $U^\# = \{0\}$, $U^\# = \mathrm{i}W'_0$, $U^\# = I'(W'_0)$ or $U^\# = I'(W'_0) \oplus \mathrm{i}W'_0$: first, the condition $W \perp U$ implies that $0 = \langle u_1, x - \mathrm{i}I'x\rangle = \langle u_1, x\rangle$ for all $u_1 \in U_1$ and $x \in W'_0$. Hence $U_1 \subset W_0'^{\perp}$, thus $U_1 \cap W' \subset I'(W'_0)$. Similarly, we can show that $U_2 \cap W' \subset W'_0$. Thus, on the one hand,

$$U^\# = U \cap \mathbb{C}W' = U_1 \cap W' \oplus \mathrm{i}(U_2 \cap W') \subset I'(W'_0) \oplus \mathrm{i}W'_0. \tag{55}$$

Moreover, according to (51), each of the linear spaces W', U_1 and U_2 is invariant under the action of $\mathfrak{h}_W$. Therefore, on the other hand, since (52) gives a decomposition of $\mathbb{C}W' \cap W^\perp$ into irreducible $\mathfrak{h}_W$-modules, we conclude that $U_1 \cap W' \in \{\{0\}, I'(W'_0)\}$ and $U_2 \cap W' \in \{\{0\}, W'_0\}$. Our claim follows from (55).

Next, we claim that $U^\# = \{0\}$: assume, by contradiction, that $I'(W'_0) \subset U$. Since $\dim(W'_0) \geq 2$, there exists an orthonormal pair $x, y \in W'_0$. Then $\{I'x, I'y\} \subset U \cap \mathfrak{R} =$

U_1, hence $A := R^N_{I'x,I'y}$ leaves W invariant (since (W, U) is a curvature invariant pair). Further, by means of (39), we have $A = -I'x \wedge I'y$. It follows, in particular, that $A \in \mathfrak{so}(W')$ and $A|_{W_0'} = 0$. Therefore, applying Lemma 6 (c) (with $a = 0$), we obtain that $A = 0$ (since W_0' is a real form of (W', I')), a contradiction. A similar argument shows that neither $\mathrm{i}W_0'$ is contained in U. We conclude that $U^\# = \{0\}$, i.e. $U \perp \mathbb{C}W'$. Clearly, this implies that $U_1 \oplus U_2 \perp W'$, which finishes our proof. □

Spaces of Type (tr_k') are neither 1-dimensional nor do they contain any complex subspaces. Hence Lemma 5 (b) implies:

Corollary 14. *There are no orthogonal curvature invariant pairs (W, U) of Type $(\mathrm{tr}_k', \mathrm{c}_l')$.*

Corollary 15. *Let W and U be of Types (tr_k') and (tr_l') defined by the data $(\mathfrak{R}, W', I', W_0')$ and $(\mathfrak{R}^*, U', J', U_0')$, respectively. Further, in case $k = 2$, let $\{e_1, e_2\}$ be an orthonormal basis of W_0' and $\tilde{I}$ be the Hermitian structure of W' defined by* (53).

If $\mathfrak{R} = \mathfrak{R}^*$ and one of the following conditions holds, then (W,U) is an orthogonal curvature invariant pair:

- *we have $U' \perp W'$;*
- *$U' = W'$, $I' = J'$ and $U_0' = I'(W_0')$;*
- *$U' = W'$, $I' = -J'$ and $U_0' = \exp(\theta I')(W_0')$ for some $\theta \in \mathbb{R}$;*
- *$k = l = 2$, $U' = W'$ and there exists some $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{so}(W')$ such that $U_0' = \tilde{J}(W_0')$ and $J' = \tilde{J} \circ I' \circ \tilde{J}^{-1}$.*

Conversely, every orthogonal curvature invariant pair of Type $(\mathrm{tr}_k', \mathrm{tr}_l')$ can be obtained in this way.

Proof. In the one direction, in order to see that the given pairs (W,U) are actually curvature invariant, we proceed as follows: the case $U' \perp W'$ is handled by means of Lemma 6, (a) and (c). In the other cases, we have $U = J^N(W)$, $U = \exp(-\theta I')(\overline{W})$ or $U = \tilde{J}(W)$, respectively. If $U = \mathrm{i}W$, then

$$\mathfrak{h}_U = \{J^N \circ A \circ J^N \mid A \in \mathfrak{h}_W\} = \mathfrak{h}_W, \tag{56}$$

where the first equality is straightforward and the second uses that J^N commutes with any curvature endomorphism of T. If $U = \exp(-\theta I')(\overline{W})$, then

$$\mathfrak{h}_U = \{\exp(-\theta I') \circ A \circ \exp(\theta I') \mid A \in \mathfrak{h}_{\overline{W}}\} = \mathfrak{h}_{\overline{W}} = \mathfrak{h}_W, \tag{57}$$

where the first equality is again straightforward and the second as well as the last one follow immediately from Lemma 6 (a). If $k = 2$, then $\mathfrak{h}_W = \mathbb{R}\tilde{I}$ according to Lemma 6 (a) and (53), hence, with $U = \tilde{J}(W)$,

$$\mathfrak{h}_U = \{\tilde{J} \circ A \circ \tilde{J} \mid A \in \mathfrak{h}_W\} = \mathbb{R}\tilde{J} \circ \tilde{I} \circ \tilde{J} \overset{\tilde{J}\in \mathrm{SU}(W',\tilde{I})}{=} \mathbb{R}\tilde{I} = \mathfrak{h}_W. \tag{58}$$

Clearly, if $\mathfrak{h}_W = \mathfrak{h}_U$, then (W, U) is a curvature invariant pair (by the curvature invariance of both W and U). This shows that the pairs in question are actually curvature invariant pairs.

It remains to verify that $U \perp W$. This is straightforward in case $U' \perp W'$. Further, we have $W \perp \mathrm{i}W$ (since W is totally real) and $e^{-\mathrm{i}\theta}\overline{W} \perp W$ for any θ (since even $\mathbb{C}\overline{W} \perp \mathbb{C}W$, see Corollary 11). If $k = 2$, then $f_1 := e_1$ and $f_2 := I'e_1$ defines a Hermitian basis of $(W', \tilde{I})$. Consider the complex matrix (g_{ij}) defined by

$$g_{ij} := \langle f_i, \tilde{J} f_j\rangle + \mathrm{i}\langle \tilde{I} f_i, \tilde{J} f_j\rangle: \tag{59}$$

Then (g_{ij}) belongs to $\mathrm{SU}(2) \cap \mathfrak{su}(2)$, hence there exist $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $t^2 + |w|^2 = 1$ such that

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \mathrm{i}t & -\bar{w} \\ w & -\mathrm{i}t \end{pmatrix} \tag{60}$$

holds. Using the skew-symmetry of $\tilde{J}$ and (60), we calculate

(61)
$$\langle e_i - J^N I'e_i, \tilde{J}(e_i - J^N I'e_i)\rangle = \langle e_i, \tilde{J}e_i\rangle + \langle I'e_i, \tilde{J} I'e_i\rangle = 0 + 0 = 0 \quad \text{for} \quad i = 1, 2,$$

(62)
$$\langle e_2 - J^N I'e_2, \tilde{J}(e_1 - J^N I'e_1)\rangle = \langle \tilde{I} f_1, \tilde{J} f_1\rangle + \langle \tilde{I} f_2, \tilde{J} f_2\rangle = \Im(g_{11} + g_{22}) = 0,$$

(63)
$$\langle e_1 - J^N I'e_1, \tilde{J}(e_2 - J^N I'e_2)\rangle = -\langle e_2 - J^N I'e_2, \tilde{J}(e_1 - J^N I'e_1)\rangle = 0.$$

This shows that $W \perp \tilde{J}(W)$.

In the other direction, let (W, U) be an orthogonal curvature invariant pair of Type $(\mathrm{tr}'_k, \mathrm{tr}'_l)$ defined by the data $(\mathfrak{R}, W', I', W'_0; \mathfrak{R}^*, U', J', U'_0)$. Then we can assume that $\mathfrak{R} = \mathfrak{R}^*$ (cf. the proof of Corollary 13). Clearly, we can also suppose that $l \leq k$. Therefore, since U is $\mathfrak{h}_W$-invariant with $\dim(U) \leq k$, either $U \perp \mathbb{C}W'$ or there exists $(c_0 : c_1 : c_2) \in \mathbb{K}\mathrm{P}^2$ with $\mathbb{K} = \mathbb{R}$ (for $k \geq 3$) or $\mathbb{K} = \mathbb{C}$ (for $k = 2$) such that U is given by r.h.s. of (54) according to Lemma 6 (b).

Suppose that $U \perp \mathbb{C}W'$. Then

$$0 = \langle u - J^N I'u, x\rangle = \langle u, x\rangle, \tag{64}$$

$$0 = \langle u - J^N I'u, J^N x\rangle = -\langle I'u, x\rangle \tag{65}$$

for all $u \in U'_0$ and $x \in W'$, i.e. we obtain that $W' \perp U'$.

We are left with the case that there exists $(c_0 : c_1 : c_2) \in \mathbb{K}\mathrm{P}^2$ such that U is given by r.h.s. of (54). In particular, then $U \subset \mathbb{C}W'$, i.e. $U'_0 \subset \mathfrak{R} \cap \mathbb{C}W' = W'$ and $J'(U'_0) \subset W'$, hence $U' = U'_0 \oplus J'(U'_0) = W'$ since automatically $k = l$ in this case. Furthermore, we claim that here $\mathfrak{h}_W = \mathfrak{h}_U$: given $A \in \mathfrak{h}_U$, by means of Lemma 6 (a), we have, in particular, $A \in \mathfrak{so}(W')$. Further, we have $A(W) \subset W$ since (W, U) is assumed to be a curvature invariant pair. Thus, we obtain from Lemma 6 (c) (with $a = 0$) that $A \in \mathfrak{u}(W, I')$ and $A(W_0') \subset W_0'$. Then $A \in \mathfrak{h}_W$ again by means of Lemma 6 (a). This shows that $\mathfrak{h}_U \subset \mathfrak{h}_W$ holds. The other inclusion is proved in a similar way. This gives our claim.

For $k \geq 3$, we claim that $U = \mathrm{i}W$ or $U = \mathrm{e}^{\mathrm{i}\theta}\overline{W}$ for some $\theta \in \mathbb{R}$: taking real and imaginary parts in (54), we obtain that

$$u := c_0 x + c_2 I' x \tag{66}$$

belongs to U_0' for every $x \in W_0'$ and

$$J' u = -c_1 x - c_0 I' x. \tag{67}$$

Moreover, any $u \in U_0'$ can be uniquely obtained from some $x \in W_0'$ via (66). Now assume that x is a unit vector. Then,

$$c_0^2 + c_2^2 \overset{(66)}{=} |u|^2 = |J'u|^2 \overset{(67)}{=} c_0^2 + c_1^2, \tag{68}$$

$$-c_0 c_1 - c_2 c_0 \overset{(66),(67)}{=} \langle u, J'u \rangle = 0. \tag{69}$$

Note, $c := c_0^2 + c_1^2$ does not vanish (since otherwise $c_0 = c_1 = c_2 = 0$ according to (68) which is not allowed). Thus, we can assume that $c = 1$ (because we consider only the ratio $(c_0 : c_1 : c_2)$). Then (68) implies

$$c_0^2 + c_2^2 = c_0^2 + c_1^2 = 1. \tag{70}$$

Therefore, by means of (69), (70), the real matrix (g_{ij}) defined by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} := \begin{pmatrix} c_0 & -c_1 \\ c_2 & -c_0 \end{pmatrix} \tag{71}$$

belongs to O(2). If $g \in \mathrm{SO}(2)$, then $c_0 = 0$ and $c_1 = c_2 = \pm 1$. Hence (66) and (67) together imply that $U_0' = I'(W_0')$ and $J' = I'$. Otherwise, there exists $\theta \in \mathbb{R}$ such that $c_0 = \cos(\theta)$ and $c_2 = -c_1 = \sin(\theta)$, thus $J' = -I'$ and $U_0' = \{\cos(\theta) + \sin(\theta) I' x \mid x \in W_0'\}$ according to (66), (67). This finishes the proof for $k \geq 3$.

For $k = 2$, we first recall that $\tilde{I}$ equips the linear space W' with a second Hermitian structure such that $\tilde{I}(W_0') \subset W_0'$ and I' belongs to $\mathrm{U}(W', \tilde{I})$. Now it is straightforward by means of (66), (67) that also $\tilde{I}(U_0') \subset U_0'$ and $J' \in \mathrm{U}(W', \tilde{I})$. Then it follows on the analogy of (68)–(70) that

$$|c_0|^2 + |c_2|^2 = |c_0|^2 + |c_1|^2 = 1, \tag{72}$$

$$-\bar{c}_0 c_1 - \bar{c}_2 c_0 = 0. \tag{73}$$

Thus, the complex matrix (g_{ij}) defined by (71) belongs to U(2). Moreover, since our considerations depend only on the complex ratio $(c_0 : c_1 : c_2)$, we can even assume that (g_{ij}) belongs to SU(2). Then necessarily $c_0 = -\bar{c}_0$ and $c_1 = \bar{c}_2$, hence (g_{ij}) takes the form (60) which implies that $(g_{ij}) \in \mathrm{SU}(2) \cap \mathfrak{su}(2)$. Further, recall that $f_1 := e_1$ and $f_2 := I' e_1$ defines a Hermitian basis of $(W', \tilde{I})$. Thus, we obtain a unique element of $\mathrm{SU}(W', \tilde{I}) \cap \mathfrak{so}(W')$ via $\tilde{J} f_i := g_{1i} f_1 + g_{2i} f_2$. Then, using the previous and (66), (67), we conclude that $U_0' = \tilde{J}(W_0')$ and $\tilde{J} \circ I' = J' \circ \tilde{J}$. The details of this part of the proof are left to the reader. □

Corollary 16. *Let W and U be of Types* (tr_k') *and* (tr_1) *defined by the data* $(\mathfrak{R}, W', I', W_0')$ *and a unit vector u of T, respectively. Then (W, U) is an orthogonal curvature invariant pair if and only if u belongs to* $(\mathbb{C}W')^\perp$.

Lemma 7. *Let W be of Type* (ex_3) *defined by the data* $(\mathfrak{R}, \{e_1, e_2\})$.
(a) *The Lie algebra $\mathfrak{h}_W$ is the linear space which is generated by $J^N + e_1 \wedge e_2$.*
(b) *A subspace of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is the* 1*-dimensional space* $\mathbb{R}(e_2 - \mathrm{i}e_1)$, *a complex subspace of the orthogonal complement of* $\{e_1, e_2\}_{\mathbb{C}}$, *or a sum of such spaces.*
(c) *Let $A \in \mathfrak{so}(\mathfrak{R})$ and $a \in \mathbb{R}$ be given. The linear map $aJ^N + A$ leaves W invariant if and only if $A - ae_1 \wedge e_2$ vanishes on* $\{e_1, e_2\}_{\mathbb{R}}$.

Proof. Consider the Hermitian structure $I' := e_1 \wedge e_2$ on $W' := \{e_1, e_2\}_{\mathbb{R}}$ and put $x_1 := e_1 - \mathrm{i}e_2$, $x_2 := e_2 + \mathrm{i}e_1$ and $x_3 := e_1 + \mathrm{i}e_2$. A straightforward calculation shows that $R^N_{x_1,x_3} = R^N_{x_2,x_3} = 0$. Further, let $\tilde{W}$ be the curvature invariant space of Type (c_1') defined by $(\mathfrak{R}, W', I')$. Thus $W = \tilde{W} \oplus \mathbb{R}x_3$, hence $\mathfrak{h}_W = \mathfrak{h}_{\tilde{W}}$. Now part (a) follows from Lemma 5 (a) (with $k = 1$). Clearly, the intersection $\mathbb{C}W' \cap W^\perp$ is given by $\mathbb{R}(e_2 - \mathrm{i}e_1)$. Thus part (b) follows from Lemma 5 (b) (with $k = 1$). For (c), since $J^N + I'$ leaves W invariant (by means of (a) and since W is curvature invariant), we can assume that $a = 0$. If A leaves W invariant, then $Ax_1 = Ae_1 - \mathrm{i}Ae_2$ is necessarily a linear combination of x_2 and x_3, say $Ax_1 = cx_2 + dx_3$. It follows that

$$d = \langle cx_2 + dx_3, e_1\rangle = \langle Ax_1, e_1\rangle = \langle Ae_1 - \mathrm{i}Ae_2, e_1\rangle = \langle Ae_1, e_1\rangle = 0,$$

hence $Ax_1 = cx_2$, i.e. $Ae_1 = ce_2$ and $Ae_2 = -ce_1$. Thus

$$W \ni Ax_3 = Ae_1 + \mathrm{i}Ae_2 = c(e_2 - \mathrm{i}e_1) \in W^\perp,$$

hence $Ax_3 \in W \cap W^\perp = \{0\}$. It follows that $c = 0$. This implies that $Ae_1 = Ae_2 = 0$ which proves our claim. □

Corollary 17. *Let W and U be of Types* (ex_3) *and* (tr_1) *defined by the data* $(\mathfrak{R}, \{e_1, e_2\})$ *and a unit vector u of T, respectively. Then (W, U) is an orthogonal curvature invariant pair if and only if* $u = \pm(1/\sqrt{2})(\mathrm{e}_2 - \mathrm{i}e_1)$.

Corollary 18. *There do not exist any orthogonal curvature invariant pairs of Types* $(\mathrm{ex}_3,\mathrm{c}'_k)$, $(\mathrm{ex}_3,\mathrm{tr}'_k)$, $(\mathrm{ex}_3,\mathrm{tr}_{k,l})$ *and* $(\mathrm{ex}_3,\mathrm{ex}_3)$.

Proof. Let W be of Type (ex_3) defined by the data $(\mathfrak{R},\{e_1,e_2\})$ and U be a subspace of $W^\perp$ such that (W,U) is a curvature invariant pair. Recall that the $\mathfrak{h}_W$-invariance of U implies that there is the splitting $U=\tilde{U}\oplus U^\#$ into a totally real space $U^\#\subset\mathbb{R}(e_2-\mathrm{i}e_1)$ and a complex subspace $\tilde{U}$ of the orthogonal complement of $\{e_1,e_2\}_{\mathbb{C}}$ according to Lemma 7 (b).

Hence, if U is of Type (c'_k) defined by the data $(\mathfrak{R}^*,U',I')$, then $U^\#=\{0\}$ (since U is complex) and thus $U\perp\{e_1,e_2\}_{\mathbb{C}}$. Further, we can assume that $\mathfrak{R}^*=\mathfrak{R}$. Thus $\{e_1,e_2\}_{\mathbb{R}}\perp U'$ (see (48)). Therefore, we obtain that $W\perp\mathbb{C}U'$ and whence the $\mathfrak{h}_U$-invariant space W is complex according to Lemma 5 (b), which is not given.

Furthermore, if U is of Type (tr'_k) or $(\mathrm{tr}_{k,l})$, then $\tilde{U}=\{0\}$ (since U is totally real) and hence U is at most 1-dimensional, which is not given.

If U is of Type (ex_3), too, defined by $(\mathfrak{R}^*,\{f_1,f_2\})$, then U is defined also by $(e^{\mathrm{i}\varphi}\mathfrak{R}^*,\{f_1(\varphi),f_2(\varphi)\})$ with $f_1(\varphi):=e^{\mathrm{i}\varphi}(\cos(\varphi)f_1+\sin(\varphi)f_2)$ and $f_2(\varphi):=e^{\mathrm{i}\varphi}(-\sin(\varphi)f_1+\cos(\varphi)f_2)$. Hence we can assume that $\mathfrak{R}=\mathfrak{R}^*$. Further, an orthogonal decomposition $U=U^\#\oplus\tilde{U}$ into a totally real subspace $U^\#$ and a complex subspace $\tilde{U}$ is unique (if it exists). We conclude that $U^\#=\mathbb{R}(\mathrm{i}f_2+f_1)$ and $\tilde{U}=\{f_1-\mathrm{i}f_2,f_2+\mathrm{i}f_1\}_{\mathbb{R}}$. Thus, on the one hand, $\{f_1,f_2\}_{\mathbb{R}}\perp\{e_1,e_2\}_{\mathbb{R}}$. On the other hand, $\mathrm{i}f_2+f_1=\pm(e_2-\mathrm{i}e_1)$, a contradiction. □

Lemma 8. *Let W be of Type* (ex_2) *defined by the data* $(\mathfrak{R},\{e_1,e_2,e_3\})$.
(a) *The Lie algebra $\mathfrak{h}_W$ is the linear space which is generated by* $J^N+e_1\wedge e_2+\sqrt{3}e_2\wedge e_3$.
(b) *A subspace U of $W^\perp$ is $\mathfrak{h}_W$-invariant if and only if it is the complex space* $\mathbb{C}(-e_1+\sqrt{3}e_3+2\mathrm{i}e_2)$, *belongs to a distinguished family of (real) 2-dimensional subspaces of the linear space*

$$\left\{2e_2+\mathrm{i}\left(-3e_1+\frac{1}{\sqrt{3}}e_3\right),\,e_1+\frac{5}{\sqrt{3}}e_3-2\mathrm{i}e_2\right\}_{\mathbb{R}}\oplus(\{e_1,e_2,e_3\}_{\mathbb{C}})^\perp,\tag{74}$$

or is a sum of such spaces.
(c) *Let $A\in\mathfrak{so}(\mathfrak{R})$ and $a\in\mathbb{R}$. The linear map aJ^N+A leaves W invariant if and only if* $A-a(e_1\wedge e_2+\sqrt{3}e_2\wedge e_3)$ *vanishes on* $\{e_1,e_2,e_3\}_{\mathbb{R}}$.

Proof. For (a), set $x_1:=2e_1+\mathrm{i}e_2$ and $x_2:=e_2+\mathrm{i}(e_1+\sqrt{3}e_3)$. A straightforward calculation shows that the curvature endomorphism $R^N_{x_1,x_2}$ is given by $-J^N-e_1\wedge e_2-\sqrt{3}e_2\wedge e_3$.

For (b), we first verify that the eigenvalues of $A:=R^N_{x_1,x_2}$ (seen as a complex-linear endomorphism of T) are given by $\{\mathrm{i},-\mathrm{i},-3\mathrm{i}\}$. The complex eigenspace for the eigenvalue $-3\mathrm{i}$ is a subspace of $W^\perp$, given by $\mathbb{C}(-e_1+\sqrt{3}e_3+\mathrm{i}2e_2)$. Furthermore,

$A^2 = -\mathrm{Id}$ on the $(2n-4)$-dimensional linear space (74), i.e. the linear map A defines a second complex structure on (74). This proves (b).

For (c): since W is curvature invariant, the endomorphism $J^N + e_1 \wedge e_2 + \sqrt{3} e_2 \wedge e_3$ leaves W invariant. This reduces the problem to the case $a = 0$. If $A(W) \subset W$, then $Ax_1 = cx_2$ and $Ax_2 = -cx_1$ for some $c \in \mathbb{R}$ (since A is skew-symmetric and $\|x_1\| = \sqrt{5} = \|x_2\|$). Considering the action of A on the real and imaginary parts of x_1 and x_2, respectively, this implies that $Ae_2 = -2ce_1$ and $Ae_2 = c(e_1 + \sqrt{3}e_3)$, a contradiction unless $c = 0$. Thus $Ax_1 = Ax_2 = 0$ and hence $A|_{\{e_1,e_2,e_3\}_{\mathbb{R}}} = 0$ since $A \in \mathfrak{so}(\mathfrak{R})$. This finishes the proof. □

Corollary 19. *If W is of Type* (ex$_2$), *then there are no orthogonal curvature invariant pairs* (W, U) *at all.*

Proof. Let $\mathfrak{R} \in \mathcal{U}$ and an orthonormal system $\{e_1, e_2, e_3\}$ of $\mathfrak{R}$ be given such that W is spanned by $x_1 := 2e_1 + \mathrm{i}e_2$ and $x_2 := e_2 + \mathrm{i}(e_1 + \sqrt{3}e_3)$. Suppose further, by contradiction, that there exists some curvature invariant subspace U of T such that (W, U) is an orthogonal curvature invariant pair.

For Type (c$_k$, ex$_2$), see Corollary 7. If U is of Type (c$'_k$) or (ex$_3$), then W is a 2-dimensional $\mathfrak{h}_U$-invariant subspace of $U^\perp$ but not a complex subspace of T according to Lemma 8 (c). However, this is not possible, because of parts (b) of Lemmas 5 and 7, respectively.

Now suppose that U is of Type (tr$_{i,j}$) determined by the data $(\mathfrak{R}^*, U_1, U_2)$. Using Lemmas 4 (c) and 8 (a), we see that $\mathfrak{h}_W(U) \subset U$ does not hold.

Similarly, the case that U is of Type (tr$_1$) can not occur.

Suppose that U is of Type (tr$'_k$) determined by the quadruple $(\mathfrak{R}^*, U', I', U'_0)$. Then we can assume that $\mathfrak{R} = \mathfrak{R}^*$. Using Lemma 6 (b), the fact that W is 2-dimensional linear subspace of T which is invariant under $\mathfrak{h}_U$ implies that either $W \subset \mathbb{C}U'^\perp$ or W is a 2-dimensional $\mathfrak{h}_U$-invariant subspace of $\mathbb{C}U'$.

In the first case, we have $\langle \Re(x_i), u\rangle = \langle \Im(x_i), u\rangle = 0$ for all $u \in U'$ and $i = 1, 2$. With $i = 1$, it follows that $\langle e_1, u\rangle = \langle e_2, u\rangle = 0$, then the previous with $i = 2$ implies that also $\langle e_3, u\rangle = 0$ for all $u \in U'$. Thus Lemma 8 (a) and the fact that $\mathfrak{h}_W(U) \subset U$ show that U is a complex subspace of T, a contradiction.

In the second case, we have $\dim(U'_0) = 2$, hence $\dim(U') = 4$. Further, both $\Re(x_i)$ and $\Im(x_i)$ belong to U' for $i = 1, 2$. Thus we conclude that $\{e_1, e_2, e_3\}_{\mathbb{R}} \subset U'$. Let $\{u_1, u_2\}$ be an orthonormal basis of U'_0. According to Lemma 6, the curvature endomorphism $R^N_{u-\mathrm{i}I'u,\xi-\mathrm{i}I'\xi}$ is given by $A := -u \wedge \xi - I'u \wedge I'\xi$ for all $u, \xi \in U$. Hence, since (W, U) is a curvature invariant pair, we obtain that $A(W) \subset W$. Using Lemma 8 (c) (with $a = 0$), we obtain that A vanishes on $\{e_1, e_2, e_3\}_{\mathbb{R}}$. Therefore, since $A \in \mathfrak{so}(U')$, the rank of A would be at most one, which is not possible unless $A = 0$, a contradiction.

Consider the case that U is of Type (ex$_2$), too. Then there exists some $\mathfrak{R}^* \in \mathcal{U}$ and an orthonormal system $\{f_1, f_2, f_3\}$ of $\mathfrak{R}^*$ such that U is spanned by $u_1 := 2f_1 + \mathrm{i}f_2$

and $u_2 := f_2 + \mathrm{i}(f_1 + \sqrt{3} f_3)$. Let φ be chosen such that $e^{\mathrm{i}\varphi}\mathfrak{R}^* = \mathfrak{R}$. In accordance with Lemma 8, the curvature endomorphism $R_{1,2} := R^N_{u_1,u_2}$ is given by $-J^N + A$ with $A := -f_1 \wedge f_2 - \sqrt{3} f_2 \wedge f_3$. We decompose $f_i = f_i^\top + f_i^\perp$ such that $f_i^\top \in e^{-\mathrm{i}\varphi}\{e_1, e_2, e_3\}_\mathbb{R}$ and $f_i^\perp \perp e^{-\mathrm{i}\varphi}\{e_1, e_2, e_3\}_\mathbb{R}$. Since $R_{1,2}(W) \subset W$, Lemma 8 (c) (with $a = -1$) shows that

$$e_1 \wedge e_2 + \sqrt{3} e_2 \wedge e_3 = f_1^\top \wedge f_2^\top + \sqrt{3} f_2^\top \wedge f_3^\top$$

(both sides seen as elements of $\mathfrak{u}(T)$). Comparing the length of the tensors on the left and right hand side above, we see that

$$\| f_1^\top \| = \| f_2^\top \| = \| f_3^\top \| = 1,$$

i.e. $e^{\mathrm{i}\varphi} f_i \in \{e_1, e_2, e_3\}_\mathbb{R}$ for $i = 1, 2, 3$. Hence we can assume that $n = 3$. Since U is $\mathfrak{h}_W$-invariant but not complex, it follows from Lemma 8 (b) that U is the linear space spanned by $\tilde{u}_1 := 2e_2 + \mathrm{i}(-3e_1 + (1/\sqrt{3})e_3)$ and $\tilde{u}_2 := e_1 + (5/\sqrt{3})e_3 - 2\mathrm{i}e_2$. A short calculation shows that the curvature endomorphism $R^N_{\tilde{u}_1,\tilde{u}_2}$ is given by $(8/3)J^N - 4(e_1 \wedge e_2 + \sqrt{3}e_2 \wedge e_3)$. Thus we obtain that $\mathfrak{h}_U$ does not leave W invariant. Therefore, (W, U) is not a curvature invariant pair. □

3.2. Integrability of the curvature invariant pairs of $\mathbf{G}_2^+(\mathbb{R}^{n+2})$. Let (W, U) be an orthogonal curvature invariant pair of $G_2^+(\mathbb{R}^{n+2})$ such that $\dim(W) \geq 2$. It remains the question whether (W, U) or (U, W) is integrable. By means of a case by case analysis of the possible pairs (see Table 1), we will show that the answer is "no" unless $V := W \oplus U$ is curvature invariant.

Let $\mathfrak{k}$ denote the isotropy Lie algebra of $N := G_2^+(\mathbb{R}^{n+2})$ and $\rho\colon \mathfrak{k} \to \mathfrak{so}(T)$ be the linearized isotropy representation. Recall that $\rho(\mathfrak{k}) = \mathbb{R} J^N \oplus \mathfrak{so}(\mathfrak{R})$. Further, by definition, the Lie algebra $\mathfrak{k}_V$ is the maximal subalgebra of $\mathfrak{k}$ such that $\rho(\mathfrak{k}_V)|_V$ is a subalgebra of $\mathfrak{so}(V)$, see (21).

Type ($\mathbf{c}_k, \mathbf{c}_l$). Let W and U be of Types (c_k) and (c_l) defined by the data $(\mathfrak{R}, W_0)$ and $(\mathfrak{R}^*, U_0)$, respectively. If (W, U) is a curvature invariant pair, then the only possibility is $\mathfrak{R} = \mathfrak{R}^*$ and $W_0 \perp U_0$. Then V is curvature invariant of Type (c_{k+l}) defined by the data $(\mathfrak{R}, W_0 \oplus U_0)$. Thus, there is nothing to prove.

Type ($\mathbf{tr}_{i,j}, \mathbf{tr}_{k,l}$). Let W and U be of Types ($\mathrm{tr}_{i,j}$) and ($\mathrm{tr}_{k,l}$) defined by the data $(\mathfrak{R}, W_1, W_2)$ and $(\mathfrak{R}^*, U_1, U_2)$, respectively. Let φ be chosen such that $\mathfrak{R} = e^{\mathrm{i}\varphi}\mathfrak{R}^*$. Substituting, if necessary, $\mathrm{i}\mathfrak{R}^*$ for $\mathfrak{R}^*$, we can assume that $\varphi \in [-\pi/4, \pi/4]$.

- Case $i = j = 1$. Suppose that (W, U) is integrable and let M be a simply connected complete parallel submanifold through p such that $T_pM = W$ and $\perp^1_p M = U$. Since W is curvature isotropic, we have $M \cong \mathbb{R}^2$ according to Corollary 5 and there exists some totally geodesic submanifold $\bar{M} \subset N$, a Riemannian splitting of its universal covering space $\bar{M}^{\mathrm{uc}} = M_1 \times M_2$ with simply connected factors M_i of dimension at least two and there exist extrinsic circles $c_i\colon \mathbb{R} \to M_i$ such that the isometric immersion of M into $\bar{M}$ is given by the product map $c_1 \times c_2$ followed by the covering map $\bar{M}^{\mathrm{uc}} \to \bar{M}$. In particular, $\dim(U) = 2$ and hence $k = l = 1$. Further, recall that $G_2^+(\mathbb{R}^4) \cong S^2_{\sqrt{2}} \times S^2_{\sqrt{2}}$ whereas the symmetric space $G_2^+(\mathbb{R}^{n+2})$ is irreducible if $n \geq 3$ (since then its root-system is of Type B_n, see [14]) and that any rank-one symmetric space is irreducible, too. Therefore, using the classification of totally geodesic submanifolds in N from [14, Section 5], the only possibilities are $\bar{M}^{\mathrm{uc}} = S^a \times S^b$ with $a, b \geq 2$ or $\bar{M} = \bar{M}^{\mathrm{uc}} = S^2_{\sqrt{2}} \times S^2_{\sqrt{2}}$ such that $T_p\bar{M}$ is curvature invariant of Types $(\mathrm{tr}_{a,b})$ or (c_2), respectively. In the first case, applying reduction of the codimension to each factor, we can even assume that $a = b = 2$. Therefore, $\dim(\bar{M}) = 4$ anyway and hence $V = T_p\bar{M}$ is curvature invariant of Types $(\mathrm{tr}_{2,2})$ or (c_2).

In the remaining cases, at least one of the indices $\{i, j\}$ is strictly greater than 1 and hence (possibly after substituting $\mathrm{i}\mathfrak{R}$ for $\mathfrak{R}$), we can suppose that $i \geq 2$. Then we have to consider the possibilities $\mathfrak{R} = \mathfrak{R}^*$ and $W_1 = U_2$, or $W_1 \perp e^{\mathrm{i}\varphi}U_2$.

- Case $i = l \geq 2$, $\mathfrak{R} = \mathfrak{R}^*$ and $W_1 = U_2$. Here we have $W_2 \perp U_1$, or $W_2 = U_1$, or $j = k = 1$. In case $W_2 = U_1$, the linear space V is curvature invariant of Type c_{k+l} defined by $(\mathfrak{R}, W_1 \oplus U_1)$. Otherwise, we claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$: let $a \in \mathbb{R}$, $B \in \mathfrak{so}(\mathfrak{R})$, set $A := aJ^N + B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$ holds. Then $A(W) \subset U$ and $A(U) \subset W$. We aim to show that $A = 0$. Let $x_2 \in W_2$. Thus $A\mathrm{i}x_2 \in U$. It follows that $ax_2 \in U_1$ and $Bx_2 \in U_2$. In the same way, $au_1 \in W_2$ and $Bu_1 \in W_1$ for all $u_1 \in U_1$. Hence $a = 0$, since $W_2 = U_1$ would be a different case. Further, setting $V_i := W_i \oplus U_i$ for $i = 1, 2$, we have $A|_{V_i} \in \mathfrak{so}(V_i)_-$. Since the maps $\mathfrak{so}(V_1)_- \to \mathrm{Hom}(W_1, U_1)$, $A \mapsto A|_{W_1}$ and $\mathfrak{so}(V_2)_- \to \mathrm{Hom}(U_2, W_2)$, $A \mapsto A|_{U_2}$ both are linear isomorphisms according to (27), for the vanishing of A it suffices to show that $A|_{W_1} = 0$ and $A|_{U_2} = 0$: on the one hand, $A(W_1) = A(U_2) \subset W_2$ since $A \in \mathfrak{so}(V_2)_-$. On the other hand, $A(W_1) \subset U_1$ because $A \in \mathfrak{so}(V_1)_-$. Hence $A(W_1) \subset W_2 \cap U_1$. Further, the linear space $W_2 \cap U_1$ is trivial if $W_2 \perp U_1$, or if $j = k = 1$ and $W_2 \neq U_1$. Therefore, $A|_{W_1} = 0$ unless $W_2 = U_1$. Similar considerations show that also $A|_{U_2} = 0$ unless $W_2 = U_1$. This establishes our claim.

Assume that $W_2 \neq U_1$ and, by contradiction, that (W, U) is integrable. Thus there exists an integrable symmetric bilinear map $h\colon W \times W \to W^\perp$ whose image spans U. Further, since the previous discussion shows that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$, Corollary 4 implies that h satisfies (30). According to Lemma 4, the Lie algebra $\mathfrak{h}$ (24) is given by $\mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2) \oplus \mathfrak{so}(U_1)$ (note, the last two summands are trivial in case $j = k = 1$). Anyway, the direct sum Lie algebra $\mathfrak{so}(W_2) \oplus \mathfrak{so}(U_1)$ gives the direct sum representation on $\mathrm{i}W_2 \oplus U_1$ whereas $\mathfrak{so}(W_1)$ acts diagonally on $W_1 \oplus \mathrm{i}W_1$ (i.e. $A(x_1 + \mathrm{i}y_1) = Ax_1 + \mathrm{i}Ay_1$ for all $A \in \mathfrak{so}(W_1)$ and $(x_1, y_1) \in W_1 \times W_1$). In particular, the induced action of $\mathfrak{so}(W_1)$ is non-trivial and irreducible on both W_1 and $\mathrm{i}W_1$ (since $i \geq 2$) and trivial on both $\mathrm{i}W_2$ and U_1. Therefore, Schur's Lemma implies that $\mathrm{Hom}_{\mathfrak{h}}(W, U) \subset \mathrm{Hom}_{\mathfrak{h}}(W_1, \mathrm{i}W_1) \oplus \mathrm{Hom}_{\mathfrak{h}}(\mathrm{i}W_2, U_1)$. We conclude from the previous that $h(x, y_1) \in \mathrm{i}W_1$ and $h(x, \mathrm{i}y_2) \in U_1$

for all $x \in W$, $y_1 \in W_1$ and $y_2 \in W_2$, hence

$$h(W_1 \times \mathrm{i}W_2) = h(\mathrm{i}W_2 \times W_1) \subset U_1 \cap \mathrm{i}W_1 = \{0\}, \tag{75}$$

$$\mathrm{i}W_1 = \{h(x_1, x_1) \mid x_1 \in W_1\}_{\mathbb{R}} \text{ and } U_1 = \{h(\mathrm{i}x_2, \mathrm{i}x_2) \mid x_2 \in W_2\}_{\mathbb{R}}. \tag{76}$$

We claim that $W_2 = \{0\}$: let $x_1, x_2 \in W_1 \times W_2$ with $x_1 \neq 0$. Then $R^N_{x_1,\mathrm{i}x_2} = 0$ (by the condition $W_1 \perp W_2$, see (39)) and hence (5) (with $k = 1$) yields

$$0 = \big[\boldsymbol{h}_{x_1}, R^N_{x_1,\mathrm{i}x_2}\big|_V\big] = R^N_{h(x_1,x_1),\mathrm{i}x_2}\big|_V + R^N_{x_1,h(x_1,\mathrm{i}x_2)}\big|_V \overset{(75)}{=} R^N_{h(x_1,x_1),\mathrm{i}x_2}\big|_V + 0. \tag{77}$$

Using (76), it follows that

$$0 = R^N_{\mathrm{i}x_1,\mathrm{i}x_2}|_V \overset{(39)}{=} -x_1 \wedge x_2 \tag{78}$$

for all $(x_1, x_2) \in W_1 \times W_2$. Thus (78) implies that $x_1 = 0$ or $x_2 = 0$ by the condition $W_1 \perp W_2$. Since $x_1 \neq 0$, this gives our claim.

But then also $U_1 = \{0\}$ by means of (76), hence $W_2 = U_1$, a contradiction.

- Case $i \geq 2$ and $W_1 \perp e^{\mathrm{i}\varphi} U_2$. The subcase $j = k \geq 2$, $\varphi = 0$ and $W_2 = U_1$ follows from the previous one (by means of interchanging W with U). In the remaining cases, we have $W_2 \perp e^{\mathrm{i}\varphi} U_1$, or $j = k = 1$. In case $\varphi = 0$ and $W_2 \perp U_1$, we obtain that V is curvature invariant of Type $(\mathrm{tr}_{i+k,j+l})$ defined by $(\Re, W_1 \oplus U_1, W_2 \oplus U_2)$. Otherwise, we claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$: let $a \in \mathbb{R}$, $B \in \mathfrak{so}(\Re)$ be given such that $A := aJ^N + B$ satisfies $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. Thus $Ax_1 \in U$ for every unit vector $x_1 \in W_1$, i.e. $Ax_1 = u_1 + \mathrm{i}u_2$ for suitable $u_1 \in U_1$ and $u_2 \in U_2$. Since

$$e^{\mathrm{i}\varphi}(Ax_1) = e^{\mathrm{i}\varphi}(a\mathrm{i}x_1 + Bx_1) = a\mathrm{i}\cos(\varphi)x_1 - a\sin(\varphi)x_1 + \cos(\varphi)Bx_1 + \mathrm{i}\sin(\varphi)Bx_1,$$

we see that

$$e^{\mathrm{i}\varphi}u_1 = \Re(e^{\mathrm{i}\varphi}(Ax_1)) = -a\sin(\varphi)x_1 + \cos(\varphi)Bx_1, \tag{79}$$

$$e^{\mathrm{i}\varphi}u_2 = \Im(e^{\mathrm{i}\varphi}(Ax_1)) = a\cos(\varphi)x_1 + \sin(\varphi)Bx_1. \tag{80}$$

The condition $W_1 \perp e^{\mathrm{i}\varphi} U_2$ implies that

$$0 = \langle x_1, e^{\mathrm{i}\varphi\rangle u_2} \overset{(80)}{=} a\cos(\varphi)\langle x_1, x_1\rangle + \sin(\varphi)\langle Bx_1, x_1\rangle = a\cos(\varphi),$$

since x_1 is a unit vector and $B \in \mathfrak{so}(\Re)$. Thus $a = 0$, because $\varphi \in [-\pi/4, \pi/4]$. Therefore, $A = B \in \mathfrak{so}(\Re)$ anyway. In particular,

$$\cos(\varphi)Ax_1 \overset{(79)}{=} e^{\mathrm{i}\varphi}u_1,$$

$$\sin(\varphi)Ax_1 \overset{(80)}{=} e^{\mathrm{i}\varphi}u_2.$$

We conclude that

$$0 = \langle u_1, u_2\rangle = \langle e^{i\varphi}u_1, e^{i\varphi}u_2\rangle = \sin(\varphi)\cos(\varphi)\langle Ax_1, Ax_1\rangle.$$

Hence $\varphi = 0$ or $Ax_1 = 0$ for all $x_1 \in W_1$. In the same way, we can show that $\varphi = 0$ or $Ax_2 = 0$ for all $x_2 \in W_2$. By means of (27), we conclude that $A|_V = 0$ unless $\varphi = 0$.

In case $\varphi = 0$ and $j = k = 1$, set $V_1 := W_1 \oplus U_1$ and $V_2 := W_2 \oplus U_2$. Note that $A|_{V_1} \in \mathfrak{so}(V_1)_-$ and $A|_{V_2} \in \mathfrak{so}(V_2)_-$. Thus, using that $W_1 \perp U_2$,

$$\langle Ax_1, x_2\rangle = -\langle x_1, Ax_2\rangle = 0$$

for all $x_1 \in W_1$ and $x_2 \in W_2$. Further, the linear form $\langle x_2, \cdot\,\rangle$ defines an isomorphism $U_1 \to \mathbb{R}$ for every $x_2 \in W_2$ which is not equal to zero (since $W_2 \perp U_1$ is a different case). Therefore, we conclude that $A|_{W_1} = 0$ and hence $A|_{V_1} = 0$, since $\mathfrak{so}(V_1)_- \to \mathrm{Hom}(W_1, U_1)$, $A \mapsto A|_{W_1}$ is a linear isomorphism according to (27). For the same reason, $A|_{U_2} = 0$ and hence $A|_{V_2} = 0$. We conclude that $A|_V = 0$. This establishes our claim.

Assume that one of the cases $W_2 \perp e^{i\varphi}U_1$ or $j = k = 1$, but not $\varphi = 0$ and $W_2 \perp U_1$ holds, and, by contradiction, that (W, U) is integrable. We have just seen that this implies that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$. Thus, there exists a symmetric bilinear map $h\colon W \times W \to U$ whose image spans U and which satisfies (30). Note, the Lie algebra $\mathfrak{h}$ defined in (24) is given by $\mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2) \oplus \mathfrak{so}(U_1) \oplus \mathfrak{so}(U_2)$ (in case $j = 1$ or $k = 1$ the second or the third summand, respectively, is trivial) and acts as a direct sum representation on $W_1 \oplus iW_2 \oplus U_1 \oplus iU_2$, where $\mathfrak{so}(W_1)$ acts non-trivially and irreducibly on W_1 anyway (since $i \geq 2$). Therefore, by means of Schur's lemma, $\mathrm{Hom}_{\mathfrak{h}}(W_1, U) = \{0\}$, i.e. $\mathrm{Hom}_{\mathfrak{h}}(W, U) \subset \mathrm{Hom}(iW_2, U)$. If $j \neq 1$, then we even have $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \{0\}$, hence $h = 0$ which is not possible. Otherwise, if $j = 1$, we thus see that $h(x, y) = h(y, x) = 0$ for all $x \in W_1$ and $y \in W$, i.e. $h(W \times W) = h(iW_2 \times iW_2)$ which spans a 1-dimensional space, a contradiction (since $k + l \geq 2$).

Type ($\mathbf{tr}_{k,l}, \mathbf{tr}_1$). Suppose that W is of Type ($\mathrm{tr}_{k,l}$) defined by the data $(\Re, W_1, W_2)$ and U is spanned by a unit vector u.

- Case $k = l = 1$. Similar as for Type ($\mathrm{tr}_{1,1}$, $\mathrm{tr}_{1,1}$), if (W, U) is integrable, then the corresponding simply connected parallel submanifold M is given by the product of the real line with an extrinsic circle in the totally geodesic Riemannian product space $\bar{M}^{\mathrm{uc}} = \mathbb{R} \times S^2$ or $\bar{M}^{\mathrm{uc}} = \mathbb{R} \times S^2_{\sqrt{2}}$ followed by a covering map onto some totally geodesic submanifold $\bar{M} \subset N$ such that $V = T_p\bar{M}$ is of Types ($\mathrm{tr}_{2,1}$) or (ex_3).
- Case $k \geq l$ with $k \geq 2$. Let us write $u = u_1 + iu_2$ with $u_1, u_2 \in \Re$. Then we have $u_1 \perp W_1$ and $u_2 \perp W_1 \oplus W_2$. Further, if $u_1 = 0$, or if $u_2 = 0$ and $u_1 \perp W_2$, then $W \oplus U$ is curvature invariant of Types ($tr_{k,l+1}$) or ($\mathrm{tr}_{k+1,l}$) defined by the triples $(\Re, W_1, W_2 \oplus \mathbb{R}u_2)$ or $(\Re, W_1 \oplus \mathbb{R}u_1, W_2)$, respectively. Otherwise, we claim that the linear space $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ is trivial: let $a \in \mathbb{R}$ and $B \in \mathfrak{so}(\Re)$ be given and suppose that $A := aJ^N \oplus B$ satisfies $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. Then there exists a linear

form λ on W_1 such that

$$\forall x_1 \in W_1 \colon Ax_1 = a\mathrm{i}x_1 + Bx_1 = \lambda(x_1)(u_1 + \mathrm{i}u_2). \tag{81}$$

Comparing the imaginary parts of the last equation, we obtain that $ax_1 = \lambda(x_1)u_2$ for all $x_1 \in W_1$, hence $a = 0$ (since $k \geq 2$). Thus there exists $\mu \colon W_2 \to \mathbb{R}$ such that

$$\forall x_2 \in W_2 \colon \mathrm{i}Bx_2 = \mu(x_2)(u_1 + \mathrm{i}u_2). \tag{82}$$

Comparing the real parts of the previous equation and recalling that $u_1 \neq 0$, we obtain that $\mu = 0$, i.e. $B|_{W_2} = 0$. Suppose now, by contradiction, that there exists $x_1 \in W_1$ with $Bx_1 \neq 0$. Then $\lambda(x_1) \neq 0$ and hence $u_2 = 0$ by means of (81). Further,

$$0 = \langle x_1, Bx_2 \rangle = -\langle Bx_1, x_2 \rangle \overset{(81)}{=} -\lambda(x_1)\langle u_1, x_2 \rangle \tag{83}$$

for all $x_2 \in W_2$. Since $\lambda(x_1) \neq 0$, we obtain that u_1 belongs to the orthogonal complement of W_2, i.e. we have shown that $u_2 = 0$ and $u_1 \perp W_2$, which is a different case. This proves our claim.

Assume that neither the case $u_1 = 0$ nor the case $u_2 = 0$ and $u_1 \perp W_2$ holds but, by contradiction, that there exists an integrable symmetric bilinear map h: $W \times W \to U$ whose image spans U. Since we have already shown that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$, Corollary 4 implies that h satisfies (30). Note, the Lie algebra $\mathfrak{h}$ defined in Corollary 4 is given by $\mathfrak{so}(W_1) \oplus \mathfrak{so}(W_2)$ where the first summand acts irreducibly and non-trivially on W_1 (since $k \geq 2$) and trivially on U. Hence $\mathrm{Hom}_{\mathfrak{h}}(W, U) \subset \mathrm{Hom}_{\mathbb{R}}(\mathrm{i}W_2, U)$ anyway. If $l \neq 1$, then $\mathrm{Hom}_{\mathfrak{h}}(W, U)$ is trivial, thus $h = 0$, a contradiction. This finishes the proof unless $l = 1$. Further, by means of (30), we obtain that

$$h(W_1 \times \mathrm{i}W_2) = h(\mathrm{i}W_2 \times W_1) = h(W_1 \times W_1) = \{0\}, \tag{84}$$

In case $l = 1$, using the previous equation, there exists $x_2 \in W_2$ such that

$$h(\mathrm{i}x_2, \mathrm{i}x_2) = u. \tag{85}$$

Therefore, by means of (5) and (85), we have for all $x_1 \in W_1$

$$R^N_{u,x_1}\big|_V = \big[\boldsymbol{h}_{\mathrm{i}x_2}, R^N_{\mathrm{i}x_2,x_1}\big|_V\big] - R^N_{\mathrm{i}x_2,h(\mathrm{i}x_2,x_1)}\big|_V = 0 - 0$$

according to (84) and since $R^N_{x_1,\mathrm{i}x_2} = 0$. Hence,

$$0 = R^N_{u,x_1}x_1 \overset{(39)}{=} (\langle u_2, x_1\rangle J^N - u_1 \wedge x_1)x_1 \overset{u_2 \perp W_1}{=} -(u_1 \wedge x_1)x_1 = u_1,$$

for any unit vector $x_1 \in W_1$, thus $u_1 = 0$, a contradiction.

Type ($\mathbf{c}_k'$, $\mathbf{c}_l'$). Let W and U be of Types (c_k') and (c_l') defined by the data ($\mathfrak{R}$, W', I') and ($\mathfrak{R}^*$, U', J') with $\mathfrak{R} = \mathfrak{R}^*$. If $U' = W'$ and $J' = -I'$, then $U = \overline{W}$ and $V = W \oplus \overline{W} = \mathbb{C}W'$ is curvature invariant of Type (c_{2k}). If $W' \perp U'$, then V is curvature invariant of Type (c_{k+l}') defined by ($\mathfrak{R}$, $W' \oplus U'$, $I' \oplus J'$).

Type ($\mathbf{c}_1'$,$\mathbf{tr}_1$). Let W and U be of Types (c_1') and (tr_1), respectively, with $U \subset \overline{W}$. The action of $\mathfrak{h}_W$ on W is given by $\mathfrak{so}(W)$ and hence W is an irreducible $\mathfrak{h}_W$-module (see Lemma 5 (a)). Therefore, if (W, U) is integrable, then the linear space $W \oplus U$ is curvature invariant according to Proposition 4.

Type ($\mathbf{c}_k$, $\mathbf{c}_l'$). Let W and U be of Types (c_k) and (c_l') determined by the data ($\mathfrak{R}$, W_0) and ($\mathfrak{R}^*$, U', I') respectively. Suppose further that $\mathfrak{R} = \mathfrak{R}^*$ and $W_0 \perp U'$ holds. We claim that $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \{0\}$: let $a \in \mathbb{R}$ and $B \in \mathfrak{so}(\mathfrak{R})$ be given, set $A := aJ^N \oplus B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. If x is a unit vector of W_0, then x, $\mathrm{i}x \in W$ and thus the condition $A(W) \perp W$ implies

$$0 = \langle Ax, \mathrm{i}x\rangle = a\langle \mathrm{i}x, \mathrm{i}x\rangle,$$

i.e. $a = 0$. Hence $A \in \mathfrak{so}(\mathfrak{R})$ and Ax belongs to $U \cap \mathfrak{R} = \{0\}$, i.e. $A\mathrm{i}x = \mathrm{i}Ax = 0$ for all $x \in W_0$. Therefore, $A|_V = 0$ because of (27).

Further, recall that $\mathfrak{h}_W$ and $\mathfrak{h}_U$ are given by $\mathbb{R}J^N + \mathfrak{so}(W_0)$ and $\mathbb{R}(I' + lJ^N) \oplus \mathfrak{su}(U', I')$. Hence I' can be written as $A + B$ with $A \in \mathfrak{h}_W$ and $B \in \mathfrak{h}_U$. Thus I' belongs to the Lie algebra $\mathfrak{h}$ defined in Corollary 4. Since the action of I' is trivial on W whereas I' is an isomorphism on U, we see that the linear spaces $\mathrm{Hom}_{\mathfrak{h}}(W, U)$ and $\mathrm{Hom}_{\mathfrak{h}}(U, W)$ both are trivial. Therefore, Corollary 4 implies that neither (W, U) nor (U, W) is integrable.

Type ($\mathbf{tr}_{j,k}$, $\mathbf{tr}_l'$). Let (W, U) be an integrable orthogonal curvature invariant pair with W and U of Types ($\mathrm{tr}_{j,k}$) and (tr_l') determined by the data ($\mathfrak{R}$, W_1, W_2) and ($\mathfrak{R}^*$, U', I', U_0'), respectively. By means of Corollary 13, we can assume that $\mathfrak{R} = \mathfrak{R}^*$ and that $W_1 \oplus W_2$ is contained in the orthogonal complement of U' in $\mathfrak{R}$. We claim that the linear space $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ is trivial: let $a \in \mathbb{R}$ and $B \in \mathfrak{so}(\mathfrak{R})$ be given, set $A := aJ^N \oplus B$ and suppose that $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$ holds. If $x_1 \in W_1$, then $a\mathrm{i}x_1$ is the imaginary part of Ax_1. Since $Ax_1 \in U$, we see that $Ax_1 = a(I'x_1 + \mathrm{i}x_1)$. In particular, $aI'x_1 \in U_0' \subset U'$. Because $W_1 \cap U' = \{0\}$, this implies $a = 0$, i.e. A vanishes on W_1. In the same way, we can show that A vanishes on $\mathrm{i}W_2$, too. Hence, we see that $A|_V = 0$, since (27) is a linear isomorphism. This establishes our claim.

Further, according to Lemma 6 (a), the action of $\mathfrak{h}_U$ on U is given by $\mathfrak{so}(U)$ and $\mathfrak{h}_U$ acts trivially on $(\mathbb{C}U')^\perp$. Thus, $\mathrm{Hom}_{\mathfrak{h}}(W, U) = \{0\}$. Therefore, Corollary 4 implies that neither (W, U) nor (U, W) is integrable.

Type ($\mathbf{tr}_k'$, $\mathbf{tr}_l'$). Let W and U be of Types (tr_k') and (tr_l') defined by the data ($\mathfrak{R}$, W', I', W_0') and ($\mathfrak{R}^*$, U', J', U_0'), respectively. We can assume that $\mathfrak{R} = \mathfrak{R}^*$.

If W' is orthogonal to U', then $W \oplus U$ is curvature invariant of Type (tr_{k+l}') defined by ($\mathfrak{R}$, $W' \oplus U'$, $I' \oplus J'$, $W_0' \oplus U_0'$). If $U = \mathrm{i}W$, then $W \oplus U$ is curvature invariant of Type (c_k') defined by ($\mathfrak{R}$, W', I').

Suppose that $k = l \geq 3$ and $U = e^{-\mathrm{i}\theta}\overline{W}$ for some $\theta \in \mathbb{R}$. We claim that neither (W, U) nor (U, W) is integrable. Since $W = e^{-\mathrm{i}\theta}\bar{U}$, it suffices to prove the first assertion. In order to explain the idea of our proof, first consider the case $\theta = 0$. Then, the linear space V is curvature invariant of Type $(\mathrm{tr}_{k,k})$ defined by $(\mathfrak{R}, W'_0, I'(W'_0))$ and the totally geodesic submanifold $\exp^N(V)$ is the Riemannian product $\mathrm{S}^k \times \mathrm{S}^k$ through $p = (o, o)$ and the linear space W is given by $\{(x, x) \mid x \in T_o\mathrm{S}^k\}$. If we assume, by contradiction, that (W, U) is integrable, then the corresponding complete parallel submanifold through p would be contained in $\mathrm{S}^k \times \mathrm{S}^k$ via reduction of the codimension and, moreover, it would be even a symmetric submanifold of $\mathrm{S}^k \times \mathrm{S}^k$ according to Corollary 3. However, this is not possible, since a symmetric submanifold $M \subset \mathrm{S}^k \times \mathrm{S}^k$ through the point $p = (o, o)$ with $T_pM = \{(x, x) \mid x \in T_o\mathrm{S}^k\}$ is totally geodesic according to Theorem 4. In the general case, the linear space V is not curvature invariant, but a similar idea shows that (W, U) is not integrable, as follows.

DEFINITION 6. Let $A \in \mathfrak{so}(W')$ be given. We say that A is real, holomorphic or anti-holomorphic if $A(W'_0) \subset W'_0$, $A \circ I' = I' \circ A$ or $A \circ I' = -I' \circ A$, respectively.

Consider the linear map J_θ on $W' \oplus \mathrm{i}W'$ which is given on $W'_0 \oplus \mathrm{i}I'(W'_0)$ by $J_\theta(x - \mathrm{i}I'x) := e^{-\mathrm{i}\theta}(x + \mathrm{i}I'x)$ and $J_\theta(x + \mathrm{i}I'x) := -e^{\mathrm{i}\theta}(x - \mathrm{i}I'x)$ for all $x \in W'_0$ and which is extended to $W' \oplus \mathrm{i}W'$ by $\mathbb{C}$-linearity (note, $W'_0 \oplus \mathrm{i}I'(W'_0)$ is a real form of $W' \oplus \mathrm{i}W'$).

Lemma 9. *Let W be of Type (tr'_k) defined by the data $(\mathfrak{R}, W', I', W'_0)$. Set $U := e^{-\mathrm{i}\theta}\overline{W}$ and $V := W \oplus U$.*
(a) *J_θ is a Hermitian structure on $W' \oplus \mathrm{i}W'$ such that W gets mapped onto U and vice versa. In particular, V is a complex subspace of $(W' \oplus \mathrm{i}W', J_\theta)$ and $J_\theta|_V$ belongs to $\mathfrak{so}(V)_-$.*
(b) *Let $A \in \mathfrak{so}(W')$ and suppose that A is real. As usual, we extend both A and I' to complex linear maps on $W' \oplus \mathrm{i}W'$ via complexification. If A is holomorphic, then A commutes with J_θ for all $\theta \in \mathbb{R}$. If A is anti-holomorphic, then $\exp(\theta I') \circ A$ anti-commutes with J_θ for all $\theta \in \mathbb{R}$.*
(c) *Let $\mathfrak{h}_W$ be the Lie algebra described in (51). Then $\mathfrak{h} := \mathfrak{h}_W|_V$ defines a subalgebra of $\mathfrak{so}(V)_+$. Moreover, the Lie algebra $\mathfrak{h}$ acts irreducibly (in case $k \geq 3$ even over $\mathbb{C}$) on both W and U.*
(d) *Further, let $Z(\mathfrak{h})$ denote the centralizer of $\mathfrak{h}$ in $\mathfrak{so}(V)$, see (22). If $k \geq 3$, then $Z(\mathfrak{h}) \cap \mathfrak{so}(V)_- = \mathbb{R}J_\theta|_V$.*

Proof. Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of W'_0 and set $x_i := (1/\sqrt{2})(e_i - \mathrm{i}I'e_i)$. Then $\{x_1, \ldots, x_k, \bar{x}_1, \ldots, \bar{x}_k\}$ is a Hermitian basis of $W' \oplus \mathrm{i}W'$. We define a unitary map J on $W' \oplus \mathrm{i}W'$ via $J(x_i) := \bar{x}_i$ and $J(\bar{x}_i) := -x_i$. Further, set $I := I'$ and $K := I \circ J$. Then $I^2 = J^2 = -\mathrm{Id}$ and $I \circ J = -J \circ I$ by means of (46), i.e. the usual quaternionic relations hold. Furthermore, $Kx = -\mathrm{i}\bar{x}$ for all $x \in W' \oplus \mathrm{i}W'$. Note that

$$(86) \qquad J_\theta = \exp(\theta I) \circ J = J \circ \exp(-\theta I) = \exp\left(\frac{\theta}{2} I\right) \circ J \circ \exp\left(-\frac{\theta}{2} I\right).$$

It follows that J_θ defines another Hermitian structure on $W' \oplus \mathrm{i}W'$. Since $W = \{x_1, \ldots, x_k\}_{\mathbb{R}}$ and $\overline{W} = \{\bar{x}_1, \ldots, \bar{x}_k\}_{\mathbb{R}}$, we see from (46) that $J_\theta(W) = e^{-\mathrm{i}\theta}\overline{W}$ and $J_\theta(\overline{W}) = \mathrm{e}^{\mathrm{i}\theta} W$, i.e. $J_\theta(V) = V$ and $J_\theta \in \mathfrak{so}(V)_-$. This proves the first part of the lemma.

Moreover, if $A \in \mathfrak{so}(W')$ is holomorphic or anti-holomorphic, then A commutes or anti-commutes with I on $W' \oplus \mathrm{i}W'$, respectively. If A is additionally real, then the same is true for J instead of I: in fact, since A is real, we have $A(\bar{x}) = \overline{Ax}$ for all $x \in W'$, hence $A \circ K = K \circ A$ on $W' \oplus \mathrm{i}W'$ and thus

$$(87) \qquad A \circ J = A \circ K \circ I = K \circ A \circ I = \pm K \circ I \circ A = \pm J \circ A,$$

where the sign $\pm$ is chosen according to whether A is holomorphic $(+)$ or anti-holomorphic $(-)$. Our claim follows.

Therefore, if A is real and holomorphic, then A commutes with both J and I, hence A commutes also with J_θ according to (86). Suppose that $A \in \mathfrak{so}(W')$ is real and anti-holomorphic. Using (86) and (87) (with the negative sign), we have

$$\begin{aligned} J_\theta \circ \exp(\theta I) \circ A &\overset{(86)}{=} J \circ \exp(-\theta I) \circ \exp(\theta I) \circ A = J \circ A \\ &= -A \circ J = -A \circ \exp(-\theta I) \circ J_\theta = -\exp(\theta I) \circ A \circ J_\theta. \end{aligned}$$

Thus, $\exp(\theta I') \circ A$ anti-commutes with J_θ for any real and anti-holomorphic $A \in \mathfrak{so}(W')$. This gives (b).

For (c): from (51) we immediately obtain that $\mathfrak{h}$ acts via $\mathfrak{so}(W)$ and $\mathfrak{so}(U)$ on W and U, respectively. The result follows.

For (d): recall that A is real and holomorphic on W' for each $A \in \mathfrak{h}_W$ as a consequence of (51). Furthermore, recall that $J_\theta|_V \in \mathfrak{so}(V)_-$ as was shown in part (a). Thus $J_\theta|_V \in Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$ according to part (b). Moreover, since $k \geq 3$, the Lie algebra $\mathfrak{h}$ acts irreducibly on both W and U even over $\mathbb{C}$ by means of part (c), hence $\dim(\mathrm{Hom}_{\mathfrak{h}}(W, U)) \leq 1$. Therefore, because of (28) for every subalgebra $\mathfrak{h} \subset \mathfrak{so}(V)_+$, the linear space $Z(\mathfrak{h}) \cap \mathfrak{so}(V)_-$ is spanned by J_θ. □

Lemma 10. *Let W be of Type (tr'_k) defined by the data $(\mathfrak{R}, W', I', W'_0)$. Set $U := e^{-\mathrm{i}\theta}\overline{W}$ for some $\theta \in \mathbb{R}$ and $V := W \oplus U$.*

(a) *The linear map*

$$(88) \qquad \begin{aligned} &F\colon W'_0 \oplus W'_0 \to V, \\ &(x, y) \mapsto \frac{1}{2}[x - \mathrm{i}I'x + J_\theta(x - \mathrm{i}I'x) + y - \mathrm{i}I'y - J_\theta(y - \mathrm{i}I'y)] \end{aligned}$$

is an isometry such that the linear spaces $\{(x, x) \mid x \in W_0'\}$ *and* $\{(x, -x) \mid x \in W_0'\}$ *get identified with* W *and* U*, respectively.*
(b) *By means of* (88)*, the direct sum Lie algebra* $\mathfrak{so}(W_0') \oplus \mathfrak{so}(W_0')$ *gets identified with the Lie algebra* $\rho(\mathfrak{k}_V)|_V$ *such that* $(A, A) \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ *and* $(A, -A) \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ *for every* $A \in \mathfrak{so}(W_0')$*.*
(c) *The complex structure* $J_\theta|_V$ *commutes with every element of* $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ *whereas it anti-commutes with every element of* $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$*.*

Proof. For (a): we have $F(x, x) = x - \mathrm{i}I'x \in W$ and $F(x, -x) = J_\theta(x - \mathrm{i}I'x) = e^{-\mathrm{i}\theta}(x + \mathrm{i}I'x) \in \mathrm{e}^{-\mathrm{i}\theta}\overline{W} = U$. Since $\dim(W) = \dim(U) = \dim(W_0')$, we conclude that F is actually a linear isometry onto V with the properties described above.

For (b): given $A \in \mathfrak{so}(W_0')$, we associate therewith linear maps $\hat{A}$ and $\tilde{A}$ on W' defined by $\hat{A}(x + I'x) := Ax + I'Ax$ and $\tilde{A}(x + I'x) := Ax - I'Ax$ for every $x \in W_0'$. By definition, both $\hat{A}$ and $\tilde{A}$ are real, further, $\hat{A}$ is holomorphic whereas $\tilde{A}$ is anti-holomorphic. Furthermore, we consider the second splitting $V = V_1 \oplus V_2$ with $V_1 = \{x - \mathrm{i}I'x + J_\theta(x - \mathrm{i}I'x) \mid x \in W_0'\}$ and $V_2 = \{x - \mathrm{i}I'x - J_\theta(x - \mathrm{i}I'x) \mid x \in W_0'\}$. Note that both V_1 and V_2 are naturally isomorphic to W_0'. Hence, this splitting induces a monomorphism of Lie algebras $\mathfrak{so}(W_0') \oplus \mathfrak{so}(W_0') \hookrightarrow \mathfrak{so}(V)$. We claim that this monomorphism is explicitly given by

$$(89) \qquad (A, B) \mapsto \frac{1}{2}[(\widehat{A + B}) + \exp(\theta I') \circ (\widetilde{A - B})] :$$

for each $A \in \mathfrak{so}(W_0')$ and all $x \in W_0'$ we have

$$\begin{aligned}
&\frac{1}{2}(\hat{A} + \exp(\theta I') \circ \tilde{A})(x - \mathrm{i}I'x + J_\theta(x - \mathrm{i}I'x)) \\
&= \frac{1}{2}[(Ax - \mathrm{i}I'Ax) + e^{-\mathrm{i}\theta}(Ax + \mathrm{i}I'Ax) + e^{-\mathrm{i}\theta}(Ax + \mathrm{i}I'Ax) + e^{\mathrm{i}\theta}e^{-\mathrm{i}\theta}(Ax - \mathrm{i}I'Ax)] \\
&= (Ax - \mathrm{i}I'Ax) + e^{-\mathrm{i}\theta}(Ax + \mathrm{i}I'Ax), \\
&\frac{1}{2}(\hat{A} + \exp(\theta I') \circ \tilde{A})(x - \mathrm{i}I'x - J_\theta(x - \mathrm{i}I'x) \\
&= \frac{1}{2}[(Ax - \mathrm{i}I'Ax) - e^{-\mathrm{i}\theta}(Ax + \mathrm{i}I'Ax) + e^{-\mathrm{i}\theta}(Ax + \mathrm{i}I'Ax) - e^{\mathrm{i}\theta}e^{-\mathrm{i}\theta}(Ax - \mathrm{i}I'Ax)] \\
&= 0.
\end{aligned}$$

This establishes our claim in case $B = 0$. For $A = 0$, a similar calculation works.

Further, we claim that in this way $\mathfrak{so}(W_0') \oplus \mathfrak{so}(W_0') \cong \rho(\mathfrak{k}_V)|_V$ such that $(A, A) \cong \hat{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$ and $(A, -A) \cong \exp(\theta I') \circ \tilde{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$.

For "$\subseteq$": we have $\hat{A}(x \pm \mathrm{i}I'x) = Ax \pm \mathrm{i}I'Ax$ for all $x \in W_0'$ and $A \in \mathfrak{so}(W_0')$, thus $\hat{A}$ maps W to W and U to U. Further, $\hat{A} \in \mathfrak{so}(\mathfrak{R})$. This shows that $\hat{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_+$. Furthermore, $\tilde{A}(x \pm \mathrm{i}I'x) = Ax \mp \mathrm{i}I'Ax$ and $I'(x \pm \mathrm{i}I'x) = \mp\mathrm{i}(x \pm \mathrm{i}I'x)$ for all $x \in W_0'$, thus $\exp(\theta I') \circ \tilde{A}$ maps W to U and vice versa. Finally, note that $\exp(\theta I') \circ \tilde{A}$ is in fact skew-symmetric since

$$(\exp(\theta I') \circ \tilde{A})^* = \tilde{A}^* \circ \exp(\theta I')^* = -\tilde{A} \circ \exp(-\theta I') = -\exp(\theta I') \circ \tilde{A}.$$

Hence $\exp(\theta I') \circ \tilde{A} \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$.

For "$\supseteq$": conversely, let some $A \in \rho(\mathfrak{k}_V)|_V$ be given. We distinguish the cases $A \in \mathfrak{so}(V)_+$ and $A \in \mathfrak{so}(V)_-$. Anyway, we have $A = aJ^N|_V + B|_V$ with $B \in \mathfrak{so}(W')$ and $a \in \mathbb{R}$. Set $B' := aI'|_V + B|_V$. Note that $I' = J^N$ on $W + \mathrm{i}W$ and $I' = -J^N$ on $\overline{W} + \mathrm{i}\overline{W}$, hence

$$A = B' \quad \text{on} \quad W + \mathrm{i}W, \tag{90}$$

$$A = B' + 2aJ^N \quad \text{on both } \overline{W} \text{ and } U. \tag{91}$$

If $A \in \mathfrak{so}(V)_+$, then $A(W) \subset W$ and hence we conclude from (90) that B' is real and holomorphic; thus $B' = C$ with $C := B'|_{W_0'}$. In particular, $B'(\overline{W}) \subset \overline{W}$ and hence $B'(U) \subset U$. Thus $a = 0$ because of (91) and since $J^N(U) \subset U^\perp$ where $U^\perp$ denotes the orthogonal complement of U in T.

If $A \in \mathfrak{so}(V)_-$, then A maps W to U and vice versa, hence $B'(W) \subset U$ because of (90), thus $e^{\mathrm{i}\theta} B'(W) \subset \overline{W}$ which shows that the linear endomorphism $C := \exp(-\theta I') \circ B'$ is real and anti-holomorphic on W'. Hence B' is anti-holomorphic, too. Therefore, also $\exp(-\theta I')(B'(\overline{W})) \subset W$, thus $B'(U) \subset W$. We conclude that $a = 0$ according to (91) (since $J^N(U) \subset \overline{W} + \mathrm{i}\overline{W} \subset W^\perp$). This establishes our claim. Part (b) follows.

For (c), recall that $\hat{A}$ commutes with $J_\theta|_V$ whereas $\tilde{A}$ anti-commutes with $J_\theta|_V$ according to Lemma 9 for every $A \in \mathfrak{so}(W_0')$. Hence, the result is a consequence of Part (b) and (89). □

Corollary 20. *Suppose that W is of Type* (tr_k') *with $k \geq 3$. The curvature invariant pair $(W, \mathrm{e}^{-\mathrm{i}\theta}\overline{W})$ is not integrable.*

Proof. Set $U := \mathrm{e}^{-\mathrm{i}\theta}\overline{W}$ and $V := W \oplus U$. Suppose, by contradiction, that (W, U) is integrable. Let $\mathfrak{g}$ be the subalgebra of $\rho(\mathfrak{k}_V)|_V$ described in Theorem 3 (a). Recall that $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded Lie algebra such that the Lie algebra $\mathfrak{h} := \mathfrak{h}_W|_V$ considered in Lemma 9 (c) is contained in $\mathfrak{g}_+$ according to Theorem 3 (b). Recall further that there exist $A_x \in \mathfrak{g}_-$ and $B_x \in Z(\mathfrak{g}) \cap \mathfrak{so}(V)_-$ such that $\boldsymbol{h}_x = A_x + B_x$ for every $x \in W$ according to Theorem 3 (c). First, we claim that for every $x \in W$ there exists some $b \in \mathbb{R}$ such that $B_x = bJ_\theta|_V$: we have $Z(\mathfrak{g}) \subset Z(\mathfrak{h})$ (since $\mathfrak{h} \subset \mathfrak{g}$), thus $B_x \in \mathbb{R}J_\theta|_V$ by means of Lemma 9 (d). This gives our first claim.

Next, we claim that $b = 0$: by definition, we have $R^N_{y,z}|_V \in \mathfrak{h}$ for all $y, z \in W$ and we also have $A_x \in \mathfrak{g}_-$. Furthermore, $\mathfrak{h} \subset \mathfrak{g}_+$, see above. Therefore, following the rules for $\mathbb{Z}_2$-graded Lie algebras, we have $[A_x, R^N_{y,z}|_V] \in \mathfrak{g}_-$. Thus, on the one hand, since $\mathfrak{g}_- \subset \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$, the complex structure J_θ anti-commutes with $[A_x, R^N_{y,z}|_V]$ on V

for all $x, y, z \in W$ according to Lemma 10 (c). Assume, by contradiction, that $b \neq 0$. Then, on the other hand, $J_\theta|_V = (1/b)B_x$ would also commute with $[A_x, R^N_{y,z}|_V]$ by the very definition of B_x (recall that $[A_x,\, R^N_{y,z}|_V] \in \mathfrak{g}$). Moreover, since J_θ is a complex structure, this is not possible unless $[A_x,\, R^N_{y,z}|_V] = 0$. But then also

$$[\boldsymbol{h}_x,\, R^N_{y,z}|_V] = [A_x,\, R^N_{y,z}|_V] + [B_x,\, R^N_{y,z}|_V] = 0 + 0.$$

In other words, since the Lie algebra $\mathfrak{h}$ is spanned by the endomorphisms of V which are given by $R^N_{y,z}|_V$ with $y, z \in W$, we obtain that

$$\forall x \in W\colon \boldsymbol{h}_x \in Z(\mathfrak{h}) \cap \mathfrak{so}(V)_- = \mathbb{R}\, J_\theta|_V$$

where the latter equality follows from Lemma 9 (d) again. Thus the rank of the linear map $\boldsymbol{h}\colon W \to \mathfrak{so}(V)$, $x \mapsto \boldsymbol{h}_x$ is zero or one. In particular, since $\dim(W) > 1$, it is not possible that $\boldsymbol{h}$ is injective. Therefore, because $\mathfrak{h}_W$ acts irreducibly on W (see Lemma 9 (c)) and since $\mathrm{Kern}(\boldsymbol{h})$ is a non-trivial proper subspace of W which is invariant under the action of $\mathfrak{h}_W$ on W according to (18), (19) and Proposition 2, we necessarily have $\boldsymbol{h}_x = 0$ for all $x \in W$, a contradiction. Our second claim follows.

Thus $B_x = 0$, which implies that $\boldsymbol{h}_x \in \rho(\mathfrak{k}_V)|_V$ for all $x \in W$. Let us choose some $o \in \mathrm{S}^k$, a linear isometry $f\colon T_o\mathrm{S}^k \to W'_0$ and consider the Riemannian product $\tilde{N} := \mathrm{S}^k \times \mathrm{S}^k$ whose curvature tensor will be denoted by $\tilde{R}$. On the analogy of (88),

$$\begin{aligned} &F\colon T_{(o,o)}\tilde{N} \to T,\\ &(x, y) \mapsto \frac{1}{2}[f(x) + f(y) - \mathrm{i}I'(f(x) + f(y)) + J_\theta(f(x) - f(y) - \mathrm{i}I'(f(x) - f(y)))] \end{aligned}$$

is an isometry onto V such that $\{F^{-1} \circ A|_V \circ F \mid A \in \rho(\mathfrak{k}_V)\}$ is the direct sum Lie algebra $\mathfrak{so}(T_o\mathrm{S}^k) \oplus \mathfrak{so}(T_o\mathrm{S}^k)$. Note, the latter is the image $\tilde{\rho}(\tilde{\mathfrak{k}})$ of the linearized isotropy representation of $\tilde{N}$. Put $\tilde{W} := F^{-1}(W)$, $\tilde{h} := F^{-1} \circ h \circ F \times F$ and $\tilde{U} := \{\tilde{h}(x, y) \mid x, y \in \tilde{W}\}_{\mathbb{R}}$. Then $\tilde{U} = F^{-1}(U)$ and hence $T_{(o,o)}\tilde{N} = \tilde{W} \oplus \tilde{U}$ holds. Furthermore, we claim that $(\tilde{W}, \tilde{h})$ is an integrable 2-jet in $T_{(o,o)}\tilde{N}$: Let $\iota\colon \mathrm{S}^k_{\sqrt{2}} \to \mathrm{S}^k \times \mathrm{S}^k$, $p \mapsto (p/\|p\|, p/\|p\|)$. Then $T_{(o,o)}\iota\big(\mathrm{S}^k_{\sqrt{2}}\big) = \{(x, x) \mid x \in T_o\mathrm{S}^k\}$, hence $F\big(T_{(o,o)}\iota\big(\mathrm{S}^k_{\sqrt{2}}\big)\big) = W$, i.e. $T_{(o,o)}\iota\big(\mathrm{S}^k_{\sqrt{2}}\big) = \tilde{W}$. Further, on the one hand, we have

$$\begin{aligned} R^N(F(x, x), F(y, y), F(z, z)) &= R^N(f(x) - \mathrm{i}I' f(x),\ f(y) - \mathrm{i}I' f(y),\ f(z) - \mathrm{i}I' f(z))\\ &= -f(x) \wedge f(y) f(z) + \mathrm{i}(I' f(x) \wedge I' f(y)) I' f(z) \end{aligned}$$

for all $x, y, z \in \tilde{W}$ according to Lemma 6 (a). On the other hand,

$$\begin{aligned} F(x \wedge yz, x \wedge yz) &= f(x) \wedge f(y) f(z) - \mathrm{i}I'(f(x) \wedge f(y) f(z))\\ &= f(x) \wedge f(y) f(z) - \mathrm{i}I' f(x) \wedge I' f(y) I' f(z). \end{aligned}$$

This shows that $F \circ \tilde{R}^N_{(x,x),(y,y)}\big|_{\tilde{W}} = R^N_{F(x,x),F(y,y)} \circ F|_{\tilde{W}}$. Furthermore, (14), (20) and Lemma 10 (b) show that $\tilde{R}^N_{(x,x),(y,y)}$ and $F^{-1} \circ R^N_{F(x,x),F(y,y)} \circ F$ both belong to $\tilde{\rho}(\tilde{\mathfrak{k}})_+$, i.e. there exist $A, B \in \mathfrak{so}(T_o\mathrm{S}^k)$ with $\tilde{R}^N_{(x,x),(y,y)} = A \oplus A$ and $F^{-1} \circ R^N_{F(x,x),F(y,y)} \circ F = B \oplus B$. Thus, since the direct sum endomorphism $A \oplus A$ is uniquely determined by its restriction to $\tilde{W}$ for every $A \in \mathfrak{so}(T_o\mathrm{S}^k)$, we conclude that $F \circ \tilde{R}^N_{(x,x),(y,y)} = R^N_{F(x,x),F(y,y)} \circ F$. Therefore, $\tilde{W}$ is curvature invariant and $\tilde{h}$ is semi-parallel in $\tilde{N}$. Moreover, since $\boldsymbol{h}_x \in \rho(\mathfrak{k}_V)|_V$ for all $x \in W$, we have $\tilde{\boldsymbol{h}}_x \in \tilde{\rho}(\tilde{\mathfrak{k}})$ for all $x \in \tilde{W}$ which shows that Equation 5 for $(\tilde{W}, \tilde{h})$ is implicitly given for all k. Hence, by means of Theorem 2, we obtain that $(\tilde{W}, \tilde{h})$ is an integrable 2-jet in $\tilde{N}$.

Thus, there exists a complete parallel submanifold $\tilde{M}$ through (o, o) whose 2-jet is given by $(\tilde{W}, \tilde{h})$. The fact that $T_{(o,o)}\tilde{N} = \tilde{W} \oplus \tilde{U}$ holds implies that $\tilde{M}$ is 1-full in $\tilde{N}$, i.e. extrinsically symmetric according to Corollary 3. Further, since $\tilde{M}$ is tangent to $\iota\big(\mathrm{S}^k_{\sqrt{2}}\big)$ at (o, o), there do not exist submanifolds $\tilde{M}_1 \subset \mathrm{S}^k$ and $\tilde{M}_2 \subset \mathrm{S}^k$ such that $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2$. Therefore, by means of Theorem 4, $\tilde{M}$ is totally geodesic, i.e. $h = 0$, a contradiction. □

Suppose that W is of Type (tr_2') defined by the data $(\mathfrak{R}, W', I', W_0')$. Let $\{e_1, e_2\}$ be an orthonormal basis of W_0' and $\tilde{I}$ be defined according to (53). Further, let $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{so}(W')$ be given and set $U := \tilde{J}(W)$. We will show that neither (W, U) nor (U, W) is integrable unless $V := W \oplus U$ is curvature invariant. First, we claim that it suffices to prove the first assertion: recall that here U is also of Type (tr_2'), defined by the triple $(\mathfrak{R}, U', U_0', J')$ with $U' := W'$, $J' := \tilde{J} \circ I' \circ \tilde{J}^{-1}$ and $U_0' := \tilde{J}(W_0')$. Further,

$$\tilde{I} = \tilde{J} \circ \tilde{I} \circ \tilde{J}^{-1} \overset{(53)}{=} \tilde{J}e_1 \wedge \tilde{J}e_2 + \tilde{J}I'e_1 \wedge \tilde{J}I'e_2 = \tilde{J}e_1 \wedge \tilde{J}e_2 + J'\tilde{J}e_1 \wedge J'\tilde{J}e_2.$$

Hence, since $\{\tilde{J}e_1, \tilde{J}e_2\}$ is an orthonormal basis of U_0', the Hermitian structure $\tilde{I}$ may also be defined on the analogy of (53) via the triple $(U_0', \{\tilde{J}e_1, \tilde{J}e_2\}, J')$. Furthermore, we have $U = \tilde{J}(W)$, hence also $W = \tilde{J}(U)$ (since $\tilde{J}^2 = -\mathrm{Id}$). This proves the claim.

Lemma 11. *Suppose that W is of Type (tr_2') defined by the data $(\mathfrak{R}, W', I', W_0')$. Let $\{e_1, e_2\}$ be an orthonormal basis of W_0' and $\tilde{I}$ be defined according to (53). Further, let $\tilde{J} \in \mathrm{SU}(W', \tilde{I}) \cap \mathfrak{so}(W')$ be given, set $U := \tilde{J}(W)$ and $V := W \oplus U$. Then we have $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \mathbb{R}\tilde{J}|_V$ unless $\tilde{J} = \pm I'$.*

Proof. First, we claim that $\tilde{J} \in \rho(\mathfrak{k}_V)$ and $\tilde{J}|_V \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$: we have $\tilde{J} \in \mathfrak{so}(W') \subset \mathfrak{so}(\mathfrak{R}) \subset \rho(\mathfrak{k})$. Further, $\tilde{J}(W) = U$ and $\tilde{J}(U) = W$, hence $\tilde{J}(V) = V$ and $\tilde{J}|_V \in \mathfrak{so}(V)_-$. This gives our claim.

Conversely, let $a \in \mathbb{R}$ and $B \in \mathfrak{so}(\mathfrak{R})$ be given such that $A := aJ^N + B$ satisfies $A(V) \subset V$ and $A|_V \in \mathfrak{so}(V)_-$. We aim to show that A is a multiple of $\tilde{J}|_V$ unless

$\tilde{J} = \pm I'$. With $\tilde{A} := aI' + B|_{W'}$, we have $\tilde{A} \in \mathfrak{so}(\mathfrak{R})$ and $A|_W = \tilde{A}|_W$ according to (46), hence

$$\tilde{A}(x - \mathrm{i}I'x) \in \{\tilde{J}y - \mathrm{i}\tilde{J}y \mid y \in W_0'\} \tag{92}$$

for all $x \in W_0$. Thus, using (92) and passing to real and imaginary parts, we conclude that $\tilde{A}|_{W'} \in \mathfrak{so}(W')$ such that $\tilde{A}(W_0') \subset \tilde{J}(W_0')$. In particular, the endomorphism $C := \tilde{J} \circ \tilde{A}$ on W' is real and holomorphic (see Definition 6). Further, $\tilde{A}^* = -\tilde{A}$ which implies that

$$\tilde{J} \circ C = C^* \circ \tilde{J}. \tag{93}$$

We claim that $C = c\,\mathrm{Id}$ for some $c \in \mathbb{R}$ or $\tilde{J} = \pm I'$: for this, let RH denote the algebra of real and holomorphic maps on W'. Note, $\tilde{I}$ is real and holomorphic, hence there is the splitting $\mathrm{RH} = \mathrm{RH}_+ \oplus \mathrm{RH}_-$ with

$$\mathrm{RH}_+ := \{A \in \mathrm{RH} \mid A \circ \tilde{I} = \tilde{I} \circ A\},$$
$$\mathrm{RH}_- := \{A \in \mathrm{RH} \mid A \circ \tilde{I} = -\tilde{I} \circ A\}.$$

Then $\mathrm{RH}_+ = \{\mathrm{Id}, \tilde{I}\}_{\mathbb{R}}$ and $\mathrm{RH}_- = \{\sigma, \sigma \circ \tilde{I}\}_{\mathbb{R}}$, where σ denotes the conjugation of $(W', \tilde{I})$ with respect to the real form $\{e_1, I'e_1\}_{\mathbb{R}}$. Further, consider the involution on $\mathrm{End}(W')$ defined by $C \mapsto -\tilde{J} \circ C^* \circ \tilde{J}$. This map preserves both RH_+ and RH_- and its fixed points in RH are the solutions to (93). It follows that a solution to (93) with $C \in \mathrm{RH}$ decomposes as $C = C_+ + C_-$ such that $C_\pm \in \mathrm{RH}_\pm$ and $C_\pm$ is a solution to (93), too.

Then we have $\tilde{J} \circ \tilde{I} = \tilde{I} \circ \tilde{J} = -\tilde{I}^* \circ \tilde{J}$ since $\tilde{I}$ is skew-symmetric and commutes with $\tilde{J}$. Hence a solution to (93) with $C \in \mathrm{RH}_+$ is given only if C is a multiple of Id. If $C \in \mathrm{RH}_-$ is a solution to (93), then $C \circ \tilde{I}$ is a solution to this equation, too, since

$$\begin{aligned}\tilde{J} \circ C \circ \tilde{I} &= C^* \circ \tilde{J} \circ \tilde{I} = C^* \circ \tilde{I} \circ \tilde{J}\\ &= (\tilde{I}^* \circ C)^* \circ \tilde{J} = (-\tilde{I} \circ C)^* \circ \tilde{J} = (C \circ \tilde{I})^* \circ \tilde{J}.\end{aligned}$$

Thus, since RH_- is invariant under right multiplication by $\tilde{I}$, the intersection of the solution space to (93) with RH_- is either trivial or all of RH_-. Hence, to finish the proof of our claim, it suffices to show that $C := \sigma$ is not a solution to (93) unless $\tilde{J} = \pm I'$: for this, recall that there exist $t \in \mathbb{R}$ and $w \in \mathbb{C}$ with $t^2 + |w|^2 = 1$ such that the matrix of $\tilde{J}$ with respect to the Hermitian basis $\{e_1, I'e_1\}$ of $(W', \tilde{I})$ is given by Equations (59), (60). Clearly, $\sigma = \sigma^{-1}$ and $\sigma^* = \sigma$. Hence, if (93) holds for $C := \sigma$, then $\sigma \circ \tilde{J} \circ \sigma = \tilde{J}$, i.e.

$$\begin{pmatrix} \mathrm{i}t & -\bar{w} \\ w & -\mathrm{i}t \end{pmatrix} = \begin{pmatrix} -\mathrm{i}t & -w \\ \bar{w} & \mathrm{i}t \end{pmatrix}. \tag{94}$$

Thus $t = 0$ and $w = \pm 1$, i.e. $\tilde{J} = \pm I'$.

This proves our claim. Therefore, if $\tilde{J} \neq \pm I'$, then $\tilde{A} = -c\tilde{J}$. Hence $A|_V = -c\tilde{J}|_V + a(J^N|_V - I'|_V)$. It remains to show that $a(J^N|_V - I'|_V) = 0$: we have $J^N|_W = I'|_W$ and $a(J^N|_V - I'|_V) = A|_V + c\tilde{J}|_V \in \mathfrak{so}(V)_-$. Since $\mathfrak{so}(V)_- \to \mathrm{Hom}(W, U)$, $A \mapsto A|_W$ is an isomorphism, $a(J^N|_V - I'|_V) = 0$. This finishes our proof. □

Corollary 21. *In the situation of* Lemma 11, *the curvature invariant pair* (W, U) *is not integrable unless* V *is a curvature invariant subspace of* T.

Proof. Note, if $\tilde{J} = \pm I'$, then $V := W \oplus U$ is curvature invariant of Type (c_2') defined by $(\mathfrak{R}, W', I')$. Otherwise, if $\tilde{J} \neq \pm I'$, then we will show that (W, U) is not integrable. Assume, by contradiction, that (W, U) is integrable but $\tilde{J} \neq \pm I'$. Thus, there exists an integrable symmetric bilinear map $h\colon W \times W \to U$ such that $U = \{h(x, y) \mid x, y \in W\}_{\mathbb{R}}$. Further, let $\{I, J, K\}$ be a quaternionic basis of $\mathfrak{su}(V, \tilde{I})$ defined as follows: set $I|_W := \tilde{I}|_W$, $I|_U := -\tilde{I}|_U$, $J := \tilde{J}|_V$ and $K := I \circ J$. Since $\tilde{I}$ commutes with $\tilde{J}$, we have $I \circ J = -J \circ I$ and then the usual quaternionic relations hold, i.e. $I^2 = J^2 = K^2 = -\mathrm{Id}$, $J \circ K = -K \circ J = I$ and $K \circ I = -I \circ K = J$. We claim that $\boldsymbol{h}_x \in \{J, K\}_{\mathbb{R}}$ for all $x \in W$: note, the set $\{\tilde{I}, I, J, K\}$ is a basis of $\mathfrak{u}(V, \tilde{I})$ and $I, \tilde{I} \in \mathfrak{so}(V)_+$ whereas $J, K \in \mathfrak{so}(V)_-$. Hence

$$\mathfrak{u}(V, \tilde{I}) \cap \mathfrak{so}(V)_- = \{J, K\}_{\mathbb{R}}. \tag{95}$$

Moreover, recall that $\mathfrak{h}_W = \mathbb{R}\tilde{I}$ according to Lemma 6. Therefore, by virtue of Theorem 3, there exist $A_x \in \rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_-$ and $B_x \in \mathfrak{u}(V, \tilde{I}) \cap \mathfrak{so}(V)_-$ with $\boldsymbol{h}_x = A_x + B_x$. Furthermore, since $\tilde{J} \neq \pm I'$, we have $\rho(\mathfrak{k}_V)|_V \cap \mathfrak{so}(V)_- = \mathbb{R}\tilde{J}$ as a consequence of Lemma 11. Thus both A_x and B_x belong to $\mathfrak{u}(V, \tilde{I})$, hence $\boldsymbol{h}_x \in \mathfrak{u}(V, \tilde{I}) \cap \mathfrak{so}(V)_-$ for all $x \in W$, too. The claim follows by means of (95).

Further, $\mathfrak{h}_W$ acts on W via $\mathfrak{so}(W)$ which implies that $\boldsymbol{h}\colon W \to \mathfrak{so}(V)_-$ is injective according to (18), (19) and Proposition 2. Hence there exists some $x \in W$ with $\boldsymbol{h}_x = K$. Furthermore, set $y := e_1 - \mathrm{i}I'e_1$, $z := e_2 - \mathrm{i}I'e_2$ and recall that $R^N_{y,z} = -e_1 \wedge e_2 - I'e_1 \wedge I'e_2 = -\tilde{I}$. Therefore, with x, y, z chosen as above, Equation (5) with $k = 1$ means that

$$[K, \tilde{I}] = -R^N(Ky \wedge z + y \wedge Kz)|_V. \tag{96}$$

Since K belongs to $\mathfrak{u}(V, \tilde{I})$, l.h.s. of the last Equation vanishes. In order to evaluate r.h.s. of (96), note that $z = \tilde{I}y$, hence

$$Ky = IJy = -JIy = -\tilde{I}\tilde{J}y = -\tilde{J}\tilde{I}y = -\tilde{J}z,$$

thus $z = JKy = -KJy$ and $Kz = Jy = \tilde{J}y$, which gives

$$Ky \wedge z + y \wedge Kz = z \wedge \tilde{J}z + y \wedge \tilde{J}y.$$

Let $c \in \mathbb{R}$ and $A \in \mathfrak{so}(\Re)$ be given such that $R^N(Ky \wedge z + y \wedge Kz) = cJ^N + A$. Using Equation (39), the real part A is given as follows,

$$A = \tilde{J}e_1 \wedge e_1 + \tilde{J}I'e_1 \wedge I'e_1 + \tilde{J}e_2 \wedge e_2 + \tilde{J}I'e_2 \wedge I'e_2 = -2\tilde{J},$$

(the last equality uses that $\{e_1, e_2, I'e_1, I'e_2\}$ is an orthonormal basis of W' and that $\tilde{J} \in \mathfrak{so}(W')$). Therefore, since r.h.s. of (96) vanishes, we conclude that $2\tilde{J} = cJ^N$ on V. Hence $c = \pm 2$ (since both $\tilde{J}$ and $J^N|_{\mathbb{C}W'}$ are isometries of $\mathbb{C}W'$), i.e. $\tilde{J} = \pm J^N$ on V. In particular, $\mp J^N + \tilde{J}$ vanishes on W. With $B := \mp I' + \tilde{J}$ and $a := \mp 1$, Lemma 6 (c) in combination with (46) implies that B vanishes identically on W'. This shows that $\tilde{J} = \pm I'$, a contradiction. □

Type ($\mathbf{tr}'_k, \mathbf{tr}_1$). Let (W, U) be an integrable orthogonal curvature invariant pair of Type $(\mathrm{tr}'_k, \mathrm{tr}_1)$. Since the action of $\mathfrak{h}_W$ on W is given by $\mathfrak{so}(W)$ (see Lemma 6 (a)), Proposition 4 shows that here the linear space $W \oplus U$ is curvature invariant.

Type ($\mathbf{ex}_3, \mathbf{tr}_1$). Let W and U be of Types (ex_3) and (tr_1) defined by the data $(\Re, \{e_1, e_2\})$ and a unit vector $u \in T$, respectively. Then $u = \pm(1/\sqrt{2})(e_2 - \mathrm{i}e_1)$ and the linear space $W \oplus U$ is curvature invariant of Type (c_2) defined by the data $(\Re, \{e_1, e_2\}_{\mathbb{R}})$.

ACKNOWLEDGMENTS. The results on parallel submanifolds with curvature isotropic tangent spaces of maximal possible dimension in Section 2.4 greatly benefited from a conversation with Jost Eschenburg from Augsburg. Further, the author wants to thank the referee for his thorough work and his patience.

References

[1] J. Berndt, S. Console and C. Olmos: Submanifolds and Holonomy, Chapman & Hall/CRC Research Notes in Mathematics **434**, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[2] E. Backes and H. Reckziegel: *On symmetric submanifolds of spaces of constant curvature*, Math. Ann. **263** (1983), 419–433.
[3] B. Chen and T. Nagano: *Totally geodesic submanifolds of symmetric spaces* I, Duke Math. J. **44** (1977), no. 4, 745–755.
[4] B. Chen and T. Nagano: *Totally geodesic submanifolds of symmetric spaces* II, Duke Math. J. **45** (1978), 405–425.
[5] P. Dombrowski: *Differentiable maps into Riemannian manifolds of constant stable osculating rank* I, J. Reine Angew. Math. **274/275** (1975), 310–341.
[6] D. Ferus: *Immersions with parallel second fundamental form*, Math. Z. **140** (1974), 87–93.
[7] D. Ferus: *Symmetric submanifolds of Euclidean space*, Math. Ann. **247** (1980), 81–93.
[8] D. Ferus and F. Pedit: *Curved flats in symmetric spaces*, Manuscripta Math. **91** (1996), 445–454.
[9] S. Helgason: Differential Geometry, Lie Groups, and Symmetric Spaces, Graduate Studies in Mathematics **34**, Amer. Math. Soc., Providence, RI, 2001.
[10] T. Jentsch: *The extrinsic holonomy Lie algebra of a parallel submanifold*, Ann. Global Anal. Geom. **38** (2010), 335–371.
[11] T. Jentsch: *Extrinsic homogeneity of parallel submanifolds*, Manuscripta Math. **137** (2012), 347–382.

[12] T. Jentsch: *Extrinsic homogeneity of parallel submanifolds* II, Differential Geom. Appl. **29** (2011), 214–232.
[13] T. Jentsch and H. Reckziegel: *Submanifolds with parallel second fundamental form studied via the Gauss map*, Ann. Global Anal. Geom. **29** (2006), 51–93.
[14] S. Klein: *Totally geodesic submanifolds of the complex quadric*, Differential Geom. Appl. **26** (2008), 79–96.
[15] S. Klein: *Totally geodesic submanifolds of the complex and the quaternionic* 2*-Grassmannians*, Trans. Amer. Math. Soc. **361** (2009), 4927–4967.
[16] S. Klein: *Reconstructing the geometric structure of a Riemannian symmetric space from its Satake diagram*, Geom. Dedicata **138** (2009), 25–50.
[17] S. Klein: *Totally geodesic submanifolds in Riemannian symmetric spaces*; in Differential Geometry, World Sci. Publ., Hackensack, NJ, 2009, 136–145.
[18] S. Klein: *Totally geodesic submanifolds of the exceptional Riemannian symmetric spaces of rank* 2, Osaka J. Math. **47** (2010), 1077–1157.
[19] B. Mendonca and R. Tojeiro: *Submanifolds of products of space forms*, to appear in Indiana Univ. Math. J (preprint available at arXiv:1203.3790v3 [MathDG]).
[20] H. Naitoh: *Isotropic submanifolds with parallel second fundamental form in* $P^m(c)$, Osaka J. Math. **18** (1981), 427–464.
[21] H. Naitoh: *Symmetric submanifolds of compact symmetric spaces*, Tsukuba J. Math. **10** (1986), 215–242.
[22] H. Naitoh: *Compact simple Lie algebras with two involutions and submanifolds of compact symmetric spaces* I, Osaka J. Math. **30** (1993), 653–690.
[23] H. Naitoh: *Grassmann geometries on compact symmetric spaces of general type*, J. Math. Soc. Japan **50** (1998), 557–592.
[24] K. Nomizu and K. Yano: *On circles and spheres in Riemannian geometry*, Math. Ann. **210** (1974), 163–170.
[25] W. Strübing: *Symmetric submanifolds of Riemannian manifolds*, Math. Ann. **245** (1979), 37–44.
[26] M. Takeuchi: *Parallel submanifolds of space forms*; in Manifolds and Lie Groups (Notre Dame, Ind., 1980), Progr. Math. **14**, Birkhäuser, Boston, MA, 1981, 429–447.

Tillmann Jentsch
University of Stuttgart
Department of Mathematics
Institute for Geometry and Topology
Pfaffenwaldring 57
D-70569 Stuttgart
Germany
e-mail: tilljentsch@gmx.de