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GRADIENT ESTIMATES FOR A SIMPLE PARABOLIC LICHNEROWICZ EQUATION

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Abstract

In this paper, we study the gradient estimates for positive solutions to the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \triangle_f u + cu^{-\alpha}$$

on complete noncompact manifolds with Bakry–Émery Ricci curvature bounded below, where α, c are two real constants and $\alpha > 0$.

1. Introduction

Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold. For a smooth real-valued function f on M, the drifting Laplacian is defined by

$$\triangle_f = \triangle - \nabla f \cdot \nabla.$$

There is a naturally associated measure $d\mu = e^{-f}\,dV$ on M, which makes the operator $\triangle_f$ self-adjoint. The N-Bakry–Émery Ricci tensor is defined by

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \nabla^2 f - \frac{1}{N}\,df \otimes df$$

for $0 \leq N \leq \infty$ and $N = 0$ if and only if $f = 0$. Here ∇^2 is the Hessian and Ric is the Ricci tensor.

Recently, there has been an active interest in the study of gradient estimates for the partial differential equation. Wu [16] gave a local Li–Yau type gradient estimate for the positive solutions to a general nonlinear parabolic equation

$$u_t = \triangle u - \nabla\varphi\nabla u - au\log u - qu$$

in $M \times [0, \tau]$, where $a \in R$, ϕ is a C^2-smooth function and $q = q(x, t)$ is a function, which generalizes many previous well-known gradient estimates results. Zhu [18]

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investigated the fast diffusion equation

$$u_t = \triangle u^\alpha \tag{1.1}$$

and the author got the following results:

Theorem 1.1 (Zhu [18]). *Let M be a Riemannian manifold of dimension $n \geq 2$ with* $\mathrm{Ric}(M) \geq -k$ *for some* $k \geq 0$. *Suppose that* $v = -(\alpha/(\alpha-1))u^{\alpha-1}$ *is any positive solution to the equation* (1.1) *in* $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. *Suppose also that* $v \leq \tilde{M}$ *in* $Q_{R,T}$. *Then there exists a constant* $C = C(\alpha, M)$ *such that*

$$\frac{|\nabla v|}{v^{1/2}} \leq C\tilde{M}^{1/2}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\right)$$

in $Q_{R/2,T/2}$.

Later, Huang and Li [5] considered the generalized equation

$$u_t = \triangle_f u^\alpha$$

on Riemannian manifolds and got some interesting gradient estimates. Zhang and Ma [17] considered gradient estimates for positive solutions to the following nonlinear equation

$$\triangle_f u + cu^{-\alpha} = 0 \tag{1.2}$$

on complete noncompact manifolds. When N is finite and the N-Bakry–Émery Ricci tensor is bounded from below, the authors in [17] got a gradient estimate for positive solutions of the above equation (1.2).

Theorem 1.2 (Zhang and Ma [17]). *Let* (M, g) *be a complete noncompact n-dimensional Riemannian manifold with N-Bakry–Émery Ricci tensor bounded from below by the constant* $-K =: -K(2R)$, *where* $R > 0$ *and* $K(2R) > 0$ *in the metric ball* $B_{2R}(p)$ *around* $p \in M$. *Let u be a positive solution of* (1.2). *Then*
(1) *if* $c > 0$, *we have*

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq{}& \frac{(N+n)(N+n+2)c_1^2}{R^2} + \frac{(N+n)[(N+n-1)c_1 + c_2]}{R^2}\\ &+ \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + 2(N+n)K.\end{aligned}$$

(2) *if $c < 0$, we have*

$$\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \le (A+\sqrt{A})|c|\left(\inf_{B_p(2R)} u\right)^{-\alpha-1} + \frac{(N+n)[(N+n-1)c_1+c_2]}{R^2}$$
$$+ \frac{(N+n)c_1^2}{R^2}\left(n+N+2+\frac{n+N}{2\sqrt{A}}\right) + \frac{(N+n)\sqrt{(N+n)K}c_1}{R}$$
$$+\left(2+\frac{1}{\sqrt{A}}\right)(n+N)K,$$

where $A = (N+n)(\alpha+1)(\alpha+2)$ and c_1, c_2 are absolute positive constants.

For interesting gradient estimates in this direction, we can refer to [1] [2] [7] [8] [9].

Recently, a simple Lichnerowicz equation

$$\triangle u = u^{p-1} - u^{-p-1},$$

where $p > 1$, was studied by Ma [10]. The author obtained a Liouville type result for smooth positive solutions for the Lichnerowicz equation in a complete non-compact Riemannian manifold with the Ricci curvature bounded from below. Later, Sun and Zhao [14] studied a generalized elliptic Lichnerowicz equation

$$\triangle u(x) + h(x)u(x) = A(x)u^p(x) + \frac{B(x)}{u^q(x)}$$

on compact manifold (M, g). The authors in [14] got the local gradient estimate for the positive solutions of the above equation. Moreover, they considered the following parabolic Lichnerowicz equation

$$u_t(x,t) + \triangle u(x,t) + h(x)u(x,t) = A(x)u^p(x,t) + B(x)u^{-q}(x,t)$$

on manifold (M, g) and obtained the Harnack differential inequality.

From the above work, we can see gradient estimates for positive solutions to nonlinear heat equations are interesting subjects to researchers. Gradient estimates often lead to Liouville type theorems and Harnack inequalities. For nonlinear heat equations with drifting Laplacians on manifolds, to get good controls of suitable Harnack quantities (depending on nonlinear terms), one may need the key lower bounds assumption about Bakry–Émery Ricci curvatures. Without the drifting term, the nature assumptions are about the Ricci curvatures. These are the main geometric differences caused by drifting terms. A new research direction is the nonlinear heat equation with negative power, which has its root from the Einstein-scalar Lichnerowicz equation. In this paper, we study the following parabolic equation

$$\frac{\partial u}{\partial t} = \triangle_f u + cu^{-\alpha}, \tag{1.3}$$

where α, c are two real constants and $\alpha > 0$, f is a smooth real-valued function on M.

We state our main results as follows.

Theorem 1.3. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry–Émery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_{2R}(p)$ around $p \in M$. Let u be a positive solution of* (1.3). *Then*

(1) *if $c > 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2}{4\delta\beta(1-\beta)R^2} + A + \frac{1}{t}\right);$$

(2) *if $c < 0$ and $u^{-(\alpha+1)} \leq \tilde{M}$ for all $(x, t) \in B_{2R}(p) \times [0, \infty)$. We have*

$$\begin{aligned}\beta\frac{|\nabla u|^2}{u^2} &+ cu^{-(\alpha+1)} - \frac{u_t}{u} \\ &\leq \frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2}{4\delta\beta(1-\beta)R^2} + A - \frac{(\alpha+2-\beta)}{2(1-\beta)}c\tilde{M}(\alpha+1) + \frac{1}{t}\right),\end{aligned}$$

where $A = ((n-1+\sqrt{nK}R)c_1 + c_2 + 2c_1^2)/R^2$, c_1, c_2, δ are positive constants with $0 < \delta < 1$ and $\beta = e^{-2Kt}$.

Let $R \to \infty$, we can get the following global gradient estimates for the nonlinear parabolic equation (1.3).

Corollary 1.4. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry–Émery Ricci tensor bounded from below by the constant $-K$, where $K > 0$. Let u be a positive solution of* (1.3). *Then*

(1) *if $c > 0$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\frac{1}{t};$$

(2) *if $c < 0$ and $u^{-(\alpha+1)} \leq \tilde{M}$ for all $(x, t) \in M \times [0, \infty)$. We have*

$$\beta\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta}\left(-\frac{(\alpha+2-\beta)}{2(1-\beta)}c\tilde{M}(\alpha+1) + \frac{1}{t}\right),$$

here $0 < \delta < 1$ and $\beta = e^{-2Kt}$.

As an application, we get the following Harnack inequality.

Theorem 1.5. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with $\mathrm{Ric}_f^N > -K$, where $K > 0$. Let $u(x, t)$ be a positive smooth solution to the equation*

$$u_t = \triangle_f u$$

on $M \times [0, +\infty)$. Then for any points (x_1, t_1) and (x_2, t_2) on $M \times [0, +\infty)$ with $0 < t_1 < t_2$, we have the following Harnack inequality:

$$u(x_1, t_1) \leq u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^{(N+n)/2} e^{\phi(x_1,x_2,t_1,t_2)+B},$$

where $\phi(x_1, x_2, t_1, t_2) = inf_\gamma \int_{t_1}^{t_2} 4e^{2Kt}|\dot{\gamma}|^2 dt$, $B = ((N + n)/2)(e^{2Kt_2} - e^{2Kt_1})$ and γ is any space time path joining (x_1, t_2) and (x_2, t_2).

REMARK 1.6. The above Theorem 1.5 has been proved in [6], we can also get this result by letting $c = 0$ and $\delta \to 0$ in Corollary 1.4. We can refer to [6] for detailed proof.

2. Proof of Theorem 1.3

Let u be a positive solution to (1.3). Set $w = \ln u$, then w satisfies the equation

$$w_t = \triangle_f w + |\nabla w|^2 + ce^{-w(\alpha+1)}. \tag{2.1}$$

Theorem 2.1. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry–Émery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_{2R}(p)$ around $p \in M$. For a smooth function w defined on $M \times [0, +\infty)$ satisfies the equation* (2.1), *we have*

$$\begin{aligned}\left(\triangle_f - \frac{\partial}{\partial t}\right)F \geq & -2\nabla w \cdot \nabla F \\ & + t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2 + c(\beta+\alpha)(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2\right\} \\ & + c(\alpha+1)e^{-w(\alpha+1)}F - \frac{F}{t},\end{aligned}$$

where

$$F = t(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t),$$

and $\beta = e^{-2Kt}$.

Proof. Define

$$F = t(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t),$$

where $\beta = e^{-2Kt}$. It is well known that for the N-Bakry–Émery Ricci tensor, we have the Bochner formula:

$$\triangle_f |\nabla w|^2 \geq \frac{2}{N+n} |\triangle_f w|^2 + 2\nabla w \nabla(\triangle_f w) - 2K|\nabla w|^2.$$

Noticing $\triangle_f w_t = (\triangle_f w)_t = -2\nabla w \nabla w_t + c(\alpha+1)e^{-w(\alpha+1)} w_t + w_{tt}$ and

$$\begin{aligned}\triangle_f w &= -|\nabla w|^2 - ce^{-w(\alpha+1)} + w_t \\ &= \left(1 - \frac{1}{\beta}\right)(-ce^{-w(\alpha+1)} + w_t) - \frac{F}{\beta t},\end{aligned}$$

we have

$$\begin{aligned}\triangle_f F &= t(\beta \triangle_f |\nabla w|^2 + c\triangle_f e^{-w(\alpha+1)} - \triangle_f w_t) \\ &= t(\beta \triangle_f |\nabla w|^2) + tc((\alpha+1)^2 e^{-w(\alpha+1)} |\nabla w|^2 - (\alpha+1)e^{-w(\alpha+1)} \triangle_f w) - t\triangle_f w_t \\ &\geq t\bigg\{\frac{2\beta}{N+n}|\triangle_f w|^2 + 2\beta \nabla w \nabla(\triangle_f w) - 2K\beta|\nabla w|^2 + c(\alpha+1)^2 e^{-w(\alpha+1)}|\nabla w|^2 \\ &\qquad - c(\alpha+1)e^{-w(\alpha+1)}\left[\left(1-\frac{1}{\beta}\right)(-ce^{-w(\alpha+1)} + w_t) - \frac{F}{\beta t}\right] \\ &\qquad - (-2\nabla w \nabla w_t + c(\alpha+1)e^{-w(\alpha+1)} w_t + w_{tt})\bigg\} \\ &= t\bigg\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2 - \frac{2}{t}\nabla w \nabla F + 2\beta \nabla w \nabla w_t \\ &\qquad + [(2\beta + \alpha - 1)c(\alpha+1)e^{-w(\alpha+1)} - 2K\beta]|\nabla w|^2 \\ &\qquad + c^2(\alpha+1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} \\ &\qquad + c(\alpha+1)\left(\frac{1}{\beta} - 2\right)w_t - w_{tt} + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta t}\bigg\}\end{aligned}$$

and

$$\begin{aligned}F_t &= (\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t) \\ &\qquad + t(2\beta \nabla w \nabla w_t - c(\alpha+1)e^{-w(\alpha+1)} w_t - w_{tt} - 2K\beta|\nabla w|^2) \\ &= \frac{F}{t} + t(2\beta \nabla w \nabla w_t - c(\alpha+1)e^{-w(\alpha+1)} w_t - w_{tt} - 2K\beta|\nabla w|^2).\end{aligned}$$

Therefore, it follows that

$$
\begin{aligned}
&\left(\triangle_f - \frac{\partial}{\partial t}\right)F \\
&\geq -2\nabla w \nabla F + t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2\right. \\
&\qquad\qquad + [(2\beta+\alpha-1)c(\alpha+1)e^{-w(\alpha+1)}]|\nabla w|^2 \\
&\qquad\qquad \left. + c^2(\alpha+1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} + \frac{\beta-1}{\beta}c(\alpha+1)e^{-w(\alpha+1)}w_t\right\} \\
&\quad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&= -2\nabla w \nabla F + t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2\right. \\
&\qquad\qquad + ((\beta-1)c(\alpha+1)e^{-w(\alpha+1)})\left[|\nabla w|^2 + \frac{ce^{-w(\alpha+1)}}{\beta} - \frac{1}{\beta}w_t\right] \\
&\qquad\qquad \left. + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2\right\} \\
&\quad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&= -2\nabla w \nabla F + t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2\right. \\
&\qquad\qquad + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 \\
&\qquad\qquad \left. + (\beta-1)c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta t}\right\} \\
&\quad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&= -2\nabla w \cdot \nabla F + t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2\right. \\
&\qquad\qquad \left. + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2\right\} \\
&\quad + c(\alpha+1)e^{-w(\alpha+1)}F - \frac{F}{t}.
\end{aligned}
$$

We complete the proof of Theorem 2.1. □

We take a C^2 cut-off function $\tilde{\varphi}$ defined on $[0,\infty)$ such that $\tilde{\varphi}(r) = 1$ for $r \in [0,1]$, $\tilde{\varphi}(r) = 0$ for $r \in [2, \infty)$, and $0 \le \tilde{\varphi}(r) \le 1$. Furthermore $\tilde{\varphi}$ satisfies

$$-\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{1/2}(r)} \le c_1$$

and

$$\tilde{\varphi}''(r) \ge -c_2$$

for some absolute constants $c_1, c_2 > 0$. Denote by $r(x)$ the distance between x and p in M. Set

$$\varphi(x) = \tilde{\varphi}\left(\frac{r(x)}{R}\right).$$

Using an argument of Cheng and Yau [3], we can assume $\varphi(x) \in C^2(M)$ with support in $B_p(2R)$. Direct calculation shows that on $B_p(2R)$

$$\frac{|\nabla\varphi|^2}{\varphi} \le \frac{c_1^2}{R^2}. \tag{2.2}$$

It has been shown by Qian [13] that

$$\triangle_f(r^2) \le n\left(1 + \sqrt{1 + \frac{4Kr^2}{n}}\right).$$

Hence, we have

$$\begin{aligned}\triangle_f(r) &= \frac{1}{2r}(\triangle_f(r^2) - 2|\nabla r|^2)\\ &\le \frac{n-2}{2r} + \frac{n}{2r}\left(1 + \sqrt{1 + \frac{4Kr^2}{n}}\right)\\ &= \frac{n-1}{r} + \sqrt{nK}.\end{aligned}$$

It follows that

$$\triangle_f\varphi = \frac{\tilde{\varphi}''(r)|\nabla r|^2}{R^2} + \frac{\tilde{\varphi}'(r)\triangle_f r}{R} \ge -\frac{(n-1+\sqrt{nK}\,R)c_1 + c_2}{R^2}. \tag{2.3}$$

For $T \ge 0$, let (x,s) be a point in $B_{2R}(p)\times[0,T]$ at which φF attains its maximum value P, and we assume that P is positive (otherwise the proof is trivial). At the point (x, s), we have

$$\nabla(\varphi F) = 0, \quad \triangle_f(\varphi F) \le 0, \quad F_t \ge 0.$$

It follows that

$$\varphi\triangle_f F + F\triangle_f\varphi - 2F\varphi^{-1}|\nabla\varphi|^2 \le 0.$$

This inequality together with the inequalities (2.2) and (2.3) yields

$$\varphi\triangle_f F \le AF, \tag{2.4}$$

where

$$A = \frac{(n-1+\sqrt{nK}R)c_1 + c_2 + 2c_1^2}{R^2}.$$

At (x,s), by Theorem 2.1, we have

$$\begin{aligned}
\varphi\triangle_f F \ge{}& -2\varphi\nabla w\nabla F + s\varphi\bigg\{\frac{2\beta}{N+n}\bigg((\beta-1)|\nabla w|^2 - \frac{F}{s}\bigg)^2 \\
&\qquad + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2\bigg\} \\
&+ c\varphi(\alpha+1)e^{-w(\alpha+1)}F - \varphi\frac{F}{s} \\
\ge{}& -\frac{2c_1}{R}\varphi^{1/2}F|\nabla w| + s\varphi\bigg\{\frac{2\beta}{N+n}\bigg((\beta-1)|\nabla w|^2 - \frac{F}{s}\bigg)^2 \\
&\qquad + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2\bigg\} \\
&+ c\varphi(\alpha+1)e^{-w(\alpha+1)}F - \varphi\frac{F}{s},
\end{aligned}$$

where the last inequality used

$$-2\varphi\nabla w\nabla F = 2F\nabla w\nabla\varphi \ge -2F|\nabla w|\,|\nabla\varphi| \ge -\frac{2c_1}{R}\varphi^{1/2}F|\nabla w|.$$

Therefor, by (2.4), we obtain

$$\begin{aligned}
&\frac{2s\varphi\beta}{N+n}\bigg((\beta-1)|\nabla w|^2 - \frac{F}{s}\bigg)^2 \\
&\le \frac{2c_1}{R}\varphi^{1/2}F|\nabla w| + AF - (\beta+\alpha)cs\varphi(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 \\
&\quad - c\varphi(\alpha+1)e^{-w(\alpha+1)}F + \frac{\varphi F}{s}.
\end{aligned}$$

Following Davies [4] (see also Negrin [12]), we set

$$\mu = \frac{|\nabla w|^2}{F}.$$

Then we have

$$\frac{2\varphi\beta((\beta-1)s\mu-1)^2F^2}{(N+n)s} \le \frac{2c_1}{R}\varphi^{1/2}\mu^{1/2}F^{3/2} + AF - (\beta+\alpha)cs\varphi(\alpha+1)e^{-w(\alpha+1)}\mu F - c\varphi(\alpha+1)e^{-w(\alpha+1)}F + \frac{\varphi F}{s}.$$

Next, we consider the following two cases:

(1) $c > 0$;

(2) $c < 0$.

(1) When $c > 0$, then we have

$$\frac{2\varphi\beta((\beta-1)s\mu-1)^2F^2}{(N+n)s} \le \frac{2c_1}{R}\varphi^{1/2}\mu^{1/2}F^{3/2} + AF + \frac{\varphi F}{s},$$

multiplying both sides of the above inequality by $s\varphi$, we have

$$\begin{aligned}\frac{2\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2 &\le \frac{2c_1}{R}\varphi^{1/2}\mu^{1/2}(\varphi F)^{3/2} + As\varphi F + \varphi F \\ &\le \frac{2\delta\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2 + \frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}\varphi F \\ &\quad + As\varphi F + \varphi F.\end{aligned}$$

So, it follows that

$$P \le \frac{N+n}{2(1-\delta)\beta((\beta-1)s\mu-1)^2}\left(\frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2} + As + 1\right).$$

Since

$$((\beta-1)s\mu-1)^2 \ge 2(1-\beta)s\mu + 1 \ge 2(1-\beta)s\mu,$$

we get

$$P \le \frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2 s}{4\delta\beta(1-\beta)R^2} + As + 1\right).$$

Now, (1) of Theorem 1.3 can be easily deduced from the inequality above.

(2) When $c < 0$, then we have

$$\frac{2\varphi\beta((\beta-1)s\mu-1)^2F^2}{(N+n)s} \le \frac{2c_1}{R}\varphi^{1/2}\mu^{1/2}F^{3/2} + AF - (\beta+\alpha)cs\varphi(\alpha+1)\tilde{M}\mu F - c\tilde{M}(\alpha+1)\varphi F + \frac{\varphi F}{s},$$

multiplying both sides of the above inequality by $s\varphi$, we have

$$\begin{aligned}
&\frac{2\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2\\
&\le \frac{2c_1}{R}\varphi^{1/2}\mu^{1/2}(\varphi F)^{3/2}+As\varphi F-(\beta+\alpha)cs^2\varphi^2(\alpha+1)\tilde{M}\mu F\\
&\quad -c\tilde{M}(\alpha+1)\varphi sF+\varphi F\\
&\le \frac{2\delta\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2+\frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}\varphi F\\
&\quad +As\varphi F-(\beta+\alpha)cs^2\varphi(\alpha+1)\tilde{M}\mu F-c\tilde{M}(\alpha+1)\varphi sF+\varphi F.
\end{aligned}$$

So, it follows that

$$P\le\frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2s}{4\delta\beta(1-\beta)R^2}+As-\frac{(\alpha+2-\beta)}{2(1-\beta)}c\tilde{M}(\alpha+1)s+1\right).$$

Similarly, we can obtain (2) of Theorem 1.3.

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