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L^p-ESTIMATES FOR THE ROUGH SINGULAR INTEGRALS ASSOCIATED TO SURFACES

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Abstract

In this paper we obtain the L^p-boundedness for the maximal functions and the singular integrals associated to surfaces $(y, \phi(|y|))$ with rough kernels, $1 < p < \infty$. The analogue estimate is also established for the corresponding maximal singular integrals.

1. Introduction

Let $K\colon \mathbb{R}^n \to \mathbb{R}$ be a Calderón–Zygmund standard kernel in $\mathbb{R}^n$ $(n \geq 2)$, that is, $K(y) = \Omega(y)/|y|^n$ with $y \neq 0$, where $\Omega(y)$ satisfies

$$\Omega(y) \in C^\infty(\mathbf{S}^{n-1}),$$
$$\Omega(\lambda y) = \Omega(y), \quad \lambda > 0,$$

and

$$\int_{\mathbf{S}^{n-1}} \Omega(y)\, d\sigma(y) = 0. \tag{1.1}$$

Let Γ: $\mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Then, we define the singular integrals $\mathcal{T}$ associated with Γ by the principal-value integral

$$\mathcal{T} f(x) = p.v. \int_{\mathbb{R}^n} f(x - \Gamma(y)) K(y)\, dy, \tag{1.2}$$

where $x \in \mathbb{R}^m$ and $f \in \mathscr{S}(\mathbb{R}^m)$. Similar to the case of classical singular integrals theory, one can define the corresponding maximal functions as

$$\mathcal{M} f(x) = \sup_{h>0} \frac{1}{h^n} \int_{|y|\leq h} |f(x - \Gamma(y))|\, dy.$$

The boundedness of the two operators $\mathcal{T}$ and $\mathcal{M}$ above on $L^p(\mathbb{R}^m)$ has been well studied. We begin with the classical results by Stein, which can be found in [15].

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Theorem A (See [15])**.** *If Γ is any polynomial map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the operators $\mathcal{T}$ and $\mathcal{M}$ are both bounded on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$.*

Moreover, if Γ is a smooth mapping from the unit ball in $\mathbb{R}^n$ to $\mathbb{R}^m$, and of finite type at the origin, then $\mathcal{T}$ and $\mathcal{M}$ are bounded operators on $L^p(\mathbb{R}^m)$ for $1 < p < \infty$.

Later, the theorem above was extended. That is, even in the case Ω is rough, the two results above still holds (see [9] and [10]). Furthermore, $\mathcal{T}$ is bounded on $\dot{F}_\alpha^{p,q}$ for $1 < p, q < \infty$ and $\alpha \in \mathbb{R}$, where Ω is rough and Γ is a polynomial map or a smooth mapping of finite type. More details can be found in [6] and [12].

For $\Gamma(y) = (y, \phi(|y|))$, $y \in \mathbb{R}^n$ and $\phi \in C(\mathbb{R}^+)$, Kim, Wainger, Wright and Ziesler proved the following result in [11].

Theorem B (See [11])**.** *Let $\phi(t)$ be a C^2 function on $[0, \infty)$, and assume that ϕ is convex and increasing on $[0, \infty)$, and $\phi(0) = 0$. Then, for $1 < p < \infty$, there exists a positive constant A_p such that*

$$\|\mathcal{T} f\|_{L^p} \le A_p \|f\|_{L^p} \quad \textit{and} \quad \|\mathcal{M} f\|_{L^p} \le A_p \|f\|_{L^p} \quad (f \in L^p).$$

In this case, the L^p-boundedness for the singular integrals in (1.2) with rough kernel is studied by Chen–Fan [5] and Lu–Pan–Yang [13].

Let $P(t)$ be a real-valued polynomial of t in $\mathbb{R}$, and assume that γ satisfies

$$\gamma \in C^2[0, \infty), \quad \text{convex on} \quad [0, \infty) \quad \text{and} \quad \gamma(0) = 0.$$

In this paper, we consider the hypersurface parameterized by $\Gamma \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$, where Γ is given by

$$\Gamma(y) = (y, P(\gamma(|y|))), \quad y \in \mathbb{R}^n.$$

Then, the operators $\mathcal{T}$ and $\mathcal{M}$ above take the form

$$\mathcal{T} f(u) = p.v. \int_{\mathbb{R}^n} f(x - y, s - P(\gamma(|y|)))K(y)\, dy \tag{1.3}$$

and

$$\mathcal{M} f(u) = \sup_{h>0} \frac{1}{h^n} \int_{|y|\le h} |f(x - y, s - P(\gamma(|y|)))|\, |\Omega(y)|\, dy, \tag{1.4}$$

where $x \in \mathbb{R}^n$, $s \in \mathbb{R}$ and $u = (x, s)$, K is the Calderón–Zygmund standard kernel as before.

For the L^p-boundedness of the singular integrals $\mathcal{T}$ in (1.3) and the maximal functions $\mathcal{M}$ in (1.4), Bez proved the following theorem in [1].

Theorem C (See [1]). *For* $\mathcal{T}$ *in* (1.3) *and* $\mathcal{M}$ *in* (1.4), *if* $\gamma'(0) \geq 0$, $\Omega \in C^\infty(\mathbf{S}^{n-1})$, *then, for* $1 < p < \infty$, *there exists a positive constant* C *only dependent on* p, n, γ *and the degree of* P *such that*

$$\|\mathcal{T} f\|_{L^p} \leq C\|f\|_{L^p} \quad \textit{and} \quad \|\mathcal{M} f\|_{L^p} \leq C\|f\|_{L^p} \quad (f \in L^p).$$

REMARK 1.1. One may notice that there is a little difference between the maximal function in (1.4) and that in Bez's paper [1], we represent the maximal function in this form just for convenient. But Bez's results still hold, since $C^\infty(\mathbf{S}^{n-1}) \subset L^\infty(\mathbf{S}^{n-1})$.

Besides the operators $\mathcal{T}$ and $\mathcal{M}$ above, we also consider the corresponding maximal singular integrals

$$\mathcal{T}^* f(u) = \sup_{\varepsilon>0}\left|\int_{|y|\geq\varepsilon} f(x-y, s-P(\gamma(|y|)))K(y)\,dy\right|. \tag{1.5}$$

Appropriate estimates for the maximal singular integrals give the pointwise existence of the principle value singular integrals.

REMARK 1.2. For $n = 1$, if Γ satisfies a 'finite type condition' at origin in $\mathbb{R}^m$, the L^p-estimates for the Hilbert transform, the maximal function and the maximal Hilbert transform can be found in the survey [14] of results through 1978. For other one-dimensional curves Γ, there are considerable results about the L^p-estimates for the Hilbert transform and the maximal function, see [2], [7] and [8] for example. Specially, the maximal Hilbert transform has been discussed in detail in [8].

The purpose of this note is to study the L^p-boundedness for $\mathcal{T}$ in (1.3) and $\mathcal{M}$ in (1.4), also, the analogue estimate for the maximal singular integrals $\mathcal{T}^*$ in (1.5) is considered. Main results are presented as follows.

Theorem 1.3. *Let* $\mathcal{T}$ *and* $\mathcal{M}$ *be given as in* (1.3) *and* (1.4), *respectively. If* $\gamma'(0) \geq 0$ *and* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $1 < q \leq \infty$, *then* $\mathcal{T}$ *and* $\mathcal{M}$ *are bounded on* $L^p(\mathbb{R}^{n+1})$ *for* $1 < p < \infty$.

REMARK 1.4. Note that $C^\infty(\mathbf{S}^{n-1}) \subset L^q(\mathbf{S}^{n-1})$ for $1 < q \leq \infty$, so, Theorem 1.3 improves and extends Theorem C. Also, Theorem B is a special case of Theorem 1.3 for $P(t) = t$. Further, the L^p-boundedness for $\mathcal{M}$ can be proved by using Calderón–Zygmund's rotation method with $\Omega \in L^1(\mathbf{S}^{n-1})$, if either
(1) $P'(0) = 0$, or
(2) $P'(0) \neq 0$ and $\gamma'(\lambda t) \geq 2\lambda'(t)$ for some $\lambda > 1$.

Theorem 1.5. *Let* $\mathcal{T}^*$ *be given as in* (1.5). *If* $\gamma'(0) \geq 0$ *and* $\Omega \in L^q(\mathbf{S}^{n-1})$ *for some* $1 < q \leq \infty$, *then* $\mathcal{T}^*$ *is bounded on* $L^p(\mathbb{R}^{n+1})$ *for* $1 < p < \infty$.

This paper is organized as follows. In Section 2 we list some key properties concerning polynomials of one variable and give some fundamental lemmas for the proof of main results. The L^p-boundedness of $\mathcal{M}$ and $\mathcal{T}$ is proved following the arguments of Bez [1] and Carbery et al. [2] in Section 3 and Section 4, respectively. The last section contains the proof of Theorem 1.5, where we use the ideas of Córdoba and Rubio de Francia [8].

2. Preliminaries

Without loss of generality, we suppose that $P(t) = \sum_{k=1}^{d} p_k t^k$, where $d \geq 2$. Let $z_1, z_2, \cdots, z_d$ be the d complex roots of P ordered as

$$0 = |z_1| \leq |z_2| \leq \cdots \leq |z_d|.$$

Let $A > 1$, whose value we fix in Lemma 2.1. Define $G_j = (A|z_j|, A^{-1}|z_{j+1}|]$ if it is nonempty for $1 \leq j < d$ and $G_d = (A|z_d|, \infty)$. Let $\mathcal{J} = \{j \colon G_j \neq \emptyset\}$, then, $(0, \infty) \setminus \bigcup_{j\in\mathcal{J}} G_j$ can be decomposed as $\bigcup_{k\in\mathcal{K}} D_k$, where D_k is the interval between G_k and adjacent G_{k+l} for some $l \geq 1$, it it obvious that D_k's are disjoint. Then, we can split $(0, \infty)$ as

$$(0, \infty) = \bigcup_{j\in\mathcal{J}} \gamma^{-1}(G_j) \cup \bigcup_{k\in\mathcal{K}} \gamma^{-1}(D_k),$$

where $\gamma^{-1}(I) = \{t \in (0, \infty) \colon \gamma(t) \in I\}$.

The properties of P on D_k and G_j are important for our proof, the following related lemma can be found in [1] and [3].

Lemma 2.1. *There exists a constant $C_d > 1$ such that for any $A \geq C_d$ and any $j \in \mathcal{J}$,*
(1) $|P(t)| \sim |p_j|\,|t|^j$ *for* $|t| \in G_j$;
(2) $P'(t)/P(t) > 0$ *for* $t \in G_j$, $P'(t)/P(t) < 0$ *for* $-t \in G_j$;
(3) $|P'(t)/P(t)| \sim 1/|t|$ *for* $|t| \in G_j$;
(4) $P''(t)/P(t) > 0$ *and* $P''(t)/P(t) \sim 1/t^2$ *for* $|t| \in G_j$, $j \in \mathcal{J} \setminus \{1\}$.

The following trivial fact follows the proof of Lemma 2.1 (see [1]), that is, we can choose $A > 0$ such that for $|t| \in G_j$,

$$|P(t)| \leq 2|p_j|\,|t|^j \quad \text{and} \quad \frac{1}{2} j|p_j|\,|t|^{j-1} \leq |P'(t)| \leq 2j|p_j|\,|t|^{j-1}. \tag{2.1}$$

Let $\rho = n + 2$, for $I \subset (0, \infty)$, $\mathcal{M}_I$ and $\mathcal{T}_I$ are given by

$$\mathcal{M}_I f(u) = \sup_{k\in\mathbb{Z}} \frac{1}{\rho^{nk}} \int_{|y|\in\gamma^{-1}(I)\cap(\rho^k,\rho^{k+1}]} |f(x-y, s-P(\gamma(|y|)))|\,|\Omega(y)|\,dy,$$

and

$$\mathcal{T}_I f(u) = \int_{|y|\in\gamma^{-1}(I)} f(x-y, s-P(\gamma(|y|)))K(y)\,dy.$$

For $k \in \mathbb{Z}$ and $j \in \mathcal{J}$, let

$$A_{k,j} = \begin{pmatrix} \rho^k & 0 & \cdots & 0 \\ 0 & \rho^k & 0 & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & |p_j|\gamma^j(\rho^k) \end{pmatrix}_{(n+1)\times(n+1)},$$

then, $A_{k,j}$ satisfies Rivière condition, that is $\|A_{k+1,j}^{-1}A_{k,j}\| \le \alpha < 1$. In fact,

$$A_{k+1,j}^{-1}A_{k,j} = \begin{pmatrix} \rho^{-1}I_n & 0 \\ 0 & \left(\frac{\gamma(\rho^k)}{\gamma(\rho^{k+1})}\right)^j \end{pmatrix}.$$

Note that γ is convex, $\gamma(t)/t \le \gamma(s)/s$ for $0 < t \le s$, therefore,

$$\left(\frac{\gamma(\rho^k)}{\gamma(\rho^{k+1})}\right) \le \frac{1}{\rho} < 1.$$

We choose $\phi \in C^\infty(\mathbb{R}^{n+1})$ such that $\hat{\phi}(\zeta) = 1$ for $|\zeta| \le 1$ and $\hat{\phi}(\zeta) = 0$ for $|\zeta| \ge 2$. For $k \in \mathbb{Z}$ and $j \in \mathcal{J}$, the multiplier $m_{k,j}$ is defined by

$$m_{k,j}(\zeta) = \hat{\phi}(A_{k,j}^*\zeta) - \hat{\phi}(A_{k+1,j}^*\zeta),$$

where $A_{k,j}^*$ is the adjoint of $A_{k,j}$. Then, we define the operator $S_{k,j}$ by

$$(S_{k,j} f)^\wedge(\zeta) = m_{k,j}(\zeta)\hat{f}(\zeta).$$

In the next proposition, we state a useful result for future reference.

Proposition 2.2. *For any $j \in \mathcal{J}$, if $m_{l+k,j}(\zeta) \ne 0$ for some $k, l \in \mathbb{Z}$, then*

(2.2) $$|A_{k,j}^*\zeta| \ge C\rho^{-l}, \quad l < 0;$$

and

(2.3) $$|A_{k+1,j}^*\zeta| \le C\rho^{-l}, \quad l > 0.$$

Proof. If $m_{l+k,j}(\zeta) \neq 0$, then $|A^*_{l+k,j}\zeta| \leq 2$ and $|A^*_{l+k+1,j}\zeta| > 1$. For $l < 0$, by the convexity of γ,

$$1 < |A^*_{l+k+1,j}\zeta| \leq \rho^{l+1}|A^*_{k,j}\zeta|,$$

that is (2.2). When $l > 0$,

$$2 \geq |A^*_{l+k,j}\zeta| \geq \rho^{l-1}|A^*_{k+1,j}\zeta|,$$

then, (2.3) is obtained. □

We need the following Littlewood–Paley theorem, which can be found in [2] and [4].

Lemma 2.3. *For $m_{k,j}$ and $S_{k,j}$ above, we have the following properties*:
(i) *for each ζ at most C_0 of the $m_{k,j}(\zeta)$ are not zero*;
(ii) *for each $\zeta \neq 0$, $\sum_{k\in\mathbb{Z}} m_{k,j}(\zeta) = 1$*;
(iii) $\big\|\big(\sum_{k\in\mathbb{Z}}|S_{k,j}f|^2\big)^{1/2}\big\|_{L^p} \leq C_p\|f\|_{L^p}$, $1 < p < \infty$;
(iv) $\big\|\sum_{k\in\mathbb{Z}} S_{k,j}f_k\big\|_{L^p} \leq C_p\big\|\big(\sum_{k\in\mathbb{Z}}|S_{k,j}f_k|^2\big)^{1/2}\big\|_{L^p}$, $1 < p < \infty$.

3. The L^p-boundedness for $\mathcal{M}$

It is trivial that

$$\mathcal{M}f(u) \leq C\Bigg[\sum_{k\in\mathcal{K}}\mathcal{M}_{D_k}f(u) + \sum_{j\in\mathcal{J}}\mathcal{M}_{G_j}f(u)\Bigg].$$

Note that the cardinalities of $\mathcal{K}$ and $\mathcal{J}$ are less than d, so we just need to verify that $\mathcal{M}_{D_k}$ and $\mathcal{M}_{G_j}$ are L^p-bounded for each $k \in \mathcal{K}$ and $j \in \mathcal{J}$.

3.1. The L^p-bounedness for $\mathcal{M}_{D_k}$. For any $u \in \mathbb{R}^{n+1}$, there exists an integer $j(u)$ such that

$$\mathcal{M}_{D_k}f(u) \leq \frac{2}{\rho^{nj(u)}}\int_{|y|\in\gamma^{-1}(D_k)\cap(\rho^{j(u)},\rho^{j(u)+1}]}|f(x-y, s-P(\gamma(|y|)))|\,|\Omega(y)|\,dy.$$

Then, by Minkowski's inequality, the L^p-norm of $\mathcal{M}_{D_k}f$ can be dominated by

$$\Bigg(\int_{\mathbb{R}^{n+1}}\Bigg[\frac{1}{\rho^{nj(u)}}\int_{|y|\in\gamma^{-1}(D_k)\cap(\rho^{j(u)},\rho^{j(u)+1}]}|f(x-y, s-P(\gamma(|y|)))|\,|\Omega(y)|\,dy\Bigg]^p du\Bigg)^{1/p}$$

$$\leq \int_{|y|\in\gamma^{-1}(D_k)}\frac{|\Omega(y)|}{|y|^n}\Bigg(\int_{\mathbb{R}^{n+1}}|f(x-y, s-P(\gamma(|y|)))|^p\,du\Bigg)^{1/p} dy$$

$$\leq C\|f\|_{L^p}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\int_{r\in\gamma^{-1}(D_k)}\frac{1}{r}\,dr.$$

Let $D_k = (A^{-1}|z_j|, A|z_{j+l}|]$ for some $2 \le j \le d$ and $0 \le l \le d - j$, then

$$A^{-1}|z_j| \le A^{-1}|z_{j+1}| \le A|z_j| \le \cdots \le A|z_{j+l}| < A^{-1}|z_{j+l+1}|$$

and

$$A^2 \le \frac{A|z_{j+l}|}{A^{-1}|z_j|} \le \frac{A|z_{j+l}|}{A^{-2l-1}|z_{j+l}|} \le A^{2l+2}.$$

Notice that γ is convex and $\gamma(0) = 0$, so, $\gamma(t) \le t\gamma'(t)$ for $t > 0$. Thus,

$$\begin{aligned}\int_{r\in\gamma^{-1}(D_k)} \frac{1}{r}\,dr &= \int_{\gamma^{-1}(A^{-1}|z_j|)}^{\gamma^{-1}(A|z_{j+l}|)} \frac{1}{r}\,dr = \int_{A^{-1}|z_j|}^{A|z_{j+l}|} \frac{1}{\gamma^{-1}(r)\gamma'(\gamma^{-1}(r))}\,dr\\ &\le \int_{A^{-1}|z_j|}^{A|z_{j+l}|} \frac{1}{r}\,dr \le 2d \ln A,\end{aligned}$$

where $\gamma^{-1}(t)$ is the inverse function of $\gamma(t)$.

According to the calculation above, the L^p-bounedness for $\mathcal{M}_{D_k}$ is established,

$$\|\mathcal{M}_{D_k} f\|_{L^p} \le C\|f\|_{L^p}, \quad \text{for} \quad 1 < p < \infty,\ k \in \mathcal{K}.$$

3.2. The L^p-bounedness for $\mathcal{M}_{G_j}$. Next, we verify that $\mathcal{M}_{G_j}$ is L^p-bounded for $j \in \mathcal{J}$. The maximal operators $\mathcal{M}_{G_j}$ can be expressed as

$$\mathcal{M}_{G_j} f(u) = \sup_{k\in\mathbb{Z}} \int_{|y|\in\rho^{-k}\gamma^{-1}(G_j)\cap(1,\rho]} |f(x - \rho^k y, s - P(\gamma(|\rho^k y|)))|\,|\Omega(y)|\,dy.$$

Set $I_{k,j} = (1, \rho] \cap \rho^{-k}\gamma^{-1}(G_j)$, and define the measure $\mu_{k,j}$ by

$$\langle \mu_{k,j}, \psi\rangle = \int_{|y|\in I_{k,j}} \psi(\rho^k y, P(\gamma(|\rho^k y|)))|\Omega(y)|\,dy$$

for $\psi \in \mathscr{S}(\mathbb{R}^{n+1})$. Then, for $j \in \mathcal{J}$, $\mathcal{M}_{G_j} f$ also can be rewritten as

$$\mathcal{M}_{G_j} f(u) = \sup_{k\in\mathbb{Z}} \mu_{k,j} * |f|(u).$$

We also need to define the measure $\sigma_{k,j}$ by

$$\langle \sigma_{k,j}, \psi\rangle = \frac{\hat{\mu}_{k,j}(0)}{|A_{k+1,j}B|} \int_{A_{k+1,j}B} \psi(u)\,du,$$

where $B = \{u \in \mathbb{R}^{n+1} : |u| \le n + 1\}$.

3.2.1. Fourier transform estimates for related measures.

Proposition 3.1. *For $j \in \mathcal{J}$ and $k \in \mathbb{Z}$, then there exists $C > 0$ and $\beta > 0$ independent of j and k such that*

$$|\hat{\mu}_{k,j}(\zeta)|, |\hat{\sigma}_{k,j}(\zeta)| \le C \max\{|A^*_{k,j}\zeta|^{-1}, |A^*_{k,j}\zeta|^{-\beta}\} \tag{3.1}$$

and

$$|\hat{\mu}_{k,j}(\zeta) - \hat{\sigma}_{k,j}(\zeta)| \le C|A^*_{k+1,j}\zeta|. \tag{3.2}$$

Proof. The main idea of the following proof comes from the work of Bez (see [1]). For completeness, we show more details.

Let $\zeta = (\xi, \eta)$, where $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$. For $k \in \mathbb{Z}$ and $j \in \mathcal{J}$, we have

$$\begin{aligned} |\hat{\mu}_{k,j}(\zeta)| &= \left| \int_{|y| \in I_{k,j}} e^{-i[\rho^k y \cdot \xi + \eta P(\gamma(\rho^k |y|))]} |\Omega(y)|\, dy \right| \\ &\le \int_{I_{k,j}} \left| \int_{\mathbf{S}^{n-1}} e^{-i\rho^k t y' \cdot \xi} |\Omega(y')|\, d\sigma(y') \right| dt. \end{aligned}$$

Set $I_k(t) = \int_{\mathbf{S}^{n-1}} e^{-i\rho^k t y' \cdot \xi} |\Omega(y')|\, d\sigma(y')$, by Hölder's inequality,

$$\begin{aligned} |\hat{\mu}_{k,j}(\zeta)|^2 &\le C \int_{I_{k,j}} |I_k(t)|^2\, dt \\ &\le C \int_{(\mathbf{S}^{n-1})^2} |\Omega(y')|\, |\Omega(z')| \left| \int_{I_{k,j}} e^{i\rho^k t \xi \cdot (y' - z')}\, dt \right| d\sigma(y')\, d\sigma(z'). \end{aligned}$$

By van der Corput's lemma, for any $\alpha \in (0, 1)$, we have

$$\begin{aligned} \left| \int_{I_{k,j}} e^{i\rho^k t \xi \cdot (y' - z')}\, dt \right| &\le C \min\{1, |\rho^k \xi \cdot (y' - z')|^{-1}\} \\ &\le C(\rho^k |\xi|)^{-\alpha} |\xi' \cdot (y' - z')|^{-\alpha}. \end{aligned}$$

If $q = \infty$, it is trivial, we set $\beta = 1/2$. For $q \in (1, \infty)$, specially, we choose a positive constant α so that $\alpha q' < 1$. By Hölder's inequality, we get

$$\begin{aligned} |\hat{\mu}_{k,j}(\zeta)|^2 &\le C(\rho^k |\xi|)^{-\alpha} \int_{(\mathbf{S}^{n-1})^2} |\Omega(y')|\, |\Omega(z')| \frac{d\sigma(y')\, d\sigma(z')}{|\xi' \cdot (y' - z')|^{\alpha}} \\ &\le C(\rho^k |\xi|)^{-\alpha} \left(\int_{(\mathbf{S}^{n-1})^2} |\Omega(y')|^q |\Omega(z')|^q\, d\sigma(y')\, d\sigma(z') \right)^{1/q} \\ &\quad \times \left(\int_{(\mathbf{S}^{n-1})^2} \frac{d\sigma(y')\, d\sigma(z')}{|\xi' \cdot (y' - z')|^{\alpha q'}} \right)^{1/q'} \\ &\le C \|\Omega\|^2_{L^q(\mathbf{S}^{n-1})} (\rho^k |\xi|)^{-\alpha}. \end{aligned}$$

Finally, there exists a constant $\beta \in (0, 1/(2q'))$ such that

$$|\hat{\mu}_{k,j}(\zeta)| \leq C(\rho^k|\xi|)^{-\beta}. \tag{3.3}$$

CASE 1. $j \in \mathcal{J} \setminus \{1\}$. If ζ satisfies $4\rho^k|\xi| \geq |p_j|\gamma^j(\rho^k)|\eta|$, then, $|A^*_{k,j}\zeta| \leq \sqrt{17}\rho^k|\xi|$. Therefore, (3.3) implies $|\hat{\mu}_{k,j}(\zeta)| \leq C|A^*_{k,j}\zeta|^{-\beta}$.

If ζ satisfies $4\rho^k|\xi| < |p_j|\gamma^j(\rho^k)|\eta|$, in order to estimate $|\hat{\mu}_{k,j}(\zeta)|$, we need the following lemma which is Lemma 2.2 in [1].

Lemma 3.2. *For all $j \in \mathcal{J} \setminus \{1\}$, the function*

$$t \mapsto P''(\gamma(\rho^k t))\gamma'(\rho^k t)^2 + P'(\gamma(\rho^k t))\gamma''(\rho^k t)$$

is singled-signed on $I_{k,j}$.

On the other hand,

$$|\hat{\mu}_{k,j}(\zeta)| \leq \int_{\mathbf{S}^{n-1}} \left| \int_{I_{k,j}} e^{-i[\rho^k t y'\cdot\xi + \eta P(\gamma(\rho^k t))]}\, dt \right| |\Omega(y')|\, d\sigma(y').$$

For fixed $y' \in \mathbf{S}^{n-1}$, let $h_k(t) = \rho^k t y' \cdot \xi + \eta P(\gamma(\rho^k t))$. For $t \in I_{k,j}$, by (2.1) and the convexity of γ, we have

$$\begin{aligned} |h_k'(t)| &\geq |\rho^k P'(\gamma(\rho^k t))\gamma'(\rho^k t)\eta| - |\rho^k \xi| \\ &\geq \frac{1}{2} j|p_j|\rho^k \gamma^{j-1}(\rho^k t)\gamma'(\rho^k t)|\eta| - \rho^k|\xi| \geq \frac{1}{2} j|p_j|\gamma^j(\rho^k)|\eta| - \rho^k|\xi|. \end{aligned} \tag{3.4}$$

Note that $4\rho^k|\xi| < |p_j|\gamma^j(\rho^k)|\eta|$ and $|A^*_{k,j}\zeta| \leq (\sqrt{17}/|p_j|)\gamma^j(\rho^k)|\eta|$. Hence,

$$|h_k'(t)| \geq \frac{1}{4}|p_j|\gamma^j(\rho^k)|\eta| \geq \frac{1}{\sqrt{17}}|A^*_{k,j}\zeta|. \tag{3.5}$$

For $j \in \mathcal{J} \setminus \{1\}$, $h_k'(t)$ is monotone on $I_{k,j}$ by Lemma 3.2. By van der Corput's lemma and (3.5), we get

$$|\hat{\mu}_{k,j}(\zeta)| \leq C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}(|p_j|\gamma^j(\rho^k)|\eta|)^{-1} \leq C|A^*_{k,j}\zeta|^{-1}.$$

CASE 2. $j = 1$. If ζ satisfies $|\xi| \geq (1/4)|p_1|\gamma'(\rho^k)|\eta|$, by the convexity of γ, then, $\rho^k|\xi| \geq (1/4)|p_1|\gamma(\rho^k)|\eta|$ and $|A^*_{k,1}\zeta| \leq \sqrt{17}\rho^k|\xi|$. According to (3.3), we obtain

$$|\hat{\mu}_{k,1}(\zeta)| \leq C|A^*_{k,1}\zeta|^{-\beta}.$$

If ζ satisfies $|\xi| < (1/4)|p_1|\gamma'(\rho^k)|\eta|$, (3.4) implies

$$|h_k'(t)| \geq \frac{1}{2}|p_1|\rho^k\gamma'(\rho^k t)|\eta| - \rho^k|\xi| \geq \frac{1}{4}|p_1|\rho^k\gamma'(\rho^k t)|\eta| \geq \frac{1}{4}|p_1|\rho^k\gamma'(\rho^k)|\eta|. \tag{3.6}$$

Integration by parts and (3.6) show that

$$\begin{aligned}\left|\int_{I_{k,1}} e^{-i[\rho^k t y'\cdot\xi + \eta P(\gamma(\rho^k t))]}\,dt\right| &= \left|\int_{I_{k,1}} e^{-ih_k(t)}h_k'(t)\frac{dt}{h_k'(t)}\right| \\ &\leq 8(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1} + \int_{I_{k,1}} \frac{|h_k''(t)|}{[h_k'(t)]^2}\,dt.\end{aligned}$$

Essentially, we just need to consider the second term, which can be dominated by

$$\int_{I_{k,1}} \frac{\rho^{2k}|\eta|\,|P'(\gamma(\rho^k t))|\gamma''(\rho^k t)}{h_k'(t)^2}\,dt + \int_{I_{k,1}} \frac{\rho^{2k}|\eta|\,|P''(\gamma(\rho^k t))|\gamma'(\rho^k t)^2}{h_k'(t)^2}\,dt := \alpha_1 + \alpha_2.$$

In order to estimate the term α_1, we define $\varphi_k(t) = \rho^k t|\xi| + |p_1|\gamma(\rho^k t)|\eta|$, then, $\varphi_k'(t) = \rho^k|\xi| + |p_1|\rho^k\gamma'(\rho^k t)|\eta|$. By (3.6), for $t \in I_{k,1}$, it is obvious that

$$|\varphi_k'(t)| \leq \frac{5}{4}|p_1|\gamma'(\rho^k t)\rho^k|\eta| \leq 5h_k'(t). \tag{3.7}$$

On the other hand, for $t \in I_{k,1}$,

$$|\varphi_k'(t)| \geq |p_1|\rho^k\gamma'(\rho^k t)|\eta| - \rho^k|\xi| \geq \frac{3}{4}|p_1|\rho^k\gamma'(\rho^k t)|\eta|. \tag{3.8}$$

Also, by (2.1), for $t \in I_{k,1}$,

$$\varphi_k''(t) = |p_1|\rho^{2k}\gamma''(\rho^k t)|\eta| \geq \frac{1}{2}\rho^{2k}|\eta|\,|P'(\gamma(\rho^k t))|\gamma''(\rho^k t). \tag{3.9}$$

Thus, in view of (3.7), (3.9) and (3.8), we have

$$\alpha_1 \leq C\int_{I_{k,1}} \frac{\varphi_k''(t)}{\varphi_k'(t)^2}\,dt \leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1}. \tag{3.10}$$

For α_2, by (3.6) and (2.1),

$$\begin{aligned}\alpha_2 &\leq C\int_{I_{k,1}} \frac{\rho^{2k}|\eta|\,|P''(\gamma(\rho^k t))|\gamma'(\rho^k t)^2}{[|p_1|\rho^k\gamma'(\rho^k t)|\eta|]^2}\,dt \\ &\leq C\int_{I_{k,1}} |p_1|^{-1}|P''(\gamma(\rho^k t))|\rho^k\gamma'(\rho^k t)\frac{1}{|p_1|\rho^k\gamma'(\rho^k t)|\eta|}\,dt \\ &\leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1}\int_{G_1} |p_1|^{-1}|P''(t)|\,dt \\ &\leq C(|p_1|\rho^k\gamma'(\rho^k)|\eta|)^{-1}.\end{aligned} \tag{3.11}$$

Note that $|A^*_{k,1}\zeta| \le (\sqrt{17}/4)|p_1|\rho^k\gamma'(\rho^k)|\eta|$. Then, (3.10) and (3.11) imply

$$|\hat{\mu}_{k,1}(\zeta)| \le C|A^*_{k,1}\zeta|^{-1}.$$

For $\hat{\sigma}_{k,j}$, we have

$$|\hat{\sigma}_{k,j}(\zeta)| = \frac{\hat{\mu}_{k,j}(0)}{|B|}\left|\int_B e^{-iu\cdot A^*_{k+1,j}\zeta}\,du\right| \le C|A^*_{k,j}\zeta|^{-1}.$$

According to the estimates for $\hat{\mu}_{k,j}$ and $\hat{\sigma}_{k,j}$ above, we obtain (3.1). (3.2) can be proved as follows,

$$\begin{aligned}|\hat{\mu}_{k,j}(\zeta)-\hat{\sigma}_{k,j}(\zeta)| &\le |\hat{\mu}_{k,j}(\zeta)-\hat{\mu}_{k,j}(0)| + |\hat{\mu}_{k,j}(0)|\,|\hat{\sigma}_{k,j}(\zeta)-1|\\ &\le \int_{|y|\in I_{k,j}} |e^{-i[\rho^k y\cdot\xi+\eta P(\gamma(\rho^k|y|))]}-1|\,|\Omega(y)|\,dy\\ &\quad+\frac{\|\Omega\|_{L^1(S^{n-1})}}{|B|}\int_B |e^{-iu\cdot A^*_{k+1,j}\zeta}-1|\,du\\ &\le C|A^*_{k+1,j}\zeta|.\end{aligned}$$

□

3.2.2. The L^p-norm of $\mathcal{M}_{G_j}f$. For the maximal operators $\mathcal{M}_{G_j}$, it can be dominated by

$$\begin{aligned}\mathcal{M}_{G_j}f(u) &\le \sup_{k\in\mathbb{Z}}\sigma_{k,j}*f(u)+\sup_{k\in\mathbb{Z}}|(\mu_{k,j}-\sigma_{k,j})*f|(u)\\ &\le \mathcal{M}_s f(u)+\sup_{k\in\mathbb{Z}}|(\mu_{k,j}-\sigma_{k,j})*f|(u),\end{aligned}$$

where $\mathcal{M}_s$ denotes the strong maximal function.

We first consider the L^2-estimates for $\mathcal{M}_{G_j}$. It is known that $\mathcal{M}_s$ is L^p bounded for $1<p\le\infty$, thus, it suffices to consider the L^2-norm of $\sup_{k\in\mathbb{Z}}|(\mu_{k,j}-\sigma_{k,j})*f|$. In view of Lemma 2.3, we have

$$\begin{aligned}&|(\mu_{k,j}-\sigma_{k,j})*f|\\ &\le \left|\sum_{l\le 0}\mu_{k,j}*S_{l+k,j}f\right|+\left|\sum_{l\le 0}\sigma_{k,j}*S_{l+k,j}f\right|+\left|\sum_{l=1}^{\infty}(\mu_{k,j}-\sigma_{k,j})*S_{l+k,j}f\right|\\ &:=\mathcal{A}_{k,j}+\mathcal{B}_{k,j}+\mathcal{C}_{k,j}.\end{aligned}\tag{3.12}$$

The L^2-norm of the supremums of $\mathcal{A}_{k,j}$, $\mathcal{B}_{k,j}$ and $\mathcal{C}_{k,j}$ are considered separately. Now, the supremum of $\mathcal{A}_{k,j}$ is controlled by

$$\sup_{k\in\mathbb{Z}}\mathcal{A}_{k,j} \le \sum_{l\le 0}\sup_{k\in\mathbb{Z}}|\mu_{k,j}*S_{l+k,j}f| \le \sum_{l\le 0}\left(\sum_{k\in\mathbb{Z}}|\mu_{k,j}*S_{l+k,j}f|^2\right)^{1/2} := \sum_{l=-\infty}^{0}\mathcal{E}_{l,j}f.$$

For each integer $l \le 0$, by Plancherel's theorem, (3.1) and (2.2),

$$\|\mathcal{E}_{l,j} f\|_{L^2} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^{n+1}} |\hat{\mu}_{k,j}(\zeta)|^2 |m_{l+k,j}(\zeta)|^2 |\hat{f}(\zeta)|^2 d\zeta\right)^{1/2} \le C\rho^{\beta l}\|f\|_{L^2}. \tag{3.13}$$

Then, by the triangle inequality in L^2, we have

$$\left\|\sup_{k\in\mathbb{Z}} \mathcal{A}_{k,j}\right\|_{L^2} \le C\|f\|_{L^2}. \tag{3.14}$$

The L^2-norm of $\sup_{k\in\mathbb{Z}} \mathcal{B}_{k,j}$ can be considered in the same way, therefore,

$$\left\|\sup_{k\in\mathbb{Z}} \mathcal{B}_{k,j}\right\|_{L^2} \le C\|f\|_{L^2}. \tag{3.15}$$

Similarly, for $\sup_{k\in\mathbb{Z}} \mathcal{C}_{k,j}$, we have

$$\sup_{k\in\mathbb{Z}} \mathcal{C}_{k,j} \le \sum_{l=1}^{\infty}\left(\sum_{k\in\mathbb{Z}} |(\mu_{k,j} - \sigma_{k,j}) * S_{l+k,j} f|^2\right)^{1/2} := \sum_{l=1}^{\infty} \mathcal{F}_{l,j} f.$$

For each integer $l \ge 1$, by Plancherel's theorem, (3.2) and (2.3), $\|\mathcal{F}_{l,j} f\|_{L^2} \le C\rho^{-l}\|f\|_{L^2}$. Furthermore,

$$\left\|\sup_{k\in\mathbb{Z}} \mathcal{C}_{k,j}\right\|_{L^2} \le C\|f\|_{L^2}. \tag{3.16}$$

Then, combining (3.12), (3.14), (3.15) with (3.16), we have

$$\|\mathcal{M}_{G_j} f\|_{L^2} \le C\|f\|_{L^2}. \tag{3.17}$$

For the L^p-boundedness of $\mathcal{M}_{G_j}$ with $p \ne 2$, we need the following lemma, which is Lemma 4 in [8].

Lemma 3.3. *Suppose that $U_k f = u_k * f$ is a sequence of positive operators uniformly bounded on L^∞ and $U^* f = \sup_{k\in\mathbb{Z}} |u_k * f|$ is bounded on L^r, then, for $p > 2r/(1+r)$, there exists a positive constant C_p such that*

$$\left\|\left(\sum_{k\in\mathbb{Z}} |u_k f_k|^2\right)^{1/2}\right\|_{L^p} \le C_p \left\|\left(\sum_{k\in\mathbb{Z}} |f_k|^2\right)^{1/2}\right\|_{L^p}, \quad \{f_k\} \in L^p(l^2).$$

By (3.17), Lemma 3.3 and Lemma 2.3, for $p > 4/3$, we get

$$\begin{aligned} \|\mathcal{E}_{l,j}\|_{L^p} &= \left\| \left(\sum_{k\in\mathbb{Z}} |\mu_{k,j} * S_{l+k,j} f|^2 \right)^{1/2} \right\|_{L^p} \\ &\le C \left\| \left(\sum_{k\in\mathbb{Z}} |S_{l+k,j} f|^2 \right)^{1/2} \right\|_{L^p} \le C\|f\|_{L^p}. \end{aligned} \tag{3.18}$$

Interpolation between (3.13) and (3.18), and the triangle inequality in L^p imply that

$$\left\| \sup_{k\in\mathbb{Z}} \mathcal{A}_{k,j} \right\|_{L^p} \le C\|f\|_{L^p}, \quad p > \frac{3}{4}. \tag{3.19}$$

For $\sup_{k\in\mathbb{Z}} \mathcal{B}_{k,j}$ and $\sup_{k\in\mathbb{Z}} \mathcal{C}_{k,j}$, by the same argument as we used for $\sup_{k\in\mathbb{Z}} \mathcal{A}_{k,j}$, we obtain

$$\left\| \sup_{k\in\mathbb{Z}} \mathcal{B}_{k,j} \right\|_{L^p} \le C\|f\|_{L^p} \quad \text{and} \quad \left\| \sup_{k\in\mathbb{Z}} \mathcal{C}_{k,j} \right\|_{L^p} \le C\|f\|_{L^p}, \quad p > \frac{3}{4}. \tag{3.20}$$

So, according to the L^p-boundedness of $\mathcal{M}_s$, (3.19) and (3.20), we have $\|\mathcal{M}_{G_j} f\|_{L^p} \le C\|f\|_{L^p}$ for $p > 4/3$.

Finally, by a bootstrap argument, we can apply Lemma 3.3 inductively to show that

$$\|\mathcal{M}_{G_j} f\|_{L^p} \le C\|f\|_{L^p}, \quad 1 < p < \infty.$$

4. The L^p-boundedness for $\mathcal{T}$

Similar to the maximal functions $\mathcal{M}$, the singular integrals $\mathcal{T}$ can be decomposed as

$$\mathcal{T} f(u) = \sum_{k\in\mathcal{K}} \mathcal{T}_{D_k} f(u) + \sum_{j\in\mathcal{J}} \mathcal{T}_{G_j} f(u).$$

Then, the L^p-boundedness for $\mathcal{T}_{D_k}$ and $\mathcal{T}_{G_j}$ will be considered separately for each $k \in \mathcal{K}$ and $j \in \mathcal{J}$.

4.1. The L^p-bounedness for $\mathcal{T}_{D_k}$. For $k \in \mathcal{K}$, by Minkowski's inequality, we have

$$\begin{aligned} \|\mathcal{T}_{D_k} f\|_{L^p} &\le \int_{|y|\in\gamma^{-1}(D_k)} |K(y)| \left(\int_{\mathbb{R}^{n+1}} |f(x-y, s-P(\gamma(|y|)))|^p \, du \right)^{1/p} dy \\ &\le \|f\|_{L^p} \int_{\mathbf{S}^{n-1}} |\Omega(y')| \, d\sigma(y') \int_{r\in\gamma^{-1}(D_k)} \frac{1}{r} \, dr. \end{aligned} \tag{4.1}$$

As the L^p-estimates for $\mathcal{M}_{D_k}$ in Subsection 3.1, we get the L^p-bounedness of $\mathcal{T}_{D_k}$,

$$\|\mathcal{T}_{D_k} f\|_{L^p} \le C\|f\|_{L^p}, \quad 1 < p < \infty.$$

4.2. The L^p-bounedness for $\mathcal{T}_{G_j}$. For $j \in \mathcal{J}$, $\mathcal{T}_{G_j} f$ can be rewritten as

$$\mathcal{T}_{G_j} f(u) = \sum_{k\in\mathbb{Z}} \nu_{k,j} * f(u),$$

where the measure $\nu_{k,j}$ is given by

$$\langle \nu_{k,j}, \psi\rangle = \int_{|y|\in I_{k,j}} \psi(\rho^k y, P(\gamma(|\rho^k y|)))K(y)\,dy$$

for $\psi \in \mathscr{S}(\mathbb{R}^{n+1})$.

For the estimates of $\hat{\nu}_{k,j}$, we have the following proposition.

Proposition 4.1. *For $j \in \mathcal{J}$ and $k \in \mathbb{Z}$, then there exists $C > 0$ and $\beta > 0$ independent of j and k such that*

$$|\hat{\nu}_{k,j}(\zeta)| \le C \max\{|A^*_{k,j}\zeta|^{-1}, |A^*_{k,j}\zeta|^{-\beta}\} \tag{4.2}$$

and

$$|\hat{\nu}_{k,j}(\zeta)| \le C|A^*_{k+1,j}\zeta|. \tag{4.3}$$

Proof. (4.2) can be proved by using the same method as (3.1). It is trivial to verify (4.3). In fact, by (1.1),

$$\begin{aligned}
|\hat{\nu}_{k,j}(\zeta)| &= \left|\int_{|y|\in I_{k,j}} [e^{-i[\rho^k y\cdot\xi+\eta P(\gamma(|y|))]} - e^{-i\eta P(\gamma(|y|))}]K(y)\,dy\right| \\
&\le \int_{|y|\in I_{k,j}} |e^{-i\rho^k y\cdot\xi} - 1|\,|K(y)|\,dy \le C\|\Omega\|_{L^1(\mathbf{S}^{n-1})}\rho^{k+1}|\xi| \\
&\le C|A^*_{k+1,j}\zeta|.
\end{aligned}$$

□

By Lemma 2.3, we can decompose $\mathcal{T}_{G_j}$ as

$$\mathcal{T}_{G_j} f = \sum_{k\in\mathbb{Z}}\sum_{l\ge1} \nu_{k,j} * S_{l+k,j} f + \sum_{k\in\mathbb{Z}}\sum_{l\le0} \nu_{k,j} * S_{l+k,j} f := \mathcal{D}_j + \mathcal{G}_j. \tag{4.4}$$

By the triangle inequality in L^p and Lemma 2.3, we have

$$\|\mathcal{D}_j\|_{L^p} \le \sum_{l\ge1}\left\|\sum_{k\in\mathbb{Z}} \nu_{k,j} * S_{l+k,j} f\right\|_{L^p} \le C\sum_{l\ge1}\|\mathcal{H}_{l,j}\|_{L^p}, \tag{4.5}$$

where $\mathcal{H}_{l.j} = \left(\sum_{k\in\mathbb{Z}} |\nu_{k,j} * S_{l+k,j} f|^2\right)^{1/2}$. Plancherel's theorem, (4.3) and (2.3) give

$$\|\mathcal{H}_{l,j}\|_{L^2} = \left(\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^{n+1}} |m_{l+k,j}(\zeta)|^2 |\hat{\nu}_{k,j}(\zeta)|^2 |\hat{f}(\zeta)|^2 d\zeta\right)^{1/2} \leq C\rho^{-l} \|f\|_{L^2}. \tag{4.6}$$

On the other hand, note that $|\nu_{k,j} * g| \leq C\mu_{k,j} * |g|$. For $1 < p < \infty$, by the L^p-boundedness of $\mathcal{M}_{G_j}$, Lemma 3.3 and Lemma 2.3, we obtain

$$\|\mathcal{H}_{l,j}\|_{L^p} \leq C \left\| \left(\sum_{k\in\mathbb{Z}} |S_{l+k,j} f|^2\right)^{1/2} \right\|_{L^p} \leq C\|f\|_{L^p}. \tag{4.7}$$

Interpolation between (4.6) and (4.7), and (4.5) imply that

$$\|\mathcal{D}_j\|_{L^p} \leq C\|f\|_{L^p}, \quad 1 < p < \infty. \tag{4.8}$$

The L^p-norm of $\mathcal{G}_j$ can be obtained in the same way. For $l \leq 0$, using Plancherel's theorem, (4.2) and (2.2), we have $\|\mathcal{H}_{l,j}\|_{L^2} \leq C\rho^{\beta l}\|f\|_{L^2}$. Further, (4.7) still holds. Interpolation and the triangle inequality in L^p show that

$$\|\mathcal{G}_j\|_{L^p} \leq C\|f\|_{L^p}, \quad 1 < p < \infty. \tag{4.9}$$

Combining (4.8) and (4.9), we prove the L^p-boundedness for $\mathcal{T}_{G_j}$.

5. The L^p-boundedness for $\mathcal{T}^*$

Let $\mathcal{K}$ and $\mathcal{J}$ be given as in the second section. Then, we have the following majorization

$$\begin{aligned}\mathcal{T}^* f(u) \leq & \sum_{k\in\mathcal{K}} \sup_{\varepsilon>0} \left| \int_{|y|\in\gamma^{-1}(D_k)\cap\{t\geq\varepsilon\}} f(x-y, s-P(\gamma(|y|)))K(y)\,dy \right| \\ & + \sum_{j\in\mathcal{J}} \sup_{\varepsilon>0} \left| \int_{|y|\in\gamma^{-1}(G_j)\cap\{t\geq\varepsilon\}} f(x-y, s-P(\gamma(|y|)))K(y)\,dy \right| \\ := & \sum_{k\in\mathcal{K}} \mathcal{T}^*_{D_k} f(u) + \sum_{j\in\mathcal{J}} \mathcal{T}^*_{G_j} f(u).\end{aligned}$$

In the same way, we just need to show that $\mathcal{T}^*_{D_k}$ and $\mathcal{T}^*_{G_j}$ are L^p bounded for $k \in \mathcal{K}$ and $j \in \mathcal{J}$.

For $k \in \mathcal{K}$, let $\varepsilon(u)$ be some measurable function from $\mathbb{R}^{n+1}$ to $\mathbb{R}^+$ such that

$$\mathcal{T}^*_{D_k} f(u) \leq 2 \left| \int_{|y|\in\gamma^{-1}(D_k)\cap\{t\geq\varepsilon(u)\}} f(x-y, s-P(\gamma(|y|)))K(y)\,dy \right|.$$

Then, the L^p-boundedness for $\mathcal{T}^*_{D_k}$ can be proved in the same way as (4.1).

For $j \in \mathcal{J}$, it is trivial that

$$\mathcal{T}^*_{G_j} f(u) \le \mathcal{M}_{G_j} f(u) + \sup_{i\in\mathbb{Z}} \left| \sum_{k\ge i} \nu_{k,j} * f(u) \right|.$$

By the L^p-boundedness for $\mathcal{M}_{G_j}$, it suffices to consider the latter term. Let $\Phi \in \mathscr{S}(\mathbb{R}^n)$ be such that $\hat{\Phi}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{\Phi}(\xi) = 0$ for $|\xi| \ge 2$. Write $\hat{\Phi}_i(\xi) = \hat{\Phi}(\rho^i \xi)$, and denote by $\star$ convolution in the first n variables. For $i \in \mathbb{Z}$, the truncated singular integrals can be split as

$$\sum_{k\ge i} \nu_{k,j} * f = \Phi_i \star \left(\mathcal{T}_{G_j} f - \sum_{k<i} \nu_{k,j} * f \right) + (\delta - \Phi_i) \star \sum_{k\ge i} \nu_{k,j} * f =: \mathscr{A}_{i,j} + \mathscr{B}_{i,j},$$

where δ is the Dirac measure in $\mathbb{R}^n$. Then, we just need to estimate $\sup_{i\in\mathbb{Z}} |\mathscr{A}_{i,j}|$ and $\sup_{i\in\mathbb{Z}} |\mathscr{B}_{i,j}|$ for $j \in \mathcal{J}$.

5.1. The L^p-estimates of $\sup_{i\in\mathbb{Z}} |\mathscr{A}_{i,j}|$. By a linear transformation and (1.1), we observe that

$$\begin{aligned}
&\Phi_i \star \sum_{k<i} \nu_{k,j} * f(u) \\
&= \int_{\mathbb{R}^n} \Phi_i(x-y) \sum_{k<i} \int_{|z|\in\rho^k I_{k,j}} f(y-z, s - P(\gamma(|z|))) K(z)\, dz\, dy \\
&= \sum_{k<i} \int_{|z|\in\rho^k I_{k,j}} K(z) \int_{\mathbb{R}^n} \Phi_i(x-y-z) f(y, s-P(\gamma(|z|)))\, dy\, dz \\
&= \sum_{k<i} \int_{|z|\in\rho^k I_{k,j}} K(z) \int_{\mathbb{R}^n} [\Phi_i(x-y-z) - \Phi_i(x-y)] f(y, s-P(\gamma(|z|)))\, dy\, dz.
\end{aligned}$$

Note that $\Phi \in \mathscr{S}(\mathbb{R}^n)$, then, for any $N > 0$,

$$\begin{aligned}
&\left| \Phi_i \star \sum_{k<i} \nu_{k,j} * f(u) \right| \\
&\le \int_{|z|\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |K(z)| \int_{\mathbb{R}^n} \frac{|z|\rho^{-i}}{\rho^{in}(1+\rho^{-i}|x-y|)^N} |f(y, s-P(\gamma(|z|)))|\, dy\, dz \\
&\le \int_{\mathbb{R}^n} \frac{\rho^{-in}}{(1+|\rho^{-i}x - \rho^{-i}y|)^N} \frac{1}{\rho^i} \int_{|z|\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |f(y, s-P(\gamma(|z|)))| \frac{|\Omega(z)|}{|z|^{n-1}}\, dz\, dy.
\end{aligned}$$

For the inner integral in z, by a rotation,

$$\frac{1}{\rho^i} \int_{|z|\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |f(y, s-P(\gamma(|z|)))| \frac{|\Omega(z)|}{|z|^{n-1}}\, dz \le \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \mathcal{N}_j f(y,s),$$

where $\mathcal{N}_j$ is defined by

$$\mathcal{N}_j g(s) = \sup_{i\in\mathbb{Z}} \frac{1}{\rho^i} \int_{t\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |g(s - P(\gamma(t)))|\, dt.$$

Thus, we obtain

$$\sup_{i\in\mathbb{Z}} |\mathscr{A}_{i,j}| \leq C[\|\Omega\|_{L^1(\mathbf{S}^{n-1})}(\mathcal{N}_j f)^\star(u) + (\mathcal{T}_{G_j} f)^\star(u)], \tag{5.1}$$

where $f^\star(x, s)$ is the Hardy–Littlewood maximal function of $f(y, s)$ in the first n variables.

Proposition 5.1. *For $j \in \mathcal{J}$, $\mathcal{N}_j$ is a bounded operator on $L^p(\mathbb{R})$, $1 < p < \infty$.*

Proof. We denote $P(\gamma(t))$ by $\Upsilon(t)$ for short, then, $\Upsilon(t)' = P'(\gamma(t))\gamma'(t)$. Note that $P(s)$ has no null point on G_j, then, it is singled-signed. For $t \in \gamma^{-1}(G_j)$, $\gamma(t) \in G_j$, by (2) of Lemma 2.1, $P'(\gamma(t))$ is also singled-signed on $\gamma^{-1}(G_j)$. By $\gamma'(0) \geq 0$ and the convexity of γ, $\gamma'(t) > 0$ for $t > 0$. Then, $\Upsilon(t)$ is monotonous on $\gamma^{-1}(G_j)$. Suppose that $\Upsilon(t)$ is increasing on $\gamma^{-1}(G_j)$, then

$$\begin{aligned}\frac{1}{\rho^i} \int_{t\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |g(s - \Upsilon(t))|\, dt &= \frac{1}{\rho^i} \int_{t\in(0,\Upsilon(\rho^i)]\cap P(G_j)} |g(s - t)| \frac{dt}{\Upsilon'(\Upsilon^{-1}(t))} \\ &:= \int_0^\infty |g(s - t)|\phi_{i,j}(t)\, dt.\end{aligned}$$

For $j \in \mathcal{J} \setminus \{1\}$, by Lemma 3.2, $\Upsilon(t)'$ is monotonous on $\gamma^{-1}(G_j)$. If $\Upsilon'(t)$ is increasing on $\gamma^{-1}(G_j)$, then, for $i \in \mathbb{Z}$, $\phi_{i,j}(t)$ is nonnegative and decreasing on $P(G_j)$. Furthermore, one should note that

$$\int_0^\infty \phi_{i,j}(t)\, dt \leq \frac{1}{\rho^i} \int_{t\in(0,\Upsilon(\rho^i)]} \frac{dt}{\Upsilon'(\Upsilon^{-1}(t))} = 1.$$

Therefore, for $i \in \mathbb{Z}$, we have

$$\frac{1}{\rho^i} \int_{t\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |g(s - \Upsilon(t))|\, dt \leq CMg(s).$$

If $\Upsilon'(t)$ is decreasing on $\gamma^{-1}(G_j)$, write

$$\int_0^\infty |g(s - t)|\phi_{i,j}(t)\, dt = \int_0^\infty |\tilde{g}(-s + t)|\tilde{\phi}_{i,j}(-t)\, dt = \int_{-\infty}^0 |\tilde{g}(-s - t)|\tilde{\phi}_{i,j}(t)\, dt,$$

where $\tilde{g}$ denotes the reflection of g. Notice that $\tilde{\phi}_{i,j}(t)$ is nonnegative and decreasing on $-P(G_j)$. Also, $\|\tilde{\phi}_{i,j}\|_{L^1} \le 1$. Similarly,

$$\frac{1}{\rho^i}\int_{t\in(0,\rho^i]\cap\gamma^{-1}(G_j)} |g(s-\Upsilon(t))|\,dt \le CM\tilde{g}(-s).$$

For $j=1$, note that $\Upsilon(t)$ and $\gamma(t)$ are increasing on $\gamma^{-1}(G_1)$ and $\mathbb{R}^+$, respectively. Then, $P(s)$ is increasing on G_1, that is, $P'(s)>0$. According to (2.1), $(1/2)|p_1| \le P'(t) \le 2|p_1|$, furthermore, $(1/2)|p_1|t \le P(t) \le 2|p_1|t$ for $t\in G_1$. Therefore, combining the convexity of γ, we get

$$\begin{aligned}
&\frac{1}{\rho^i}\int_{t\in(0,\Upsilon(\rho^i)]\cap P(G_1)} |g(s-t)|\frac{dt}{\Upsilon'(\Upsilon^{-1}(t))}\\
&\le \frac{1}{\rho^i}\int_{t\in(0,2|p_1|\gamma(\rho^i)]\cap 2|p_1|G_1} |g(s-t)|\frac{dt}{(1/2)|p_1|\gamma'(\gamma^{-1}(2|p_1|^{-1}t))}\\
&\le \frac{1}{\rho^i}\int_{t\in(0,4\gamma(\rho^i)]\cap 4G_1}\left|g\left(s-\frac{t|p_1|}{2}\right)\right|\frac{dt}{\gamma'(\gamma^{-1}(t))} \le CMg_{|p_1|/2}\left(\frac{2}{|p_1|}s\right),
\end{aligned}$$

where $g_{|p_1|/2}(t) = g(|p_1|t/2)$.

Thus, for $j\in\mathcal{J}$, $\mathcal{N}_j$ is bounded on $L^p(\mathbb{R})$, $1<p<\infty$. □

Finally, by Lemma 5.1 and the L^p-boundedness for $\mathcal{T}_{G_j}$, we obtain

$$\left\|\sup_{i\in\mathbb{Z}}|\mathscr{A}_{i,j}|\right\|_{L^p} \le C\|f\|_{L^p}.$$

5.2. The L^p-estimates of $\sup_{i\in\mathbb{Z}}|\mathscr{B}_{i,j}|$. $\sup_{i\in\mathbb{Z}}|\mathscr{B}_{i,j}|$ is dominated by

$$\sup_{i\in\mathbb{Z}}|\mathscr{B}_{i,j}| \le \sum_{l\ge 0}\sup_{i\in\mathbb{Z}}|(\delta-\Phi_i)\star\nu_{l+i,j} * f| := \sum_{l\ge 0}\mathscr{P}_{l,j}.$$

The maximal operator $\mathscr{P}_{l,j}$ is uniformly bounded on L^p, $1<p<\infty$, since

$$\mathscr{P}_{l,j} \le C(\mathcal{M}_{G_j} f)^{\star}.$$

On the other hand, for $p=2$, we have

$$\begin{aligned}
\|\mathscr{P}_{l,j}\|_{L^2} &\le \left\|\left(\sum_{i\in\mathbb{Z}}|(\delta-\Phi_i)\star\nu_{l+i,j} * f|^2\right)^{1/2}\right\|_{L^2}\\
&\le \left(\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}}|1-\hat{\Phi}(\rho^i\xi)|^2|\hat{\nu}_{l+i,j}(\zeta)|^2|\hat{f}(\zeta)|^2 d\zeta\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq C\left(\sum_{i\in\mathbb{Z}}\int_{\mathbb{R}^{n+1}}\chi_{\{\rho^i|\xi|\geq 1\}}(\zeta)|\rho^{l+i}\xi|^{-2\beta}|\hat{f}(\zeta)|^2d\zeta\right)^{1/2}\\
&\leq C\rho^{-l\beta}\left(\int_{\mathbb{R}^{n+1}}\sum_{i\,:\,\rho^{-i}\leq|\xi|}|\rho^i\xi|^{-2\beta}|\hat{f}(\zeta)|^2d\zeta\right)^{1/2}\\
&\leq C\rho^{-l\beta}\|f\|_{L^2},
\end{aligned}$$

where the fact $|\hat{\nu}_{k,j}(\zeta)|\leq C(\rho^k|\xi|)^{-\beta}$ can be proved in the same way as (3.3).

Interpolation and the triangle inequality in L^p imply that

$$\left\|\sup_{i\in\mathbb{Z}}|\mathscr{B}_{i,j}|\right\|_{L^p}\leq\sum_{l\geq 0}\|\mathscr{P}_{l,j}\|_{L^p}\leq C\|f\|_{L^p},\quad 1<p<\infty.$$

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