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# ON BLOCKS OF NORMAL SUBGROUPS OF FINITE GROUPS

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**Abstract**

For a block $b$ of a normal subgroup of a finite group $G$, E.C. Dade has defined a subgroup $G[b]$ of $G$. We give a character-theoretical interpretation of his result on $G[b]$. In the course of proofs we determine a defect group of a block of $G[b]$ covering $b$. We also consider character-theoretical characterizations of isomorphic blocks with respect to normal subgroups.

## Introduction

Let $G$ be a finite group and $p$ a prime. Let $(\mathcal{K}, R, k)$ be a $p$-modular system. We assume that $\mathcal{K}$ is sufficiently large for $G$. In this paper a block of $G$ means a block ideal of $RG$. For a normal subgroup $K$ of $G$ and a block $b$ of $K$, Dade [3] has defined a normal subgroup $G[b]$ of the inertial group of $b$ in $G$ such that $G[b] \geq K$. More precisely put $C = C_{RG}(K)$. We have $C = \bigoplus_{\bar{x} \in \bar{G}} C_{\bar{x}}$, where $\bar{G} = G/K$ and $C_{\bar{x}} = C \cap RKx$. Let $e_b$ be the block idempotent of $b$. The subgroup $G[b]$ is defined by

$$G[b] = \{x \in G \mid (e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) = e_b C_{\bar{1}}\}.$$

(Strictly speaking, Dade defines a subgroup $(G/K)[b]$ of $G/K$. The subgroup $G[b]$ is the preimage of $(G/K)[b]$ in $G$.) In [3, Corollary 12.6] Dade has determined $G[b]$ in terms of $C_G(Q)$ and a root of $b$ in $C_K(Q)$, where $Q$ is a defect group of $b$. In Section 3 we shall give a character-theoretical characterization of elements of $G[b]$ and give a character-theoretical interpretation of Dade's result above. In the course of proofs we determine a defect group of a block of $G[b]$ covering $b$, which is a refinement of a result in [9]. In Section 1, we shall consider weakly regular and regular blocks with respect to normal subgroups. In Section 4, character-theoretical characterizations of isomorphic blocks with respect to normal subgroups which involve $G[b]$ will be obtained. More applications of $G[b]$ will be given in a separate paper [11].

## Notation

Let $B$ be a block of $G$. The block idempotent of $B$ will be denoted by $e_B$. For an irreducible character $\chi$ in $B$, put $\omega_B(z) = \omega_\chi(z)$, $z \in Z(RG)$. For a subset $S$ of

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$G$, let $\hat{S} = \sum_{s \in S} s \in RG$. For $x \in G$, let $K_x$ be the conjugacy class of $G$ containing $x$, and so $\hat{K}_x$ is the class sum of $K_x$. Let $D(K_x)$ be a defect group of $K_x$. Let $e_B = \sum_y a_B(K_y)\hat{K}_y$, where $y$ runs through a set of representatives of conjugacy classes of $G$.

Let $B_0(G)$ be the principal block of $G$. Let $\mathrm{Irr}(B)$ be the set of irreducible characters in $B$. Let $\mathrm{Irr}_0(B)$ be the set of irreducible characters of height 0 in $B$. $\mathrm{d}(B)$ is the defect of $B$. For a block $b$ of a normal subgroup $K$ of $G$, let $G_b$ be the inertial group of $b$ in $G$ and let $\mathrm{BL}(G \mid b)$ be the set of blocks of $G$ covering $b$. For an irreducible character $\xi$ of $K$ and a block $B$ of $G$, let $\mathrm{Irr}(B \mid \xi)$ be the set of irreducible characters in $B$ lying over $\xi$. Put

$$\mathrm{Irr}_0(B \mid \xi) = \{\chi \in \mathrm{Irr}(B \mid \xi) \mid \mathrm{ht}(\chi) = \mathrm{ht}(\xi)\},$$

where $\mathrm{ht}(\chi)$ is the height of $\chi$. Let ${}^*\colon R \to k$ be the natural map. For a function $\varphi\colon S \to R$ defined on a set $S$, the function $\varphi^*\colon S \to k$ is defined by $\varphi^*(s) = \varphi(s)^*$, $s \in S$. Let $\nu$ be the valuation of $\mathcal{K}$ normalized so that $\nu(p) = 1$.

## 1. Weakly regular and regular blocks with respect to normal subgroups

In this section we strengthen Theorem 2.1 of [9].

**Proposition 1.1.** *Let $N$ be a normal subgroup of $G$. Let $b$ be a block of $N$ covered by a block $B$ of $G$. Let $D$ be a defect group of $B$. The following conditions are equivalent.*

(i) *$B$ is a unique weakly regular block of $G$ covering $b$.*

(ii) *For a block $\hat{b}$ of $DN$, we have $\hat{b}^G = B$.*

(iii) *For any $p'$-element $x$ of $G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$, we have $x \in N$.*

(iv) *For any $p'$-element $x$ of $G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

(v) *For any $x \in G$ satisfying $a_B(K_x)^* \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

Proof. (i) $\Rightarrow$ (ii). By replacing $D$ by a conjugate, we may assume $D$ is a defect group of the Fong–Reynolds correspondent of $B$ over $b$ in the inertial group of $b$ in $G$. Let $\hat{b}$ be a unique block of $DN$ covering $b$. Then $\hat{b}^G = B$, see the proof of Theorem 2.1 of [9].

(ii) $\Rightarrow$ (iii). This is easy to see.

(iii) $\Rightarrow$ (iv). This is trivial.

(iv) $\Rightarrow$ (v). Since $a_B(K_x)^* \neq 0$, $x$ is a $p'$-element. Let $\{\varphi_i\}$ be the set of irreducible Brauer characters in $B$. Let $\Phi_i$ be the principal indecomposable character corresponding to $\varphi_i$. Let $\{\chi_j\}$ be the set of irreducible characters in $B$. Put

$$\varphi_i = \sum_j n_{ij}\chi_j \quad \text{(on the set of } p'\text{-elements of } G),$$

where $n_{ij}$ are integers. Then

$$a_B(K_x) = \frac{1}{|G|}\sum_i \Phi_i(1)\varphi_i(x^{-1}) = \sum_i \frac{\Phi_i(1)}{|G|}\sum_j n_{ij}\omega_{\chi_j}(\hat{K}_{x^{-1}})\frac{\chi_j(1)}{|K_x|}.$$

Since $\Phi_i(1)/|G|$ and $\chi_j(1)/|K_x|$ lie in $R$ for any $i$ and $j$, we obtain

$$a_B(K_x) \equiv \omega_B(\hat{K}_{x^{-1}})\sum_i \frac{\Phi_i(1)\varphi_i(1)}{|G|\,|K_x|} \mod \mathrm{J}(R).$$

Since $\Phi_i(1)\varphi_i(1)/(|G|\,|K_x|)$ lies in $R$ for any $i$, $a_B(K_x)^* \neq 0$ implies $\omega_B^*(\hat{K}_{x^{-1}}) \neq 0$. Hence $x \in N$ by (iv).

(v) $\Rightarrow$ (i). Let $K_s$ be a defect class for $B$ ([12, p. 311]). Then $K_s \subset N$ by (v). Since $\omega_B^*(\hat{K}_s) \neq 0$ and $D(K_s) =_G D$, $B$ is weakly regular with respect to $N$ by definition ([12, p. 344]). Let $B_1$ be any weakly regular block of $G$ covering $b$. Put $e_B = s_N(e_B)+a$, where $s_N(e_B) = \sum_{K_y \subset N} a_B(K_y)\hat{K}_y$. We claim $\omega_{B_1}^*(a) = 0$. Assume this were false. Then there would be an element $x \notin N$ such that $a_B(K_x)^*\omega_{B_1}^*(\hat{K}_x) \neq 0$. Since $a_B(K_x)^* \neq 0$, $D(K_x) \leq_G D$. Since $\omega_{B_1}^*(\hat{K}_x) \neq 0$, $D(K_x) \geq_G D_1$, where $D_1$ is a defect group of $B_1$. By Fong's theorem $D =_G D_1$. Thus $D(K_x) =_G D$. So $x \in N$ by (v), a contradiction, and the claim follows. Now $\omega_{B_1}^*(e_B) = \omega_{B_1}^*(s_N(e_B)) = \omega_b^*(s_N(e_B))$ by [12, Theorem 5.5.5]. Since $B$ is weakly regular, $\omega_b^*(s_N(e_B)) = \omega_b^*(s_N(e_B)e_b) \neq 0$ by [9, Theorem 1.10]. Thus $\omega_{B_1}^*(e_B) \neq 0$. Hence $B_1 = B$ and (i) follows. The proof is complete. □

REMARK 1.2. The equivalence of (i) and (ii) is proved in [4, Theorem 2.4].

**Theorem 1.3.** *Let $N$ be a normal subgroup of $G$. Let $b$ be a block of $N$ covered by a block $B$ of $G$. Let $D$ be a defect group of $B$. The following conditions are equivalent.*

(i) *$B$ is a unique weakly regular block of $G$ covering $b$ and $Z(D) \leq N$.*

(ii) *$B = b^G$.*

(iii) *For any $x \in G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$.*

(iv) (iv a) *For any $p'$-element $x$ of $G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in N$, and*

(iv b) *$Z(D) \leq N$.*

(v) (v a) *For any $x \in G$ satisfying $a_B(K_x)^* \neq 0$ and $D(K_x) =_G D$, we have $x \in N$, and*

(v b) *$Z(D) \leq N$.*

Proof. (i) $\Leftrightarrow$ (ii). This is Theorem 2.1 of [9].

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (iv). (iv a) is trivial. Let $K_s$ be a defect class for $B$ ([12, p. 311]). So $s$ is a $p'$-element. We may assume $D$ is a Sylow $p$-subgroup of $C_G(s)$. Let $u \in Z(D)$. Then, as in [13, Lemma 5.15], $D(K_{us}) =_G D$ and $\omega_B^*(\hat{K}_{us}) \neq 0$. Then $us \in N$ by (iii). So $u \in N$, and $Z(D) \leq N$.

(iv) $\Rightarrow$ (v). This follows from (iv) $\Rightarrow$ (v) of Proposition 1.1.

(v) $\Rightarrow$ (i). This follows from (v) $\Rightarrow$ (i) of Proposition 1.1. □

REMARK 1.4. The equivalences of (i), (ii), (v) have been proved in Fan [4, Theorem 2.3] in a different way.

## 2. A lemma on $G[b]$

In the rest of this paper, $K$ is a normal subgroup of a group $G$, and $b$ is a block of $K$ with a defect group $Q$. The following lemma is certainly well-known. We give a proof for completeness sake. We shall use this lemma without explicit reference.

**Lemma 2.1.** *Let $x$ be an element of $G$. The following are equivalent.*

(i) $x \in G[b]$; *that is,* $(e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) = e_b C_{\bar{1}}$.

(ii) $e_b C_{\bar{x}}$ *contains a unit of* $e_b C$.

(iii) ([6, *p.* 210]) $x \in G_b$ *and* $x$ *induces an inner automorphism of* $b$.

Proof. (i) $\Rightarrow$ (ii). This follows from [15, p. 551, ll. 5–7][1].

(ii) $\Rightarrow$ (iii). This follows from [3, Proposition 2.17] and [15, p. 551, ll. 7–9][2].

(iii) $\Rightarrow$ (i). Let $u$ be a unit of $b$ such that $v^x = v^u$ for all $v \in b$. We claim $ux^{-1} \in e_b C_{\bar{x}^{-1}}$. Indeed, $(ux^{-1})v = v(ux^{-1})$ for all $v \in b$. Let $b'$ be any block of $K$ with $b' \neq b$. Let $v' \in b'$. Then $(ux^{-1})v' = uv'^x x^{-1} = 0 = v'(ux^{-1})$. So $ux^{-1} \in C$. Then the claim follows. Let $u'$ be an element of $b$ such that $uu' = u'u = e_b$. Then we obtain similarly that $xu' \in e_b C_{\bar{x}}$. We have $(xu')(ux^{-1}) = e_b$. So $(e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) \ni e_b$, which implies $(e_b C_{\bar{x}})(e_b C_{\bar{x}^{-1}}) = e_b C_{\bar{1}}$. The proof is complete. □

REMARK 2.2. See Hida–Koshitani [5, Lemma 3.2] for a module-theoretical reformulation of the definition of $G[b]$.

## 3. The subgroup $G[b]$

Navarro [14] has obtained a relative version of a well-known theorem of Burnside as follows (letting $K = 1$, we recover the original theorem of Burnside):

**Lemma 3.1** (Navarro [14, Theorem A]). *Let $\chi$ be an irreducible character of $G$. The following are equivalent.*

[1]Note that $e_b C_{\bar{1}} = Z(b)$ is a local $R$-algebra.
[2]In l. 9 $\mathfrak{O}G$ should be $e\mathfrak{O}G$.

(i) $\chi_K$ *is irreducible.*
(ii) *For any* $x \in G$, *there is an element* $y$ *in* $xK$ *such that* $\chi(y) \neq 0$.

**Proposition 3.2.** *Assume that* $G/K$ *is abelian. Let* $B$ *be a block of* $G$ *covering* $b$. *The following are equivalent.*
(i) $G = G[b]$ *and for any irreducible character* $\chi$ *in* $B$, $\chi_K$ *is irreducible.*
(ii) *For any* $x \in G$, *there is an element* $y$ *in* $xK$ *such that* $\omega_B^*(\hat{K}_y) \neq 0$.

Proof. In both cases, the following holds:
($*$) For any irreducible character $\chi$ in $B$, $\chi_K$ is irreducible.
Indeed, if (i) holds, trivially ($*$) holds. Assume (ii) holds. Let $\chi$ be an irreducible character in $B$. Since $\omega_B^*(\hat{K}_y) \neq 0$, we have $\chi(y) \neq 0$. Then, by Lemma 3.1, $\chi_K$ is irreducible.

Let $\{B_i\}$ be the set of blocks of $G$ covering $b$. We show that ($*$) implies the following:
($**$) For any irreducible character $\chi$ in $B_i$ for any $i$, $\chi_K$ is an irreducible character in $b$.
Indeed, let $\xi \in \mathrm{Irr}(b)$ be an irreducible constituent of $\chi_K$. Let $\zeta$ be an irreducible character in $B$ lying over $\xi$. By ($*$), $\zeta_K = \xi$. Hence $\chi = \zeta \otimes \theta$ for some $\theta \in \mathrm{Irr}(G/K)$. Since $G/K$ is abelian, we have $\chi_K = \xi$. Hence ($**$) holds. Thus for the proof of proposition we may assume ($**$) holds.

Recall that $C = C_{RG}(K)$. We claim the following:
($***$) $e_bC = Z(Gb) = \bigoplus_i Z(B_i)$,
where $Gb = RGe_b$. By ($**$), $b$ is $G$-invariant. This yields the second equality. We prove the first equality. Clearly $Z(Gb) \subseteq e_bC$. To prove the reverse containment, let $a \in e_bC$ and $v \in \mathcal{K}Gb$, where $\mathcal{K}Gb = \mathcal{K}Ge_b$. Let $T$ be any irreducible matrix representation of $\mathcal{K}Gb$. By ($**$), restriction of $T$ to $\mathcal{K}b$ is irreducible, where $\mathcal{K}b = \mathcal{K}Ke_b$. Since $e_bC \subseteq \mathcal{K}Gb \cap C(\mathcal{K}b)$, $T(a)$ is a scalar matrix by Schur's lemma. So $T(av - va) = 0$. It follows that $av - va = 0$, since $\mathcal{K}Gb$ is semi-simple. Therefore, $e_bC \subseteq Z(\mathcal{K}Gb) \cap RG = Z(Gb)$. ($***$) is proved.

(i) $\Rightarrow$ (ii). Let $x \in G$. By (i), there exists a unit $u$ of $e_bC$ in $e_bC_{\bar{x}}$. Then, by ($***$), $\omega_B^*(u) \neq 0$. Since $u \in Z(RG)$ by ($***$) and $u \in RKx$, $u$ is an $R$-linear combination of $\hat{K}_z$ for $z \in xK$. Thus there is some $y \in xK$ such that $\omega_B^*(\hat{K}_y) \neq 0$. Thus (ii) follows.

(ii) $\Rightarrow$ (i). The latter part follows from ($**$). Let $\xi$ be an irreducible character in $b$. Then, by ($**$), any irreducible character of $G$ lying over $\xi$ is an extension of $\xi$. Therefore for any $i$, there is a linear character $\lambda_i : G/K \to k^*$, where $k^*$ is the multiplicative group of $k$, such that $\omega_{B_i}^*(\hat{K}_g) = \omega_B^*(\hat{K}_g)\lambda_i(gK)$ for any $g \in G$. Let $x \in G$ and let $y$ be as in (ii). Then $\omega_{B_i}^*(e_b\hat{K}_y) = \omega_{B_i}^*(\hat{K}_y) = \omega_B^*(\hat{K}_y)\lambda_i(yK) \neq 0$. Therefore, by ($***$), $e_b\hat{K}_y$ is a unit of $e_bC$. Since $G/K$ is abelian, $e_b\hat{K}_y$ lies in $e_bC_{\bar{x}}$. Thus we obtain $G = G[b]$. The proof is complete. □

The following corollary will be used repeatedly.

**Corollary 3.3.** *Assume that $G/K$ is cyclic, and let $G = \langle x, K\rangle$ for an element $x \in G$. Let $B$ be a block of $G$ covering $b$. The following are equivalent.*
(i) $x \in G[b]$; *that is,* $G = G[b]$.
(ii) *There exists an element $y$ in $xK$ such that $\omega_B^*(\hat{K}_y) \neq 0$.*

Proof. (i) $\Rightarrow$ (ii). $G$ induces inner automorphisms of $b$, so any irreducible character in $b$ is $G$-invariant. Then, since $G/K$ is cyclic, any irreducible character in $B$ restricts irreducibly to $K$. Thus (ii) holds by Proposition 3.2.

(ii) $\Rightarrow$ (i). For any positive integer $i$, $\omega_B^*((\hat{K}_y)^i) \neq 0$. Since $y \in xK$, $(\hat{K}_y)^i$ is an integral combination of $\hat{K}_z$ with $z \in x^iK$. So $\omega_B^*(\hat{K}_z) \neq 0$ for some $z \in x^iK$. Thus (i) holds by Proposition 3.2. The proof is complete. □

**Proposition 3.4.** *Assume that $G/K$ is a cyclic $p$-group. Let $b$ be $G$-invariant. Let $B$ be a unique block of $G$ covering $b$. The following are equivalent.*
(i) $G = G[b]$.
(ii) *For any defect group $S$ of $B$ with $S \geq Q$, $S = Z(S)Q$.*
(ii)′ *For some defect group $S$ of $B$, $S = Z(S)Q$.*
(iii) *For any defect group $S$ of $B$ with $S \geq Q$, $S = C_S(Q)Q$; that is, $S$ induces inner automorphisms of $Q$.*
(iii)′ *For some defect group $S$ of $B$, $S = C_S(Q)Q$.*

Proof. The assertion is trivial if $G = K$. So we assume $G \neq K$. Put $G = \langle x, K\rangle$. Let $\beta$ be a block of $\langle x^p, K\rangle$ covered by $B$.

(i) $\Rightarrow$ (ii). Assume $S \neq Z(S)Q$. Since $b$ is $G$-invariant, $G = SK$. So $S/Q \simeq G/K$ is cyclic. Therefore $Z(S) \leq \langle x^p, K\rangle$. Then $B = \beta^G$ by Theorem 1.3. Thus $\omega_B^*(K_y) = 0$ for all $y \in xK$. Then $x \notin G[b]$ by Corollary 3.3, a contradiction.

(ii) $\Rightarrow$ (i). Assume $x \notin G[b]$. Then $x^i \notin G[b]$ for any $p'$-integer $i$. Thus $\omega_B^*(\hat{K}_y) = 0$ for any $y \in G - \langle x^p, K\rangle$ by Corollary 3.3. Hence $B = \beta^G$. Then $Z(S) \leq \langle x^p, K\rangle$ by Theorem 1.3. Since $b$ is $G$-invariant, $G = SK$. Therefore $G = SK = Z(S)QK \leq \langle x^p, K\rangle < G$, a contradiction. Thus $x \in G[b]$, and $G = G[b]$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (ii). Since $b$ is $G$-invariant, $G = SK$. So $G/K \simeq S/Q \simeq C_S(Q)/Z(Q)$ is cyclic. Hence $C_S(Q)$ is abelian, and $C_S(Q) \leq Z(S)$. Thus $S = Z(S)Q$.

(iii) $\Rightarrow$ (iii)′. Trivial.

(iii)′ $\Rightarrow$ (iii). Let $U$ be any defect group of $B$ with $U \geq Q$. We have $U = S^g$ for some $g \in G$. Then $Q = U \cap K = S^g \cap K = (S \cap K)^g = Q^g$. So $Q = Q^g$. Then $C_U(Q)Q = C_{S^g}(Q^g)Q^g = S^g = U$.

(ii) $\Leftrightarrow$ (ii)′. This is proved similarly.

This completes the proof. □

**Theorem 3.5.** *Let $b$ be $G$-invariant. Let $B$ be a block of $G$ covering $b$. We choose a block $B'$ of $G[b]$ so that $B$ covers $B'$ (and $B'$ covers $b$). Let $D, S$ be defect groups of $B$, $B'$, respectively, such that $Q \leq S \leq D$. The following holds.*
(i) *$B = B'^G$. In particular, $B$ is a unique block of $G$ that covers $B'$.*
(ii) *$S = QC_D(Q)$.*

Proof. We first note that $G[b] \lhd G$, so the statement makes sense.

(i) We show $B = B'^G$. By Theorem 1.3, it suffices to show the following:
($*$) For any $x \in G$ satisfying $\omega_B^*(\hat{K}_x) \neq 0$ and $D(K_x) =_G D$, we have $x \in G[b]$.
We may assume $D$ is a Sylow $p$-subgroup of $C_G(x)$. Let $\chi$ be an irreducible character of height 0 in $B$. Put $\chi_{\langle x,K\rangle} = \sum_i n_i\zeta_i$, where $\zeta_i$ are distinct irreducible characters of $\langle x, K\rangle$ and $n_i$ are positive integers. Then

$$\omega_\chi(\hat{K}_x) = \sum_i n_i\omega_{\zeta_i}(\hat{L}_x)\frac{\zeta_i(1)|G|\,|C_K(x)|}{\chi(1)|K|\,|C_G(x)|},$$

where $L_x$ is the conjugacy class of $\langle x, K\rangle$ containing $x$. For any $i$, let $b_i$ be the block of $\langle x, K\rangle$ containing $\zeta_i$. Then $b_i$ covers $b$. We claim $\mathrm{d}(b_i) - \mathrm{d}(b) = \nu(|\langle x, K\rangle|) - \nu(|K|)$. Indeed, let $H/K$ be a (normal) Sylow $p$-subgroup of $\langle x, K\rangle/K$. Let $\hat{b}$ be a unique block of $H$ covering $b$. Then, since $b_i$ covers $\hat{b}$, $\mathrm{d}(b_i) = \mathrm{d}(\hat{b})$. Furthermore, $\mathrm{d}(\hat{b}) - \mathrm{d}(b) = \nu(|H|) - \nu(|K|)$. Thus the claim follows. On the other hand, since $D$ is a Sylow $p$-subgroup of $C_G(x)$, $D \cap K$ is a Sylow $p$-subgroup of $C_K(x)$. Furthermore $D \cap K$ is a defect group of $b$. Thus

$$\begin{aligned}\nu\left(\frac{\zeta_i(1)|G|\,|C_K(x)|}{\chi(1)|K|\,|C_G(x)|}\right) &= \nu(|\langle x, K\rangle|) - \mathrm{d}(b_i) + \mathrm{ht}(\zeta_i) + \nu(|G|) + \nu(|C_K(x)|)\\ &\quad - \{\nu(|G|) - \mathrm{d}(B) + \nu(|K|) + \nu(|C_G(x)|)\}\\ &= \nu(|\langle x, K\rangle|) - \nu(|K|) - \mathrm{d}(b_i) + \mathrm{d}(b) + \mathrm{ht}(\zeta_i)\\ &= \mathrm{ht}(\zeta_i) \geq 0.\end{aligned}$$

Since $\omega_\chi^*(\hat{K}_x) \neq 0$, there exists $i$ such that $\omega_{\zeta_i}^*(\hat{L}_x) \neq 0$. Then $x \in \langle x, K\rangle[b]$ by Corollary 3.3, and $x \in G[b]$. Thus ($*$) follows and $B = B'^G$.

If $B_1$ is another block of $G$ covering $B'$, then similarly $B_1 = B'^G$. So $B_1 = B$.

(ii) Since $Q = D \cap K$, $Q$ is a normal subgroup of $D$. Put

$$I = \{u \in D \mid u \text{ induces an inner automorphism of } Q\}.$$

Clearly $I = QC_D(Q)$, so it suffices to show $I = S$. For any $u \in D$, put $Q_u = \langle u, Q\rangle$. If $b_u$ is a unique block of $Q_uK$ covering $b$, then $Q_u$ is a defect group of $b_u$, cf. Lemma 4.13 of [9].

Let $u \in I$. Then $Q_u$ induces inner automorphisms of $Q$. Since $Q_uK = \langle u, K\rangle$, $Q_uK = (Q_uK)[b] \leq G[b]$ by Proposition 3.4. So $u \in G[b]$, and $I \leq G[b] \cap D = S$.

Conversely let $u \in S$. Then, since $u \in G[b]$ and $Q_u K = \langle u, K\rangle$, we have $Q_u K = (Q_u K)[b]$. Thus $Q_u$ induces inner automorphisms of $Q$ by Proposition 3.4. So $u \in I$, and $S \leq I$. Thus $I = S$. The proof is complete. □

REMARK 3.6. (1) Theorem 3.5 sharpens Lemma 4.14 of [9].
(2) Theorem 3.5 (i) is implicit in [3]. It follows from Lemma 3.3 and Proposition 1.9 of [3].
(3) Proposition 3.1 of [1] follows immediately from Theorem 3.5 (ii). (The assumption made there that $c$ is nilpotent is unnecessary.)

The following extends Proposition 3.4.

**Corollary 3.7.** *Assume that $G/K$ is a $p$-group. Let $B$ be a unique block of $G$ covering $b$. Let $D$ be a defect group of $B$ such that $D \geq Q$. Then the following are equivalent.*
(i) $G = G[b]$.
(ii) *$b$ is $G$-invariant and $D = QC_D(Q)$.*
*In particular, if $D$ is abelian and $b$ is $G$-invariant, then $G = G[b]$.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3.5.

(ii) $\Rightarrow$ (i). Let $B'$ be a block of $G[b]$ such that $B$ covers $B'$ and that $S := D \cap G[b]$ is a defect group of $B'$. Then $B'$ covers $b$. Since $b$ is $G$-invariant, $G = DK$ and $G[b] = SK$. By Theorem 3.5, $S = QC_D(Q) = D$. Therefore $G = G[b]$. □

REMARK 3.8. The last statement of Corollary 3.7 is implicit in the proof of Theorem of [7].

**Proposition 3.9.** *Assume that $G/K$ is a cyclic $p'$-group. The following are equivalent.*
(i) $G = G[b]$.
(ii) $|\mathrm{BL}(G \mid b)| = |G/K|$.

Proof. (i) $\Rightarrow$ (ii). Put $G = \langle x, K\rangle$. Let $B$ be a block of $G$ covering $b$. By Corollary 3.3, there exists some $y$ in $xK$ such that $\omega_B^*(K_y) \neq 0$. Let $\chi$ be an irreducible character in $B$. Let $\lambda$ be any linear character of $G/K$. Assume that $\chi \otimes \lambda$ lies in $B$. Then $\omega^*_{\chi\otimes\lambda}(\hat{K}_y) = \omega^*_\chi(\hat{K}_y)$, which implies $\lambda^*(y) = 1$. Since $G/K$ is a $p'$-group, we see that $\lambda$ is a trivial character. Therefore we obtain $|\mathrm{BL}(G \mid b)| \geq |G/K|$. To prove the reverse inequality, let $\xi \in \mathrm{Irr}(b)$. Let $m$ be the number of irreducible characters of $G$ lying over $\xi$. Any block of $G$ covering $b$ contains an irreducible character lying over $\xi$, so $|\mathrm{BL}(G \mid b)| \leq m$. On the other hand, $m \leq (\xi^G, \xi^G)_G = ((\xi^G)_K, \xi)_K \leq |G/K|$. Thus $|\mathrm{BL}(G \mid b)| \leq |G/K|$, and (ii) follows.

(ii) $\Rightarrow$ (i). We claim that any block $B$ in $\mathrm{BL}(G \mid b)$ is induced from a block in $\mathrm{BL}(G[b] \mid b)$. To see this, let $\tilde{B}$ be the Fong–Reynolds correspondent of $B$ in $G_b$. Choose a block $B'$ of $G[b]$ such that $\tilde{B}$ covers $B'$ and $B'$ covers $b$. Then $\tilde{B} = B'^{G_b}$ by Theorem 3.5. So $B = \tilde{B}^G = (B'^{G_b})^G = B'^G$. Thus the claim is proved. Then $|\mathrm{BL}(G[b] \mid b)| \geq |\mathrm{BL}(G \mid b)|$. Since $|\mathrm{BL}(G[b] \mid b)| \leq |G[b]/K|$ (as above), it follows that $|G/K| \leq |G[b]/K|$. Thus $G = G[b]$. The proof is complete. □

REMARK 3.10. Application of Theorem 3.7 of [3] would shorten the proof of Proposition 3.9.

The following gives a necessary and sufficient condition for $G$ to coincide with $G[b]$ when $G/K$ is an arbitrary group.

**Theorem 3.11.** *Let $B_w$ be a weakly regular block of $G$ covering $b$. Let $D_w$ be a defect group of $B_w$ such that $D_w \geq Q$. The following are equivalent.*

(i) $G = G[b]$.

(ii) (ii a) *$b$ is $G$-invariant*;

(ii b) *For any subgroup $L$ of $G$ such that $L \geq K$ and that $L/K$ is a cyclic $p'$-group, it holds that* $|\mathrm{BL}(L \mid b)| = |L/K|$; *and*

(ii c) $D_w = QC_{D_w}(Q)$.

Proof. (i) $\Rightarrow$ (ii). This follows from Proposition 3.9 and Theorem 3.5.

(ii) $\Rightarrow$ (i). Let $x$ be a $p'$-element of $G$ and put $H = \langle x, K\rangle$. By (ii b) and Proposition 3.9, $x \in H = H[b]$. So $x \in G[b]$. Let $x$ be a $p$-element of $G$. By (ii a) and Fong's theorem $D_wK/K$ is a Sylow $p$-subgroup of $G/K$. So $x^g \in D_wK$ for some $g \in G$. By (ii a) and [9, Lemma 2.2], $D_w$ is a defect group of a unique block of $D_wK$ covering $b$. So by (ii c) and Corollary 3.7, $(D_wK)[b] = D_wK$. Thus $x^g \in G[b]$. Since $G[b] \lhd G$ by (ii a), $x \in G[b]$. Hence $G = G[b]$. □

We introduce some notation. Let $\tilde{b}$ be the Brauer correspondent of $b$ in $N_K(Q)$ and let $\beta$ be a block of $QC_K(Q)$ covered by $\tilde{b}$. Put $L_0 = QC_K(Q)$. Let $\beta_0$ be a block of $C_K(Q)$ covered by $\beta$. Let $\theta$ be the canonical character of $\beta$ and let $\varphi$ be the restriction of $\theta$ to $C_K(Q)$. So $\varphi$ is the canonical character of $\beta_0$. Let $S = N_G(Q)_\beta$ and $T = N_K(Q)_\beta$. So $T$ is the inertial group of $\beta_0$ in $N_K(Q)$. Put $L = QC_G(Q)$ and $C = C_G(Q)$.

Noting that $T$ and $L_\beta$ are normal subgroups of $S$, we have $[T, L_\beta] \leq L_\beta \cap T = L_0$. So we can define (after Isaacs [6, Section 2]) $\langle\langle t, x\rangle\rangle_\theta \in \mathcal{K}^*$ for $(t, x) \in T \times L_\beta$, where $\mathcal{K}^*$ is the multiplicative group of $\mathcal{K}$. The definition is as follows: let $x \in L_\beta$ and let $\hat{\theta}$ be an extension of $\theta$ to $\langle x, L_0\rangle$. Let $t \in T$. Then, since $\hat{\theta}^t$ is also an extension of $\theta$ to $\langle x, L_0\rangle$, there exists a unique linear character $\lambda_t$ of $\langle x, L_0\rangle/L_0$ such that $\hat{\theta}^t = \hat{\theta} \otimes \lambda_t$. Then put $\langle\langle t, x\rangle\rangle_\theta = \lambda_t(x)$. This definition is independent of the choice of $\hat{\theta}$. It is bilinear in the sense that $\langle\langle ts, x\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta \langle\langle s, x\rangle\rangle_\theta$ for $t, s \in T$ and $x \in L_\beta$

and $\langle\langle t, xy\rangle\rangle_\theta = \langle\langle t, x\rangle\rangle_\theta\langle\langle t, y\rangle\rangle_\theta$ for $t \in T$ and $x, y \in L_\beta$, see [6, Lemma 2.1 and Theorem 2.3]. Similarly we can define $\langle\langle t, x\rangle\rangle_\varphi \in \mathcal{K}^*$ for $(t, x) \in T \times C_{\beta_0}$. It is also bilinear. Define

$$L_\omega = \{x \in L_\beta \mid \langle\langle t, x\rangle\rangle_\theta = 1 \text{ for all } t \in T\},$$
$$C_\omega = \{x \in C_{\beta_0} \mid \langle\langle t, x\rangle\rangle_\varphi = 1 \text{ for all } t \in T\}.$$

By definition, we see that for $x \in L_\beta$, the condition that $x \in L_\omega$ is equivalent to the condition that any (equivalently, some) extension of $\theta$ to $\langle x, L_0\rangle$ is $T$-invariant.

**Lemma 3.12.** (i) *$L_\omega$ is a normal subgroup of $L_\beta$ such that $L_\beta/L_\omega$ is a $p'$-group.* (ii) *$L_\omega K = C_\omega K$.*

Proof. (i) Put $\alpha_x(t) = \langle\langle t, x\rangle\rangle_\theta$ for $(t, x) \in T \times L_\beta$. Since $\alpha_x(t) = 1$ for $t \in L_0$, $\alpha_x$ may be regarded as an element of $\mathrm{Hom}(T/L_0, \mathcal{K}^*)$. Then the map $\alpha$ sending $x$ to $\alpha_x$ is a group homomorphism from $L_\beta$ to $\mathrm{Hom}(T/L_0, \mathcal{K}^*)$. Since $\mathrm{Ker}\,\alpha = L_\omega$ and $T/L_0$ is a $p'$-group, the result follows.

(ii) We have $L_\beta = C_{\beta_0}L_0$. So $L_\omega = (L_\omega \cap C_{\beta_0})L_0$. It is easy to see $\langle\langle t, x\rangle\rangle_\varphi = \langle\langle t, x\rangle\rangle_\theta$ for $t \in T$ and $x \in C_{\beta_0}$. So $L_\omega \cap C_{\beta_0} = C_\omega$. Thus $L_\omega = C_\omega L_0$, and hence $L_\omega K = C_\omega K$. □

**Theorem 3.13.** *We have $G[b] = C_\omega K$.*

Proof. By Lemma 3.12 it suffices to show $G[b] = L_\omega K$. We fix a block $B$ of $G$ covering $b$. Let $\tilde{B}$ be the Harris–Knörr correspondent of $B$ over $b$ in $N_G(Q)$.

We first claim $G[b] \leq L_\beta K$. Let $x \in G[b]$. Put $G_x = \langle x, K\rangle$ and $L_x = L \cap G_x$. Then $L_x = QC_{G_x}(Q)$. Since the condition that $x \in G[b]$ is equivalent to the condition that $b$ is $\langle x\rangle$-invariant and $\langle x\rangle$ acts on $b$ as inner automorphisms, $x \in G[b]$ if and only if $x \in G_x[b]$. Thus it suffices to show $G_x[b] \leq (L_x)_\beta K$, where $(L_x)_\beta$ is the inertial group of $\beta$ in $L_x$. Thus we may assume $G = G_x = \langle x, K\rangle$. By Corollary 3.3, there is some $y \in xK$ such that $\omega_B^*(\hat{K}_y) \neq 0$. Since $\tilde{B}$ covers $\tilde{b}$, $\tilde{B}$ covers $\beta$. So there is a block $B'$ of $L$ such that $\tilde{B}$ covers $B'$ and $B'$ covers $\beta$. Let $\beta'$ be the Fong–Reynolds correspondent of $B'$ over $\beta$ in $L_\beta$. Since a defect group of $B'$ contains $Q$, we have $B'^H = \tilde{B}$. This implies $B = \beta'^G$. So $\omega_B^*(\hat{K}_y) = \omega_{\beta'}^*(\widehat{K_y \cap L_\beta})$. Thus there is $g \in G$ such that $y^g \in L_\beta \leq L_\beta K$. Then $y \in L_\beta K$, since $G/K$ is abelian. Thus $x \in L_\beta K$, and the claim is proved.

Then $G[b] = (L_\beta \cap G[b])K$. Therefore it suffices to prove $L_\beta \cap G[b] = L_\omega$. We shall show both sides contain the same $p$-elements and $p'$-elements. It suffices to show that under the assumption that $x$ is either a $p$-element or a $p'$-element, it holds that $x \in L_\beta \cap G[b]$ if and only if $x \in L_\omega$. Since $x \in L_\beta \cap G[b]$ if and only if $x \in (L_x)_\beta \cap G_x[b]$ and $x \in L_\omega$ if and only if $x \in (L_x)_\omega$ (here $(L_x)_\omega$ is defined in a manner similar to $L_\omega$), we may assume $G = G_x$.

Let $x$ be a $p$-element. If $x \in L_\beta \cap G[b]$, then $x \in L_\omega$, since $L_\beta/L_\omega$ is a $p'$-group by Lemma 3.12. Conversely let $x \in L_\omega$. Then $L = \langle x, L_0 \rangle$. So $L = L_\beta \le S$. Then $S = \langle x, T \rangle = LT$. Thus $S/L \simeq T/L_0$, and $S/L$ is a $p'$-group. Let $B_1$ be the Fong–Reynolds correspondent of $\tilde{B}$ over $\beta$ in $S$. Let $D$ be a defect group of $B_1$. Then $D \ge Q$. Since $S/L$ is a $p'$-group, $D \le L$. So $D = QC_D(Q)$. By the Fong–Reynolds theorem, $D$ is a defect group of $\tilde{B}$. So $D$ is a defect group of $B$. Since $\beta$ is $\langle x \rangle$-invariant, $b = \beta^K$ is $G$-invariant. Therefore, $G = G[b]$ by Proposition 3.4, and $x \in L_\beta \cap G[b]$. The proof is complete in this case.

Let $x$ be a $p'$-element. It suffices to show that under the assumption that $x \in L_\beta$, $x \in G[b]$ if and only if $x \in L_\omega$. Assume $x \in L_\beta$. Then $L = \langle x, L_0 \rangle = L_\beta$. We have

$$\begin{aligned} |\mathrm{BL}(G \mid b)| &= |\mathrm{BL}(N_G(Q) \mid \tilde{b})| \quad \text{(by the Harris–Knörr theorem)} \\ &= |\mathrm{BL}(N_G(Q) \mid \beta)| \quad (\text{since } \tilde{b} \text{ is a unique block of } N_K(Q) \text{ covering } \beta) \\ &= |\mathrm{BL}(S \mid \beta)| \quad \text{(by the Fong–Reynolds theorem).} \end{aligned}$$

Since $\beta$ is $S$-invariant, if $B_1 \in \mathrm{BL}(S \mid \beta)$ covers a block $B'$ of $L$, then $B' \in \mathrm{BL}(L \mid \beta)$. If $B' \in \mathrm{BL}(L \mid \beta)$ and a block $B_1$ of $S$ covers $B'$, then $B_1 \in \mathrm{BL}(S \mid \beta)$. Further in this case $B'$ is determined up to $S$-conjugacy by $B_1$ and $B_1 = B'^S$, since $L = QC_G(Q)$. Thus $|\mathrm{BL}(S \mid \beta)| = |\mathrm{BL}(L \mid \beta)/S|$, where $\mathrm{BL}(L \mid \beta)/S$ is a set of representatives of $S$-conjugacy classes of $\mathrm{BL}(L \mid \beta)$. Since $G = \langle x, K \rangle$, we have $S = \langle x, T \rangle$. So $|\mathrm{BL}(L \mid \beta)/S| = |\mathrm{BL}(L \mid \beta)/T| \le |\mathrm{BL}(L \mid \beta)|$.

Since $L/L_0$ is cyclic and $\theta$ is $L$-invariant, there is an extension of $\theta$ to $L$. Let $\mathcal{E}$ be the set of such extensions. We show there is a bijection of $\mathrm{BL}(L \mid \beta)$ onto $\mathcal{E}$. For any $B' \in \mathrm{BL}(L \mid \beta)$, $B'$ contains an irreducible character $\hat{\theta}$ lying over $\theta$. Then $\hat{\theta} \in \mathcal{E}$. Since $L/L_0$ is a $p'$-group, $B'$ has defect group $Q$. Therefore $\hat{\theta}$ is the canonical character of $B'$ and $\hat{\theta}$ is uniquely determined. Of course any $\hat{\theta} \in \mathcal{E}$ is contained in some $B' \in \mathrm{BL}(L \mid \beta)$. Therefore the map $B' \mapsto \hat{\theta}$ is the required bijection. So $|\mathrm{BL}(L \mid \beta)| = |\mathcal{E}| = |L/L_0|$.

Since $|L/L_0| = |G/K|$, we obtain $|\mathrm{BL}(G \mid b)| \le |G/K|$. By Proposition 3.9, $x \in G[b]$ if and only if the equality holds here. The last condition is equivalent to the condition that any extension of $\theta$ to $L$ is $T$-invariant. Thus it is equivalent to the condition that $x \in L_\omega$, since $L = \langle x, L_0 \rangle$. Thus $x \in G[b]$ if and only if $x \in L_\omega$. This completes the proof. □

**Corollary 3.14.** *Our* $C_\omega$ *in* Theorem 3.13 *is the same as* $C_\omega$ ($= C(D \text{ in } H)_\omega$ *in Dade's notation*) *appearing in* Corollary 12.6 *of* [3].

Proof. If we denote by $C'_\omega$ the group $C_\omega$ defined above, then Theorem 3.13 becomes $G[b] = C'_\omega K$. Then $C'_\omega = C \cap G[b]$. From Dade's theorem that $G[b] = C_\omega K$ [3, Corollary 12.6], we also obtain $C_\omega = C \cap G[b]$. Thus (our) $C_\omega = C'_\omega =$ (Dade's) $C_\omega$. □

**Corollary 3.15** (Külshammer [8, Proposition 9])**.** $G[b] = N_G(Q)[\tilde{b}]K$.

Proof. Use Theorem 3.13 to $G[b]$ and $N_G(Q)[\tilde{b}]$. □

## 4. Isomorphic blocks

The following theorem gives characterizations of isomorphic blocks with respect to normal subgroups. For isomorphic blocks, see [5, Section 4] and references therein.

**Theorem 4.1.** *Let $B$ be a block of $G$ covering $b$. The following are equivalent.*
(i) $G = G[b]$, $\mathrm{d}(B) = \mathrm{d}(b)$ *and for some irreducible character $\chi$ in $B$, $\chi_K$ is irreducible.*
(ii) *$G/K$ is a $p'$-group and for any $x \in G$, there is an element $y$ in $xK$ such that* $\omega_B^*(\hat{K}_y) \neq 0$.
(iii) *The restriction $\chi \mapsto \chi_K$ is a bijection of* $\mathrm{Irr}(B)$ *onto* $\mathrm{Irr}(b)$.
(iv) *The restriction $\chi \mapsto \chi_K$ is a bijection of* $\mathrm{Irr}_0(B)$ *onto* $\mathrm{Irr}_0(b)$.
(v) *For some character $\xi \in$* $\mathrm{Irr}(b)$, *we have* $\mathrm{Irr}(B \mid \xi) = \{\chi\}$ *with $\chi_K = \xi$.*
(vi) *For some character $\xi \in$* $\mathrm{Irr}(b)$, *we have* $\mathrm{Irr}_0(B \mid \xi) = \{\chi\}$ *with $\chi_K = \xi$.*

Proof. (i) $\Rightarrow$ (ii). Since $\chi = \chi \otimes 1_{G/K}$, we see $B_0(G/K)$ is $\chi$-dominated by $B$ (for $\chi$-domination see [10, p. 35]). So a defect group of $B_0(G/K)$ is contained in $QK/K = 1$ by [10, Corollary 1.5]. Thus $G/K$ is a $p'$-group.

Let $x \in G$ and put $H = \langle x, K\rangle$. Since $H = H[b]$, by Corollary 3.3, there is some $y \in xK$ such that $\omega_\chi^*(\hat{L}_y) \neq 0$, where $L_y$ is the conjugacy class of $H$ containing $y$. Now $C_G(y)$ normalizes $H$. So $C_G(y)H$ is a subgroup of $G$ containing $K$. Thus $|G : C_G(y)H|$ is a $p'$-integer. On the other hand, we have $\omega_\chi(\hat{K}_y) = \omega_\chi(\hat{L}_y)|G : C_G(y)H|$. Therefore $\omega_\chi^*(\hat{K}_y) \neq 0$.

(ii) $\Rightarrow$ (iii). Let $\chi \in \mathrm{Irr}(B)$. For any $x \in G$, there is an element $y \in xK$ such that $\chi(y) \neq 0$ by (ii). Then, by Lemma 3.1, $\chi_K$ is irreducible and $\chi_K \in \mathrm{Irr}(b)$. Of course, then the restriction is surjective. Let $\chi' \in \mathrm{Irr}(B)$ such that $\chi_K = \chi'_K$. Then $\chi' = \chi \otimes \theta$ for a linear character $\theta$ of $G/K$. For any $x \in G$, let $y \in xK$ be such that $\omega_\chi^*(\hat{K}_y) \neq 0$. We have

$$\omega_\chi^*(\hat{K}_y) = \omega_{\chi'}^*(\hat{K}_y) = \omega_\chi^*(\hat{K}_y)\theta(x)^*.$$

So $\theta(x)^* = 1$. Since $G/K$ is a $p'$-group, we see that $\theta$ is the trivial character. Thus $\chi' = \chi$.

(iii) $\Rightarrow$ (iv). Put $a = \nu(|G|)$ and $a' = \nu(|K|)$. We have $a - \mathrm{d}(B) + \mathrm{ht}(\chi) = a' - \mathrm{d}(b) + \mathrm{ht}(\chi_K)$ for all $\chi \in \mathrm{Irr}(B)$. If $\mathrm{ht}(\chi) = 0$, we obtain $a - \mathrm{d}(B) \geq a' - \mathrm{d}(b)$. If $\mathrm{ht}(\chi_K) = 0$, we obtain $a' - \mathrm{d}(b) \geq a - \mathrm{d}(B)$. Thus $a - \mathrm{d}(B) = a' - \mathrm{d}(b)$. Hence $\mathrm{ht}(\chi) = \mathrm{ht}(\chi_K)$ for all $\chi \in \mathrm{Irr}(B)$. Thus (iv) follows.

(iii) $\Rightarrow$ (v). This is trivial.

(iv) $\Rightarrow$ (vi). This is trivial.

(v) $\Rightarrow$ (vi). Let $a$ and $a'$ be as above. We have $a - \mathrm{d}(B) + \mathrm{ht}(\chi) = a' - \mathrm{d}(b) + \mathrm{ht}(\xi)$. Let $B_w$ be a weakly regular block of $G$ covering $b$. Since $b$ is $G$-invariant, we have $a - \mathrm{d}(B_w) = a' - \mathrm{d}(b)$. Thus $a - \mathrm{d}(B) \geq a - \mathrm{d}(B_w) = a' - \mathrm{d}(b)$. On the other hand, we have $\mathrm{ht}(\chi) \geq \mathrm{ht}(\xi)$ by [10, Lemma 2.2]. Thus equality holds throughout and $\mathrm{ht}(\chi) = \mathrm{ht}(\xi)$. So $\mathrm{Irr}_0(B \mid \xi) = \{\chi\}$.

(vi) $\Rightarrow$ (i). Let $\theta$ be an irreducible character of $p'$-degree in $B_0(G/K)$. Then $\chi \otimes \theta \in \mathrm{Irr}(B \mid \xi)$. We have $\mathrm{ht}(\chi \otimes \theta) = \mathrm{ht}(\chi) = \mathrm{ht}(\xi)$. Thus $\chi \otimes \theta = \chi$, and $\theta$ is the trivial character. So $B_0(G/K)$ has defect 0 by the Cliff–Plesken–Weiss theorem [2, Proposition 3.3] ([13, Problem 3.11]), and $G/K$ is a $p'$-group. So $\mathrm{d}(B) = \mathrm{d}(b)$. Put $\zeta = \chi_{G[b]}$. We claim $\mathrm{Irr}(G \mid \zeta) = \{\chi\}$. Let $\chi' \in \mathrm{Irr}(G \mid \zeta)$. Then $\nu(\chi'(1)) = \nu(\zeta(1)) = \nu(\chi(1))$. Since $\chi'$ lies in $B$ by Theorem 3.5, $\mathrm{ht}(\chi') = \mathrm{ht}(\chi)$. Therefore $\chi' = \chi$ by assumption, and the claim follows. Then, by Frobenius reciprocity, $\zeta^G = \chi$. Since $\zeta(1) = \chi(1)$, we obtain $G = G[b]$.

The proof is complete. □

REMARK 4.2. The equivalence of (i) and (iii) in Theorem 4.1 follows from [5, Proposition 2.6, Theorem 3.5, and Theorem 4.1].

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